Solutions to the Tutorial Problems in the book "Magnetohydrodynamics of the Sun" by ER Priest
CHAPTER 10

PROBLEM 10.1. Connection Formulae for Resonant Absorption.

Consider a linear perturbation of the form

$$f(r)\exp[i(m\phi + kz - \omega t)]$$

to a twisted cylindrical flux tube in equilibrium having field components $[B_{0\phi}(r), B_{0z}(r)]$. Such perturbations satisfy the *Hain-Lüst equation* (4.60).

(a) Show that

$$g_B P_1 - 2 f_B B_{0\phi} B_{0z} \xi_{1r}/(\mu r) \tag{1}$$

is conserved and equal to a constant (C_A) at the Alfvén resonance point $[\omega_A(r) = \omega]$, where $g_B = (m/r)B_{0z} - kB_{0\phi}$, $f_B = (m/r)B_{0\phi} + kB_{0z}$, $P_1 = p_1 + \mathbf{B}_0 \cdot \mathbf{B}_1/\mu$ is the total pressure perturbation, and ξ_{1r} is the radial displacement perturbation.

(b) Show that the thickness of the dissipation layer is $[\omega(\nu + \omega)/|\Delta|]^{1/3}$, where $\Delta = -2\omega_A d\omega_A/dr$.

(c) Show that the jumps in ξ_{1r} and P_1' across the resonant dissipation layer are

$$[\xi_{1r}] = -i\pi \frac{\text{sign } \omega}{|\Delta|} \frac{g_B}{\rho B_0^2} C_A,$$

$$[P_1'] = -i\pi \frac{\text{sign } \omega}{|\Delta|} \frac{2B_{0\phi}B_{0z}f_B}{\rho B_0^2 \mu r} C_A,$$

and are therefore independent of the dissipation coefficients (ν, η).

SOLUTION.

We follow Sakurai et al (1981) Solar Phys. 133, 227.

(i) In the absence of dissipation, perturbations of the above form to a twisted flux rope satisfy two equations of the form

$$D\frac{d}{dr}(r\xi_{1r}) = C_1 r\xi_{1r} - C_2 r P_1' \tag{2}$$

and

$$D\frac{dP_1'}{dr} = C_3\xi_{1r} - C_1P_1', \tag{3}$$

where

$$D(r) = \rho(c_s^2 + v_A^2)(\omega^2 - \omega_A^2)(\omega^2 - \omega_c^2),$$

in terms of the Alfvén frequency (ω_A) and the cusp frequency (ω_c).

These may be combined to give the *Hain-Lüst equation* in the form

$$\frac{d}{dr}\left[f(r)\frac{d}{dr}(r\xi_{1r})\right] - g(r)r\xi_{1r} \tag{4}$$

for the radial displacement and

$$\frac{d}{dr}\left[F(r)\frac{dP_1}{dr}\right] - G(r)rP_1 \tag{5}$$

for the total pressure perturbation.

Expand about the Alfvén resonance r_A [for which $\omega(r_A) = \omega$], by putting $s = r - r_A$. Then Eqs.(4), (5) reduce to

$$a\frac{d}{ds}\left(s\frac{d\xi_{1r}}{ds}\right) + b\xi_{1r}, \tag{6}$$

$$c\frac{d}{ds}\left(s\frac{dP_1'}{ds}\right) + dPi_1', \tag{7}$$

where $a,\ b,\ c,\ d$ are constants.

Their solution has the form

$$\xi_{1r}(s) = u_1(s)\ln|s| + \begin{cases} \xi_{1A}^-(s), & s < 0 \\ \xi_{1A}^+(s), & s > 0, \end{cases} \tag{8}$$

$$P_1'(s) = u_2(s)\ln|s| + \begin{cases} P_A^-(s), & s < 0 \\ P_A^+(s), & s > 0. \end{cases} \tag{9}$$

Now, close to the resonance point Eqs.(2, (3) become

$$s\Delta\frac{d\xi_{1r}}{ds} = \frac{g_B}{\rho B_0^2}\left(g_BP_1' - 2f_BB_{0\phi}B_{0z}\frac{\xi_{1r}}{\mu r}\right) \tag{10}$$

and

$$s\Delta\frac{dP_1'}{ds}=\frac{2f_B B_{0\phi}B_{0z}}{\mu r\rho B_0^2}\left(g_B P_1' - 2f_B B_{0\phi}B_{0z}\frac{\xi_{1r}}{\mu r}\right), \tag{11}$$

which imply

$$s\frac{d}{ds}\left(g_B P_1' - 2f_B B_{0\phi}B_{0z}\frac{\xi_{1r}}{\mu r}\right)=0.$$

This implies that Eq.(1) holds, namely, that

$$g_B P_1 - 2f_B B_{0\phi}B_{0z}\xi_{1r}/(\mu r) = C_A, \tag{12}$$

say.

Eqs.(10) and (11) may then be integrated to determine the functions $u_1(s)$ and $u_2(s)$ in Eqs.(8) and (9) as $u_1(s) = g_B C_A/(\rho\Delta B_0^2)$ and $u_2(s) = 2f_B B_{0\phi}B_{0z}C_A/(\mu\rho\Delta B_0^2)r_A$.

(ii) The jumps in ξ_{1r} and P_1' are due to dissipative effects and are determined by including those effects in the basic equations, for which Eqs.(2) and (3) still hold, but with D replaced by

$$D(r)=\rho(c_s^2+v_A^2)\left(\omega^2-\omega_A^2-i\omega(\nu+\eta)\frac{d^2}{dr^2}\right)\left(\omega^2-\omega_c^2-i\omega\left(\nu+\frac{\omega_c^2}{\omega_A^2}\eta\right)\frac{d^2}{dr^2}\right).$$

Then the inclusion of diffusion modifies Eqs.(10) and (11) to

$$\left(s\Delta - i\omega(\nu+\eta)\frac{d^2}{ds^2}\right)\frac{d\xi_{1r}}{ds}=\frac{g_B}{\rho B_0^2}\left(g_B P_1' - 2f_B B_{0\phi}B_{0z}\frac{\xi_{1r}}{\mu r}\right) \tag{13}$$

and

$$\left(s\Delta - i\omega(\nu+\eta)\frac{d^2}{ds^2}\right)\frac{dP_1'}{ds}=\frac{2f_B B_{0\phi}B_{0z}}{\mu r\rho B_0^2}\left(g_B P_1' - 2f_B B_{0\phi}B_{0z}\frac{\xi_{1r}}{\mu r}\right). \tag{14}$$

The result is that the conservation law Eq.(12) still holds and so the two above equations have the form

$$\left(s\Delta - i(\omega+\nu)\frac{d^2}{ds^2}\right)\frac{dY}{ds}=R. \tag{15}$$

Dissipation is important when both terms on the left are comparable, and so equating them in order or magnitude gives the thickness of the dissipation layer as $[\omega(\nu+\omega)/|\Delta|]^{1/3}$, as required.

