Solutions to the Tutorial Problems in the book "Magnetohydrodynamics of the Sun" by ER Priest (2014)
CHAPTER 4

PROBLEM 4.1. Sound Waves in a Flowing Medium.
Show that sound waves in the presence of a uniform flow $\mathbf{v}_0$ obey a wave equation of the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\right)^2 \mathbf{v}_1 = \ c_s^2 \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}_1) \tag{1}$$

and deduce their dispersion relation.

SOLUTION.
In the absence of a magnetic field, gravity or an angular rotation, the basic MHD equations (4.1)–4.3) reduce to

$$\frac{d\rho}{dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{v} = 0, \tag{2}$$

$$\rho \ \frac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla} p, \tag{3}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0, \tag{4}$$

By writing $\rho = \rho_0 + \rho_1$, $p = p_0 + p_1$, $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1$ and linearising, where $\rho_0,\ p_0, \mathbf{v}_0$ are uniform, these reduce to

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\right) \rho_1 + \rho_0(\nabla \cdot \mathbf{v}_1) = 0, \tag{5}$$

$$\rho_0 \left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\right) \mathbf{v}_1 = -\nabla p_1, \tag{6}$$

$$\left(\frac{\partial}{\partial t} + (\mathbf{v}_0 \cdot \nabla\right) p_1 = c_s^2 \left(\frac{\partial}{\partial t} + (\mathbf{v}_0 \cdot \nabla\right) \rho_1, \tag{7}$$

Eliminating p_1 and ρ_1 gives

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \boldsymbol{\nabla}\right)^2 \mathbf{v}_1 = \ c_s^2 \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{v}_1), \tag{8}$$

as required, where $c_s^2 = \gamma p_0/\rho_0$.

Thus, plane wave solutions of the form

$$\mathbf{v}_1(\mathbf{r}, t) = \mathbf{v}_1 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

then yield

$$\omega'^2 = k^2 c_s^2,$$

where $\omega' \equiv \omega - \mathbf{k}\cdot\mathbf{v}_0$, which is just a Doppler-shifted frequency.

PROBLEM 4.2 Finite-Amplitude Alfvén Waves.

Show that an Alfvén wave of arbitrary amplitude propagating along a uniform background magnetic field ($B_0\hat{\mathbf{z}}$) with $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$, $p + B^2/(2\mu) = \textit{constant}$, $\mathbf{v} = \pm\mathbf{b}/\sqrt{(\mu\rho)}$ and $\mathbf{b} = B_0\mathbf{f}(x, y, z \pm V_A t)$ satisfies the ideal MHD equations, where $\mathbf{f}$ is an arbitrary vector with $\boldsymbol{\nabla}\cdot\mathbf{f} = 0$.

SOLUTION.

The ideal incompressible MHD equations are

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}P + \frac{1}{\mu}(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B},$$

$$\frac{\partial\mathbf{B}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B} - (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{v} = \mathbf{0},$$

where $P = p + B^2/(2\mu)$ and $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$.

Now write $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, where $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ is uniform and suppose the total pressure remains constant ($P = \textit{constant}$). Then the equation of motion and the induction equation reduce to

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \frac{1}{\mu\rho}(\mathbf{B}_0\cdot\boldsymbol{\nabla})\mathbf{b} + \frac{1}{\mu\rho}(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b},$$

and

$$\frac{\partial\mathbf{b}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{b} = (\mathbf{B}_0\cdot\boldsymbol{\nabla})\mathbf{v} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{v}.$$

Then, if $\mathbf{v} = \pm\mathbf{b}/\sqrt{(\mu\rho)}$, both of these equations reduce to

$$\frac{\partial\mathbf{b}}{\partial t} = \pm V_A\frac{\partial\mathbf{b}}{\partial z},$$

where $V_A = B_0/\sqrt{(\mu\rho)}$, which has solutions

$$\mathbf{b} = B_0\mathbf{f}(x, y, z \pm V_A t),$$

where $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ implies $\boldsymbol{\nabla} \cdot \mathbf{f} = 0$.

The corresponding plasma velocity and electric field are from $\mathbf{v} = \pm\mathbf{b}/\sqrt{(\mu\rho)}$ and $\mathbf{E} = \mathbf{v} \times \mathbf{B}$

$$\mathbf{v} = \pm V_A\mathbf{f}(x, y, z \pm V_A t),$$

and

$$\mathbf{E} = \pm V_A\mathbf{B}_0 \times \mathbf{f}(x, y, z \pm V_A t).$$

PROBLEM 4.3. Energy Flux in Alfvén Waves.
Find the magnetic and kinetic energy densities and the Poynting flux in a finite-amplitude Alfvén wave propagating in the positive z-direction.

