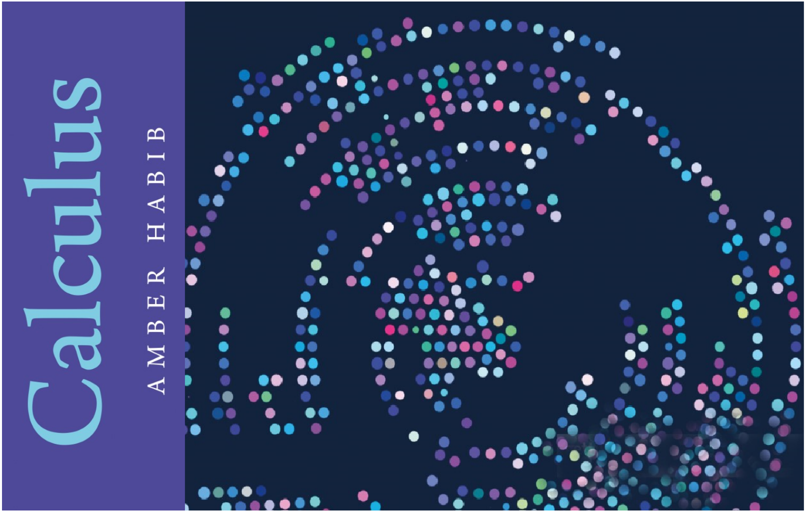

Chapter 5: Techniques of Integration

Part A: The Second Fundamental Theorem and Applications

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3. A function's anti-derivative is not unique. For example, both $\sin x$ and $1 + \sin x$ are anti-derivatives of $\cos x$.
4. On the other hand, two anti-derivatives of the same function over an interval can differ only by a constant.
5. Over non-overlapping intervals, two anti-derivatives of a function need not differ by the same constant. For example, the Heaviside step function and the zero function are anti-derivatives of the zero function over $(-\infty, 0) \cup (0, \infty)$.

Second Fundamental Theorem

Theorem 1

Suppose that $f\colon [a,b] \to \mathbb{R}$ *is a continuous function and* $F\colon [a,b] \to \mathbb{R}$ *satisfies* $F' = f$. *Then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

The difference $F(b) - F(a)$ *is denoted by* $F(x)\big|_a^b$.

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Hence, $F(x) = G(x) + c$. Therefore,

$$F(b) - F(a) = G(b) - G(a) = \int_a^b f(t)\,dt - \int_a^a f(t)\,dt = \int_a^b f(t)\,dt.$$

□

Examples

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Example 2

$$\sin' x = \cos x \implies \int_a^b \cos x \, dx = \sin x \Big|_a^b = \sin b - \sin a,$$

$$\cos' x = -\sin x \implies \int_a^b \sin x \, dx = -\cos x \Big|_a^b = \cos a - \cos b.$$

Example 3

For any integer $n \neq 0$,

$$(x^n)' = nx^{n-1} \implies \left(\frac{x^n}{n}\right)' = x^{n-1} \implies \int_a^b x^{n-1} \, dx = \frac{x^n}{n}\Big|_a^b = \frac{b^n - a^n}{n}.$$

We can allow n to be any non-zero real as well, of course restricting the domain of the integrand to $x > 0$.

Examples

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Example 4

Consider $f(x) = \log|x|$. For $x \neq 0$,

$$f(x) = \begin{cases} \log x & \text{if } x > 0 \\ \log(-x) & \text{if } x < 0 \end{cases} \implies f'(x) = \begin{cases} 1/x & \text{if } x > 0 \\ -1/(-x) & \text{if } x < 0 \end{cases} = \frac{1}{x}.$$

Hence, if a and b have the same sign,

$$\int_a^b \frac{1}{x}\, dx = \log|b| - \log|a| = \log|b/a|.$$

Task 1

We are given that f satisfies $f'(x) = 1/x$ for $x \neq 0$ and $f(1) = 1$. Can we conclude that $f(x) = \log|x| + 1$?

Integral Notation for Anti-derivative

We use the notation $\int f(x)\,dx$ for the collection of anti-derivatives of $f(x)$.

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with the C standing for an arbitrary real number.

The terms 'integral' and 'integration' are used for both definite integrals and anti-derivatives.