(iii) Integrating Eq.(15) across the dissipation layer can be shown to give the jump in Y as

$$[Y] = -\frac{iR}{\Delta}\pi \text{sign}\left(\frac{\omega}{\Delta}\right),$$

or, in terms of ξ_{1r} and P_1',

$$[\xi_{1r}] = -i\pi\frac{\text{sign }\omega}{|\Delta|}\frac{g_B}{\rho B_0^2}C_A$$

$$[P_1'] = -i\pi\frac{\text{sign }\omega}{|\Delta|}\frac{2B_{0\phi}B_{0z}f_B}{\rho B_0^2\mu r}C_A,$$

as required.

PROBLEM 10.2. Phase Mixing of Waves in Space.
Derive the solution of

$$\frac{\partial^2 v_{1y}}{\partial t^2} = v_A^2(x)\frac{\partial^2 v_{1y}}{\partial z^2} + (\nu+\eta)\frac{\partial^2}{\partial x^2}\frac{\partial v_{1y}}{\partial t} \qquad (16)$$

of the form of

$$v_{1y}(x,z,t) = V(x,z)e^{i[k_z(x)z-\omega_d t]}, \qquad (17)$$

in the case of weak damping and strong phase mixing, namely,

$$\frac{1}{k_z} \ll 1 \quad \text{and} \quad \frac{z}{k_z}\frac{dk_z}{dx} \gg 1,$$

SOLUTION.
Consider

$$\frac{\partial^2 v_{1y}}{\partial t^2} = v_A^2(x)\frac{\partial^2 v_{1y}}{\partial z^2} + (\nu+\eta)\frac{\partial^2}{\partial x^2}\frac{\partial v_{1y}}{\partial t}. \qquad (18)$$

Seek solution in the form

$$v_{1y}(x,z,t) = V(x,z)e^{i[k_z(x)z-\omega_d t]}. \qquad (19)$$

which is strongly phase-mixed in the sense of having variations that are much stronger across the field than along it. This is similar to Eq.(10.13), except that the amplitude $[V(x,z)]$ varies weakly (in the sense that $z dk_z/dx > \partial/\partial x$, $k_z > \partial/\partial z$ and $k_z z > 1$).

Then the dominant terms in Eq.(18) with $v_A = \omega_d/k_z$ are

$$0 = \left(\frac{\omega_d}{k_z}\right)^2 2ik_z \, \frac{\partial V}{\partial z} + (\nu+\eta)(i\omega_d)\left(-\frac{dk_z}{dx}\right)^2 z^2 \, V,$$

which is of the form

$$\frac{\partial V}{\partial z} + f(x)z^2 \, V,$$

with

$$f = \frac{\nu+\eta}{2}\left(\frac{dk_z}{dx}\right)^2 \frac{k_z}{\omega_d}.$$

But $k_z(x) = \omega_d/v_A(x)$ and so

$$f = \frac{\nu+\eta}{2}\left(\frac{dv_A}{dx}\right)^2 \frac{k_z^2}{v_A^3}.$$

The solution for V is

$$V = V_0 \exp(-fz^3/3)$$

or

$$V(x,z) = V(x,0)\exp\left\{-\frac{(\nu+\eta)k_z^2}{6v_A^3}\left(\frac{dv_A}{dx}\right)^2 z^3\right\},$$

which decays with height like z^3.

PROBLEM 10.3. Phase Mixing Solution by Multiple Scales

Seek solutions of Eq.(18) in the form

$$v_{1y} = v_{0y}(x,z,t_0,t_1) + \varepsilon v_{1y}(x,z,t_0,t_1) + \ldots.., \tag{20}$$

where $t_0 = t$, $t_1 = \varepsilon t$ and $\varepsilon \ll 1$.

SOLUTION.

We follow Hood et al (2002) Proc. Roy. Soc **458**, 2307–225 for this solution.

Substituting Eq.(20) into Eq.(18) gives to zeroth order

$$\frac{\partial^2 v_{0y}}{\partial t_0^2} - V_A^2(x)\frac{\partial^2 v_{0y}}{\partial z^2} = 0, \tag{21}$$

with solution

$$v_{y0} = F([z - V_A(x)t_0], x, t_1), \tag{22}$$

where the function F is determined by initial conditions and a solvability condition that comes from the order ε equations.

At first order in ε Eq.(18) becomes

$$\frac{\partial^2 v_{1y}}{\partial t_0^2} - V_A^2(x)\frac{\partial^2 v_{1y}}{\partial z^2} = -2\frac{\partial^2 v_{0y}}{\partial t_0 \partial t_1} + \frac{\eta}{\varepsilon}\frac{\partial^3 v_{0y}}{\partial t_0 \partial x^2}. \tag{23}$$

In order to eliminate secular terms, we equate the right-hand side of this equation to zero and integrate over t_0 to give

$$2\frac{\partial v_{0y}}{\partial t_1} - \frac{\eta}{\varepsilon}\frac{\partial^2 v_{0y}}{\partial x^2} = 0. \tag{24}$$

The solution is in the form

$$v_{y0} = f(t_1)F(\xi) \qquad \text{and} \qquad \xi = g(t_1)[z - V_A(x)t_0]. \tag{25}$$

Substituting into Eq.(24) gives an equation to determine the functions f and g, namely, in terms of the variables ξ and t_1 to leading order

$$2f'F + 2\left(\frac{fg'}{g}\right)\xi F' = \frac{\eta}{\varepsilon^3}(fg^2(V_A'(x))^2 t_1^2 F''. \tag{26}$$

In order to make both sides the same order, we choose $\varepsilon = \eta^{1/3}$ and we choose functions f and g to make the coefficients in this differential equation for $F(\xi)$ independent of t_1, namely,

$$2g' = -a(V_A'(x))^2 t_1^2 g^3 \ \ (a) \qquad \text{and} \qquad 2f' = -b(V_A'(x))^2 t_1^2 f g^2, \ \ (b) \tag{27}$$

where a and b are constants.

