CORRIGENDUM TO THE BOOK

Fundamental Solutions in Elastodynamics: A Compendium

by

Eduardo Kausel

VERSION: June 26, 2018

Since publication in 2006 and as of today, this book has been quoted some 251 times, i.e. some 21 times a year. This is a very high rate of quotation, and reflects on the success of the book.

As of today, no serious or "cascading" errors have been found, i.e. none of those listed here affect any of the ensuing formulas, results or programs. Thus, they are simply of the editorial type.

I. Errata

1) Page 39, footnote, in the title of the paper by Eason et al, replace "…by a variable body" with "…by variable body forces".

2) Page 41, Eq. 3.39, errors in both expressions. Change as indicated below:

$$\mathbf{G}_{yj} = \begin{cases} -\gamma_j \dfrac{\partial g_{yy}}{\partial r}\hat{\mathbf{j}} = -\gamma_j \dfrac{\mathrm{i}\Omega_S}{4\mu r} H_1^{(2)}(\Omega_S)\hat{\mathbf{j}} \\ -\gamma_j \dfrac{\partial u_{yy}}{\partial r}\hat{\mathbf{j}} = -\gamma_j \dfrac{t_S^2\, \mathcal{H}(t-t_S)}{2\pi\mu r\left(t^2 - t_S^2\right)^{3/2}}\hat{\mathbf{j}} \end{cases}$$

3) Page 45: Eq. 3.53, first term is missing a parenthesis square:

$$a^{-2} = \left(\frac{\alpha}{\beta}\right)^2$$

4) Page 71, eq. 5.12: square root missing:

$$a = \frac{\beta}{\alpha} = \sqrt{\frac{1-2\nu}{2-2\nu}}$$

5) Page 72: The vertical ordinates in the three figures 5.2 are too small by a factor 5. This resulted because my program computed the figures at x=5 and thereafter I failed to normalize the figures by r as indicated.

Also, the subindices in the ordinate labels of the second and third figures should have been zz and xz, i.e. the last factor in each of the figures should have been u_{xx}, u_{zz}, u_{xz}, respectively.

6) Page 73, equations 5.18, in both expressions, the τ factor of the second term should be deleted, i.e. these equations should read:

$$\sqrt{q_\alpha^2 + a^2} = \tau\cos\theta_z + \mathrm{i}\sin\theta_z\sqrt{\tau^2 - a^2} \qquad \sqrt{q_\beta^2 + 1} = \tau\cos\theta_z + \mathrm{i}\sin\theta_z\sqrt{\tau^2 - 1}$$

Page 77, eq. 6.28b: Change the sub-index y into z, i.e. $u_z(x,0,\infty)=\cdots$

7) Page 78, eq. 6.6 is missing a factor ξ^2 in front of the square root; also, the sign is wrong. The correct equation is

$$R(\xi^2)=(1-2\xi^2)^2-4\xi^2\sqrt{\xi^2-1}\sqrt{\xi^2-a^2} \qquad (6.6)$$

8) Page 78, second line below eq. 6.6, modify the equation and delete the fragment "and dividing by ξ^2". The whole line should read as follows:

"by $(1-2\xi^2)^2+4\xi^2\sqrt{\xi^2-1}\sqrt{\xi^2-a^2}$, ~~and dividing by ξ^2~~ one obtains the bicubic (i.e. cubic"

9) Page 79, replace eq. 6.11b by

$$u_{zz}=\frac{(1-\nu)P}{2\pi\mu r}\begin{cases} 0 & \tau<a \\ \frac{1}{2}\left[1-\frac{A_3}{\sqrt{\xi_3^2-\tau^2}}-2\,\mathrm{Re}\left[\frac{(a^2-\xi_1^2)(1-2\xi_1^2)^2}{(\xi_1^2-\xi_2^2)(\xi_1^2-\xi_3^2)(\tau^2-a^2)(Q_1^{-1}-Q_1)}\right]\right] & a<\tau<1 \\ \left[1-\frac{A_3}{\sqrt{\xi_3^2-\tau^2}}\right] & 1<\tau<\gamma \\ 1 & \tau>\gamma \end{cases}$$

Fortunately, the program Lamb3D.m implementing this formula is correct as written.

10) Page 80, eq. 6.12
A factor 2 is missing in front of the square root. The correct equation is

$$Q_1(z)=1+2z+2\sqrt{z^2+z}, \qquad z=\frac{a^2-\xi_1^2}{\tau^2-a^2}$$

11) Page 80, second line below eq. 6.12, the equation is missing a factor 2:

"that $Q_2(z)=1+2z-2\sqrt{z^2+z}$ satisfies …"

12) Page 106: In equation 8.92, in the diagonal matrix with exponentials, the first three terms should have a negative sign, the next three positive (i.e. the signs are reversed). This error did *not* affect any of the later formulas and developments. The correct expression is

$$\bar{\mathbf{u}}(x,z,\omega)=\begin{Bmatrix} u_x \\ u_y \\ -\mathrm{i}\,u_z \end{Bmatrix}$$

$$=\begin{Bmatrix} 1 & 0 & -s & 1 & 0 & s \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -p & 0 & 1 & p & 0 & 1 \end{Bmatrix}\begin{Bmatrix} e^{-kpz} & & & & & \\ & e^{-ksz} & & & & \\ & & e^{-ksz} & & & \\ & & & e^{+kpz} & & \\ & & & & e^{+ksz} & \\ & & & & & e^{+ksz} \end{Bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix} e^{-\mathrm{i}kx}$$

13) Page 107, two lines above box 8.95, change $\mathbf{D}_{zr}$ into $\mathbf{D}_{zx}$ i.e. change the sub-index $r \to x$. Also, in box 8.95, middle equation, delete the factor 2 after the equal sign, i.e. $2k\mu \to k\mu$.

14) Page 114: In eq. 8.151, the subscript on the left hand side should be *r*, not *R*:

$$u_r = \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} \left\{ H'_{\alpha n} A_\Phi + n \frac{H_{\beta n}}{k_\beta r} A_\psi + \frac{k_z}{k_\beta} H'_{\beta n} A_\chi \right\} e^{-\mathrm{i}k_z z}$$

In eq. 8.153, the subscript of the leftmost *H* function should be α, not β:

$$u_z = \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} \left\{ -\mathrm{i} \frac{k_z}{k_\alpha} H_{\alpha n} A_\Phi + \mathrm{i} H_{\beta n} A_\chi \right\} e^{-\mathrm{i}k_z z}$$

15) Page 115 In eq. 8.154, the subscript of the first element in the vector on the left hand side should be *r*, not *R*:

$$\begin{Bmatrix} u_r \\ u_\theta \\ u_z \end{Bmatrix} = \begin{Bmatrix} \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} & 0 & 0 \\ 0 & \begin{pmatrix} -\sin n\theta \\ \cos n\theta \end{pmatrix} & 0 \\ 0 & 0 & \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} \end{Bmatrix} \begin{Bmatrix} H'_{\alpha n} & n \frac{H_{\beta n}}{k_\beta r} & \frac{k_z}{k_\beta} H'_{\beta n} \\ n \frac{H_{\alpha n}}{k_\alpha r} & H'_{\beta n} & n \frac{k_z}{k_\beta} \frac{H_{\beta n}}{k_\beta r} \\ -\mathrm{i} \frac{k_z}{k_\alpha} H_{\alpha n} & 0 & \mathrm{i} H_{\beta n} \end{Bmatrix} \begin{Bmatrix} A_\Phi \\ A_\psi \\ A_\chi \end{Bmatrix} e^{-\mathrm{i}k_z z}$$

16) Page 124, equation 8.208: Some terms in the f_{11}, f_{22} functions do not explicitly indicate the type of spherical Hankel (or Bessel) functions that must be used. Here is the corrected form, which can also be found in Table 10.7 on page 177:

$$f_{11} = -k_P \left[(\lambda + 2\mu) h_{Pm} - \frac{2\mu}{z_P} \left(2h_{P,m+1} + m(m-1) \frac{h_{Pm}}{z_P} \right) \right]$$

$$f_{22} = -k_S \left[\mu h_{Sm} - \frac{2\mu}{z_S} \left(h_{S,m+1} - (m+1)(m-1) \frac{h_{Sm}}{z_S} \right) \right]$$

17) Page 132, equation 9.39: The right hand side is missing an identity matrix

$$\int_0^{2\pi} \mathbf{T}_n \mathbf{T}_j \, d\theta = \pi \delta_{(nj)} (1 + \delta_{n0} \delta_{j0}) \mathbf{I} = \delta_{(nj)} / a_n \, \mathbf{I}$$

18) Page 136, eq. 9.63, super-index of second Hankel matrix should be 2, not 1:

$$\mathbf{u}(r, k_z, \omega) = \mathbf{H}_n^{(1)} \mathbf{a}_1 + \mathbf{H}_n^{(2)} \mathbf{a}_2$$

19) Page 145, Table 10.1, first line of Note 1: change the first repeated occurrence of $\operatorname{Im} kp \geq 0$ after "Riemann sheet" into $\operatorname{Re} kp \geq 0$

20) Page 150, equations 10.32 and 10.33, delete the $\frac{1}{2}$ factor in front of the square brackets, i.e. the correct expressions are

$$u_y(z) = u_m \frac{\cosh ksz}{\cosh \frac{1}{2} ksh} + \Delta u \frac{\sinh ksz}{\sinh \frac{1}{2} ksh} \qquad -\tfrac{1}{2}h \leq z \leq \tfrac{1}{2}h$$

$$\begin{aligned} \tau_{yz}(z) &= ks\mu \left\{ u_m \frac{\sinh ksz}{\cosh \frac{1}{2} ksh} + \Delta u \frac{\cosh ksz}{\sinh \frac{1}{2} ksh} \right\} \\ &= \tau_m \frac{\cosh ksz}{\cosh \frac{1}{2} ksh} + \Delta \tau \frac{\sinh ksz}{\sinh \frac{1}{2} ksh} \end{aligned} \qquad -\tfrac{1}{2}h \leq z \leq \tfrac{1}{2}h$$

21) Page 152, eq. 10.50, delete the ending zero equality, = 0

22) Page 160, Table 10.2, first line, the left hand side has incorrect arguments, and on both sides of the equation remove the tildes, i.e. replace by:

$$\begin{aligned} \mathbf{u}(r, \theta, z, t) &= \begin{Bmatrix} u_r \\ u_\theta \\ u_z \end{Bmatrix} \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{\mathrm{i}\omega t} \sum_{n=0}^{\infty} \mathbf{T}_n \int_0^{\infty} k \, \mathbf{C}_n \tilde{\mathbf{u}}_n(k, z, \omega) \, dk \, d\omega \end{aligned}$$

The second line is correct as written.

23) Page 164, eq. 10.95, the first factor in the numerator of the integral should be *p*, not *s*:

$$u_z = \frac{1}{4\pi\mu} \int_0^{\infty} \frac{p \frac{1}{2}(1 - s^2)}{\Delta} J_0(kr) \, dk$$

24) Page 170, eq. 10.123 $(n = 1)$: In the center element in the matrix, change the subscript of the denominator from b to β. Also, in the fraction before the opening brace, change the divisor 8μ into 4μ. For the same reason, on page 172 change 8μ into 4μ in the denominator of the first line of 10.135. Also, in the leading fraction of 10.136 change the 16μ into 8μ, and do this also on page 173, unnumbered box, in the denominator of the expression for "Lateral point source".

25) Page 182, eq. 10.168: The right hand side is missing an identity matrix

$$\int_0^{2\pi} \mathbf{T}_n \mathbf{T}_j \, d\theta = \pi \delta_{(nj)} (1 + \delta_{n0} \delta_{j0}) \mathbf{I}$$

26) On page 182, in the rigid body displacement equations 10.170 and 10.171 it is convenient to replace the arbitrary constant *c* by the amplitudes of the six rigid body displacements and rotations $u_x, u_y, u_z, \vartheta_x, \vartheta_y, \vartheta_z$:

Translations:

$$U_x = -u_x \mathbf{T}_1^{(1)} \mathbf{L}_1^1 \hat{\mathbf{e}}_{12} \qquad U_y = -u_y \mathbf{T}_1^{(2)} \mathbf{L}_1^1 \hat{\mathbf{e}}_{12} \qquad U_z = u_z \mathbf{T}_0^{(1)} \mathbf{L}_1^0 \hat{\mathbf{e}}_{12}$$

Rotations

$$\Omega_x = -\vartheta_x R \mathbf{T}_1^{(2)} \mathbf{L}_1^1 \hat{\mathbf{e}}_3 \qquad \Omega_y = \vartheta_y R \mathbf{T}_1^{(1)} \mathbf{L}_1^1 \hat{\mathbf{e}}_3 \qquad \Omega_z = \vartheta_z R \mathbf{T}_0^{(2)} \mathbf{L}_1^0 \hat{\mathbf{e}}_3$$

Observe the negative signs in u_x, u_y, ϑ_x, the factor *R* in the rotations, and the correction in the sub-index of Ω_z.

27) Page 185: In equation 11.1, substitute r in lieu of x, i.e.

$$\frac{d^2 y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \left(k^2 - \frac{n^2}{r^2}\right) y = 0$$

28) Page 192, table of spheroidal harmonics, third row, first column, replace $-/+$ by —

ADDENDA to the Book

Fundamental Solutions in Elastodynamics: A Compendium

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VERSION: December 21, 2015

Included here is a list of improvements and additions:

1) On page 48, in the space below the title, annotate the *static values* of the functions ψ, χ

$$\psi = \tfrac{1}{2}\left(1+a^2\right) = \frac{3-4\nu}{4\left(1-\nu\right)}, \qquad \chi = \tfrac{1}{2}\left(1-a^2\right) = \frac{1}{4\left(1-\nu\right)}, \qquad \psi+\chi = 1 \qquad (\omega = 0)$$

2) Section 10.2.1, page 149, analytic continuation in the half-space: An explicit expression which is simpler than eq. 10.31 is as follows:

$$\mathfrak{u}(z) = \mathbf{R}_2 \mathbf{E}_z \mathbf{R}_2^{-1} \mathfrak{u}_1 = \frac{1}{1-sp}\begin{Bmatrix} e^{kpz} - sp\, e^{ksz} & s\left(e^{ksz} - e^{kpz}\right) \\ p\left(e^{kpz} - e^{ksz}\right) & e^{ksz} - sp\, e^{kpz} \end{Bmatrix} \mathfrak{u}_1, \qquad z < z_1$$

with $\mathbf{R}_2, \mathbf{E}_z$ being given by eq. 10.8 (page 143). This gives the displacement within the half-space at elevation $z < z_1$ in terms of the displacements in the half-space al elevation z_1, which may coincide with the interface of the layers with the half-space. This assumes there are no sources below z_1, and that z is positive up! In the case of half-space subjected to loads at its surface and the origin of coordinates $z = 0$ is at the free surface, the solution at depth $z < 0$ is

$$\begin{Bmatrix} \tilde{u} \\ -\mathrm{i}\,\tilde{u}_z \end{Bmatrix}_z = \frac{1}{2k\mu\left(1-sp\right)\Delta}\begin{Bmatrix} e_{\mathrm{P}} - sp\, e_{\mathrm{S}} & s\left(e_{\mathrm{S}} - e_{\mathrm{P}}\right) \\ p\left(e_{\mathrm{P}} - e_{\mathrm{S}}\right) & e_{\mathrm{S}} - sp\, e_{\mathrm{P}} \end{Bmatrix} \times$$

$$\begin{Bmatrix} s\tfrac{1}{2}\left(1-s^2\right) & ps - \tfrac{1}{2}\left(1+s^2\right) \\ ps - \tfrac{1}{2}\left(1+s^2\right) & p\tfrac{1}{2}\left(1-s^2\right) \end{Bmatrix} \begin{Bmatrix} \overline{p}_x \\ -\mathrm{i}\,\tilde{p}_z \end{Bmatrix}_{z=0}$$

$$= \frac{1}{2k\mu\Delta}\begin{Bmatrix} s\left[e_{\mathrm{P}} - \tfrac{1}{2}\left(1+s^2\right)e_{\mathrm{S}}\right] & sp\, e_{\mathrm{S}} - \tfrac{1}{2}\left(1+s^2\right)e_{\mathrm{P}} \\ sp\, e_{\mathrm{P}} - \tfrac{1}{2}\left(1+s^2\right)e_{\mathrm{S}} & p\left[e_{\mathrm{S}} - \tfrac{1}{2}\left(1+s^2\right)e_{\mathrm{P}}\right] \end{Bmatrix} \begin{Bmatrix} \overline{p}_x \\ -\mathrm{i}\,\tilde{p}_z \end{Bmatrix}_{z=0}$$

where

$$e_{\mathrm{P}} = \mathrm{e}^{kpz}, \qquad e_{\mathrm{S}} = \mathrm{e}^{ksz}, \qquad \Delta = sp - \tfrac{1}{4}\left(1+s^2\right)^2$$