SOLUTION.
We follow Roberts (1967). Consider an Alfvén wave of the form

$$\mathbf{b} = B_0\mathbf{f}(x, y, z - V_A t),\ \ \mathbf{v} = -V_A\mathbf{f}(x, y, z - V_A t),\ \ \mathbf{E} = -V_A\mathbf{B}_0 \times \mathbf{f}(x, y, z - V_A t)$$

propagating in the positive z-direction.

The total magnetic energy density is

$$\frac{(\mathbf{B}_0 + \mathbf{b})^2}{2\mu} = \frac{B_0^2}{2\mu} + \frac{b^2}{2\mu},$$

and so the magnetic energy density associated with the wave is

$$\frac{b^2}{2\mu} = \frac{B_0^2}{2\mu}f^2 = \tfrac{1}{2}\rho V_a^2 f^2 = \tfrac{1}{2}\rho v^2,$$

which is just the kinetic energy density. Thus, we have equipartition between magnetic and kinetic energy in the wave.

The total energy density is the sum of magnetic and kinetic energy density, namely,

$$\mathcal{E}_{wave} = \frac{b^2}{2\mu} + \tfrac{1}{2}\rho v^2.$$

The corresponding Poynting vector is

$$\frac{\mathbf{E} \times \mathbf{B}}{\mu} = \frac{\mathbf{E} \times (\mathbf{B}_0 + \mathbf{b})}{\mu}$$

or, after substituting for $\mathbf{E}$ and rearranging,

$$\frac{\mathbf{E}\times\mathbf{B}}{\mu} = \rho V_A^2 \mathbf{v} + V_A \mathcal{E}_{wave}\hat{\mathbf{z}}.$$

The first term is a radiation term that vanishes when integrated over a closed surface or when integrated over time at a fixed position. The second term represents a flux of energy in the z-direction.

PROBLEM 4.4. Diffusion of Linear Alfvén Waves.
By adding a term of the form $\eta\nabla^2\mathbf{B}_1$ to the right-hand side of

$$\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0), \tag{9}$$

determine the effect of magnetic diffusion on shear Alfvén waves in a uniform medium. If the magnetic Reynolds number is defined as $R_m = v_A/(k\eta)$, find the real and imaginary parts of ω and deduce that, when $R_m \gg 1$, the effect of diffusion is to produce a slow decay of the wave and a small reduction in its frequency of oscillation.

SOLUTION.
Assume $\nabla.\mathbf{v}' = 0$. Then the linearised continuity equation implies $\rho' = 0$ as usual for Alfven waves, while the equations of induction (with diffusion) and motion become

$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times (\mathbf{v}' \times \mathbf{B}_0) + \eta\nabla^2\mathbf{B}', \quad \mu\rho_0\frac{\partial \mathbf{v}'}{\partial t} = (\nabla\times\mathbf{B}')\times\mathbf{B}_0 \ ,$$

where $\nabla.\mathbf{B}' = 0$. After making the usual wave assumption, they reduce to

$$-i\omega\mathbf{B}' = i(\mathbf{B}_0.\mathbf{k})\mathbf{v}' - k^2\zeta\mathbf{B}', \quad -\mu\rho_0\omega\mathbf{v}' = \mathbf{B}'(\mathbf{B}_0.\mathbf{k}) - \mathbf{k}(\mathbf{B}'.\mathbf{B}_0) \ ,$$

where, as usual, $\mathbf{B}'.\mathbf{B}_0 = 0$.

These two equations may be combined to yield the dispersion relation

$$\omega^2 = k^2 {v_{\mathrm{A}}}^2 - i\omega\frac{k v_{\mathrm{A}}}{R_{\mathrm{m}}} \ , \tag{10}$$

where

$$R_{\mathrm{m}} = \frac{v_{\mathrm{A}}}{k\eta} \ .$$

By splitting ω into a real ($\omega_{\rm r}$) and imaginary ($\omega_{\rm i}$) part, Eq.(**??**) gives

$$\omega_{\rm i} = -k\frac{v_{\rm A}}{2R_{\rm m}}, \quad \omega_{\rm r}^2 = k^2 {v_{\rm A}}^2 \left(1 - \frac{1}{4{R_{\rm m}}^2}\right) \ . \tag{11}$$

Thus if $R_{\rm m} \gg 1$, the effect of diffusion is to produce a slow decay of the wave and a small reduction in the frequency $\omega_{\rm r}$.