Eq.(26) becomes

$$F'' + a\xi F' + bF = 0. \tag{28}$$

The solutions of Eq.(27) are

$$g(t_1) = (\sigma^2 + \frac{1}{3}aV_A' t_1^3)^{-1/2} \ \ (a) \qquad \text{and} \qquad f(t_1) = [g(t_1)]^{b/a}, \ \ (b)$$

where $\sigma = 1/g(0)$ is a constant.

Thus, the solution for v_{y0} becomes finally

$$v_{y0} = \frac{1}{[\sigma^2 + (1/3)aV_A' t_1^3]^{b/2a}} F\left[\frac{z - V_A t}{(\sigma^2 + (1/3)aV_A' t_1^3)^{1/2}}\right], \tag{29}$$

where the function F is found by solving Eq.(28), depending on the value of the constant a.

In his paper, Hood shows that the case $a = 0$ reduces to the usual Heyvaerts-Priest phase mixing case, whereas otherwise the solutions can be found in terms of an infinite series of Hermite polynomials. By picking various initial conditions he is able to treat single pulses and multiple pulses.

PROBLEM 10.4. Phase Mixing of Waves in Time.
Find the time-scale for phase mixing in time of Alfvén waves.

SOLUTION.
On a short coronal loop, standing waves have their wavenumber (k_z) fixed by the geometry, and so each magnetic surface with $x =$ constant oscillates independently of its neighbour with a frequency $\omega(x) = k_z v_A(x) \equiv \omega_A(x)$ and a velocity of

$$v_{1y}(x, z, t) = V(x)e^{i[k_z z - \omega_A(x)t]}.$$

Thus, we have *phase mixing in time* as the field lines become more and more out of phase. At the same time the x-gradients grow, since the variations in phase $\omega_A(x)t$ make a contribution

$$\frac{\partial v_{1y}}{\partial x} = i\frac{d\omega_A}{dx} t\ v_{1y}.$$

Thus, the phase mixing produces an effective wavenumber $k_{x,eff} = (d\omega_A/dx)t$, which grows in time, so that the effective wavelength becomes smaller and smaller until again dissipation converts the wave energy into heat.

The inclusion of dissipation may be modelled by seeking a strongly phase-mixed solution of

$$\frac{\partial^2 v_{1y}}{\partial t^2} = v_A^2(x)\frac{\partial^2 v_{1y}}{\partial z^2} + (\nu + \eta)\frac{\partial^2}{\partial x^2}\frac{\partial v_{1y}}{\partial t}.$$

of the form $v_{1y}(x, z, t) = V(x, t)e^{i[k_z z - \omega_A(x)t]}$, having an amplitude that is weakly varying (with $\omega_A > \partial/\partial t$ and $\omega_A > t^{-1}$). The dominant terms in this equation are

$$\frac{\partial V}{\partial t}(-2i\omega_A) = i(\nu+\eta)\omega_A \left(\frac{d\omega_A}{dx}\right)^2 t^2\, V,$$

with a solution having an amplitude that decays in time like

$$V(x,t) = V(x,0)\exp\left\{-\frac{(\nu+\eta)}{6}\left(\frac{d\omega_A}{dx}\right)^2 t^3\right\}.$$

The time-scale (typically a few periods) for dissipation by phase mixing is therefore

$$\tau_{phase} = \frac{1}{\omega_A}\left(12\pi Re^* \frac{a}{\lambda}\right)^{1/3},$$

where ω_A^{-1} is the Alfvén time. At this *phase-mixing time*, the x-scale again becomes a fraction $(k_{x,eff}a)^{-1} = (12\pi Re^* a/\lambda)^{-1/3}$ of the original scale (a).

PROBLEM 10.5. Interaction Distance of Photospheric Magnetic Fragments.

Consider a 2D potential field model of two fragments of flux $\pm F_0$ at points $(x,y) = (a,0)$ on the x-axis in an ambient uniform magnetic field $B_0\hat{\boldsymbol{x}}$. Show that:

(a) when $a > d \equiv 2F_0/(\pi B_0)$, there are two null points on the x-axis and none on the y-axis;

(b) when $a = d$, a second-order null point appears at the origin with separatrices inclined at $\pi/3$;

(c) when $a < d$, a first-order null point is present on the y-axis, and its maximum altitude is $\frac{1}{2}d$;

(d) for a 3D field the interaction distance is $d^* = \sqrt{F_0/(\pi B_0)}$.

SOLUTION.

We follow Priest et al (1994) Ap. J. 427, 459.

(a) The 2D potential magnetic field due to two fragments of flux $\pm F_0$ at points $z = a$ in an ambient uniform magnetic field $B_0\hat{\boldsymbol{x}}$, where $z = x + iy$ is given by

$$B_y + iB_x = \frac{iF_0/\pi}{z-a} - \frac{iF_0/\pi}{z+a} + iB_0, \tag{30}$$

The *interaction distance* is defined to be

$$d = \frac{2F_0}{\pi B_0}.$$

Then, in terms of this, when $a > d$, Eq.(30) becomes

$$B_y + iB_x = iB_0 \frac{z^2 - b^2}{z^2 - a^2}, \tag{31}$$

where

$$b^2 = a(a - d).$$

Thus, the magnetic field vanishes at $z = \pm a$, namely, two points on the x-axis and none on the y-axis, as required.

(b) When $a = d$, the magnetic field becomes

$$B_y + iB_x = iB_0 \frac{z^2}{z^2 - d^2},$$

or, near the origin,

$$B_y + iB_x \approx -iB_0 \frac{z^2}{d^2},$$

so that

$$B_x = \frac{B_0}{d^2}(-x^2 + y^2), \qquad B_y = \frac{2B_0}{d^2} xy.$$

In other words, the origin is a second-order null point with flux function

$$A = \frac{B_0}{d^2}\left(x^2 y - \frac{y^3}{3}\right).$$

and the field lines are given by

$$x^2 = \frac{y^3 - C^3}{3y}, \qquad B_y = \frac{2B_0}{d^2} xy.$$

where C is a constant. The separatrices are thus the positive and negative x-axis and the two field lines inclined at $\pi/3$ to the x-axis

(c) When $a < d$, Eq.(30) becomes

$$B_y + iB_x = iB_0 \frac{z^2 + c^2}{z^2 - a^2}, \tag{32}$$

where

$$c^2 = a(d - a).$$

Thus, the magnetic field vanishes at $z = ic$, namely, one point on the y-axis, as required. As a increases from 0 to $\frac{1}{2}d$, so the altitude of this null point increases from 0 to a maximum value of $\frac{1}{2}d$. As a increases further to d, so the altitude decreases to 0.