For static problems, the limit $\omega \to 0$ of the above expression is

$$\mathbf{u}(z)=\frac{e^{|k|z}}{\alpha^2+\beta^2}\begin{Bmatrix}\alpha^2+\beta^2+|k|z(\alpha^2-\beta^2) & -kz(\alpha^2-\beta^2)\\ kz(\alpha^2-\beta^2) & \alpha^2+\beta^2-|k|z(\alpha^2-\beta^2)\end{Bmatrix}\mathbf{u}_1$$

$$=e^{|k|z}\begin{Bmatrix}1+|k|zA & -kzA\\ kzA & 1-|k|zA\end{Bmatrix}\mathbf{u}_1, \qquad A=\left(\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\right)=\frac{1-a^2}{1+a^2}=\frac{1}{3-4\nu}$$

For sources at the surface of a homogeneous half-space $z\le 0$, this reduces to

$$\mathbf{u}(z)=\begin{Bmatrix}u_x\\ -\mathrm{i}u_z\end{Bmatrix}=\frac{e^{|k|z}}{2k\mu(1-a^2)}\begin{Bmatrix}1+|k|zA & -kzA\\ kzA & 1-|k|zA\end{Bmatrix}\begin{Bmatrix}\operatorname{sgn}k & -a^2\\ -a^2 & \operatorname{sgn}k\end{Bmatrix}\begin{Bmatrix}p_x\\ -\mathrm{i}\,p_z\end{Bmatrix}$$

$$\begin{Bmatrix}u_x\\ u_z\end{Bmatrix}=\frac{e^{|k|z}}{2\mu}\left[\frac{1}{k(1-a^2)}\begin{Bmatrix}\operatorname{sgn}k & \mathrm{i}a^2\\ -\mathrm{i}a^2 & \operatorname{sgn}k\end{Bmatrix}+z\begin{Bmatrix}1 & \mathrm{i}\operatorname{sgn}k\\ \mathrm{i}\operatorname{sgn}k & -1\end{Bmatrix}\right]\begin{Bmatrix}p_x\\ p_z\end{Bmatrix}$$

3) In Table 3 on page 165, the following definitions should be added next to the *H* functions:

$$k_\alpha=\sqrt{k_\mathrm{P}^2-k_z^2}\,,\quad k_\beta=\sqrt{k_\mathrm{S}^2-k_z^2}$$

$$k_\mathrm{P}=\frac{\omega}{\alpha}, \qquad k_\mathrm{S}=\frac{\omega}{\beta}$$

4) In section 10.3.2, pages 164-174, add the new supplementary material on cylindrical layers given in *Appendix 1*, which follows the example on a rigid sphere embedded in a full space. This material clarifies and simplifies the equations for the solid core and provides the analytic continuation for displacements and stresses within that core. Also, it explains the stresses in cylindrical layers and elaborates also on the functions to be used (i.e. Hankel or Bessel functions) and how these must be scaled to avoid numerical problems.

5) At the end of Chapter 10, page 184, add the example in the pages that follow. Here I evaluate the exact response of a rigid sphere with arbitrary mass embedded in a full space, subjected to either torsion or translation. I obtain closed-form results in both the frequency domain as well as in the time domain. Except for a factor $\frac{1}{2}$ in the load, the solution for the sphere in torsion is *identical* to that of a hemispherical, rigid foundation in a half-space subjected to torsion about an axis perpendicular to the surface. On the other hand, the translational case may provide a rough approximation to the response of a hemispherical foundation in a half-space subjected to either vertical or lateral loads, but it differs from it in that the free surface of the half-space acts as a wave guide for surface waves, and thus alters the response of the hemispherical foundation in comparison to the full space. For the same reason, the torsional solution for rotation about a horizontal axis may exhibit some similarities to the hemispherical foundation in a half-space subjected to rocking.

5.6 Generalized Garvin problem[1] (Alterman-Loewenthal problem)[2]

Consider a lower $(z \le 0)$ or upper $(z \ge 0)$ elastic half-plane subjected to an impulsive source in the form of a line of pressure at some depth $h \ge 0$ located directly below (or above) the origin of coordinates on the free surface. The receiver is placed at an arbitrary point in the half-plane with coordinates (x, z). Define

Geometry

$$r_1 = \sqrt{x^2 + (|z| - h)^2}\,, \qquad r_2 = \sqrt{x^2 + (|z| + h)^2} \qquad (5.29a)$$

$$\sin\theta_1 = \frac{|x|}{r_1}\,, \qquad \cos\theta_1 = \frac{|z| - h}{r_1} \qquad (5.29b)$$

$$\sin\theta_2 = \frac{|x|}{r_2}\,, \qquad \cos\theta_2 = \frac{|z| + h}{r_2} \qquad (5.29c)$$

Wave travel times:

$$t_P = \frac{r_1}{\alpha} \qquad \text{Direct P wave} \qquad (5.30a)$$

$$t_{PP} = \frac{r_2}{\alpha} \qquad \text{Reflected P wave} \qquad (5.30b)$$

$$t_{PS} = \frac{h}{\alpha\cos\theta_P} + \frac{|z|}{\beta\cos\theta_S}\,, \qquad |x| = h\tan\theta_P + |z|\tan\theta_S \qquad \text{Reflected S wave} \qquad (5.30c)$$

with θ_P, θ_S being related by Snell's law $\alpha\sin\theta_S = \beta\sin\theta_P$. Elimination of the terms in the incidence and reflection angles θ_P, θ_S between equations 5.30c is cumbersome and leads to a complicated equation. Nonetheless, an iterative solution is easily obtained by searching for the point in the interval $[x_P, |x|]$ that satisfies Snell's relationship, where $x_P > 0$ is the distance of intersection of the PP ray with the free surface. Thus, t_{PS} can readily be determined to high accuracy, and can thus be assumed to be known.

Dimensionless time

$$\tau = \frac{t\beta}{r_2} \qquad (5.31a)$$

$$\tau_P = \frac{t_P\beta}{r_2} = \frac{r_1}{r_2}\frac{\beta}{\alpha}\,, \qquad (5.31b)$$

$$\tau_{PP} = \frac{t_{PP}\beta}{r_2} = \frac{\beta}{\alpha} \qquad (5.31c)$$

$$\tau_{PS} = \frac{t_{PS}\beta}{r_2} \qquad (5.31d)$$

Auxiliary variables

[1] Sánchez-Sesma, F.J., Iturrarán, U. and Kausel, E. (2013): Garvin's Generalized Problem Revisited, *Soil Dynamics and Earthquake Engineering*, vol xxx

[2] Alterman, Z.S., and Loewenthal, D. (1969). Algebraic expressions for the impulsive motion of an elastic half-space, *Israel Journal of Technology*, **7** (6), 495-504.

The solution is expressed in terms of two auxiliary variables $q_{\alpha a}(\tau),\ q_{\alpha\beta}(\tau)$ together with their derivatives with respect to dimensionless time τ. The first of these can be given in closed form:

$$q_{\alpha\alpha}(\tau)=\cos\theta_2\sqrt{\tau^2-a^2}+\mathrm{i}\,\tau\sin\theta_2 \tag{5.32a}$$

$$\operatorname{Re} q_{\alpha\alpha}\geq 0\,,\qquad \operatorname{Re}\sqrt{q_{\alpha\alpha}^2+a^2}\geq 0\,,\qquad \operatorname{Re}\sqrt{q_{\alpha\alpha}^2+1}>0 \tag{5.32b}$$

$$\begin{aligned}\frac{\partial q_{\alpha\alpha}}{\partial\tau}&=\frac{\tau}{\sqrt{\tau^2-a^2}}\cos\theta_2+\mathrm{i}\sin\theta_2\\&=\left[\frac{q_{\alpha\alpha}}{\sqrt{q_{\alpha\alpha}^2+a^2}}\cos\theta_2-\mathrm{i}\sin\theta_2\right]^{-1}\end{aligned} \tag{5.33}$$

On the other hand, the expression for $q_{\alpha\beta}(\tau)$ requires the numerical solution to the quartic equation

$$A q_{\alpha\beta}^4-4\mathrm{i}\,B q_{\alpha\beta}^3-2C\,q_{\alpha\beta}^2+4\mathrm{i}\,D\,q_{\alpha\beta}+E=0 \tag{5.34a}$$

$$\operatorname{Re} q_{\alpha\beta}\geq 0\,,\qquad \operatorname{Re}\sqrt{q_{\alpha\beta}^2+a^2}\geq 0\,,\qquad \operatorname{Re}\sqrt{q_{\alpha\beta}^2+1}>0 \tag{5.34b}$$

with all *real* coefficients

$$A=\left[(H+Z)^2+X^2\right]\left[(H-Z)^2+X^2\right]>0 \tag{5.35a}$$

$$B=\tau X\left(H^2+X^2+Z^2\right)>0 \tag{5.35b}$$

$$C=\tau^2\left(H^2+3X^2+Z^2\right)-\left[X^2\left(H^2a^2+Z^2\right)+\left(H^2-Z^2\right)\left(H^2a^2-Z^2\right)\right] \tag{5.35c}$$

$$D=\tau X\left[\tau^2-\left(H^2a^2+Z^2\right)\right] \tag{5.35d}$$

$$E=\left(\tau^2-(Ha+Z)^2\right)\left(\tau^2-(Ha-Z)^2\right) \tag{5.35e}$$

where

$$H=\frac{h}{r_2},\ X=\frac{|x|}{r_2},\ Z=\frac{|z|}{r_2} \tag{5.35f}$$

This quartic equation admits four roots, which can show up as follows:

a) All roots are complex and appear in negative complex conjugate pairs (this is the norm when $\tau>\tau_{PS}$):
$q_1,q_2,q_3=-q_1^*,q_4=-q_2^*$

b) There exists one pair of negative complex conjugate roots and two distinct, purely imaginary roots:
$q_1,\ q_2=-q_1^*, q_3=\mathrm{i}Q_3,\ q_4=\mathrm{i}Q_4,$ (with Q_3,Q_4 being real quantities)

c) There are four distinct, purely imaginary roots:
$q_1=\mathrm{i}Q_1,\ q_2=\mathrm{i}Q_2,\ q_3=\mathrm{i}Q_3,\ q_4=\mathrm{i}Q_4,$ (All Q_j are real quantities)

d) No purely real roots can exist.

A mathematical analysis of the four solutions reveals that they define four branches, two of which have negative real parts and can thus be discarded on account of (5.34b). Of the remaining two roots, at least one of these two roots is *guaranteed* to have a positive imaginary part, and possibly even both roots have such a characteristic. Either way, choose the one with the smallest *positive* imaginary part, which is also the sole branch which starts as a purely imaginary, positive root when $\tau = \tau_{PS}$, i.e. at the arrival of the PS waves at the receiver. Having obtained $q_{\alpha\beta}$ one can proceed to obtain its derivatives as

$$\frac{dq_{\alpha\beta}}{d\tau} = \left[\left(\frac{q_{\alpha\beta}}{\sqrt{q_{\alpha\beta}^2 + a^2}} H + \frac{q_{\alpha\beta}}{\sqrt{q_{\alpha\beta}^2 + 1}} Z \right) - \mathrm{i}\, X \right]^{-1} \tag{5.36}$$

Rayleigh functions

Assuming that one has obtained $q_{\alpha\alpha}(\tau), q_{\alpha\beta}(\tau)$ together with their derivatives $\partial q_{\alpha\alpha} / \partial\tau, \partial q_{\alpha\beta} / \partial\tau$, one proceeds to use these to evaluate the Rayleigh functions

$$R_{\alpha\alpha}(\tau) = \left(1 + 2q_{\alpha\alpha}^2\right)^2 - 4q_{\alpha\alpha}^2 \sqrt{1 + q_{\alpha\alpha}^2}\sqrt{a^2 + q_{\alpha\alpha}^2} \tag{5.37a}$$

$$R_{\alpha\beta}(\tau) = \left(1 + 2q_{\alpha\beta}^2\right)^2 - 4q_{\alpha\beta}^2 \sqrt{1 + q_{\alpha\beta}^2}\sqrt{a^2 + q_{\alpha\beta}^2} \tag{5.37b}$$

Displacements:

$$\begin{aligned} u_x(x,z,t) = \frac{\operatorname{sgn}(x)}{\pi\mu} \Bigg\{ & \frac{1}{2r_1} \frac{\tau}{\sqrt{\tau^2 - \tau_P^2}} \sin\theta_1 \, \mathcal{H}(\tau - \tau_P) - \frac{1}{2r_2} \frac{\tau}{\sqrt{\tau^2 - \tau_{PP}^2}} \sin\theta_2 \, \mathcal{H}(\tau - \tau_{PP}) \\ & - \frac{4}{r_2} \operatorname{Im}\left[\frac{q_{\alpha\alpha}^3 \sqrt{q_{\alpha\alpha}^2 + 1}}{R_{\alpha\alpha}} \frac{\partial q_{\alpha\alpha}}{\partial\tau} \right] \mathcal{H}(\tau - \tau_{PP}) + \frac{2}{r_2} \operatorname{Im}\left[\frac{q_{\alpha\beta}\left(1 + 2q_{\alpha\beta}^2\right)\sqrt{1 + q_{\alpha\beta}^2}}{R_{\alpha\beta}} \frac{\partial q_{\alpha\beta}}{\partial\tau} \right] \mathcal{H}(\tau - \tau_{PS}) \Bigg\} \end{aligned} \tag{5.38a}$$

$$\begin{aligned} u_z(x,z,t) = \frac{\operatorname{sgn}(z)}{\pi\mu} \Bigg\{ & \frac{1}{2r_1} \frac{\tau}{\sqrt{\tau^2 - \tau_P^2}} \cos\theta_1 \, \mathcal{H}(\tau - \tau_P) + \frac{1}{2r_2} \frac{\tau}{\sqrt{\tau^2 - \tau_{PP}^2}} \cos\theta_2 \, \mathcal{H}(\tau - \tau_{PP}) \\ & - \frac{1}{r_2} \operatorname{Re}\left[\frac{\left(1 + 2q_{\alpha\alpha}^2\right)^2}{R_{\alpha\alpha}} \frac{\partial q_{\alpha\alpha}}{\partial\tau} \right] \mathcal{H}(\tau - \tau_{PP}) + \frac{1}{r_2} \operatorname{Re}\left[\frac{2q_{\alpha\beta}^2\left(1 + 2q_{\alpha\beta}^2\right)}{R_{\alpha\beta}} \frac{\partial q_{\alpha\beta}}{\partial\tau} \right] \mathcal{H}(\tau - \tau_{PS}) \Bigg\} \end{aligned} \tag{5.38b}$$

where $\mathcal{H}(\tau)$ is the Heaviside (unit step) function ; $\tau, \tau_P, \tau_{PP}, \tau_{PS}$ are defined by equations 5.31a-d; $q_{\alpha\alpha}(\tau)$ is given by 5.32a and $q_{\alpha\beta}(\tau)$ by the numerical solution to 5.34a; and the partial derivatives are given by 5.33 and 5.36. Observe that for a lower half-plane, $\operatorname{sgn}(z) = -1$. The vertical displacement is defined positive up for both a lower and an upper half-plane.