PROBLEM 4.5. Nonlinear Alfvén Waves with Diffusion.
Consider a finite-amplitude wave obeying the visco-resistive MHD equations and having a constant total pressure and density and a magnetic field $\mathbf{B}_0+\mathbf{b}$, where $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ is uniform. Show that, if $\mathbf{v} = -\mathbf{b}/\sqrt{(}\mu\rho)$ and $\eta = \nu$, then $\mathbf{b}$ satisfies a linear equation and solve it if $\mathbf{b} = b(z,t)\hat{\mathbf{x}}$.

SOLUTION.
With a constant total pressure, a magnetic field of the form $\mathbf{B}_0 + \mathbf{b}$ with $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ uniform, and an incompressible flow ($\boldsymbol{\nabla}\cdot\mathbf{v} = 0$), the equations of induction and motion are

$$\frac{\partial \mathbf{b}}{\partial t} = B_0\frac{\partial \mathbf{v}}{\partial z} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{b} + \eta\nabla^2\mathbf{b}$$

and

$$\mu\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}\right) = B_0\frac{\partial \mathbf{b}}{\partial z} + (\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b} + \mu\rho\nu\nabla^2\mathbf{v}.$$

Then, if we put $\mathbf{v} = -\mathbf{b}/\sqrt{(}\mu\rho)$ and $\eta = \nu$, both of these equations reduce to

$$\frac{\partial \mathbf{b}}{\partial t} = -v_A\frac{\partial \mathbf{b}}{\partial z} + \eta\nabla^2\mathbf{b},$$

where $v_A = B_0/\sqrt{(}\mu\rho)$, or, if $\mathbf{b} = b(z,t)\hat{\mathbf{x}}$,

$$\frac{\partial b}{\partial t} = -v_A\frac{\partial b}{\partial z} + \eta\frac{\partial^2 b}{\partial z^2}.$$

Since this is a linear equation with constant coefficients, we may Fourier analyse and consider each Fourier component of the form

$$b = \exp[i(\omega t - kz)],$$

for which the above equation gives the dispersion relation as

$$\omega = kv_A + i\eta k^2,$$

so that

$$b = e^{ik(v_A t - z)} e^{-\eta k^2 t}.$$

Thus, we have a nonlinear Alfvén wave of wavenumber k propagating along the z-direction and decaying over a time-scale $1/(\eta k^2)$.

PROBLEM 4.6. Internal Gravity Waves.
Use

$$\omega^2 \mathbf{v}_1 \;=\; c_s^2 \mathbf{k}(\mathbf{k}\cdot\mathbf{v}_1) \;+\; i(\gamma-1)g\hat{\mathbf{z}}(\mathbf{k}\cdot\mathbf{v}_1) \;+\; ig\mathbf{k}v_{1z} \;-\; 2i\omega\boldsymbol{\Omega}\times\mathbf{v}_1. \quad (12)$$

to establish the dispersion relation $\omega = N \sin\theta_g$ for internal gravity waves when $g/c_s \ll kc_s$. Deduce the physical behaviour of the waves and the fact that an upward propagating wave carries energy downwards.

SOLUTION.
For a plane wave of the form

$$\mathbf{v}_1(\mathbf{r},t) = \mathbf{v}_1 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

the linearised MHD equations with vanishing magnetic field and no rotation yield Eq.(**??**), namely,

$$\omega^2 \mathbf{v}_1 \;=\; c_s^2 \mathbf{k}(\mathbf{k}\cdot\mathbf{v}_1) \;+\; i(\gamma-1)g\hat{\mathbf{z}}(\mathbf{k}\cdot\mathbf{v}_1) \;+\; ig\mathbf{k}v_{1z}.$$

The intuitive discussion of Sec 4.4 led us to expect the existence of gravity waves when $N^2 > 0$ due to the tendency for plasma to oscillate slowly with frequency N. Their dispersion relation can be determined by taking the scalar product with $\mathbf{k}$ and $\hat{\mathbf{z}}$ in turn of the above equation and gathering together terms in v_{1z} and $\mathbf{k}\cdot\mathbf{v}_1$, to give

$$igk^2\ v_{1z} = (\omega^2 - c_s^2k^2 - i(\gamma-1)gk_z)\ (\mathbf{k}\cdot\mathbf{v}_1),$$

$$(\omega^2 - igk_z)\ v_{1z} = (c_s^2 k_z + i(\gamma-1)g)\ (\mathbf{k}\cdot\mathbf{v}_1).$$