(d) In 3D, the magnetic field a distance d^* from a pole of half-flux F_0 is $F_0/(2\pi d^{*2})$, and so, equating this to one-half of the ambient field B_0 gives an interaction distance of

$$d^* = \left(\frac{F_0}{\pi B_0}\right)^{1/2},$$

as required.

PROBLEM 10.6. A Smooth Force-Free Field from Imposed Footpoint Motions.

(a) Show that the small force-free perturbation $\mathbf{B}_0 + \mathbf{B}_1$ to a uniform field $\mathbf{B}_0$ produced by arbitrary perturbations $\boldsymbol{\xi} = \boldsymbol{\xi}^-$ on the plane $z = 0$ and $\boldsymbol{\xi} = \boldsymbol{\xi}^+$ on $z = d$ is determined uniquely to be

$$\boldsymbol{\xi} = \frac{\mathbf{k}_\perp \cdot \boldsymbol{\xi}^-}{k_\perp^2}\frac{\sinh[k_\perp(d-z)]}{\sinh(k_\perp d)}\mathbf{k}_\perp + \frac{\mathbf{k}_\perp \cdot \boldsymbol{\xi}^+}{k_\perp^2}\frac{\sinh(k_\perp z)}{\sinh(k_\perp d)}\mathbf{k}_\perp$$
$$-\frac{d-z}{d}\frac{\mathbf{k}_\perp \times (\mathbf{k}_\perp \times \boldsymbol{\xi}^-)}{k_\perp^2} - \frac{z}{d}\frac{\mathbf{k}_\perp \times (\mathbf{k}_\perp \times \boldsymbol{\xi}^+)}{k_\perp^2},$$

where $\mathbf{B}_1 \sim \exp i(\omega t + \mathbf{k}\cdot\mathbf{r})$ and $\boldsymbol{\xi} \sim \exp i(\omega t + \mathbf{k}\cdot\mathbf{r})$.

(b) Show that the force-free currents arise from a difference in rotational motions on the boundaries.

(c) Show also that the force-free fields here are the low-frequency limit of shear Alfvén waves.

SOLUTION.

We follow Sakurai and Levine (1981) Ap. J. 248, 817.

(a) Seek solutions of the form

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \qquad \mathbf{v} = \mathbf{v}_1 = \frac{\partial \boldsymbol{\xi}}{\partial t},$$

of the form

$$\mathbf{B}_1 \sim \exp i(\omega t + \mathbf{k}\cdot\mathbf{r}) \qquad \mathbf{v}_1 \sim \exp i(\omega t + \mathbf{k}\cdot\mathbf{r})$$

to the linearised force-free and ideal induction equations, namely,

$$\mathbf{j}_1 \times \mathbf{B}_0 = 0, \qquad \frac{\partial \mathbf{B}_1}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v}_1 \times \mathbf{B}_0), \tag{33}$$

where $\mathbf{j}_1 = \boldsymbol{\nabla} \times \mathbf{B}_1$.

Then Eqs.(33) become

$$(\mathbf{k} \times \mathbf{B}_1) \times \mathbf{B}_0 = 0 \tag{34}$$

and

$$\mathbf{B}_1 = \mathbf{k} \times (\boldsymbol{\xi} \times \mathbf{B}_0. \tag{35}$$

Now write

$$\mathbf{k} = \mathbf{k}_\perp + k_z \hat{\mathbf{z}}$$

and substitute for $\mathbf{B}_1$ from Eq.(35) into Eq.(34), which reduces to

$$k_z^2 \boldsymbol{\xi} - (\mathbf{k} \cdot \boldsymbol{\xi}) k_z \hat{\mathbf{z}} + k_\perp \xi_\perp \mathbf{k} = 0. \tag{36}$$

Taking $\mathbf{k}_\perp \cdot$ Eq.(36) gives

$$(k_z^2 + k_\perp^2) k_\perp \xi_\perp = 0, \tag{37}$$

while taking $\mathbf{k}_\perp \times$ Eq.(36) and then taking$\cdot \hat{\mathbf{z}}$ the result gives

$$\mathbf{k}_\perp \times \boldsymbol{\xi} \cdot \hat{\mathbf{z}}\ k_z^2 = 0. \tag{38}$$

Then Eqs.(37),(38) have two types of solution, namely,

$$k_\perp \xi_\perp \sim \exp i(k_\perp r_\perp \pm k_z z), \quad \text{with} \quad k_z^2 + k_\perp^2 = 0, \quad \mathbf{k}_\perp \times \boldsymbol{\xi} \cdot \hat{\mathbf{z}} = 0,$$

and

$$\mathbf{k}_\perp \times \boldsymbol{\xi} \cdot \hat{\mathbf{z}} \sim \exp i(k_\perp r_\perp) \times \text{linear function of} z, \quad \text{with} \quad k_z^2 = 0 \,(\text{i.e., } \partial^2/\partial z^2 = 0), \;\; k_\perp \xi_\perp = 0,$$

If we specify $\boldsymbol{\xi} = \boldsymbol{\xi}^-$ on the plane $z = 0$ and $\boldsymbol{\xi} = \boldsymbol{\xi}^+$ on $z = d$ and decompose $\boldsymbol{\xi}$ into the two parts

$$\boldsymbol{\xi} = \frac{\mathbf{k}_\perp \cdot \boldsymbol{\xi}}{k{-}\perp^2} \mathbf{k}_\perp - \frac{\mathbf{k}_\perp \cdot \boldsymbol{\xi}}{k\ \perp^2}, \tag{39}$$

then the solution is

$$\boldsymbol{\xi} = \frac{\mathbf{k}_\perp \cdot \boldsymbol{\xi}^-}{k_\perp^2} \frac{\sinh[k_\perp(d-z)]}{\sinh(k_\perp d)} \mathbf{k}_\perp + \frac{\mathbf{k}_\perp \cdot \boldsymbol{\xi}^+}{k_\perp^2} \frac{\sinh(k_\perp z)}{\sinh(k_\perp d)} \mathbf{k}_\perp$$
$$- \frac{d-z}{d} \frac{\mathbf{k}_\perp \times (\mathbf{k}_\perp \times \boldsymbol{\xi}^-)}{k_\perp^2} - \frac{z}{d} \frac{\mathbf{k}_\perp \times (\mathbf{k}_\perp \times \boldsymbol{\xi}^+)}{k_\perp^2}, , \tag{40}$$

as required

(b) Eq.(34) is $\boldsymbol{\nabla} \times \mathbf{B}_1 \times \mathbf{B}_0 = \mathbf{0}$, which implies

$$\boldsymbol{\nabla} \times \mathbf{B}_1 = \alpha \mathbf{B}_0$$

and

$$\mathbf{B}_0 \cdot \boldsymbol{\nabla} \alpha = 0,$$

so that α is independent of z. These imply that

$$\alpha = \frac{\bar{\mathbf{z}} \cdot (i\mathbf{k} \times \mathbf{B}_)}{B_0} = ik_z \bar{\mathbf{z}} \cdot (i\mathbf{k}_\perp \times \boldsymbol{\xi})$$

for each term in Eq.(40).