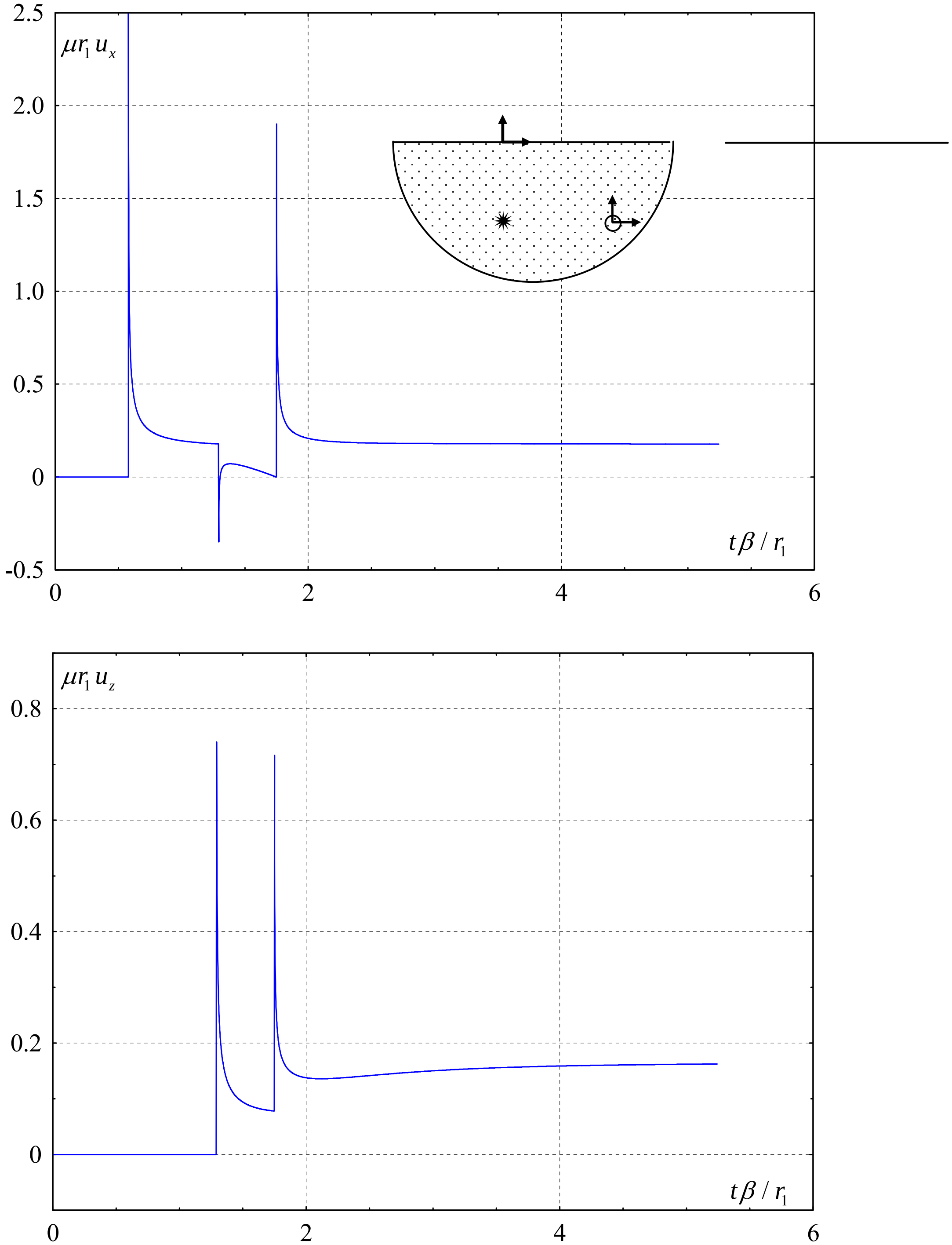


Figure 5.5: Generalized Garvin Problem, horizontal (top) and vertical (bottom) response for a source at depth $h = 1$ observed at a receiver at location $x = 1,\ |z| = 1$ for Poisson's ratio $\nu = 0.25$.

Figure 5.5 shows the horizontal and vertical displacements for a source-receiver combination $x = |z| = h = 1$ and material parameters $\beta = 1$, $\nu = 0.25$, which corresponds to Lamé parameters $\mu = \lambda = 1$. For convenience, we have chosen to display the time axis normalized with respect to r_1. The three peaks in the horizontal response correspond to the arrivals of the P, PP and PS waves. The vertical response shows only two peaks because the direct P wave travels horizontally from the source to the receiver, and thus has no vertical components.

Asymptotic(static) behavior
As time increases, the displacements approach their static values. It can be shown that at long times, the response functions approach asymptotically the following exact values:

$$u_x = \frac{\operatorname{sgn}(x)}{2\pi\mu}\left[\frac{1}{r_1}\sin\theta_1 + \frac{3-4\nu}{r_2}\sin\theta_2 - \frac{2}{r_2}\frac{|z|}{r_2}\sin 2\theta_2\right] \tag{5.39a}$$

$$u_z = \frac{\operatorname{sgn}(z)}{2\pi\mu}\left\{\frac{1}{r_1}\cos\theta_1 - \frac{3-4\nu}{r_2}\cos\theta_2 - \frac{2}{r_2}\frac{|z|}{r_2}\cos 2\theta_2\right\} \tag{5.39b}$$

When $z = 0$, the above equations reduce to the expressions 5.28a and 5.28b in section 5.5.

6. Three-dimensional problems in homogeneous half-spaces

6.1 Lamb's problem[3]

Either a vertical or horizontal point source P which varies as a step function in time, i.e. $P\mathcal{H}(t)$ is applied onto the surface of the half-space. Displacements are also observed on the surface at a range *r*.

Definitions:

$$r = \sqrt{x^2 + y^2} \tag{6.1}$$

$$a^2 = \left(\frac{\beta}{\alpha}\right)^2 = \frac{1-2\nu}{2(1-\nu)}, \qquad \tau = \frac{t\beta}{r} \tag{6.2}$$

$$\mathcal{H}(t-t_0) = \begin{cases} 1 & t > t_0 \\ \frac{1}{2} & t = t_0 \\ 0 & t < t_0 \end{cases} = \textit{Heaviside step function} \tag{6.3}$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{6.4}$$

$$\Pi(n,k) = \int_0^{\pi/2} \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \tag{6.5}$$

K and Π are complete elliptical integrals of the first and third kind, respectively. In the case of complex characteristic m, the elliptic Π function satisfies the complex conjugate symmetry $\Pi\left(m^*,n\right) = \Pi^*\left(m,n\right)$. To the best of the author's knowledge and as of this writing, only Mathematica —but not Maple or Matlab— seems to provide the capability of complex characteristic. However, it is not difficult to implement effective numerical routines which allow for complex values of these parameters (one such routine is available in the Appendix).

Consider the rationalized Rayleigh function

$$1 - 8\kappa^2 + 8\kappa^4(3-2a^2) - 16\kappa^6(1-a^2) = 0 \tag{6.6}$$

which has three roots $[\kappa_1^2, \kappa_2^2, \kappa_3^2]$, the first two of which are non-physical solutions of the rationalized Rayleigh function, while $\kappa_3 \equiv \beta / C_R$ is the actual true root. When $\nu < \nu_0 = 0.2631$, all three roots are real and satisfy $0 < \kappa_1^2 < \kappa_2^2 < a^2 < 1 < \kappa_3^2$. The transition value ν_0 is the root of the discriminant $D(\nu) = 32\nu^3 - 16\nu^2 + 21\nu - 5 = 0$ in the interval $[0 \le \nu \le 0.5]$. It defines the point beyond which the false roots turn complex. When $\nu = \nu_0$ the false roots are repeated, i.e. $\kappa_1 = \kappa_2$, and thereafter they appear in complex conjugate pairs. The ensuing formulas are valid whatever the value of Poisson's ratio.

Define the coefficients

$$A_j = \frac{\left(\kappa_j^2 - \frac{1}{2}\right)^2 \sqrt{a^2 - \kappa_j^2}}{D_j}, \qquad j = 1, 2 \qquad \tilde{A}_3 = \frac{\left(\kappa_3^2 - \frac{1}{2}\right)^2 \sqrt{\kappa_3^2 - a^2}}{D_3} \tag{6.7a}$$

[3] Kausel, E. (2012): Lamb's problem at its simplest, *Proceedings of the Royal Society*, Series A, 20120462

$$B_j = \frac{\left(1-2\kappa_j^2\right)\left(1-\kappa_j^2\right)}{D_j}, \quad j = 1,2,3 \tag{6.7b}$$

$$C_j = \frac{\left(1-\kappa_j^2\right)\sqrt{a^2-\kappa_j^2}}{D_j}, \quad j = 1, 2 \qquad\qquad \tilde{C}_3 = \frac{\left(1-\kappa_3^2\right)\sqrt{\kappa_3^2-a^2}}{D_3} \tag{6.7c}$$

$$D_j = \left(\kappa_j^2-\kappa_i^2\right)\left(\kappa_j^2-\kappa_k^2\right), \qquad\qquad i \neq j \neq k \tag{6.7d}$$

Observe that the coefficients with tilde differ from those without a tilde in the order of the arguments within the square root, which are reversed. In general, A_j, B_j, C_j may be complex (in which case they will appear in complex conjugate pairs) but $\tilde{A}_3, B_3, \tilde{C}_3$ are always real.

6.1.1 3-D half-space, suddenly applied vertical point load on its surface (Pekeris-Mooney problem)[3,4,5]

$$u_{zz}(r,\tau) = \frac{P(1-\nu)}{2\pi\mu r}\begin{cases} \frac{1}{2}\left[1-\left(\frac{A_1}{\sqrt{\tau^2-\kappa_1^2}}+\frac{A_2}{\sqrt{\tau^2-\kappa_2^2}}+\frac{\tilde{A}_3}{\sqrt{\kappa_3^2-\tau^2}}\right)\right] & a<\tau<1 \\ 1-\frac{\tilde{A}_3}{\sqrt{\kappa_3^2-\tau^2}} & 1 \le \tau < \kappa_3 \\ 1 & \tau \ge \kappa_3 \end{cases} \tag{6.8a}$$

$$u_{rz}(r,\tau) = \frac{P\tau}{8\pi\mu r}\begin{cases} \frac{1}{\pi\left(1-a^2\right)^{3/2}}\left\{2K(k)-\sum_{j=1}^{3}B_j\,\Pi\left(k^2 n_j,k\right)\right\}, & a<\tau<1 \\ \frac{k^{-1}}{\pi\left(1-a^2\right)^{3/2}}\left\{2K\left(k^{-1}\right)-\sum_{j=1}^{3}B_j\,\Pi\left(n_j,k^{-1}\right)\right\}, & 1 \le \tau < \kappa_3 \\ \frac{k^{-1}}{\pi\left(1-a^2\right)^{3/2}}\left\{2K\left(k^{-1}\right)-\sum_{j=1}^{3}B_j\,\Pi\left(n_j,k^{-1}\right)\right\}+\frac{2Q}{\sqrt{\tau^2-\kappa_3^2}}, & \tau \ge \kappa_3 \end{cases} \tag{6.8b}$$

in which

$$k^2 = \frac{\tau^2-a^2}{1-a^2}, \qquad n_j = \frac{1-a^2}{a^2-\kappa_j^2}, \qquad Q = \frac{\left(2\kappa_3^2-1\right)^3}{1-4\kappa_3^2+8(1-a^2)\kappa_3^6} \tag{6.9}$$

For complex roots, the first two terms in the summations in (6.8a,b) appear in complex conjugate pairs. Hence, one could just as well replace their sum by taking twice the real part of first term.

The displacements at depth along the epicentral axis are given later on in section 6.1.3

[4] Pekeris, C.L. (1955): The seismic surface pulse, *Proceedings of the National Academy of Sciences of the United States of America*, **41** (7), 469-480, (only for ν = ¼).
[5] Mooney, H.M. (1974), Some numerical solutions for Lamb's problem, *Bulletin of the Seismological Society of America*, **64** (2), 473-491, (any ν, but vertical displacement due to vertical load only).

6.1.2 3-D half-space, suddenly applied horizontal point load on its surface (Chao's problem)[3,6]

$$u_{rx}=\frac{P(\cos\theta)}{2\pi\mu r}\begin{cases}\frac{1}{2}(1-\nu)\tau^2\left[\frac{C_1}{\sqrt{\tau^2-\kappa_1^2}}+\frac{C_2}{\sqrt{\tau^2-\kappa_2^2}}+\frac{\tilde{C}_3}{\sqrt{\kappa_3^2-\tau^2}}\right] & a<\tau<1\\ 1+(1-\nu)\tau^2\frac{\tilde{C}_3}{\sqrt{\kappa_3^2-\tau^2}} & 1\le\tau<\kappa_3\\ 1 & \tau\ge\kappa_3\end{cases}\tag{6.10a}$$

$$u_{\theta x}=\frac{P(1-\nu)(-\sin\theta)}{2\pi\mu r}\begin{cases}\frac{1}{2}\left[1-C_1\sqrt{\tau^2-\kappa_1^2}-C_2\sqrt{\tau^2-\kappa_2^2}+\tilde{C}_3\sqrt{\kappa_3^2-\tau^2}\right] & a<\tau<1\\ 1+\tilde{C}_3\sqrt{\kappa_3^2-\tau^2} & 1\le\tau<\kappa_3\\ 1 & \tau\ge\kappa_3\end{cases}\tag{6.10b}$$

$$u_{zx}(r,\theta,\tau)=-u_{rz}(r,\tau)\cos\theta\tag{6.10c}$$

The horizontal displacements along the epicentral axis are given in the next section.

6.1.3 Pekeris-Mooney-Chao problems: Displacements at depth along the epicentral axis $(r=0,\ z\neq 0)$[3]:

With dimensionless time $\tau=t\beta/|z|$, the displacements on the axis at depth $|z|$ are

$$u_{xx}=\frac{1}{4\pi\mu|z|}\left[f_S(\tau)\mathcal{H}(\tau-1)-f_P(\tau)\mathcal{H}(\tau-a)\right]\tag{6.11a}$$

$$u_{zz}=\frac{1}{2\pi\mu|z|}\left[g_P(\tau)\mathcal{H}(\tau-a)-g_S(\tau)\mathcal{H}(\tau-1)\right]\tag{6.11b}$$

where

$$f_P=\frac{2\tau\left(\tau^2-a^2\right)S_1}{\left(2\tau^2-2a^2+1\right)^2-4\tau\left(\tau^2-a^2\right)S_1},\quad f_S=1+\frac{\left(2\tau^2-1\right)\tau^2}{\left(2\tau^2-1\right)^2-4\tau\left(\tau^2-1\right)S_2}\tag{6.12a}$$

$$g_P=\frac{\tau^2\left(2\tau^2-2a^2+1\right)}{\left(2\tau^2-2a^2+1\right)^2-4\tau\left(\tau^2-a^2\right)S_1},\quad g_S=\frac{2\tau\left(\tau^2-1\right)S_2}{\left(2\tau^2-1\right)^2-4\tau\left(\tau^2-1\right)S_2}\tag{6.12b}$$

$$S_1=\sqrt{\tau^2+1-a^2},\qquad S_2=\sqrt{\tau^2-1+a^2}\tag{6.12c}$$

[6] Chao, C.C. (1960), Dynamical Response of an elastic half-space to tangential surface loadings, *Journal of Applied Mechanics*, Vol 27, September, pp. 559-567

Although simple in appearance, at large times $\tau \gg 1$ the above representation suffers from severe cancellations. The reason is that although the sum of the two functions tends to a constant (static) value, individually each function grows without bound with time. This problem can be avoided by means of the following fully equivalent formulas when $\tau > 1$:

$$u_{xx} = \frac{1}{4\pi\mu|z|}\left\{1-\left(1-a^2\right)\left[\frac{\left(\tau^2-a^2\right)\left(2\tau^2-2a^2+1\right)^2}{\frac{1}{2}\left(1+S_1/\tau\right)D_1}+\frac{2\tau^2\left(2\tau^2-1\right)\left(\tau^2-1\right)}{\frac{1}{2}\left(1+S_2/\tau\right)D_2}\right]+\frac{1}{D_1D_2}\sum_{j=2,4,\ldots}^{12} a_j\tau^j\right\} \tag{6.13a}$$

$$u_{zz} = \frac{1}{2\pi\mu|z|}\left\{\left(1-a^2\right)\left[\frac{2\tau^2\left(2\tau^2-2a^2+1\right)\left(\tau^2-a^2\right)}{\frac{1}{2}\left(S_1/\tau+1\right)D_1}+\frac{\left(\tau^2-1\right)\left(2\tau^2-1\right)^2}{\frac{1}{2}\left(S_2/\tau+1\right)D_2}\right]+\frac{1}{D_1D_2}\sum_{j=2,4,\ldots}^{12} b_j\tau^j\right\} \tag{6.13b}$$

where

$$D_1 = 16\left(1-a^2\right)\tau^6+8\left(6a^4-8a^2+3\right)\tau^4-8\left(6a^6-10a^4+6a^2-1\right)\tau^2+\left(1-2a^2\right)^4 \tag{6.14a}$$

$$D_2 = 16\left(1-a^2\right)\tau^6-8\left(3-4a^2\right)\tau^4+8\left(1-2a^2\right)\tau^2+1 \tag{6.14b}$$

and the coefficients of the two summations are

$$\left.\begin{aligned}
a_{12} &= 128\left(1-a^2\right)\\
a_{10} &= -64\left(1+4a^2-6a^4\right)\\
a_8 &= -16\left(3-15a^2-4a^4+24a^6\right)\\
a_6 &= 16a^2\left(4-17a^2+10a^4+8a^6\right)\\
a_4 &= 16a^2\left(1-3a^2+7a^4-6a^6\right)\\
a_2 &= -\left(1-10a^2+40a^4-48a^6+16a^8\right)
\end{aligned}\right\} \tag{6.15a}$$

$$\left.\begin{aligned}
b_{12} &= 128\left(1-a^2\right)\\
b_{10} &= 64\left(1-2a^2\right)\left(2-4a^2+a^4\right)\\
b_8 &= -16\left(21-37a^2+4a^4+36a^6-16a^8\right)\\
b_6 &= 16\left(3+26a^2-78a^4+70a^6-8a^8-8a^{10}\right)\\
b_4 &= 4\left(15-87a^2+116a^4+24a^6-136a^8+64a^{10}\right)\\
b_2 &= \left(11-28a^2+16a^4\right)\left(1-2a^2\right)^3
\end{aligned}\right\} \tag{6.15b}$$

At large times, the above converge to

$$D_1 = D_2 \to 16\left(1-a^2\right)\tau^6+\cdots,\qquad \Sigma \to a_{12}\tau^{12}+\cdots \qquad \tfrac{1}{2}\left(1+S_j/\tau\right)\to 1$$

so

$$u_{xx} \to \frac{1}{4\pi\mu|z|}\left\{1 - \frac{(1-a^2)8}{16(1-a^2)} + \frac{128(1-a^2)}{16^2(1-a^2)^2}\right\} = \frac{1}{8\pi\mu|z|}\frac{2-a^2}{(1-a^2)} = \frac{3-2\nu}{8\pi\mu|z|}$$

$$u_{zz} \to \frac{1}{4\pi\mu|z|}\left(\frac{2-a^2}{1-a^2}\right) = \frac{3-2\nu}{4\pi\mu|z|}$$

which are the correct limits predicted by the Cerruti and Boussinesq theories for static tangential and vertical loads applied onto the surface of a half-space.