Linearity of Anti-derivative

1. $\int c\, f(x)\, dx = c \int f(x)\, dx.$
2. $\int \left(f(x) + g(x)\right) dx = \int f(x)\, dx + \int g(x)\, dx.$

Integral of Log

$$\begin{aligned}
(x\log x)' = \log x + 1 \implies & \int (\log x + 1)\,dx = x\log x + C \\
\implies & \int \log x\,dx + \int 1\,dx = x\log x + C \\
\implies & \int \log x\,dx + x = x\log x + C \\
\implies & \int \log x\,dx = x\log x - x + C.
\end{aligned}$$

Logarithmic Integration

$$(\log|f(x)|)' = \frac{f'(x)}{f(x)} \implies \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + C.$$

Logarithmic Integration

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Here are some applications:

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\log(x^2+1) + C,$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\log|\cos x| + C$$
$$= \log|\sec x| + C,$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \log|\sin x| + C = -\log|\csc x| + C.$$

Logarithmic Integration

Some more applications:

$$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$
$$= \log|\sec x + \tan x| + C,$$
$$\int \csc x \, dx = \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x} \, dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} \, dx$$
$$= -\log|\csc x + \cot x| + C,$$
$$\int \frac{1}{x^2 - 1} \, dx = \int \frac{1}{(x+1)(x-1)} \, dx = \frac{1}{2} \int \left(\frac{1}{x-1} - \frac{1}{x+1} \right) dx$$
$$= \frac{1}{2}(\log|x-1| - \log|x+1|) + C = \frac{1}{2} \log\left|\frac{x-1}{x+1}\right| + C.$$

Task 2

Evaluate $\int \arctan x \, dx$. *(Hint: Try the technique used to integrate* $\log x$*)*

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Substitution Method

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Theorem 5

Suppose f is continuous on an interval I while $\varphi\colon [a,b] \to I$ is continuously differentiable. Then

$$\int_a^b f(\varphi(x))\,\varphi'(x)\,dx = \int_{\varphi(a)}^{\varphi(b)} f(u)\,du.$$

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Proof. Since f is continuous it has an anti-derivative. Let $f = F'$. Then $f(\varphi(x))\,\varphi'(x) = F'(\varphi(x))\,\varphi'(x) = (F \circ \varphi)'(x)$. Therefore,

$$\begin{aligned}\int_a^b f(\varphi(x))\,\varphi'(x)\,dx &= \int_a^b (F \circ \varphi)'(x)\,dx \\ &= F(\varphi(b)) - F(\varphi(a)) = \int_{\varphi(a)}^{\varphi(b)} f(u)\,du.\end{aligned}$$

A Mnemonic

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In an integration problem we are allowed to substitute $u = \varphi(x)$ together with $du = \varphi'(x)\,dx$.

The substitution rule justifies this convention, combined with a corresponding change of limits.

$$\int_a^b f(\underbrace{\varphi(x)}_{u})\,\underbrace{\varphi'(x)\,dx}_{du}$$

Example

Consider $\int_0^{\pi} x\sin(x^2)\,dx$.

We look for a substitution that would create functions that are easier to integrate.

For example, substituting $u = x^2$ converts $\sin(x^2)$ to $\sin u$.

As discussed above, this leads to $du = 2x\,dx$ and so $\frac{1}{2}\,du = x\,dx$.

Further, the limits change as follows: $x = 0 \implies u = 0$, $x = \pi \implies u = \pi^2$. Hence

$$\int_0^{\pi} x\sin(x^2)\,dx = \int_0^{\pi^2} \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos u\Big|_0^{\pi^2} = \frac{1}{2}(1-\cos(\pi^2)).$$

Anti-derivatives

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The substitution method can also be used to find anti-derivatives. If $F' = f$ then,

$$\int f(\varphi(x))\varphi'(x)\,dx = \int (F\circ\varphi)'(x)\,dx = (F\circ\varphi)(x)+C = F(\varphi(x))+C.$$

We represent this calculation by the following abbreviation:

$$\int f(\varphi(x))\varphi'(x)\,dx = \int f(u)\,du, \quad \text{where } u = \varphi(x).$$

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Example 6

To evaluate $\int \sin(2x-\pi)\,dx$ we make the substitution $u = 2x-\pi$, and so $du = 2\,dx$. Then

$$\int \sin(2x-\pi)\,dx = \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos u + C = -\frac{1}{2}\cos(2x-\pi)+C.$$