The first two terms in Eq.(40) don't contribute to α, while the last two with $ik_z = \partial/\partial z$ give

$$\alpha = \frac{i}{d} \bar{\mathbf{z}} \cdot [\mathbf{k}_\perp \times (\boldsymbol{\xi}^+ - \boldsymbol{\xi}^-)] \tag{41}$$

for one Fourier component, or

$$\alpha = \frac{1}{d} \bar{\mathbf{z}} \cdot [\boldsymbol{\nabla}_\perp \times (\boldsymbol{\xi}^+ - \boldsymbol{\xi}^-)] \tag{42}$$

for arbitrary $\boldsymbol{\xi}^+$ and $\boldsymbol{\xi}^-$.

Eq.(41) or a superposition over all $\mathbf{k}_\perp$ implies that $\mathbf{B}_1$ is determined uniquely when an arbitrary small motion $(\boldsymbol{\xi}^+, \boldsymbol{\xi}^-)$ is imposed on the boundary. The perturbation does indeed satisfy the condition $\partial \mathbf{B}_1/\partial z = 0$ far away from the boundary, but $\boldsymbol{\xi}$ does not vanish and the system can respond with a smooth equilibrium to any small perturbations on the boundary.

Eq.(42) implies that force-free currents arise from a difference in rotational motions on the boundary. Irrotational motions generate no currents, unless the basic state is nonuniform.

(c) If we repeat the analysis of part (a) but replace the force-free equation by the equation of motion with just a magnetic force, then Eq.(33) becomes

$$\frac{\partial \mathbf{v}_1}{\partial t} = \mathbf{j}_1 \times \mathbf{B}_0, \qquad \frac{\partial \mathbf{B}_1}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v}_1 \times \mathbf{B}_0), \tag{43}$$

so that Eq.(34) is modified to

$$\mu \rho \omega \mathbf{v}_1 = (\mathbf{k} \times \mathbf{B}_1) \times \mathbf{B}_0, \tag{44}$$

which implies that $\mathbf{B}_0 \cdot \mathbf{v}_1 = 0$.

Then Eq.(44) and Eq.(35) may be combined to give

$$\frac{\omega^2}{v_A^2}\mathbf{v}_1 = k_z^2\mathbf{v}_1 + (\mathbf{k}_\perp \cdot \mathbf{v}_1)\mathbf{k}_\perp. \tag{45}$$

Taking $\mathbf{k}_\perp\cdot$ Eq.(45) gives

$$(\omega^2/v_A^2 - k_z^2 - k_\perp^2)\mathbf{k} \cdot \mathbf{v}_1 = 0, \tag{46}$$

while taking $\mathbf{k}_\perp\times$ Eq.(45) gives

$$(\omega^2/v_A^2 - k_z^2)\mathbf{k}_\perp \times \mathbf{v}_1 \cdot \hat{\mathbf{z}} = 0. \tag{47}$$

These two equations are satisfied either by shear Alfvén waves with $\omega^2 = k_z^2 v_A^2$ and $\mathbf{k} \cdot \mathbf{v}_1 = 0$ or by magnetoacoustic waves with $\omega^2 = k^2 v_A^2$ and $\mathbf{k}_\perp \times \mathbf{v}_1 \cdot \hat{\mathbf{z}} = 0$. The resulting electric current is

$$\mathbf{j}_1 = \frac{iB_0}{\mu\omega}[k_z(\mathbf{k}_\perp \times \mathbf{v}_1) - (\mathbf{k}_\perp \cdot \mathbf{v}_1)(\mathbf{k}_\perp \times \bar{\mathbf{z}}. \tag{48}$$

Therefore the current is perpendicular to $\mathbf{B}_0$ for magnetoacoustic waves but has both parallel and perpendicular components for shear Alfvén waves.

By comparing Eqs.(37),(38) with Eqs.(46),(47), it can be seen that the force-free field is the low frequency limit of shear Alfvén waves. This can also be understood from the fact that only the shear Alfvén waves have a field aligned current component (j_z) and $j_z \gg j_\perp$ when $\omega \to 0$. By comparison, the magnetoacoustic waves only survive as current-free perturbations in the limit $\omega \to 0$.

PROBLEM 10.7. Current Sheet Formation in Flux Tube Tectonics.

Consider a 2D (x, z) corona between two photospheres on $z = \pm L$, with an array of line sources of flux $[(2n+1)w, \pm L]$, where $w/(2L) \ll 1$ is the aspect ratio of the loops.

(a) Prove that the initial potential magnetic field is

$$(B_{0x}, B_{0z}) = \frac{F/(2w)}{\{\cosh[\pi(L-|z|)/w] + \cos(\pi x/w)\}}(\mathrm{sgn}(z)\sin(\pi x/w), \sin[\pi(L-|z|)/w]), \tag{49}$$

and describe its properties.

(b) Suppose the source footpoints are sheared by $Y_n = \pm(-1)^n Y$. Show that the current density is $\mathbf{j} = (db/dA_0)\mathbf{B}_0$, where A_0 is the flux function associated with the field $\mathbf{B}_0$ (Eq.49) and $b(A_0) = Y/V_0$, where

$$V_0(A_0) = \frac{4w^2}{\pi F} \log_e \frac{\sinh^2(\frac{1}{2}\pi L/w)\cos^2(\pi A_0/F) + \cosh^2(\frac{1}{2}\pi L/w)\sin^2(\pi A_0/F)}{\sin^2(\pi A_0/F)}. \tag{50}$$

(c) Show that the current flows in a narrow layer of width $w/\pi e^{-[\pi L/(2w)]}$.

SOLUTION.