6.2. Lamb dipoles

Of the nine possible point dipoles which may act within a continuous space, the explicit formulas for Lamb's problem in section 6.1 allow obtaining a subset of these, namely the solutions for the six dipoles acting at the surface of the half-space which do not depend on derivatives of the displacement functions with respect to the vertical direction, see Fig. 2.2, left and middle column. In cylindrical coordinates, these six dipoles are given by equations 2.22a,b,d,e,g,i, which involve the radial derivatives of the Pekeris-Mooney-Chao problems. The terms needed in the dipole formulas are as follows:

$$\frac{u-v}{r} = \frac{1}{2\pi\mu r^2}\begin{cases} \tfrac{1}{2}(1-\nu)\left[C_1\dfrac{2\tau^2-\kappa_1^2}{\sqrt{\tau^2-\kappa_1^2}} + C_2\dfrac{2\tau^2-\kappa_2^2}{\sqrt{\tau^2-\kappa_2^2}} + \tilde{C}_3\dfrac{2\tau^2-\kappa_3^2}{\sqrt{\kappa_3^2-\tau^2}} - 1\right] & a<\tau<1 \\ \nu + (1-\nu)\tilde{C}_3\dfrac{2\tau^2-\kappa_3^2}{\sqrt{\kappa_3^2-\tau^2}} & 1<\tau<\kappa_3 \\ \nu & \tau>\kappa_3 \end{cases}$$

(6.16a)

$$\frac{\partial u}{\partial r} = (-)\frac{1}{2\pi\mu r^2}\begin{cases} \tfrac{1}{2}(1-\nu)\tau^2\left[C_1\dfrac{2\tau^2-3\kappa_1^2}{(\tau^2-\kappa_1^2)^{3/2}} + C_2\dfrac{2\tau^2-3\kappa_2^2}{(\tau^2-\kappa_2^2)^{3/2}} - \tilde{C}_3\dfrac{2\tau^2-3\kappa_3^2}{(\kappa_3^2-\tau^2)^{3/2}}\right] & a<\tau<1 \\ 1-(1-\nu)\tau^2\tilde{C}_3\dfrac{2\tau^2-3\kappa_3^2}{(\kappa_3^2-\tau^2)^{3/2}} & 1\le\tau<\kappa_3 \\ 1+\kappa_3\delta(\tau-\kappa_3) & \tau\ge\kappa_3 \end{cases}$$

(6.16b)

$$\frac{\partial v}{\partial r} = (-)\frac{1}{2\pi\mu r^2}\begin{cases} \tfrac{1}{2}(1-\nu)\left[1 - C_1\dfrac{2\tau^2-\kappa_1^2}{\sqrt{\tau^2-\kappa_1^2}} - C_2\dfrac{2\tau^2-\kappa_2^2}{\sqrt{\tau^2-\kappa_2^2}} - \tilde{C}_3\dfrac{2\tau^2-\kappa_3^2}{\sqrt{\kappa_3^2-\tau^2}}\right] & a<\tau<1 \\ (1-\nu)\left[1-\tilde{C}_3\dfrac{2\tau^2-\kappa_3^2}{\sqrt{\kappa_3^2-\tau^2}}\right] + \delta(\tau-1) & 1\le\tau<\kappa_3 \\ 1-\nu & \tau\ge\kappa_3 \end{cases}$$

(6.16c)

$$\frac{\partial W}{\partial r} = (-)\frac{(1-\nu)}{2\pi\mu r^2}\begin{cases} \dfrac{1}{2}\left(1+\dfrac{A_1\kappa_1^2}{(\tau^2-\kappa_1^2)^{\frac{3}{2}}} + \dfrac{A_2\kappa_2^2}{(\tau^2-\kappa_2^2)^{\frac{3}{2}}} - \dfrac{\tilde{A}_3\kappa_3^2}{(\kappa_3^2-\tau^2)^{\frac{3}{2}}}\right) & a<\tau<1 \\ 1-\dfrac{\tilde{A}_3\kappa_3^2}{(\kappa_3^2-\tau^2)^{\frac{3}{2}}} & 1\le\tau<\kappa_3 \\ 1+\kappa_3\,\delta(\tau-\kappa_3) & \tau\ge\kappa_3 \end{cases}$$

(6.16d)

Also,

$$\frac{\partial w}{\partial r} = -\frac{\partial U}{\partial r} = \text{complicated} \tag{6.17}$$

The last derivative above is rather cumbersome, inasmuch as it involves derivatives of the elliptical functions in which τ appears as argument in the modulus k and k^{-1}, see eqs. 6.9. Although this could be accomplished without much ado, it is a rather lengthy and tedious task, which is thus left to the readers to carry out, should it be needed. A simple alternative would be to use numerical differentiation.

As an example of application, consider the torsional dipole (2.23c), i.e.

$$\begin{aligned}\mathbf{T}_z &= \tfrac{1}{2}\left(\mathbf{G}_{yx} - \mathbf{G}_{xy}\right) = -\frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{u-v}{r}\right)\hat{\mathbf{t}} \\ &\equiv u_\theta\hat{\mathbf{t}}\end{aligned} \tag{6.18}$$

Substituting the preceding expressions, we obtain

$$u_\theta = \frac{1}{4\pi\mu r^2}\left\{\mathcal{H}(\tau-1) + \delta(\tau-1)\right\} \tag{6.19}$$

which agrees perfectly with the formula obtained by the method of images, see section 6.4, eq. 6.25 specialized for a source and receiver at the free surface.

Example10.14: Rigid sphere in an infinite space subjected to torsion and translation

Consider a rigid, massless sphere of radius R_0 embedded in an infinite homogeneous space which is acted upon at its center by a harmonic torque $M\,e^{i\omega t}$ or by a force $P\,e^{i\omega t}$, say both in direction *z*. This causes the sphere to rotate about the vertical axis by an angle $\vartheta e^{i\omega t}$ and to displace vertically by $u_z\,e^{i\omega t}$. In spherical coordinates, both of these rigid body displacements are of the form

$$\mathbf{u}(R,\phi,\theta) = \mathbf{T}_0^{(j)}\,\mathbf{L}_1^0\,\tilde{\mathbf{u}} \qquad \text{(from 10.169 with } m=1,\ n=0\text{)}$$

$$\tilde{\mathbf{u}}(R) = \mathbf{H}_1^{(2)}\,\mathbf{c} = \begin{cases} R\vartheta_z\,\hat{\mathbf{e}}_3 & \text{torsion, } j=2 \\ u_z\hat{\mathbf{e}}_{12} & \text{displac., } j=1 \end{cases} \qquad \text{(from 10.156a, 10.170 and 10.171)}$$

$$\mathbf{T}_0^{(1)} = \mathbf{diag}\begin{bmatrix}1 & 1 & 0\end{bmatrix} \qquad \text{(from 10.172a)}$$

$$\mathbf{T}_0^{(2)} = \mathbf{diag}\begin{bmatrix}0 & 0 & -1\end{bmatrix} \qquad \text{(from 10.172b)}$$

$$\hat{\mathbf{e}}_{12} = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T \qquad \text{(from 10.173a)}$$

$$\hat{\mathbf{e}}_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T \qquad \text{(from 10.173c)}$$

$$\mathbf{L}_1^0 = \begin{Bmatrix} \cos\phi & 0 & 0 \\ 0 & -\sin\phi & 0 \\ 0 & 0 & -\sin\phi \end{Bmatrix} \qquad \text{(from Table 10.8)}$$

$$\mathbf{H}_1^{(2)} = \begin{Bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & h_{33} \end{Bmatrix} \qquad \text{(from table 10.7)}$$

which is assembled with spherical Hankel functions of the second kind

$$h_0^{(2)}(z) = \mathrm{i}\,e^{-iz}/z\,, \qquad h_1^{(2)}(z) = \mathrm{i}\,e^{-iz}(1+\mathrm{i}\,z)/z^2$$

$$h_2^{(2)}(z) = \mathrm{i}\,e^{-iz}\left(3+3\mathrm{i}\,z - z^2\right)/z^3$$

$$h_{Pj} = h_j^{(2)}(z_P), \qquad h_{Sj} = h_j^{(2)}(z_S), \qquad z_P = \frac{\omega R}{\alpha} \qquad z_S = \frac{\omega R}{\beta}$$

$$h_{11} = \frac{dh_{P1}}{dz_P} \qquad h_{12} = 2\frac{h_{S1}}{z_S}, \qquad h_{21} = \frac{h_{P1}}{z_P}, \qquad h_{22} = \frac{1}{z_S}\frac{d(z_S h_{S1})}{dz_S}, \qquad h_{33} = h_{S1}$$

On the other hand, the tractions per steradian on the surface of the sphere are

$$\tilde{\mathbf{p}}(R_0) = -R_0^2\,\mathbf{F}_1^{(2)}\left(\mathbf{H}_1^{(2)}\right)^{-1}\tilde{\mathbf{u}} \qquad \text{(from eq. 10.156)}$$

$$\mathbf{p}(R_0,\phi,\theta) = \mathbf{T}_0^{(j)}\,\mathbf{L}_1^0\,\tilde{\mathbf{p}}(R_0) \qquad \text{(from eq. 10.164)}$$

with $j=2$ for torsion and $j=1$ for translation. This leads to the definition of the stiffness or impedance matrix

$$\mathbf{K} = -R_0^2\, \mathbf{F}_1^{(2)} \left(\mathbf{H}_1^{(2)}\right)^{-1} = R_0^2 \begin{Bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & 0 \\ 0 & 0 & K_{33} \end{Bmatrix}$$

whose elements are (Note: unlike in Table 10.7, here we are defining $\mathbf{F}_1^{(2)} = -\left\{ f_{ij} \right\}$, to avoid having to carry the negative sign throughout)

$$K_{11} = f_{11} g_{11} + f_{12} g_{21}, \qquad K_{12} = f_{11} g_{12} + f_{12} g_{22}$$

$$K_{21} = f_{21} g_{11} + f_{22} g_{21} \qquad K_{22} = f_{21} g_{12} + f_{22} g_{22} \qquad K_{33} = f_{33} h_{33}^{-1}$$

$$g_{11} = h_{22} / \Delta, \quad g_{12} = -h_{12} / \Delta, \quad g_{21} = -h_{21} / \Delta, \quad g_{22} = h_{11} / \Delta, \quad \Delta = h_{11} h_{22} - h_{12} h_{21}$$

$$f_{11} = k_P \left[(\lambda + 2\mu)\, h_{P1} - \frac{4\mu}{z_P} h_{P2} \right], \; f_{12} = k_S \frac{4\mu}{z_S} h_{S2}$$

$$f_{21} = k_P \frac{2\mu}{z_P} h_{P2}, \qquad f_{22} = k_S \left[\mu\, h_{s1} - \frac{2\mu}{z_S} h_{S2} \right] \qquad f_{33} = k_S \mu\, h_{S2}$$

At this point we consider separately the two problems of torsion and displacement of the rigid sphere.

a) Torsion:

$$\begin{Bmatrix} \tilde{p}_R \\ \tilde{p}_\phi \\ \tilde{p}_\theta \end{Bmatrix} = R_0^2 \begin{Bmatrix} f_{11} & f_{12} & 0 \\ f_{21} & f_{22} & 0 \\ 0 & 0 & f_{33} \end{Bmatrix} \begin{Bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & h_{33} \end{Bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} R_0 \vartheta_z = R_0^3\, \vartheta_z \begin{Bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & 0 \\ 0 & 0 & K_{33} \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} = R_0^3\, \vartheta_z \begin{Bmatrix} 0 \\ 0 \\ K_{33} \end{Bmatrix}$$

so the tractions per steradian in the space domain are

$$\mathbf{p}\left(R_0, \phi, \theta\right) = \mathbf{T}_0^{(2)}\, \mathbf{L}_1^0\, \tilde{\mathbf{p}}\left(R_0\right) = \begin{Bmatrix} p_R \\ p_\phi \\ p_\theta \end{Bmatrix} = R_0^3\, \vartheta_z \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{Bmatrix} \begin{Bmatrix} \cos\phi & 0 & 0 \\ 0 & -\sin\phi & 0 \\ 0 & 0 & -\sin\phi \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ K_{33} \end{Bmatrix} = R_0^3\, K_{33}\, \vartheta_z \sin\phi \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

that is

$$p_\theta = R_0^3\, K_{33}\, \vartheta_z \sin\phi = R_0^3 \frac{k_S \mu\, h_{S2}}{h_{S1}} \vartheta_z \sin\phi = \mu R_0^2 \frac{\Omega_S\, h_{S2}}{h_{S1}} \vartheta_z \sin\phi$$

The net torsional moment exerted by p_θ with moment arm $R_0 \sin\phi$ and elementary area $dA = \sin\phi\, d\phi\, d\theta$ per steradian is

$$M_z = \iint\limits_{Surface} R_0 \sin^2\phi\, p_\theta\, d\phi\, d\theta = \tfrac{8}{3}\pi R_0^4\, K_{33} \vartheta_z = \tfrac{8}{3}\pi\mu\, R_0^3 \frac{\Omega_S\, h_{S2}}{h_{S1}} \vartheta_z = \tfrac{8}{3}\pi\mu\, R_0^3 \frac{3 + 3\mathrm{i}\,\Omega_S - \Omega_S{}^2}{1 + \mathrm{i}\,\Omega_S} \vartheta_z, \qquad \Omega_S = \frac{\omega R_0}{\beta}$$

The frequency response function is the inverse relationship, namely

$$\boxed{\vartheta_z(\omega) = \frac{3}{8\pi\mu R_0^3} \frac{1 + \mathrm{i}\Omega_S}{3 + 3\mathrm{i}\Omega_S - \Omega_S{}^2} M_z}$$

which is identical to the transfer function due to support motion in a 1-DOF system with resonant frequency $\Omega_n = \sqrt{3}$ and fraction of viscous damping $\xi = \frac{1}{2}\sqrt{3} = 0.866$, which is less than critical. Hence, the impulse response function for a torsional moment $\delta(t)$ is the Fourier transform of this function, which can be shown to be

$$\boxed{\vartheta_z(t) = \frac{3}{8\pi\rho\beta R_0^4} \exp\left(-\frac{3}{2}\tau\right)\left[\cos\tfrac{\sqrt{3}}{2}\tau - \tfrac{\sqrt{3}}{3}\sin\tfrac{\sqrt{3}}{2}\tau\right]}, \qquad \tau = \frac{\beta t}{R_0} \geq 0$$

Using the formulas 4.50 or 4.51 in chapter 4.6 on the response of a full space to a torsional source with vertical axis and deriving from these an expression for the torsional stresses, it can be shown that that the above flexibility functions are correct.

b) Translation

$$\begin{Bmatrix} \tilde{p}_R \\ \tilde{p}_\phi \\ \tilde{p}_\theta \end{Bmatrix} = R_0^2 \begin{Bmatrix} f_{11} & f_{12} & 0 \\ f_{21} & f_{22} & 0 \\ 0 & 0 & f_{33} \end{Bmatrix} \begin{Bmatrix} h_{11} & h_{12} & 0 \\ h_{21} & h_{22} & 0 \\ 0 & 0 & h_{33} \end{Bmatrix}^{-1} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} u_z = R_0^2 u_z \begin{Bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & 0 \\ 0 & 0 & K_{33} \end{Bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = R_0^2 u_z \begin{Bmatrix} K_{11} + K_{12} \\ K_{21} + K_{22} \\ 0 \end{Bmatrix}$$