Example

To evaluate $\int \sin^3 x \, dx$ we first rearrange it as follows.

$$\int \sin^3 x \, dx = \int \sin^2 x \sin x \, dx = \int (1 - \cos^2 x) \sin x \, dx.$$

Now we substitute $y = \cos x$, so that $dy = -\sin x \, dx$.

$$\int (1-\cos^2 x) \sin x \, dx = -\int (1-y^2) \, dy = \frac{y^3}{3} - y + C = \frac{\cos^3 x}{3} - \cos x + C.$$

Hence,

$$\int_0^{\pi/2} \sin^3 x \, dx = \left(\frac{\cos^3 x}{3} - \cos x \right) \Big|_0^{\pi/2} = \frac{2}{3}.$$

Example

To evaluate $\int_0^\pi \sin^2 x \, dx$ we first use the half-angle formula to calculate the anti-derivative:

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

Hence

$$\int_0^\pi \sin^2 x \, dx = \left(\frac{x}{2} - \frac{\sin 2x}{4} \right) \Big|_0^\pi = \frac{\pi}{2}.$$

Trigonometric Substitution

Trigonometric identities can help in integrating algebraic functions.
For example,
$\cos^2 x = 1 - \sin^2 x$ when the integrand contains $\sqrt{a^2 - x^2}$,
$\sec^2 x = 1 + \tan^2 x$ when the integrand contains $\sqrt{a^2 + x^2}$.

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Example 7

To evaluate $\int_0^2 x^3\sqrt{4-x^2}\,dx$ we substitute $x = 2\sin\theta$. Then $dx = 2\cos\theta\,d\theta$ and we get

$$\begin{aligned}\int_0^2 x^3\sqrt{4-x^2}\,dx &= 32\int_0^{\pi/2} \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta\\ &= 32\int_0^{\pi/2}(\cos^2\theta - \cos^4\theta)\sin\theta\,d\theta\\ &= -32\int_1^0 (u^2-u^4)\,du = 32\left(\frac{u^3}{3} - \frac{u^5}{5}\right)\Big|_0^1 = \frac{64}{15}.\end{aligned}$$

Trigonometric Substitution

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To evaluate $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$, substitute $x = \tan\theta$.
Then $dx = \sec^2\theta\,d\theta$, and

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx &= \int_0^{\pi/4} \sec\theta\,d\theta \\ &= \log(\sec\theta + \tan\theta)\Big|_0^{\pi/4} \\ &= \log(1+\sqrt{2}).\end{aligned}$$

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Integration by Parts

Consider the Product Rule, $\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$.

Its anti-derivative version is $f(x)g(x) = \int (f'(x)g(x) + f(x)g'(x))\, dx$.

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Theorem 8 (Integration by Parts)

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Proof. The Second Fundamental Theorem gives

$$f(x)g(x)\Big|_a^b = \int_a^b (f(x)g(x))'\,dx = \int_a^b f(x)g'(x)\,dx + \int_a^b f'(x)g(x)\,dx.$$

Integration by Parts

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Writing $df(x) = f'(x)\,dx$ and $dg(x) = g'(x)\,dx$, we can express integration by parts as

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$$\int_a^b f(x)\,dg(x) = f(x)g(x)\Big|_a^b - \int_a^b g(x)\,df(x).$$

Or, even more briefly, as

$$\int_a^b f\,dg = fg\Big|_a^b - \int_a^b g\,df.$$

Examples

Example 9

Consider $\int x \sin x \, dx$. Set $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. Hence,

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C.$$

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Consider $\int x \sin x \, dx$. Set $f(x) = x$ and $g'(x) = \sin x$. Then $f'(x) = 1$ and $g(x) = -\cos x$. Hence,

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Example 10

Consider $\int xe^x \, dx$. Set $f(x) = x$ and $g'(x) = e^x$. Then $f'(x) = 1$ and $g(x) = e^x$. Hence,

$$\int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C.$$

Examples

Example 11

Consider $\int x \log x \, dx$. Set $f(x) = \log x$ and $g'(x) = x$. Then $f'(x) = 1/x$ and $g(x) = x^2/2$. Hence,

$$\int x \log x \, dx = \frac{x^2}{2} \log x - \int \frac{x}{2} \, dx = \frac{x^2}{2} \log x - \frac{x^2}{4} + C.$$

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Task 3

Use integration by parts twice to evaluate the given integrals.