We follow Priest, Heyvaerts and Title (2002) Ap J 576. 533.

(a) For the considered boundary conditions the field components have the forms

$$B_{0x} = \sum_1^\infty B_{xn}(z)\sin\left(\frac{n\pi x}{w}\right), \quad B_{0z} = \tfrac{1}{2}B_{z0} + \sum_1^\infty B_{zn}(z)\cos\left(\frac{n\pi x}{w}\right) \tag{51}$$

Since $B_{0z}(x, \pm L)$ is a set of Dirac peaks, we have $B_{zn}(\pm L) = (F/w)(-1)^n$, so that

$$B_{zn}(z) = \frac{F}{w}(-1)^n \frac{\cosh(n\pi z/w)}{\cosh(n\pi L/w)} \tag{52}$$

Then $\boldsymbol{\nabla} \cdot \mathbf{B}_0 = 0$ determines $B_{xn}(z)$ and Eqs.(51) become

$$B_{0x} = -\frac{F}{w}\sum_1^\infty (-1)^n \sin\left(\frac{n\pi x}{w}\right)\frac{\sinh(n\pi z/w)}{\cosh(n\pi L/w)}, \tag{53}$$

$$B_{0z} = \frac{F}{2w} + \frac{F}{w}\sum_1^\infty (-1)^n \cos\left(\frac{n\pi x}{w}\right)\frac{\cosh(n\pi z/w)}{\cosh(n\pi L/w)}. \tag{54}$$

In addition, the flux function $A_0(x,z)$ defined by $B_{0x} = -\partial A_0/\partial z, B_{0z} = \partial A_0/\partial x$ is given by

$$\frac{\pi A_0(x,z)}{F} = \frac{\pi x}{2w} + \sum_1^\infty \frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{w}\right)\frac{\cosh(n\pi z/w)}{\cosh(n\pi L/w)}. \tag{55}$$

Now in the close-packing limit ($w \ll L$), these reduce to

$$(B_{0x}, B_{0z}) = \frac{F/(2w)}{[\cosh[\pi(L-|z|)/w] + \cos(\pi x/w)]}(\mathrm{sgn}(z)\sin(\pi x/w), \sin[\pi(L-|z|)/w]), \tag{56}$$

as required, whereas the equation for A_0 becomes

$$\tan\left[\frac{\pi A_0(x,z)}{F}\right] = \tanh\left[\frac{\pi(L-|z|)}{2w}\right]\tan\left(\frac{\pi x}{2w}\right). \tag{57}$$

The field is essentially uniform as soon as z departs from the boundary by more than w. The potential field has null points on the boundaries at $z = \pm L$, $x = 2nw$.

(b) The perturbed flux function has the form $A(x,z) = A_0(x,z) + a(x,z)$, where we assume $Y \ll w$, $a \ll A_0$. The corresponding field has a y-component $[b(A_0)]$ that is determined by

$$\frac{dy}{b} = \frac{ds}{B_0}, \tag{58}$$

so that

$$b(A_0) = \frac{Y}{V_0(A_0)}, \tag{59}$$

where the specific volume is

$$V_0(A_0) = \int_{A_0} \frac{ds_0}{B_0}. \tag{60}$$

The resulting current density is

$$\mathbf{j} = \frac{db/dA_0}{\mu}\mathbf{B}_0. \tag{61}$$

Thus, if V_0 diverges on field lines that pass close to a null point, then b vanishes at such surfaces. In terms of variables $X = \pi x/w$, $Z = \pi(L-|z|)/w$, the equation for magnetic surfaces becomes

$$\tan\left(\frac{\pi A_0}{F}\right) = \tan\left(\frac{X}{2}\right)\tanh\left(\frac{Z}{2}\right), \tag{62}$$

while the specific volume is given by

$$V_0(A_0) = \frac{4w^2}{\pi F}\int_0^{\pi L/c} \frac{\cos X + \cosh Z}{\sinh Z} dZ, \tag{63}$$

or

$$V_0(A_0) = \frac{4w^2}{\pi F}\log_e \frac{\sinh^2(\frac{1}{2}\pi L/w)\cos^2(\pi A_0/F) + \cosh^2(\frac{1}{2}\pi L/w)\sin^2(\pi A_0/F)}{\sin^2(\pi A_0/F)}, \tag{64}$$

as required

(c) Thus, V_0 depends on two parameters, namely,

$$\psi = \frac{\pi A_0}{F}, \qquad q = \frac{\pi L}{2w}. \tag{65}$$

For small ψ, i.e., field lines passing close to the separatrix, V_0 diverges as

$$V_0(A_0) \approx \frac{8w^2}{\pi F} \log_e \left(\frac{1}{\psi}\right). \tag{66}$$

On the other hand, for field lines that are far from the separatrix, V_0 is constant

$$V_0(A_0) \approx V_{0\infty} = \frac{8w^2}{\pi F} q. \tag{67}$$

b behaves like $1/V_0$ and so is essentially constant except near separatrices. Thus, the current is concentrated near the separatrices. The integral of the current density between a separatrix and another magnetic surface, namely, the current surface density ($I(\psi)$) is given from Eq.(61) by

$$I(\psi) = \frac{\pi F Y}{4w^2\mu} \frac{1}{[2q - \log_e(4\sin^2\psi)]}. \tag{68}$$

I vanishes at $x = 0$ and rapidly increases to a constant value

$$I_\infty = \frac{FY}{4Lw\mu}. \tag{69}$$

The width (δ) of the layer where the current is flowing is from Eq.(68)

$$\delta = \frac{w}{\pi} e^{-q}, \tag{70}$$

as required. Since $q \gg 1$, the thickness of the current layer is very small.

PROBLEM 10.8. Binary Reconnection.
(a) Suppose the position of one magnetic source rotates relative to another, and calculate the resulting electric field and field line velocity.

(b) Show that, if the mean photospheric flux densities of positive and negative flux are $\bar{B}_+$ and $\bar{B}_-$, then the heat flux due to the source motion (v_0) is

$$F_{heat} = \frac{\bar{B}_+^2 v_0}{3\pi\mu}.$$

SOLUTION.

We follow Priest, Longcope and Titov (2003) Ap J 598, 203.