The tractions in space are then

$$\mathbf{p}(R_0, \phi, \theta) = \mathbf{T}_0^{(1)} \mathbf{L}_1^0 \tilde{\mathbf{p}}(R_0) = \begin{Bmatrix} p_R \\ p_\phi \\ p_\theta \end{Bmatrix} = R_0^2 u_z \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Bmatrix} \begin{Bmatrix} \cos\phi & 0 & 0 \\ 0 & -\sin\phi & 0 \\ 0 & 0 & -\sin\phi \end{Bmatrix} \begin{Bmatrix} K_{11} + K_{12} \\ K_{21} + K_{22} \\ 0 \end{Bmatrix} = R_0^2 u_z \begin{Bmatrix} (K_{11} + K_{12})\cos\phi \\ -(K_{21} + K_{22})\sin\phi \\ 0 \end{Bmatrix}$$

The vertical component of the tractions per steradian is

$$p_z = p_R \cos\phi - p_\phi \sin\phi = R_0^2 u_z \left[(K_{11} + K_{12})\cos^2\phi + (K_{21} + K_{22})\sin^2\phi \right]$$

which has a total resultant

$$\begin{aligned} P &= \iint_{Surface} p_z \sin\phi \, d\phi \, d\theta = 2\pi R_0^2 u_z \int_0^\pi \left[(K_{11} + K_{12})\cos^2\phi + (K_{21} + K_{22})\sin^2\phi \right] \sin\phi \, d\phi \\ &= \tfrac{4}{3}\pi R_0^2 (K_{11} + K_{12} + 2K_{21} + 2K_{22}) u_z \end{aligned}$$

Evaluation of the previous expressions with Matlab's symbolic tool yields the total force

$$P = \mu R_0 \, u_z \, \frac{4\pi}{3} \left[\frac{9\alpha^2\beta^2 - \omega^2 R^2 \left(\alpha^2 + 9\alpha\beta + 2\beta^2\right) + \mathrm{i}\,\omega R \left[9\alpha\beta\left(\alpha+\beta\right) - \omega^2 R^2 \left(\alpha + 2\beta\right)\right]}{\beta^2 \left[2\alpha^2 + \beta^2 - \omega^2 R^2 + \mathrm{i}\,\omega R \left(2\alpha + \beta\right)\right]} \right]$$

which can be written as

$$P = \mu R_0 \, u_z \, \frac{4\pi}{3} \left[\frac{9 - \Omega_S^2 \left(1 + 9a + 2a^2\right) + \mathrm{i}\,\Omega_S \left[9\left(1+a\right) - \Omega_S^2 \left(1+2a\right) a \right]}{2 + a^2 - a^2\,\Omega_S^2 + \mathrm{i}\,\Omega_S \left(2+a\right) a} \right]$$

in which μ is the shear modulus

$$\boxed{a = \frac{\beta}{\alpha} = \sqrt{\frac{1-2\nu}{2-2\nu}}}$$ = ratio of S to P wave velocity (ν = Poisson's ratio)

and

$$\boxed{\Omega_S = \frac{\omega R_0}{\beta}}$$ = dimensionless frequency

Observe that at zero frequency, $\Omega_S = 0$, this simplifies to

$$P = \mu \, R_0 \, u_z \, \frac{12\pi}{2 + a^2} = \frac{24\pi\left(1-\nu\right)}{5 - 6\nu} \mu \, R_0 \, u_z$$

and for an incompressible solid ($a = 0$ or $\nu = 0.5$), $P = 6\pi \, \mu \, R_0 \, u_z$, which is finite, as expected.

The flexibility function in the frequency domain is then

$$\boxed{u_z\left(\Omega_S\right) = \frac{1}{R_0 \, \mu} \frac{3}{4\pi} \left[\frac{2 + a^2 - a^2\,\Omega_S^2 + \mathrm{i}\,\Omega_S \left(2+a\right) a}{9 - \Omega_S^2 \left(1 + 9a + 2a^2\right) + \mathrm{i}\,\Omega_S \left[9\left(1+a\right) - \Omega_S^2 \left(1+2a\right) a \right]} \right]}$$

In the case of an incompressible solid $a = 0$, we obtain from the expression above

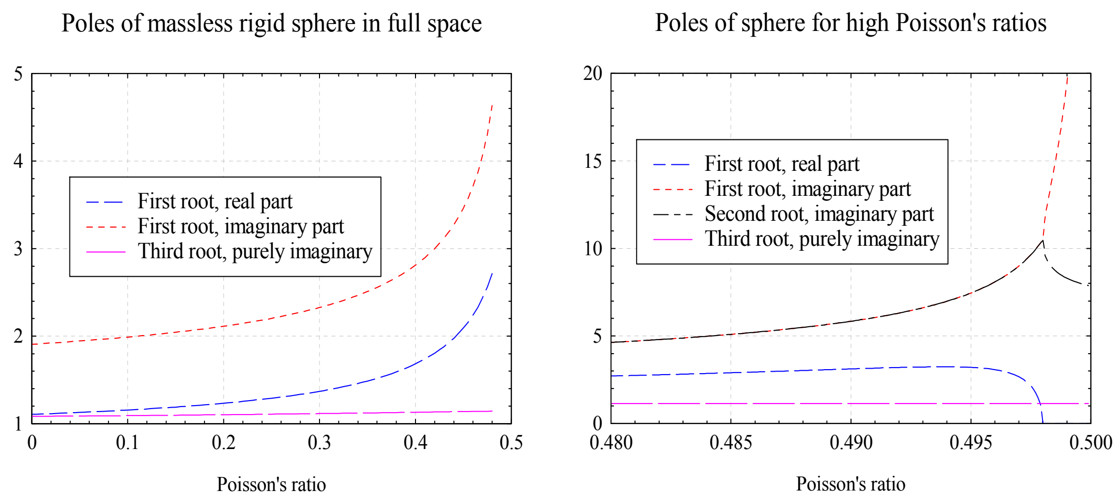
$$u_z\left(\Omega_S\right) = \frac{1}{R_0 \, \mu} \frac{1}{6\pi} \left[\frac{1}{1 - \left(\frac{1}{3}\Omega_S\right)^2 + 3\mathrm{i}\left(\frac{1}{3}\Omega_S\right)} \right]$$

which is identical to the transfer function for a 1-DOF system with undamped natural frequency $\Omega_n = 3$, that is $\omega_n = 3\beta / R_0$, and a fraction of critical damping $\xi = \frac{3}{2}$. Thus, the vibration is highly damped and not oscillatory.

Time domain:
For an impulsive load $P\,\delta\left(t\right)$, the response in time is qualitatively similar to that of a 1-DOF system with supercritical damping. To obtain this response, it suffices to find the poles of the denominator of the flexibility function, i.e.

$$\mathrm{i}\Omega^3(1+2a)a+\Omega^2(1+9a+2a^2)-9\mathrm{i}(1+a)\Omega-9=0$$

and use these in the context of a simple contour integration. The only difficulty here is that the denominator is cubic in the frequency, so finding the poles usually requires a numerical solution (the exact cubic solution is too complicated to be practical). The plot below shows the first complex root (blue and red), the imaginary part of the second root (black), and the *purely imaginary* third root



(magenta). For Poisson's ratios less than $\nu=0.498$, the second root is the negative complex conjugate of the first root, so it is not shown in the plot on left.

Above the $\nu=0.498$ threshold, all three roots are purely imaginary, and the first one grows as $a^{-1}\to\infty$ as Poisson's ratio goes to ½ in the neighborhood of an incompressible solid. This can be demonstrated by finding the characteristic roots while neglecting terms in $a^2\approx 0$:

$$\mathrm{i}\Omega^3(1+2a)a+\Omega^2(1+9a)-9\mathrm{i}(1+a)\Omega-9=0$$

whose solutions are $\Omega_1=\mathrm{i}/a$, $\Omega_{2,3}=\frac{3}{2}\mathrm{i}\left(3\pm\sqrt{5}\right)$. These agree with the values in the plot above on the right in the close neighborhood of $\nu=0.5$.

Sphere with mass

Addition of a mass to the sphere changes these results only slightly. If ρ_s is the mass density of the sphere and ρ is the mass density of the full space, then $m=\frac{4}{3}\pi R_0^3\rho_s$, so

$$\begin{aligned}P&=\mu R_0 u_z\frac{4\pi}{3}\left[\frac{9-\Omega_S^2(1+9a+2a^2)+\mathrm{i}\Omega_S\left[9(1+a)-\Omega_S^2(1+2a)a\right]}{2+a^2-a^2\Omega_S^2+\mathrm{i}\Omega_S(2+a)a}\right]-\frac{4\pi}{3}\rho_s R_0^3\omega^2 u_z\\&=\mu R_0 u_z\frac{4\pi}{3}\left[\frac{9-\Omega_S^2(1+9a+2a^2)+\mathrm{i}\Omega_S\left[9(1+a)-\Omega_S^2(1+2a)a\right]}{2+a^2-a^2\Omega_S^2+\mathrm{i}\Omega_S(2+a)a}-\frac{\rho_s}{\rho}\Omega_S^2\right]\end{aligned}$$

whose inverse is

$$u_z(\Omega_S)=\frac{1}{R_0\,\mu}\frac{3}{4\pi}\left[\frac{2+a^2-a^2\,\Omega_S^2+\mathrm{i}\Omega_S\,(2+a)\,a}{9-\Omega_S^2\left(1+9a+2a^2\right)+\mathrm{i}\Omega_S\left[9(1+a)-\Omega_S^2\,(1+2a)\,a\right]-(\rho_s\,/\,\rho)\Omega_S^2\left[2+a^2-a^2\,\Omega_S^2+\mathrm{i}\Omega_S\,(2+a)\,a\right]}\right]$$

$$=\frac{1}{R_0\,\mu}\frac{3}{4\pi}\left[\frac{2+a^2-a^2\,\Omega_S^2+\mathrm{i}\Omega_S\,(2+a)\,a}{9-\Omega_S^2\left(1+9a+2a^2+r\left(2+a^2\right)\right)+\Omega_S^4 r\,a^2+\mathrm{i}\Omega_S\left[9(1+a)-\Omega_S^2\,a\left(1+2a+r(2+a)\right)\right]}\right]$$

with

$$r=\frac{\rho_s}{\rho}$$

Expanding the denominator above, we obtain the fourth order polynomial

$$ra^2\Omega^4-ia(2a+ar+1+2r)\Omega^3-\left(1+9a+ra^2+2a^2+2r\right)\Omega^2+9i(1+a)\Omega+9=0$$

or

$$ra^2(\Omega-\Omega_1)(\Omega-\Omega_2)(\Omega-\Omega_3)(\Omega-\Omega_4)=0$$

Using contour integration we obtain the response in the time domain as

$$u_z(t)=\frac{1}{R_0^2\,\rho\beta}\frac{3}{4\pi}\left\{\frac{1}{2\pi}\left[\frac{2\pi\mathrm{i}}{r\,a^2}\sum_{j=1,4\neq k,l,m}\frac{2+a^2-a^2\,\Omega_j^2+\mathrm{i}\Omega_j\,(2+a)\,a}{\left(\Omega_j-\Omega_k\right)\left(\Omega_j-\Omega_l\right)\left(\Omega_j-\Omega_m\right)}\exp\left(\mathrm{i}\Omega_j\tau\right)\right]\right\}$$

which can readily be evaluated for any mass ratio r. When the sphere has the same density as the surrounding soil, i.e. $r=1$, the poles of the flexibility function can be found in closed form. Indeed, the characteristic equation in this case is

$$a^2\,\Omega^4-3\mathrm{i}\,a(1+a)\Omega^3-3\left(1+3a+a^2\right)\Omega^2+9\mathrm{i}(1+a)\Omega+9=0$$

$$\Omega_{1,2}=\frac{1}{2a}\left(\mathrm{i}\pm\sqrt{3}\right),\qquad\qquad \Omega_{3,4}=\frac{1}{2}\left(\mathrm{i}\pm\sqrt{3}\right)$$

Using the above roots for $r=1$ as well as $m=\frac{4}{3}\rho_s\pi R_0^3$, we obtain

$$\boxed{u_z(t)=\frac{1}{R_0^2\,\rho\,\beta}\frac{1}{2\pi}\frac{\sqrt{3}}{3}\left[a\,\exp\left(-\tfrac{3}{2a}\tau\right)\sin\left(\tfrac{\sqrt{3}}{2a}\tau\right)+2\exp\left(-\tfrac{3}{2}\tau\right)\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)\right]},\quad r=\frac{\rho_s}{\rho}=1\text{`},$$

$$\tau=\frac{\beta t}{R_0}\geq 0$$

For convenience, we list also the velocity and acceleration:

$$\frac{\partial}{\partial t}u_z(t)=\frac{1}{3m}\left\{\exp\left(-\tfrac{3}{2a}\tau\right)\left[\cos\left(\tfrac{\sqrt{3}}{2a}\tau\right)-\sqrt{3}\sin\left(\tfrac{\sqrt{3}}{2a}\tau\right)\right]+2\exp\left(-\tfrac{3}{2}\tau\right)\left[\cos\left(\tfrac{\sqrt{3}}{2}\tau\right)-\sqrt{3}\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)\right]\right\}$$

$$\frac{\partial^2}{\partial t^2}u_z(t)=\frac{\beta}{R_0\,m}\left\{\tfrac{1}{a}\exp\left(-\tfrac{3}{2a}\tau\right)\left[\sin\left(\tfrac{\sqrt{3}}{2a}\tau\right)-\sqrt{3}\cos\left(\tfrac{\sqrt{3}}{2a}\tau\right)\right]+2\exp\left(-\tfrac{3}{2}\tau\right)\left[\sin\left(\tfrac{\sqrt{3}}{2}\tau\right)-\sqrt{3}\cos\left(\tfrac{\sqrt{3}}{2}\tau\right)\right]\right\}$$

Observe that the velocity satisfies the condition $m\dot{u}_z(0)=1$, which demonstrates that the impulse is of unit magnitude.

Displacements beyond sphere

Once $u_z = u_z(\Omega_S)$ is known as a function of frequency, it can be used to find the frequency response at arbitrary points $R > R_0$ beyond the surface of the sphere. The displacements there are simply

$$\boxed{\mathbf{u}(R,\phi,\theta) = u_z\, \mathbf{T}_0^{(1)}\, \mathbf{L}_1^0\; \mathbf{H}_1^{(2)}(R)\left[\mathbf{H}_1^{(2)}(R_0)\right]^{-1} \hat{\mathbf{e}}_{12}}$$

where $\mathbf{H}_1^{(2)}(R)$ is of the same form as $\mathbf{H}_1^{(2)}(R_0)$, but computed at a different radial distance. This can be used to obtain the displacements at points beyond the sphere.