1. $\int x^2 e^x \, dx$.
2. $\int x^2 \sin x \, dx$.

Examples

Integration by parts can be useful even when the integrand is not in the form of a product. We just introduce a factor of 1.

Example 12

Consider $\int \log x \, dx$. Set $f(x) = \log x$ and $g'(x) = 1$. Then $f'(x) = 1/x$ and $g(x) = x$. Hence,

$$\int \log x \, dx = x \log x - \int 1 \, dx = x \log x - x + C.$$

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Consider $\int \log x \, dx$. Set $f(x) = \log x$ and $g'(x) = 1$. Then $f'(x) = 1/x$ and $g(x) = x$. Hence,

$$\int \log x \, dx = x \log x - \int 1 \, dx = x \log x - x + C.$$

Example 13

Consider $\int \arctan x \, dx$. Set $f(x) = \arctan x$ and $g'(x) = 1$. Then $f'(x) = 1/(1+x^2)$ and $g(x) = x$. Hence,

$$\begin{aligned} \int \arctan x \, dx &= x \arctan x - \int \frac{x}{1+x^2} \, dx \\ &= x \arctan x - \frac{1}{2} \log(1+x^2) + C. \end{aligned}$$

Example

The integral $\int \sec^3 x\, dx$ has a habit of cropping up in integrations of trigonometric functions or in trigonometric substitutions.
We can tackle it by integration by parts as follows.

$$\begin{aligned}\int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx &= \sec x \tan x - \int \sec x \tan^2 x\, dx \\ &= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.\end{aligned}$$

Hence, $\displaystyle\int \sec^3 x\, dx = \frac{1}{2}\big(\sec x \tan x + \log|\sec x + \tan x|\big) + C$

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$$= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.$$

Hence, $\displaystyle\int \sec^3 x\, dx = \frac{1}{2}\big(\sec x \tan x + \log|\sec x + \tan x|\big) + C$

Task 4

Prove the following:

$$\int e^x \cos x\, dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + C,$$

$$\int e^x \sin x\, dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + C.$$

Reduction Formulas

$$\begin{aligned}
\int \sin^n x\,dx &= \int \underbrace{\sin^{n-1} x}_{f}\,\underbrace{\sin x\,dx}_{dg} \\
&= \underbrace{(\sin^{n-1} x)}_{f}\underbrace{(-\cos x)}_{g} - \int \underbrace{(-\cos x)}_{g}\underbrace{(n-1)\sin^{n-2} x\cos x\,dx}_{df} \\
&= -\cos x\sin^{n-1} x + (n-1)\int \cos^2 x\sin^{n-2} x\,dx \\
&= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx
\end{aligned}$$

$$\implies n\int \sin^n x\,dx = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx$$

$$\implies \int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

Reduction Formulas

The reduction formula on the previous slide allows us to obtain the integral of any $\sin^n x$ in terms of the integral of $\sin^{n-2} x$, then in terms of $\sin^{n-4} x$ and so on, till we reach a power of 0 or 1.

Example 14

$$\begin{aligned}\int \sin^4 x\,dx &= -\frac{\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x\,dx \\ &= -\frac{\cos x \sin^3 x}{4} + \frac{3}{4}\left[-\frac{\cos x \sin x}{2} + \frac{1}{2}\int 1\,dx\right] \\ &= -\frac{\cos x \sin^3 x}{4} - \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C.\end{aligned}$$

Reduction Formulas

Let us apply this reduction formula to definite integrals over $[0, \pi/2]$.

$$\begin{aligned}
\int_0^{\pi/2} \sin^n x\, dx &= -\frac{1}{n}\cos x \sin^{n-1} x \Big|_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx \\
&= \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx. \\
\int_0^{\pi/2} \sin^{2n} x\, dx &= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{1}{2}\int_0^{\pi/2} 1\, dx \\
&= \frac{(2n-1)(2n-3)\cdots 1}{(2n)(2n-2)\cdots 2}\,\frac{\pi}{2} = \frac{(2n)!}{4^n (n!)^2}\,\frac{\pi}{2}, \\
\int_0^{\pi/2} \sin^{2n+1} x\, dx &= \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{2}{3}\int_0^{\pi/2} \sin x\, dx \\
&= \frac{(2n)(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3} = \frac{4^n (n!)^2}{(2n+1)!}.
\end{aligned}$$