(a) Let us seek to determine the nature of the electric field (**E**) and plasma velocity (**v**) outside the vicinity of the null point (where the electric currents are concentrated). In the steady-state kinematic approximation for a given magnetic field, **E** and **v** are determined by

$$\nabla \times \mathbf{E} = 0, \tag{71}$$

which implies that

$$\mathbf{E} = \nabla \Phi, \tag{72}$$

and by

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0. \tag{73}$$

Equation (73) has two implications. Taking the scalar product with **B** and using (72) gives

$$\mathbf{B} \cdot \nabla \Phi = 0,$$

so that Φ is constant along a magnetic field line. In particular, if such a field line is characterised by constant values for two functions ($c(x, y, z) =$ constant and $k(x, y, z) =$ constant, say) that define surfaces, then

$$\Phi = \Phi(c, k). \tag{74}$$

The second consequence of (73) arises by taking the vector product with **B**, namely, that the plasma velocity ($\mathbf{v}_1$) normal to the magnetic field is

$$\mathbf{v}_{\perp} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}. \tag{75}$$

Thus our task is to determine the function $\Phi(c, k)$, and then **E** and $\mathbf{v}_{\perp}$ follow from (72) and (75), respectively.

Suppose now that there is one source of unit strength at the origin and a smaller source of strength $-\epsilon$ at a point A a distance a along the y-axis (see Figure 4b of Priest, Longcope, Titov, 2003). The resulting magnetic field is

$$\mathbf{B} = \frac{\hat{\mathbf{r}}}{r^2} - \frac{\epsilon \hat{\mathbf{r}}}{r_1^2}, \tag{76}$$

and contains a null at a distance

$$y_N = \frac{a}{1-\sqrt{\epsilon}}$$

along the y-axis. Here r, θ, ϕ and r_1, θ_1, ϕ are the polar coordinates of a general point P with respect to the origin O and the position of the smaller source at A.

The field lines are given elegantly by constant values of

$$c = \epsilon \cos\theta_1 - \cos\theta, \qquad k = \phi, \tag{77}$$

where the coordinates (r, θ) and (r_1, θ_1) are related by

$$r_1 \sin\theta_1 = r \sin\theta, \qquad r_1 \cos\theta_1 = r\cos\theta - a,$$

so that

$$\tan\theta_1 = \frac{r\sin\theta}{r\cos\theta - a} \tag{78}$$

and

$$r_1^2 = r^2 - 2ar\cos\theta + a^2. \tag{79}$$

Furthermore, the unit vector along OP is

$$\hat{\mathbf{r}}_1 = \frac{r\cos\theta - a}{r_1^2}\left\{(r - a\cos\theta)\,\hat{\mathbf{r}} + a\sin\theta\,\hat{\boldsymbol{\theta}}\right\}. \tag{80}$$

Thus, in (76) the magnetic field is given in terms of s and θ with $\hat{\mathbf{r}}_1$ given by (80) and r_1^2 by (79), while the equation of the field lines is given by $c = c(r, \theta)$ in (77) with θ_1 given by (78).

The question now arises: how can we choose $\Phi = \Phi(c, \phi)$, with c given by (77), in such a way that

$$\Phi \to e\Omega\cos\phi' \text{ as } r \to 0,$$

so as to give uniform rotation of the source at the origin? The variables (θ', ϕ') and (θ, ϕ) in Figure 4 are related by

$$\begin{aligned} \cos\theta' &= \sin\theta\cos\phi, \left(= \frac{z}{r}\right) \\ \sin\theta'\sin\phi' &= \cos\theta\left(= \frac{y}{r}\right) \end{aligned} \tag{81}$$

Thus, the above question is equivalent to finding $\Phi(c, \phi)$ such that

$$\Phi(c, \phi) \ \rightarrow \ e\omega \sin\theta \cos\phi \text{ as } r \rightarrow 0,$$

where $c = \epsilon \cos\theta_1 - \cos\theta$.

Since $\theta_1 \rightarrow \pi$ and so $c \rightarrow -\epsilon - \cos\theta$ as $r \rightarrow 0$, the function we seek is

$$\Phi(c, \phi) = [1 - (c + \epsilon)^2]^{1/2} \cos\phi. \tag{82}$$

Singularities in $\mathbf{v}_\perp$ (given by (75)) occur where $\mathbf{B} = 0$ and also where $\partial\Phi/\partial c$ is singular, namely, at $c = -\epsilon \pm 1$, which represents the spine of the null. Thus the rotation of the dominant source drives spine reconnection at the null point.

In cartesian coordinates the potential becomes

$$\Phi(x, y, z) = \frac{z}{(x^2 + z^2)^{1/2}} \left\{ 1 - \left[\frac{y}{x^2 + y^2 + z^2} - \frac{\epsilon(y - 1)}{\sqrt{[(y - 1)^2 + x^2 + z^2]}} - \epsilon \right]^2 \right\}^{1/2}.$$

Furthermore, near the null point the ϕ-component of the electric field component behaves like

$$E_\phi = 2\sqrt{\epsilon(1 - \epsilon)} \frac{\cos\phi}{R},$$

where $R^2 = x^2 + z^2$, which is characteristic of spine reconnection.

In this analysis, we imposed a solid-body rotation of one source and had no control over the motion of the other source. In general, for different source motions, the reconnection will be of spine-fan type (see Priest, Longcope, Titov, 2003 and chapter 6.13.2).

(b) Now consider the energy contributed by turning a flux source F through an angle $\Delta\phi$ over a time Δt at a constant rotation rate $\omega = \Delta\phi/\Delta t$. Suppose the field of the source is

$$\mathbf{B}_0 = \frac{F}{2\pi r^2} \hat{\mathbf{r}}$$

and that the rotation produces an Alfvén wave with $v_{1\phi} = r\omega \sin\theta$ and $B_{1\theta} = B_0 v_{1\theta}/v_{A0}$ in terms of the local Alfvén speed (v_{A0}). Then the Poynting flux is

$$\mathbf{S} = \frac{-(\mathbf{v}_1 \times \mathbf{B}_0) \times \mathbf{B}_1}{\mu} = \frac{B_0^2}{\mu v_{A0}} v_{1\phi}^2 \hat{\mathbf{r}} = \frac{B_0^2}{\mu v_{A0}} \omega^2 \sin^2\theta \ r^2 \hat{\mathbf{r}}.$$

The resulting power produced is

$$P = \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \; r^2 S d\theta = \frac{B_0^2}{3\mu v_{A0}} r^4 \omega^2,$$

or, in terms of the flux (F) and angle ($\Delta\phi$),

$$P = \frac{F^2}{12\pi^2\mu} \frac{\omega^2}{v_{A0}} = \frac{F^2}{12\pi^2\mu} \frac{(\Delta\phi)^2}{v_{A0}(\Delta t)^2}.$$