Appendix 1: Further details on the theory of cylindrical layers

Stresses in cylindrical layers

The book gives only final results for the displacements and stresses in cylindrical layers, most of which is not self-evident, so here are some further details. From section 1.4, eq. 1.79, the stresses in cylindrical surfaces are

$$\mathbf{s}_r = \begin{bmatrix} \sigma_r & \sigma_{r\theta} & \sigma_{rz} \end{bmatrix}^T = \mathbf{D}_{rr} \frac{\partial \mathbf{u}}{\partial r} + \mathbf{D}_{r\theta} \frac{1}{r} \frac{\partial \mathbf{u}}{\partial \theta} + \mathbf{D}_{rz} \frac{\partial \mathbf{u}}{\partial z} + \mathbf{D}_{r1} \frac{\mathbf{u}}{r} \tag{1}$$

in which $\mathbf{u} = \begin{bmatrix} u_r & u_\theta & u_z \end{bmatrix}^T$ and

$$\mathbf{D}_{rr} = \begin{Bmatrix} \lambda + 2\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{Bmatrix}, \qquad \mathbf{D}_{r\theta} = \mathbf{D}_{\theta r}^T = \begin{Bmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} \tag{2a-d}$$

$$\mathbf{D}_{rz} = \mathbf{D}_{zr}^T = \begin{Bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{Bmatrix}, \qquad \mathbf{D}_{r1} = \mathbf{D}_{1r} = \begin{Bmatrix} \lambda & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$

We now define the transformation matrix $\mathbf{Q}$ together with its inverse

$$\mathbf{Q} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\mathrm{i} \end{Bmatrix} \qquad \mathbf{Q}^{-1} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{i} \end{Bmatrix} \tag{3}$$

with which we can write the transformed displacement and stresses as

$$\bar{\mathbf{u}} = \mathbf{Q}\,\mathbf{u} = \begin{bmatrix} u_r & u_\theta & -\mathrm{i}\,u_z \end{bmatrix}^T \tag{4a}$$

$$\bar{\mathbf{s}}_r = \mathbf{Q}\,\mathbf{s}_r = \begin{bmatrix} \sigma_r & \sigma_{r\theta} & -\mathrm{i}\,\sigma_{rz} \end{bmatrix}^T \tag{4b}$$

Why add the imaginary unit? Because it renders the stiffness matrices symmetric and also makes them real when the frequency is zero. Pre-multiplying eq. 1 by $\mathbf{Q}$ and expressing $\mathbf{u}$ as $\mathbf{u} = \mathbf{Q}^{-1}\bar{\mathbf{u}}$, we obtain

$$\bar{\mathbf{s}}_r = \mathbf{Q}\,\mathbf{D}_{rr}\mathbf{Q}^{-1} \frac{\partial \bar{\mathbf{u}}}{\partial r} + \mathbf{Q}\mathbf{D}_{r\theta}\mathbf{Q}^{-1} \frac{1}{r} \frac{\partial \bar{\mathbf{u}}}{\partial \theta} + \mathbf{Q}\mathbf{D}_{rz}\mathbf{Q}^{-1} \frac{\partial \bar{\mathbf{u}}}{\partial z} + \mathbf{Q}\mathbf{D}_{r1}\mathbf{Q}^{-1} \frac{\bar{\mathbf{u}}}{r} \tag{5}$$

But

$$\mathbf{Q}\,\mathbf{D}_{rr}\mathbf{Q}^{-1} = \mathbf{D}_{rr}, \qquad \mathbf{Q}\mathbf{D}_{r\theta}\mathbf{Q}^{-1} = \mathbf{D}_{r\theta}, \qquad \mathbf{Q}\mathbf{D}_{r1}\mathbf{Q}^{-1} = \mathbf{D}_{r1} \tag{6}$$

The sole exception is

$$\mathbf{Q}\mathbf{D}_{rz}\mathbf{Q}^{-1} = \mathrm{i}\,\bar{\mathbf{D}}_{rz} = \mathrm{i}\begin{Bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ -\mu & 0 & 0 \end{Bmatrix} \tag{7}$$

so

$$\bar{\mathbf{s}}_r = \mathbf{D}_{rr}\frac{\partial \bar{\mathbf{u}}}{\partial r} + \mathbf{D}_{r\theta}\frac{1}{r}\frac{\partial \bar{\mathbf{u}}}{\partial \theta} + \mathrm{i}\,\bar{\mathbf{D}}_{rz}\frac{\partial \bar{\mathbf{u}}}{\partial z} + \mathbf{D}_{r1}\frac{\bar{\mathbf{u}}}{r} \tag{8}$$

Now, in section 8.8, eq. 8.155 we found particular solutions to the wave equation of the form

$$\bar{\mathbf{u}}(r,\theta,z) = \mathbf{T}_n\,\mathbf{H}_n\,\mathbf{a}\,e^{-\mathrm{i}k_z z} \tag{9a}$$

$$\mathbf{T}_n = \mathbf{diag}\left[\begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} \quad \begin{pmatrix} -\sin n\theta \\ \cos n\theta \end{pmatrix} \quad \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix}\right] \tag{9b}$$

$$\mathbf{a} = \text{vector of arbitrary constants} \tag{9c}$$

$$\mathbf{H}_n = \begin{Bmatrix} H'_{\alpha n} & n\dfrac{H_{\beta n}}{k_\beta r} & \dfrac{k_z}{k_\beta}H'_{\beta n} \\ n\dfrac{H_{\alpha n}}{k_\alpha r} & \left(H_{\beta n}\right)' & n\dfrac{k_z}{k_\beta}\dfrac{H_{\beta n}}{k_\beta r} \\ -\dfrac{k_z}{k_\alpha}H_{\alpha n} & 0 & H_{\beta n} \end{Bmatrix} \tag{9d}$$

in which the $H_{\alpha n} = H_n(z_\alpha), H_{\beta n} = H_n(z_\beta)$ are *any* of the Bessel functions of order n and arguments

$$z_\alpha = k_\alpha r = r\sqrt{k_P^2 - k_z^2}\,, \qquad z_\beta = k_\beta r = r\sqrt{k_S^2 - k_z^2} \tag{10a,b}$$

Also,

$$\begin{aligned} \mathbf{D}_{r\theta}\frac{\partial \mathbf{T}_n}{\partial \theta} &= n\begin{Bmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}\mathrm{diag}\left\{\begin{pmatrix} -\sin n\theta \\ \cos n\theta \end{pmatrix} \quad \begin{pmatrix} -\cos n\theta \\ -\sin n\theta \end{pmatrix} \quad \begin{pmatrix} -\sin n\theta \\ \cos n\theta \end{pmatrix}\right\} \\ &= n\ \mathrm{diag}\left\{\begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix} \quad \begin{pmatrix} -\sin n\theta \\ \cos n\theta \end{pmatrix} \quad \begin{pmatrix} \cos n\theta \\ \sin n\theta \end{pmatrix}\right\}\begin{Bmatrix} 0 & -\lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} = n\,\mathbf{T}_n\,\bar{\mathbf{D}}_{r\theta} \end{aligned} \tag{11a}$$

with

$$\bar{\mathbf{D}}_{r\theta} = \begin{Bmatrix} 0 & -\lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} \tag{11b}$$

On the other hand, from 9a,

$$\mathrm{i}\,\bar{\mathbf{D}}_{rz}\frac{\partial \bar{\mathbf{u}}}{\partial z} = \mathrm{i}\left(-\mathrm{i}\,k_z\right)\bar{\mathbf{D}}_{rz}\mathbf{T}_n\mathbf{H}_n\,\mathbf{a}\,e^{-\mathrm{i}k_z z} = k_z\,\bar{\mathbf{D}}_{rz}\mathbf{T}_n\mathbf{H}_n\,\mathbf{a}\,e^{-\mathrm{i}k_z z} \tag{12}$$

It is also easy to see that

$$\mathbf{D}_{rr}\mathbf{T}_n = \mathbf{T}_n\mathbf{D}_{rr}, \qquad \mathbf{D}_{r\theta}\mathbf{T}_n = \mathbf{T}_n\mathbf{D}_{r\theta}, \qquad \mathbf{D}_{r1}\mathbf{T}_n = \mathbf{T}_n\mathbf{D}_{r1} \tag{13}$$

Hence, from 8,11,12,13, we obtain the stresses in cylindrical planes as

$$\bar{\mathbf{s}}_r = \mathbf{T}_n\left[\mathbf{D}_{rr}\frac{\partial}{\partial r} + \bar{\mathbf{D}}_{r\theta}\frac{n}{r} + k_z\,\bar{\mathbf{D}}_{rz} + \frac{1}{r}\mathbf{D}_{r1}\right]\mathbf{H}_n\,\mathbf{a}\,e^{-\mathrm{i}k_z z} \tag{14}$$

Defining the tractions per unit radian as $\bar{\mathbf{p}} = r\,\bar{\mathbf{s}}_r$, then

$$\bar{\mathbf{p}} = \mathbf{T}_n\left[\mathbf{D}_{rr}\,r\frac{\partial}{\partial r} + n\,\bar{\mathbf{D}}_{r\theta} + k_z r\,\bar{\mathbf{D}}_{rz} + \mathbf{D}_{r1}\right]\mathbf{H}_n\,\mathbf{a}\,e^{-\mathrm{i}k_z z} \tag{15}$$

Finally, we can write 9a and 15 compactly as

$$\bar{\mathbf{u}}(r,\theta,z) = \mathbf{T}_n\,\mathfrak{u}\,e^{-\mathrm{i}k_z z}, \qquad \boxed{\mathfrak{u} = \mathbf{H}_n\,\mathbf{a}} \tag{16a}$$

$$\bar{\mathbf{p}} = r\,\bar{\mathbf{s}}_r = \mathbf{T}_n\,\mathfrak{p}\,e^{-\mathrm{i}k_z z}, \qquad \boxed{\mathfrak{p} = \left[\mathbf{D}_{rr}\,r\frac{\partial}{\partial r} + n\,\bar{\mathbf{D}}_{r\theta} + k_z r\,\bar{\mathbf{D}}_{rz} + \mathbf{D}_{r1}\right]\mathbf{H}_n\,\mathbf{a}} \tag{16b}$$

Now, considering that $r\frac{\partial}{\partial r}J_n\left(kr\right) = z\frac{\partial}{\partial z}J_n\left(z\right) = z\,J_n'$ (and similarly for other Bessel functions), then

$$r\frac{\partial}{\partial r}\mathbf{H}_n = \begin{Bmatrix} z_\alpha H_{\alpha n}'' & n\,z_\beta\left(\dfrac{H_{\beta n}}{z_\beta}\right)' & \dfrac{k_z}{k_\beta}z_\beta H_{\beta n}'' \\ n z_\alpha\left(\dfrac{H_{\alpha n}}{k_\alpha r}\right)' & z_\beta H_{\beta n}'' & n\,k_z r\left(\dfrac{H_{\beta n}}{z_\beta}\right)' \\ -k_z r\,H_{\alpha n}' & 0 & z_\beta H_{\beta n}' \end{Bmatrix} \tag{17}$$

In addition, from the differential equation for Bessel functions,

$$H_n'' + \tfrac{1}{z}H_n' + \left(1-\tfrac{n^2}{z^2}\right)H_n = 0, \qquad z = kr, \qquad H_n' = \tfrac{\partial}{\partial z}H_n \tag{18}$$

so

$$z\,H_n'' = -\left[H_n' + z\left(1-\tfrac{n^2}{z^2}\right)H_n\right] \tag{19}$$

Also,

$$z\left(\frac{H_n}{z}\right)' = z\left(\frac{H_n'}{z} - \frac{H_n}{z^2}\right) = H_n' - \frac{H_n}{z} \tag{20}$$

Hence

$$\mathbf{D}_{rr}\left(r\frac{\partial}{\partial r}\mathbf{H}_n\right)=$$

$$\begin{Bmatrix} -(\lambda+2\mu)\left[H'_{\alpha n}+z_\alpha\left(1-\frac{n^2}{z_\alpha^2}\right)H_{\alpha n}\right] & (\lambda+2\mu)n\left(H'_{\beta n}-\dfrac{H_{\beta n}}{z_\beta}\right) & -(\lambda+2\mu)\dfrac{k_z}{k_\beta}\left[H'_{\beta n}+z_\beta\left(1-\frac{n^2}{z_\beta^2}\right)H_{\beta n}\right] \\ \mu n\left(H'_{\alpha n}-\dfrac{H_{\alpha n}}{z_\alpha}\right) & -\mu\left[H'_{\beta n}+z_\beta\left(1-\frac{n^2}{z_\beta^2}\right)H_{\beta n}\right] & \mu n\dfrac{k_z}{k_\beta}\left(H'_{\beta n}-\dfrac{H_{\beta n}}{z_\beta}\right) \\ -\mu r k_z H'_{\alpha n} & 0 & \mu z_\beta H'_{\beta n} \end{Bmatrix} \tag{21}$$

Furthermore

$$\begin{aligned} \bar{\mathbf{D}}_{r\theta}+k_z r\,\bar{\mathbf{D}}_{rz}+\mathbf{D}_{r1} &= n\begin{Bmatrix} 0 & -\lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}+k_z r\begin{Bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ -\mu & 0 & 0 \end{Bmatrix}+\begin{Bmatrix} \lambda & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & 0 \end{Bmatrix} \\ &= \begin{Bmatrix} \lambda & -n\lambda & \lambda k_z r \\ n\mu & -\mu & 0 \\ -\mu k_z r & 0 & 0 \end{Bmatrix} \end{aligned} \tag{22}$$

so

$$\left(n\bar{\mathbf{D}}_{r\theta}+k_z r\,\bar{\mathbf{D}}_{rz}+\mathbf{D}_{r1}\right)\mathbf{H}_n=\begin{Bmatrix} \lambda & -n\lambda & \lambda k_z r \\ n\mu & -\mu & 0 \\ -\mu k_z r & 0 & 0 \end{Bmatrix}\begin{Bmatrix} H'_{\alpha n} & n\dfrac{H_{\beta n}}{z_\beta} & \dfrac{k_z}{k_\beta}H'_{\beta n} \\ n\dfrac{H_{\alpha n}}{z_\alpha} & H'_{\beta n} & n\dfrac{k_z}{k_\beta}\dfrac{H_{\beta n}}{z_\beta} \\ -\dfrac{k_z}{k_\alpha}H_{\alpha n} & 0 & H_{\beta n} \end{Bmatrix}$$

$$=\begin{Bmatrix} \lambda\left(H'_{\alpha n}-n^2\dfrac{H_{\alpha n}}{z_\alpha}-k_z r\dfrac{k_z}{k_\alpha}H_{\alpha n}\right) & -\lambda n\left(H'_{\beta n}-\dfrac{H_{\beta n}}{z_\beta}\right) & \lambda\dfrac{k_z}{k_\beta}\left[H'_{\beta n}+z_\beta\left(1-\frac{n^2}{z_\beta^2}\right)H_{\beta n}\right] \\ \mu n\left(H'_{\alpha n}-\dfrac{H_{\alpha n}}{z_\alpha}\right) & \mu\left(n^2\dfrac{H_{\beta n}}{z_\beta}-H'_{\beta n}\right) & \mu n\dfrac{k_z}{k_\beta}\left(H'_{\beta n}-\dfrac{H_{\beta n}}{z_\beta}\right) \\ -\mu k_z r\,H'_{\alpha n} & -\mu k_z r\,n\dfrac{H_{\beta n}}{z_\beta} & -\mu k_z r\,\dfrac{k_z}{k_\beta}H'_{\beta n} \end{Bmatrix} \tag{23}$$

Combining 21 and 23, we obtain

$$
\begin{Bmatrix}
-(\lambda+2\mu)\left[H'_{\alpha n}+z_\alpha\left(1-\frac{n^2}{z_\alpha^2}\right)H_{\alpha n}\right] & (\lambda+2\mu)n\left(H'_{\beta n}-\frac{H_{\beta n}}{z_\beta}\right) & -(\lambda+2\mu)\frac{k_z}{k_\beta}\left[H'_{\beta n}+z_\beta\left(1-\frac{n^2}{z_\beta^2}\right)H_{\beta n}\right] \\
\mu n\left(H'_{\alpha n}-\frac{H_{\alpha n}}{z_\alpha}\right) & -\mu\left[H'_{\beta n}+z_\beta\left(1-\frac{n^2}{z_\beta^2}\right)H_{\beta n}\right] & \mu n\frac{k_z}{k_\beta}\left(H'_{\beta n}-\frac{H_{\beta n}}{z_\beta}\right) \\
-\mu r k_z H'_{\alpha n} & 0 & \mu z_\beta H'_{\beta n}
\end{Bmatrix}
$$

$$
+\begin{Bmatrix}
\lambda\left(H'_{\alpha n}-n^2\frac{H_{\alpha n}}{z_\alpha}-k_z r\frac{k_z}{k_\alpha}H_{\alpha n}\right) & -\lambda n\left(H'_{\beta n}-\frac{H_{\beta n}}{z_\beta}\right) & \lambda\frac{k_z}{k_\beta}\left(H'_{\beta n}-n^2\frac{H_{\beta n}}{z_\beta}+z_\beta H_{\beta n}\right) \\
\mu n\left(H'_{\alpha n}-\frac{H_{\alpha n}}{z_\alpha}\right) & \mu\left(n^2\frac{H_{\beta n}}{z_\beta}-H'_{\beta n}\right) & \mu n\frac{k_z}{k_\beta}\left(H'_{\beta n}-\frac{H_{\beta n}}{z_\beta}\right) \\
-\mu k_z r H'_{\alpha n} & -\mu k_z r n\frac{H_{\beta n}}{z_\beta} & -\mu k_z r\frac{k_z}{k_\beta}H'_{\beta n}
\end{Bmatrix}
$$

$$
= r\,\mathbf{F} = r\begin{Bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{Bmatrix} \tag{24}
$$

$$
r\,f_{11} = -\left\{\lambda z_\alpha\left(1+\left(\frac{k_z}{k_\alpha}\right)^2\right)H_{\alpha n}+2\mu\left[H'_{\alpha n}+z_\alpha\left(1-\frac{n^2}{z_\alpha^2}\right)H_{\alpha n}\right]\right\}, \qquad r\,f_{12} = 2\mu n\left(H'_{\beta n}-\frac{H_{\beta n}}{z_\beta}\right)
$$

$$
r\,f_{13} = -2\mu\frac{k_z}{k_\beta}\left[H'_{\beta n}+z_\beta\left(1-\frac{n^2}{z_\beta^2}\right)H_{\beta n}\right], \qquad r\,f_{21} = 2\mu n\left(H'_{\alpha n}-\frac{H_{\alpha n}}{z_\alpha}\right)
$$

$$
r\,f_{22} = -\mu\left[2H'_{\beta n}+z_\beta\left(1-2\frac{n^2}{z_\beta^2}\right)H_{\beta n}\right], \qquad r\,f_{23} = \mu n\frac{k_z}{k_\beta}\left(H'_{\beta n}-\frac{H_{\beta n}}{z_\beta}\right)
$$

$$
r\,f_{31} = -2\mu r k_z H'_{\alpha n}, \qquad r\,f_{32} = -\mu k_z r n\frac{H_{\beta n}}{z_\beta}
$$

$$
r\,f_{33} = \mu z_\beta\left(1-\left(\frac{k_z}{k_\beta}\right)^2\right)H'_{\beta n} \tag{25a-i}
$$

which agrees with the material in the book on pages 165, 166.