Then an elimination of $(\mathbf{k}\cdot\mathbf{v}_1)/v_{1z}$ between these two yields

$$(\omega^2 - igk_z)\ (\omega^2 - c_s^2k^2 - i(\gamma-1)gk_z) = igk^2\ (c_s^2k_z + 1(\gamma-1)g).$$

The object is to seek waves with a frequency of the order of the Brunt-Vaisala frequency (N) and much slower than that of sound waves, so that

$$\omega \approx \frac{g}{c_s} \ll kc_s;$$

this implies that the wavelength is much smaller than a scale-height and the above full dispersion relation reduces to

$$\omega^2 c_s^2 \approx (\gamma - 1) g^2 (1 - k_z^2/k^2).$$

In terms of N and the inclination ($\theta_g = \cos^{-1}(k_z/k)$) between the direction of propagation and the z-axis, this may be rewritten

$$\omega = N \ \sin \ \theta_g$$

for (*internal*) *gravity waves*. The word 'internal' is sometimes added to distinguish them from *surface gravity waves* propagating along an interface between two fluids.

Several properties of this mode are of note. A typical value for N^{-1} is 50 s, so the gravity mode tends to be rather slow by comparison with the other waves (except for the inertial wave). Gravity waves do not propagate in the vertical direction ($\theta_g = 0$), since that would not allow a horizontal interaction with elements at the same height. Furthermore, the dispersion relation implies that $\omega \leqslant N$, so that the waves cannot propagate faster than the Brunt-Väisälä frequency.

For a given ω and N, it also means that they propagate along two cones centred about the z-axis with $\theta_g = \sin^{-1}(\omega/N)$. For the upward-propagating wave with

$$\omega = N(1 - k_z^2/k^2)^{1/2},$$

the z-component of the group velocity is

$$v_{gz} = \frac{\partial \omega}{\partial k_z} = -\frac{\omega \ k_z}{k^2},$$

which is negative. Thus gravity waves have the unusual characteristic that a group of *upward* propagating waves carries energy *downward* and vice versa! In fact, the group velocity is in a direction normal to the surface of the cone with angle θ_g.

PROBLEM 4.7. Entropy Waves.

Show that, in the absence of gravity, the linearised MHD equations possess a solution with $\omega = 0$, $\mathbf{v}_1 = p_1 = \mathbf{B}_1 = 0$ and an arbitrary ρ_1 and entropy.

SOLUTION.
The linearised MHD equations about a uniform state are

$$\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + \rho_0(\nabla \cdot \mathbf{v}_1) &&= 0,\\
&\rho_0\frac{\partial \mathbf{v}_1}{\partial t} + \nabla p_1 - (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0/\mu &&= 0,\\
&\frac{\partial p_1}{\partial t} - c_s^2\frac{\partial \rho_1}{\partial t} &&= 0,\\
&\frac{\partial \mathbf{B}_1}{\partial t} - \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) &&= 0,\\
&\nabla \cdot \mathbf{B}_1 &&= 0.
\end{aligned}$$

Plane-wave solutions of the form

$$\mathbf{v}_1(\mathbf{r}, t) = \mathbf{v}_1 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

reduce these equations to

$$\begin{aligned}
&-\omega\rho_1 + \rho_0(\mathbf{k} \cdot \mathbf{v}_1) &&= 0, && (13)\\
&-\rho_0\omega\mathbf{v}_1 + \mathbf{k}p_1 - (\mathbf{k} \times \mathbf{B}_1) \times \mathbf{B}_0/\mu &&= 0, && (14)\\
&\omega p_1 - c_s^2\omega\rho_1 &&= 0, && (15)\\
&-\omega\mathbf{B}_1 - \mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0) &&= 0, && (16)\\
&\mathbf{k} \cdot \mathbf{B}_1 &&= 0. && (17)
\end{aligned}$$

Then, if $\omega = 0$, Eq.(**??**) implies $\mathbf{v}_1 = 0$, Eq.(**??**) implies $\mathbf{B}_1 = 0$, Eq.(**??**) implies $p_1 = 0$, and Eq.(**??**) implies ρ_1 is arbitrary, which implies that the perturbation $s_1 = s_0(p_1 - \gamma\rho_1)$ in entropy is arbitrary, as required for an entropy wave.