The energy injected in time Δt is then

$$\Delta E = P\Delta t = \frac{F^2}{12\pi^2\mu} \frac{(\Delta\phi)^2}{v_{A0}\Delta t}, \tag{83}$$

or, in terms of the self-helicity ($\Delta H = F^2 \Delta\phi/\pi$)

$$\Delta E = \frac{\Delta H}{12\pi\mu} \frac{\Delta\phi}{v_{A0}\Delta t}, \tag{84}$$

Now the free energy (ΔE_{fff}) in a linear force-free field is

$$\Delta E_{fff} = \frac{\Delta H}{2\mu}\alpha, \tag{85}$$

which provides a lower bound on the energy release. Equating Eqs.(84) and (85) gives an expression

$$\Delta t_{fff} = \frac{\Delta\phi}{6\pi v_{A0}\alpha} \tag{86}$$

for the maximum injection time beyond which Eq.(84) fails and the energy conversion becomes quasi-static in nature rather than wave-like.

Equating ΔH to the helicity H_{fff} in the linear force-free field determines the value of α. Changeover between a wave process and a quasi-static process occurs when wave reflections occur, namely, when

$$\Delta t = \frac{2L_{eff}}{v_{A0}},$$

where L_{eff} is the effective length of the field lines. Thus, equating this timescale to Δt_{fff} gives an alpha value of

$$\alpha = \frac{\Delta\phi}{12\pi L_{eff}}. \tag{87}$$

The distinction between impulsive and quasi-static heating is therefore determined by the parameter

$$\xi = \frac{v_{A0}\Delta t}{2L_{eff}}. \tag{88}$$

For impulsive generation ($\xi < 1$) the energy release (ΔE) is given by Eq.(84), whereas for quasi-static energization ($\xi > 1$) the energy release follows from Eqs.(85) and (87) as

$$\Delta E = \frac{\Delta H}{24\pi\mu}\frac{\Delta\phi}{L_{eff}}. \tag{89}$$

Now, consider a binary collision between two sources A_1 and A_2 with fluxes F_1 and F_2, respectively, and suppose that initially A_2 is at distance d from A_1. In other words, in the frame of reference of A_1, the source A_2 lies on a circle of radius d. Suppose that during the collision A_2 approaches within a distance b of A_1. Thus, in terms of the impact parameter (b), the change ($\Delta\phi$) in angle of orientation of A_1A_2 is given by

$$\frac{\Delta\phi}{2} = \cos^{-1}\left(\frac{b}{d}\right), \tag{90}$$

while the duration (Δt) of the interaction is

$$\Delta t = \frac{2d}{v_0}\sin\left(\frac{\Delta\phi}{2}\right). \tag{91}$$

The parameter ξ can then be rewritten, after identifying L_{eff} with b and substituting for Δt from Eq.(91), as

$$\xi = \frac{v_{A0}d}{v_0 b}\sin\left(\frac{\Delta\phi}{2}\right),$$

or, after using Eq.(90) for $\Delta\phi$,

$$\xi = \frac{v_{A0}}{v_0}\left(\frac{d^2}{b^2} - 1\right)^{1/2}. \tag{92}$$

However, the motion of magnetic sources is in practice highly sub-Alfvénic ($v_0 \ll v_{A0}$), and so we deduce from Eq.(92) that in the vast majority of interactions (i.e., unless $b \approx d$) $\xi \gg 1$, so that the interactions are quasi-static (through a series of force-free states) rather than being impulsive (mediated by wave motions).

The energy release given by Eq.(83) can be written in the form

$$\Delta E = \frac{F_{min}^2}{24\pi^2\mu} \frac{(\Delta\phi)^2}{L_{eff}},$$

where F_{min} is the smaller of the two fluxes F_1 and F_2 involved in an interaction. Replacing L_{eff} by b and substituting for $\Delta\phi$ from Eq.(90) then gives

$$\Delta E = \frac{F_{min}^2}{6\pi^2\mu b} \left[\cos^{-1}\left(\frac{b}{d}\right)\right]^2. \tag{93}$$

The rate of collisions (ν) with a source of a given sign is

$$\nu = 2\pi d \; N_{opp} \; v_0,$$

where N_{opp} is the density of sources of opposing polarity and d is the radius of a circle that contacts one source on average, so that $\pi N_{pop} d^2 = 1$. Thus,

$$d = \frac{1}{(\pi N_{opp})^{1/2}},$$

and the collision rate becomes

$$\nu = 2v_0 \left(\frac{N_{opp}}{\pi}\right)^{1/2}. \tag{94}$$

Using the expressions for ΔE and ν from Eqs.(93) and (94), and the fact that the mean value of $(d/b)[\cos^{-1}(b/d)]^2$ is about $\sqrt{2}(\pi/4)^2 \approx 0.9 \approx 1$, the mean rate of energy release is

$$\nu\langle\Delta E\rangle = \frac{1}{3\pi\mu} F_{min}^2 N_{opp} v_0.$$

The heat flux (F_{heat}) from a distribution of (majority species) sources of density N therefore becomes

$$F_{heat} = \frac{1}{3\pi\mu} F_{min}^2 N N_{opp} v_0. \tag{95}$$

If the minority species has a fraction (f) of the total source density ($N_+ + N_-$), then $NN_{pop} = f(1-f)$ and the heat flux becomes

$$F_{heat} = \frac{1}{3\pi\mu} F_{min}^2 f(1-f) v_0.$$

An alternate way of writing this is in terms of the mean flux densities

$$\bar{B}_+ = \langle F_+ \rangle N_+, \qquad \bar{B}_- = \langle F_- \rangle N_-,$$

where N_+ and N_- are the numbers of these flux elements per unit area, while $\langle F_+ \rangle$ and $\langle F_- \rangle$ are the mean fluxes per element. The heat flux is then

$$F_{heat} = \frac{\zeta}{3\pi\mu} \bar{B}_+ \bar{B}_- v_0, \tag{96}$$

where $\zeta = F_{min}/F_{maj}$ is the ratio of minority to majority flux. If, in particular $\bar{B}_+ \approx \bar{B}_-$, the heat flux becomes finally

$$F_{heat} = \frac{1}{3\pi\mu} \bar{B}_+^2 v_0, \tag{97}$$

which is enough to heat the corona provided $\bar{B}_+$ is large enough.

For further details, see Priest, Longcope and Titov (2003) Ap J 598, 203.