A relevant question might be now, what kind of Bessel functions should one use? We take up this issue next.

Cylindrical layers: Which functions should one use?

Consider a cylindrical layer bounded by finite surface with radii r_1, r_2 (outer and inner radii, respectively). A general solution valid at any interior point $r_1 \ge r \ge r_2$ is then

$$\mathfrak{u}(r) = \mathbf{H}_n^{(1)}\,\mathbf{a}_1 + \mathbf{H}_n^{(2)}\,\mathbf{a}_2$$

$$\mathfrak{p}(r) = r\,\mathfrak{s}(r) = r\left(\mathbf{F}_n^{(1)}\,\mathbf{a}_1 + \mathbf{F}_n^{(2)}\,\mathbf{a}_2\right)$$

where $\mathbf{H}_n = \mathbf{H}_n\left(H_n^{(j)}(kr)\right)$ and $\mathbf{F}_n = \mathbf{F}_n\left(H_n^{(j)}(kr)\right)$,with $H_n^{(j)}(kr)$ being either Hankel functions of the first and second kind and n^{th} order, or alternatively, Bessel and Neumann functions of order *n*. This accounts for waves that propagate both inwards and outwards. A question is then, how do we decide which functions to use? To answer this question, let us assume that we decided at the outset to use first and second Hankel functions, and that we can express these as

$$\mathbf{H}_n^{(1)} = \mathbf{J}_n + \mathrm{i}\,\mathbf{Y}_n\,, \qquad \mathbf{H}_n^{(2)} = \mathbf{J}_n - \mathrm{i}\,\mathbf{Y}_n$$

where the matrices $\mathbf{J}_n, \mathbf{Y}_n$ have the same structure as the $\mathbf{H}_n$, except that Bessel and Neumann functions are used in place of Hankel functions. Hence

$$\begin{aligned}\mathfrak{u} &= \left(\mathbf{J}_n + \mathrm{i}\,\mathbf{Y}_n\right)\mathbf{a}_1 + \left(\mathbf{J}_n - \mathrm{i}\,\mathbf{Y}_n\right)\mathbf{a}_2 \\ &= \mathbf{J}_n\left(\mathbf{a}_1 + \mathbf{a}_2\right) + \mathrm{i}\;\mathbf{Y}_n\left(\mathbf{a}_1 - \mathbf{a}_2\right) \\ &= \mathbf{J}_n\mathbf{c}_1 + \mathbf{Y}_n\,\mathbf{c}_2\end{aligned}$$

where

$$\mathbf{c}_1 = \mathbf{a}_1 + \mathbf{a}_2\,, \qquad \mathbf{c}_2 = \mathrm{i}\left(\mathbf{a}_1 - \mathbf{a}_2\right)$$

Thus, as far as displacements is concerned, switching from Hankel functions to Bessel & Neumann functions is merely a matter of changing the constants of integration. Let's examine more carefully what effect such a switch has on the stresses. From the previous developments, a generic expression for the tractions per unit radian at some location is

$$\mathfrak{p} = \left[\mathbf{D}_{rr}\,r\frac{\partial}{\partial r} + n\,\bar{\mathbf{D}}_{r\theta} + k_z r\,\bar{\mathbf{D}}_{rz} + \mathbf{D}_{r1}\right]\mathbf{H}_n\,\mathbf{a} = \left(\mathbf{D}_{rr}\mathbf{H}_n'\mathbf{Z} + \mathbf{B}\mathbf{H}\right)\mathbf{a}$$

where $\mathbf{B} = n\,\bar{\mathbf{D}}_{r\theta} + k_z r\,\bar{\mathbf{D}}_{rz} + \mathbf{D}_{r1}$, $\mathbf{Z} = \operatorname{diag}\left\{z_\alpha \quad z_\beta \quad z_\beta\right\}$, and $r\frac{\partial}{\partial r}\mathbf{H}_n = \mathbf{H}_n'\mathbf{Z}$. It follows that

$$\begin{aligned}\mathfrak{p} &= \left(\mathbf{D}_{rr}\mathbf{H}_n'^{(1)}\mathbf{Z} + \mathbf{D}\mathbf{H}_n^{(1)}\right)\mathbf{a}_1 + \left(\mathbf{D}_{rr}\mathbf{H}_n'^{(2)}\mathbf{Z} + \mathbf{D}\,\mathbf{H}_n\right)\mathbf{a}_2 \\ &= \left[\mathbf{D}_{rr}\left(\mathbf{J}_n' + \mathrm{i}\,\mathbf{Y}_n'\right)\mathbf{Z} + \mathbf{D}\left(\mathbf{J}_n + \mathrm{i}\,\mathbf{Y}_n\right)\right]\mathbf{a}_1 + \left[\mathbf{D}_{rr}\left(\mathbf{J}_n' - \mathrm{i}\,\mathbf{Y}_n'\right)\mathbf{Z} + \mathbf{D}\left(\mathbf{J}_n - \mathrm{i}\,\mathbf{Y}_n\right)\right]\mathbf{a}_2 \\ &= \mathbf{D}_{rr}\left[\mathbf{J}_n'\mathbf{Z}\left(\mathbf{a}_1 + \mathbf{a}_2\right) + \mathrm{i}\,\mathbf{Y}_n'\mathbf{Z}\left(\mathbf{a}_1 - \mathbf{a}_2\right)\right] + \mathbf{D}\left[\mathbf{J}_n\left(\mathbf{a}_1 + \mathbf{a}_2\right) + \mathrm{i}\;\mathbf{Y}_n\left(\mathbf{a}_1 - \mathbf{a}_2\right)\right] \\ &= \left(\mathbf{D}_{rr}\mathbf{J}_n'\mathbf{Z} + \mathbf{D}\,\mathbf{J}_n\right)\mathbf{c}_1 + \left(\mathbf{D}_{rr}\,\mathbf{Y}_n'\mathbf{Z} + \mathbf{D}\,\mathbf{Y}_n\right)\mathbf{c}_2\end{aligned}$$

Hence, tractions in cylindrical surfaces follow the same combination rule as displacements, and they too can be expressed either in terms of Hankel functions, or of Bessel & Neumann functions. Hence, we have shown that at least in principle, the final results of the formulation for the stiffness (impedance) matrix for a cylindrical layer does not depend on which pair of Bessel functions is being used. In general, it is best to use Hankel functions, even when the arguments z_α, z_β of these functions should be real. Although Hankel functions are complex, final results will still be real, because the stiffness matrices could equally well have been obtained with Bessel and Neumann functions, which would have been purely real. However, the reason for choosing Hankel functions is that they are more robust and numerically stable when the argument is complex, because $J_n(z), Y_n(z)$ are nearly proportional when z is strongly imaginary. But this also brings up an important issue, namely the computational strategy in the assembly of the stiffness matrix, which we describe in the next section.

Observe that exterior, unbounded regions will always be constructed solely with second Hankel functions, because only these will satisfy the radiation conditions at infinity. Conversely, in the case of a solid cylindrical region (solid core), one must employ Bessel functions, for only these will avoid the singularity at the axis. In the case of sources placed at the axis, where a singularity then develops, one must either use the method described in the book (as amended in the section later on "*Analytic continuation in cylindrical layers*"), or simply avoid these problems altogether by replacing the solid core with a solid with a very small cylindrical borehole.

Scaling of Hankel and Bessel functions

When the arguments are complex or purely imaginary, the Bessel functions can attain very large values. To avoid ill-conditioning and a total breakdown in the computations, it behooves to scale these functions appropriately.

Matlab offers a very convenient scaling through an optional, additional argument, of which we shall take advantage. Indeed,

a) Hankel functions

$$\text{besselh}(\text{n},1,\text{z},1) \rightarrow H_n^{(1)}(z)\,\exp(-\mathrm{i}\,z) \underset{def}{=} \widehat{H}_n^{(1)}(z)$$

$$\text{besselh}(\text{n},2,\text{z},1) \rightarrow H_n^{(2)}(z)\,\exp(+\mathrm{i}\,z) \underset{def}{=} \widehat{H}_n^{(2)}(z)$$

b) Bessel and Neumann functions

$$\text{besselj}(\text{n},\text{z},1) \rightarrow J_n(z)\,\exp[-\text{abs}(\text{Im}(z)]$$

$$\text{bessely}(\text{n},\text{z},1) \rightarrow Y_n(z)\,\exp[-\text{abs}(\text{Im}(z)]$$

In the light of the above, we proceed to write the *actual* Hankel functions as

$$\boxed{H_n^{(1)}(z) = \widehat{H}_n^{(1)}(z)\exp(\mathrm{i}\,z)} \qquad \boxed{H_n^{(2)}(z) = \widehat{H}_n^{(2)}(z)\exp(-\mathrm{i}\,z)}$$

so the Hankel matrices will attain the form

$$\mathbf{H}_n^{(1)} = \widehat{\mathbf{H}}_n^{(1)}\,\mathbf{E}\,, \qquad \mathbf{H}_n^{(2)} = \widehat{\mathbf{H}}_n^{(2)}\,\mathbf{E}^{-1}\,, \qquad \mathbf{E} = \text{diag}\left\{\exp(\mathrm{i}\,z_\alpha) \quad \exp(\mathrm{i}\,z_\beta) \quad \exp(\mathrm{i}\,z_\beta)\right\}$$

Similarly

$$r\,\mathbf{F}_n^{(1)} = r\,\widehat{\mathbf{F}}_n^{(1)}\mathbf{E}\,, \qquad\qquad r\,\mathbf{F}_n^{(2)} = r\,\widehat{\mathbf{F}}_n^{(2)}\mathbf{E}^{-1}$$

in which case

$$\begin{aligned}\mathfrak{u}(r) &= \mathbf{H}_n^{(1)}\,\mathbf{a}_1 + \mathbf{H}_n^{(2)}\,\mathbf{a}_2 \\ &= \widehat{\mathbf{H}}_n^{(1)}\,\mathbf{E}\,\mathbf{a}_1 + \widehat{\mathbf{H}}_n^{(2)}\,\mathbf{E}^{-1}\mathbf{a}_2\end{aligned} \qquad\qquad \begin{aligned}\mathfrak{p}(r) &= r\left(\mathbf{F}_n^{(1)}\,\mathbf{a}_1 + \mathbf{F}_n^{(2)}\mathbf{a}_2\right) \\ &= r\left(\widehat{\mathbf{F}}_n^{(1)}\,\mathbf{E}\,\mathbf{a}_1 + \widehat{\mathbf{F}}_n^{(2)}\mathbf{E}^{-1}\mathbf{a}_2\right)\end{aligned}$$

Defining $\mathbf{E}_1 = \mathbf{E}(r_1)$, $\mathbf{E}_2 = \mathbf{E}(r_2)$, the equations for the layer expressed in terms of these expansions are

$$\begin{aligned}\begin{Bmatrix}\mathfrak{u}_1 \\ \mathfrak{u}_2\end{Bmatrix} &= \begin{Bmatrix}\widehat{\mathbf{H}}_{11} & \widehat{\mathbf{H}}_{12}\mathbf{E}_1^{-1}\,\mathbf{E}_2 \\ \widehat{\mathbf{H}}_{21}\mathbf{E}_2\mathbf{E}_1^{-1} & \widehat{\mathbf{H}}_{22}\end{Bmatrix}\begin{Bmatrix}\mathbf{E}_1\,\mathbf{a}_1 \\ \mathbf{E}_2^{-1}\mathbf{a}_2\end{Bmatrix} \\ &= \begin{Bmatrix}\widehat{\mathbf{H}}_{11} & \widehat{\mathbf{H}}_{12}\mathbf{E}_1^{-1}\,\mathbf{E}_2 \\ \widehat{\mathbf{H}}_{21}\mathbf{E}_2\mathbf{E}_1^{-1} & \widehat{\mathbf{H}}_{22}\end{Bmatrix}\begin{Bmatrix}\mathbf{b}_1 \\ \mathbf{b}_2\end{Bmatrix}\end{aligned}$$

and

$$\begin{aligned}\begin{Bmatrix}\mathfrak{p}_1 \\ \mathfrak{p}_2\end{Bmatrix} &= \begin{Bmatrix}r_1\widehat{\mathbf{F}}_{11} & r_1\widehat{\mathbf{F}}_{12}\mathbf{E}_1^{-1}\mathbf{E}_2 \\ -r_2\widehat{\mathbf{F}}_{21}\mathbf{E}_2\mathbf{E}_1^{-1} & -r_2\widehat{\mathbf{F}}_{22}\end{Bmatrix}\begin{Bmatrix}\mathbf{E}_1\,\mathbf{a}_1 \\ \mathbf{E}_2^{-1}\,\mathbf{a}_2\end{Bmatrix} \\ &= \begin{Bmatrix}r_1\widehat{\mathbf{F}}_{11} & r_1\widehat{\mathbf{F}}_{12}\mathbf{E}_1^{-1}\mathbf{E}_2 \\ -r_2\widehat{\mathbf{F}}_{21}\mathbf{E}_2\mathbf{E}_1^{-1} & -r_2\widehat{\mathbf{F}}_{22}\end{Bmatrix}\begin{Bmatrix}\mathbf{b}_1 \\ \mathbf{b}_2\end{Bmatrix}\end{aligned}$$

Eliminating the integration constants between these two, the dynamic stiffness matrix is

$$\mathbf{K} = \begin{Bmatrix}r_1\widehat{\mathbf{F}}_{11} & r_1\widehat{\mathbf{F}}_{12}\mathbf{E}_1^{-1}\mathbf{E}_2 \\ -r_2\widehat{\mathbf{F}}_{21}\mathbf{E}_2\mathbf{E}_1^{-1} & -r_2\widehat{\mathbf{F}}_{22}\end{Bmatrix}\begin{Bmatrix}\widehat{\mathbf{H}}_{11} & \widehat{\mathbf{H}}_{12}\mathbf{E}_1^{-1}\,\mathbf{E}_2 \\ \widehat{\mathbf{H}}_{21}\mathbf{E}_2\mathbf{E}_1^{-1} & \widehat{\mathbf{H}}_{22}\end{Bmatrix}^{-1}$$

which is computationally well behaved. The exponential matrix in the coupling terms is

$$\mathbf{E}_1^{-1}\,\mathbf{E}_2 = \mathbf{E}_2\,\mathbf{E}_1^{-1} = \operatorname{diag}\left\{\exp\left[-\mathrm{i}\,k_\alpha\left(r_1 - r_2\right)\right] \quad \exp\left[-\mathrm{i}\,k_\beta\left(r_1 - r_2\right)\right] \quad \exp\left[-\mathrm{i}\,k_\beta\left(r_1 - r_2\right)\right]\right\}$$

Inasmuch as our computations in elastodynamics satisfy $\operatorname{Im}(k_\alpha) \le 0$, $\operatorname{Im}(k_\beta) \le 0$, each of the exponential terms will be of the form

$$\exp\left[-\mathrm{i}(a - ib)(r_1 - r_2)\right] = \exp\left[-\mathrm{i}\,a(r_1 - r_2)\right]\exp\left[-b(r_1 - r_2)\right]$$

where $k = a - \mathrm{i}b$ is a generic wavenumber. The absolute value of this expression is less than unity because

$$\left|\exp\left[-\mathrm{i}\,a(r_1 - r_2)\right]\exp\left[-b(r_1 - r_2)\right]\right| = \left|\exp\left[-b(r_1 - r_2)\right]\right| = \frac{1}{\exp\left[b(r_1 - r_2)\right]} < 1$$

Thus, the computations are now well behaved.

Observe that in the case of an exterior, unbounded region of inner radius r, the stiffness matrix will be of the form

$$\mathbf{K}_{ext} = -r\widehat{\mathbf{F}}_n \widehat{\mathbf{H}}_n^{-1}$$

in which $\widehat{\mathbf{F}}_n, \widehat{\mathbf{H}}_n$ must be assembled with second Hankel functions, to satisfy the radiation conditions at infinity. It too is obtained with the aid of scaled matrices.

At the other extreme, in the case of a solid core of radius r where Bessel functions must used, its stiffness matrix will be of the form

$$\mathbf{K} = r\,\widehat{\mathbf{F}}_n \widehat{\mathbf{H}}_n^{-1}$$

in which both $\widehat{\mathbf{F}}_n, \widehat{\mathbf{H}}_n$ are constructed with scaled Bessel functions. In this case the exponential matrix will change into

$$\mathbf{E} = \text{diag}\left\{\exp\left[-\text{abs}\left(\text{Im}\, k_\alpha r\right)\right] \quad \exp\left[-\text{abs}\left(\text{Im}\, k_\beta r\right)\right] \quad \exp\left[-\text{abs}\left(\text{Im}\, k_\beta r\right)\right]\right\}$$

but this is irrelevant here because the above matrix cancels out and is not needed anywhere.

Analytic continuation within the solid core

On page 168 and on, I describe how to extirpate the axis of a solid cylinder to avoid the singularity which arises when sources are placed there. However, I do not provide any information as to how to obtain the motions within the core after the motions have been obtained on the periphery. Here is a remedy to this situation, but before an important comment.

For now unfathomable reasons, in pages 168-173 describing the solid core, I switched the usual external and internal radii r_1, r_2, which everywhere else in the book—including the chapter on layered spheres— I consistently numbered from the outside to the inside. This introduced confusion not only with respect to the appropriate choice of Bessel and Neumann functions, but also in the sign of the tractions acting on the inner surface before taking the limit of a vanishingly small radius. Although the final equations in the book are still correct, here I re-derive the relevant equations while reverting to the usual convention where r_1 is the outer radius and $r_2 \to 0$ is the inner radius. In addition, I dispense entirely with the various elements of the global stiffness matrix for the annular region before condensation of the axis (equations 10.114-10.120), and provide instead a new derivation which is far more transparent. To distinguish the equations herein from those in the book, I shall label these with the prefix A, for "added" equations.

From eq. 10.96, 10.97, the displacements and stresses anywhere within a cylinder with external and internal radii r_1, r_2 are

$$\mathbf{u}\left(r, n, k_z, \omega\right) = \mathbf{H}_n^{(1)}\,\mathbf{a}_1 + \mathbf{H}_n^{(2)}\,\mathbf{a}_2 \qquad (10.97)$$

$$\mathbf{s}\left(r, n, k_z, \omega\right) = \mathbf{F}_n^{(1)}\,\mathbf{a}_1 + \mathbf{F}_n^{(2)}\,\mathbf{a}_2 \qquad (10.98)$$

with matrices assembled from tables 10.3, 10.4 with arguments

$$\mathbf{H}_n^{(1)}\left(r\right) = \mathbf{H}\left(J_n\left(kr\right)\right), \qquad \mathbf{H}_n^{(2)}\left(r\right) = \mathbf{H}\left(Y_n\left(kr\right)\right) \qquad \text{(A10.1a)}$$

$$\mathbf{F}_n^{(1)}(r) = \mathbf{F}(J_n(kr)), \qquad \mathbf{F}_n^{(2)}(r) = \mathbf{F}(Y_n(kr)) \tag{A10.1b}$$

meaning that either Bessel or Neumann functions must be used for the matrices in tables 10.3, 10.4 (but observe the difference with 10.110–10.113, and also the fact that r is a variable).

In particular, the displacements and external tractions per radian at the outer and inner surfaces of the cylinder are

$$\mathfrak{u}_1 = \mathbf{H}_n^{(1)}(r_1)\mathbf{a}_1 + \mathbf{H}_n^{(2)}(r_1)\mathbf{a}_2 \equiv \mathbf{H}_{11}\,\mathbf{a}_1 + \mathbf{H}_{12}\,\mathbf{a}_2 \tag{A10.2a}$$
$$\mathfrak{u}_2 = \mathbf{H}_n^{(1)}(r_2)\mathbf{a}_1 + \mathbf{H}_n^{(2)}(r_2)\mathbf{a}_2 \equiv \mathbf{H}_{21}\,\mathbf{a}_1 + \mathbf{H}_{22}\,\mathbf{a}_2 \tag{A10.2b}$$

$$\mathfrak{p}_1 = r_1\left(\mathbf{F}_n^{(1)}(r_1)\mathbf{a}_1 + \mathbf{F}_n^{(2)}(r_1)\mathbf{a}_2\right) \equiv r_1\left(\mathbf{F}_{11}\,\mathbf{a}_1 + \mathbf{F}_{12}\,\mathbf{a}_2\right) \tag{A10.3a}$$
$$\mathfrak{p}_2 = -r_2\left(\mathbf{F}_n^{(1)}(r_2)\mathbf{a}_1 + \mathbf{F}_n^{(2)}(r_2)\mathbf{a}_2\right) = -r_2\left(\mathbf{F}_{21}\,\mathbf{a}_1 + \mathbf{F}_{22}\,\mathbf{a}_2\right) \tag{A10.3b}$$

In the limit of a solid core when $r_2 \to 0$, the elements of $\mathbf{H}_{22}$ and $\mathbf{F}_{22}$ become infinitely large, but $r_2\mathbf{F}_{21} \to \mathbf{O}$ (the null matrix), in which case A10.3b approaches the limit

$$\mathfrak{p}_2 = -\lim_{r_2 \to 0}\left(r_2\mathbf{F}_{22}\right)\mathbf{a}_2 \tag{A10.4}$$

This leads us to what in page 173 in the text we referred to as $\mathfrak{q}_{axis}$:

$$\mathbf{a}_2 = -\lim_{r_2 \to 0}\left(r_2\mathbf{F}_{22}\right)^{-1}\mathfrak{p}_2 \equiv -\mathfrak{q}_{axis} \tag{A10.5}$$

On the other hand, from equations A10.2a, A10.3a, and A10.5 we can write

$$\mathfrak{u}_1 = \mathbf{H}_{11}\,\mathbf{a}_1 - \mathbf{H}_{12}\,\mathfrak{q}_{\text{axis}} \tag{A10.6a}$$

$$\mathfrak{p}_1 = r_1\left(\mathbf{F}_{11}\,\mathbf{a}_1 - \mathbf{F}_{12}\,\mathfrak{q}_{\text{axis}}\right) \tag{A10.6b}$$

that is

$$\mathbf{a}_1 = \left(r_1\mathbf{F}_{11}\right)^{-1}\left(\mathfrak{p}_1 + r_1\mathbf{F}_{12}\,\mathfrak{q}_{\text{axis}}\right) \tag{A10.7}$$

and

$$\begin{aligned}\mathfrak{u}_1 &= \mathbf{H}_{11}\left(r_1\mathbf{F}_{11}\right)^{-1}\left(\mathfrak{p}_1 + r_1\mathbf{F}_{12}\,\mathfrak{q}_{\text{axis}}\right) - \mathbf{H}_{12}\,\mathfrak{q}_{\text{axis}} \\ &= \mathbf{H}_{11}\left(r_1\mathbf{F}_{11}\right)^{-1}\left(\mathfrak{p}_1 + r_1\left(\mathbf{F}_{12} - \mathbf{F}_{11}\mathbf{H}_{11}^{-1}\mathbf{H}_{12}\right)\mathfrak{q}_{\text{axis}}\right) \\ &= \mathbf{K}_{\text{core}}^{-1}\mathfrak{p}_{\text{equiv}}\end{aligned} \tag{A10.8}$$

In summary

$$\boxed{\mathfrak{p}_{\text{equiv}} = \mathfrak{p}_1 + r_1\left(\mathbf{F}_{12} - \mathbf{F}_{11}\mathbf{H}_{11}^{-1}\mathbf{H}_{12}\right)\mathfrak{q}_{\text{axis}}} \tag{A10.9a}$$

$$\boxed{\mathfrak{p}_{\text{equiv}} = \mathbf{K}_{\text{core}}\mathfrak{u}_1 = \left(r_1\mathbf{F}_{11}\mathbf{H}_{11}^{-1}\right)\mathfrak{u}_1} \tag{A10.9b}$$

The elements of $\mathfrak{q}_{\text{axis}}$ coincide with those given in the unnumbered table on page 173. Having found both $\mathbf{a}_1, \mathbf{a}_2$ in A10.5, A10.7, we can proceed to find the displacements and stresses anywhere within the solid core by means of 10.97 and 10.98, which now change into:

$$\boxed{\begin{aligned}\mathfrak{u}(r,n,k_z,\omega) &= \mathbf{H}_n^{(1)}(r)\left(r_1\mathbf{F}_{11}\right)^{-1}\left(\mathfrak{p}_1 + r_1\mathbf{F}_{12}\,\mathfrak{q}_{\text{axis}}\right) - \mathbf{H}_n^{(2)}(r)\,\mathfrak{q}_{\text{axis}} \\ &= \mathbf{H}_n^{(1)}(r)\mathbf{H}_{11}^{-1}\,\mathfrak{u}_1 + \left\{\mathbf{H}_n^{(1)}(r)\mathbf{H}_{11}^{-1}\mathbf{H}_{12} - \mathbf{H}_n^{(2)}(r)\right\}\mathfrak{q}_{\text{axis}}\end{aligned}} \qquad \text{(A10.10)}$$

$$\boxed{\begin{aligned}\mathfrak{s}(r,n,k_z,\omega) &= \mathbf{F}_n^{(1)}(r)\left(r_1\mathbf{F}_{11}\right)^{-1}\left\{\mathfrak{p}_1 + r_1\mathbf{F}_{12}\,\mathfrak{q}_{\text{axis}}\right\} - \mathbf{F}_n^{(2)}(r)\,\mathfrak{q}_{\text{axis}} \\ &= \mathbf{F}_n^{(1)}(r)\mathbf{H}_{11}^{-1}\mathfrak{u}_1 + \left\{\mathbf{F}_n^{(1)}(r)\mathbf{H}_{11}^{-1}\mathbf{H}_{12} - \mathbf{F}_n^{(2)}(r)\right\}\mathfrak{q}_{\text{axis}}\end{aligned}} \qquad \text{(A10.11)}$$

These equations provide the analytic continuation into the body of the solid core.

When *no* external loads are applied at the axis, then $\mathfrak{q}_{\text{axis}} = \mathbf{0}$ and

$$\mathfrak{u}(r,n,k_z,\omega) = \mathbf{H}_n^{(1)}(r)\mathbf{H}_{11}^{-1}\,\mathfrak{u}_1 \qquad \text{(A10.12a)}$$

$$\begin{aligned}\mathfrak{s}(r,n,k_z,\omega) &= \mathbf{F}_n^{(1)}(r)\mathbf{H}_{11}^{-1}\mathfrak{u}_1 \\ &= \tfrac{1}{r_1}\mathbf{F}_n^{(1)}(r)\mathbf{F}_{11}^{-1}\mathfrak{p}_{\text{equiv}}\end{aligned} \qquad \text{(A10.12b)}$$

In this case, the displacements and stresses on the axis are well defined and the requisite matrices depend on the value of the azimuthal index n:

$n = 0$:

$$\mathbf{H}_n^{(1)}(0) = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{k_z}{k_\alpha} & 0 & 1 \end{Bmatrix} \qquad \text{(A10.13a)}$$

$$\mathbf{F}_n^{(1)}(0) = -k_\alpha \begin{Bmatrix} \lambda\left(1 + \left(\frac{k_z}{k_\alpha}\right)^2\right) + \mu & 0 & \mu\frac{k_z}{k_\alpha} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} \qquad \text{(A10.13b)}$$

$n = 1$:

$$\mathbf{H}_n^{(1)}(0) = \tfrac{1}{2}\begin{Bmatrix} 1 & 1 & \frac{k_z}{k_\beta} \\ 1 & 1 & \frac{k_z}{k_\beta} \\ 0 & 0 & 0 \end{Bmatrix} \qquad \text{(A10.14a)}$$

$$\mathbf{F}_n^{(1)}(0) = -\tfrac{1}{2}\begin{Bmatrix} 2\lambda k_\alpha\left(1+\left(\frac{k_z}{k_\alpha}\right)^2\right) & 0 & 0 \\ 0 & 0 & 0 \\ 2\mu k_z & \mu k_z & \mu k_\beta\left(1-\left(\frac{k_z}{k_\beta}\right)^2\right) \end{Bmatrix} \qquad \text{(A10/14b)}$$

Limiting matrices, page 170:

Note: Corrected errors are marked in red, while equivalent forms are in blue. For example, the term in square bracket in element 2,3 of the third matrix is

$$\frac{k_\alpha^2+k_z^2}{k_\beta^2} = \frac{k_P^2-k_z^2+k_z^2}{k_\beta^2} = \frac{k_P^2}{k_\beta^2} = \frac{k_S^2}{k_\beta^2}a^2 = \frac{k_S^2-k_z^2+k_z^2}{k_\beta^2}a^2 = \frac{k_\beta^2+k_z^2}{k_\beta^2}a^2 = \left[1+\frac{k_z^2}{k_\beta^2}\right]a^2$$

Observe that for the same reason, the denominators of these matrices could be replaced by

$$1+\left(\frac{k_z}{k_\beta}\right)^2 = \left(\frac{k_S}{k_\beta}\right)^2 \qquad \text{(for } n=0,1\text{)} \qquad \text{and} \qquad 1-\left(\frac{k_\alpha}{k_\beta}\right)^2 = -\left(1-a^2\right)\left(\frac{k_S}{k_\beta}\right)^2 \qquad \text{(for } n\ge 2\text{)}$$

Eq. 10.122 , $n=0$

$$\lim_{r_2\to 0}\left(r_2\mathbf{F}_{22}\right)^{-1} \to \frac{\pi}{4\mu\left(1+\left(\frac{k_z}{k_\beta}\right)^2\right)}\begin{Bmatrix} -\frac{k_\alpha}{k_\beta}\left(1-\left(\frac{k_z}{k_\beta}\right)^2\right)z_\beta & 0 & -2\frac{k_\alpha}{k_\beta}\frac{k_z}{k_\beta} \\ 0 & -\left(1+\left(\frac{k_z}{k_\beta}\right)^2\right)z_\beta & 0 \\ -2\frac{k_z}{k_\beta}z_\beta & 0 & 2 \end{Bmatrix}$$

Eq. 10.123, $n=1$

$$\lim_{r_2\to 0}\left(r_2\mathbf{F}_{22}\right)^{-1} \to \frac{\pi}{4\mu\left[1+\left(\frac{k_z}{k_\beta}\right)^2\right]}\begin{Bmatrix} \left(\frac{k_\alpha}{k_\beta}\right)^2 & \left(\frac{k_\alpha}{k_\beta}\right)^2 & -3\left(\frac{k_\alpha}{k_\beta}\right)^2\frac{k_z}{k_\beta}z_\beta \\ 1+\left(\frac{k_z}{k_\beta}\right)^2 & 1+\left(\frac{k_z}{k_\beta}\right)^2 & -\left[1+\left(\frac{k_z}{k_\beta}\right)^2\right]\frac{k_z}{k_\beta}z_\beta \\ \frac{k_z}{k_\beta} & \frac{k_z}{k_\beta} & \left[2-\left(\frac{k_z}{k_\beta}\right)^2\right]z_\beta \end{Bmatrix}$$

Eq. 10.124, $n\ge 2$

$$\lim_{r_2\to 0}\left(r_2\mathbf{F}_{22}\right)^{-1} \to \frac{\pi\, z_\beta^{n-1}}{2^n\, n!\,\mu\left(1-\left(\frac{k_\alpha}{k_\beta}\right)^2\right)}\begin{Bmatrix} n\left(\frac{k_\alpha}{k_\beta}\right)^{n+1} & n\left(\frac{k_\alpha}{k_\beta}\right)^{n+1} & -\left(\frac{k_\alpha}{k_\beta}\right)^{n+1}\frac{k_z}{k_\beta}z_\beta \\ n\left[1+\left(\frac{k_z}{k_\beta}\right)^2\right] & n\left[1+\left(\frac{k_z}{k_\beta}\right)^2\right] & -\left[\left(\frac{k_\alpha}{k_\beta}\right)^2+\left(\frac{k_z}{k_\beta}\right)^2\right]\frac{k_z}{k_\beta}z_\beta \\ n\frac{k_z}{k_\beta} & n\frac{k_z}{k_\beta} & \left[1-\left(\frac{k_\alpha}{k_\beta}\right)^2-\left(\frac{k_z}{k_\beta}\right)^2\right]z_\beta \end{Bmatrix}$$

Stiffness matrix of axis (limit):

$n=0$

$$\mathbf{K} = 2\mu\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{Bmatrix} \qquad \mathbf{K}^{-1} = \frac{1}{2\mu}\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \infty \end{Bmatrix}$$

$n=1$

$$\mathbf{K} = \mu\begin{Bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \qquad \mathbf{K}^{-1} = \frac{1}{\mu}\begin{Bmatrix} 2\times\infty & 2\times\infty & 0 \\ 2\times\infty & 2\times\infty & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$