9

Solutions to Exercises

9.1 Solutions to exercises from chapter 1

Exercise 1.1. Let $X_1, \ldots, X_N$ be a sequence of independent random variables with $P(X_i = 1 + U) = p$, $P(X_i = 1 + D) = 1 - p$. We shall use the fact that $S(1)$ has the same distribution as $S(0)X_1X_2 \ldots X_N$. Since X_i are independent,

$$
\begin{aligned}
\mathbb{E}(S(1)) &= \mathbb{E}(S(0)X_1X_2 \ldots X_N) \\
&= S(0)\mathbb{E}(X_1)\mathbb{E}(X_2) \ldots \mathbb{E}(X_N) \\
&= S(0)[(1 + U)p + (1 + D)(1 - p)]^N .
\end{aligned}
$$

We compute the variance using

$$
\mathrm{Var}(S(1)) = E\left(S(1)^2\right) - E(S(1))^2 .
$$

Since $X_1, \ldots, X_N$ are independent, so are $X_1^2, \ldots, X_N^2$. Using the independence we compute

$$
\begin{aligned}
\mathbb{E}\left(S(1)^2\right) &= \mathbb{E}\left(S(0)^2 X_1^2 X_2^2 \ldots X_N^2\right) \\
&= S(0)^2 \mathbb{E}\left(X_1^2\right)\mathbb{E}\left(X_2^2\right) \ldots \mathbb{E}\left(X_N^2\right) \\
&= S(0)^2 \left[(1 + U)^2 p + (1 + D)^2 (1 - p)\right]^N ,
\end{aligned}
$$

hence

$$
\begin{aligned}
\mathrm{Var}(S(1)) &= \mathbb{E}\left(S(1)^2\right) - \mathbb{E}(S(1))^2 \\
&= S(0)^2 \left[(1 + U)^2 p + (1 + D)^2 (1 - p)\right]^N \\
&\quad - S(0)^2 [(1 + U)p + (1 + D)(1 - p)]^{2N} .
\end{aligned}
$$

Exercise 1.2. We start by computing $\mathbb{E}(S(1)^2)$:

$$
\begin{aligned}
\mathbb{E}(S(1)^2) &= \int_0^\infty x^2 f(x)dx \\
&= \int_0^\infty x \frac{1}{s\sqrt{2\pi}} e^{-\frac{\left(\ln\frac{x}{S(0)}-m\right)^2}{2s^2}} dx \\
&= \int_{-\infty}^\infty S(0)^2 e^{2sy+2m} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \qquad \left(\text{taking } y = \frac{1}{s}\left(\ln\frac{x}{S(0)} - m\right)\right) \\
&= S(0)^2 e^{2m} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2-4sy+4s^2}{2}+\frac{4s^2}{2}} dy \\
&= S(0)^2 e^{2m+2s^2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-2s)^2}{2}} dy \\
&= S(0)^2 e^{2m+2s^2}.
\end{aligned}
$$

Using the formula for $\mathbb{E}(S(1))$ from Example 1.2, we can now compute

$$
\begin{aligned}
\mathrm{Var}(S(1)) &= \mathbb{E}\left(S(1)^2\right) - \mathbb{E}(S(1))^2 \\
&= S(0)^2 e^{2m+2s^2} - \left(S(0)e^{m+\frac{s^2}{2}}\right)^2 \\
&= S(0)^2 e^{2m+s^2}\left(e^{s^2}-1\right).
\end{aligned}
$$

Exercise 1.3. We use the formula

$$
\mathbb{E}(K) = \frac{\mathbb{E}(S(1)) - S(0)}{S(0)}.
$$

For stock from Exercise 1.1 this gives

$$
\begin{aligned}
\mathbb{E}(K) &= \frac{\mathbb{E}(S(1)) - S(0)}{S(0)} \\
&= \frac{S(0)\left[(1+U)p + (1+D)(1-p)\right]^N - S(0)}{S(0)} \\
&= \left[(1+U)p + (1+D)(1-p)\right]^N - 1.
\end{aligned}
$$

For stock from Example 1.2,

$$
\begin{aligned}
\mathbb{E}(K) &= \frac{\mathbb{E}(S(1)) - S(0)}{S(0)} \\
&= \frac{S(0)e^{m+\frac{s^2}{2}} - S(0)}{S(0)} \\
&= e^{m+\frac{s^2}{2}} - 1.
\end{aligned}
$$

Exercise 1.4. Since $\mathbb{E}(K) = \frac{\mathbb{E}(S(1))+\mathrm{Div}(1)-S(0)}{S(0)}$,

$$\begin{aligned}\mathrm{Div}(1) &= S(0)(1+\mathbb{E}(K)) - \mathbb{E}(S(1)) \\ &= 80(1+0.2) - (\frac{1}{6}60 + \frac{3}{6}80 + \frac{2}{6}90) \\ &= 16.\end{aligned}$$

Exercise 1.5. Let $V(0)$ and $V(1)$ denote the value of the position at time zero and one, respectively. We have $V(0) = wS(0)$, since this is the amount of our cash we invest, and

$$\begin{aligned}V(1) &= S(1) - (1-w)(1+R)S(0) \\ &= S(0)[(1+K_S) - (1-w)(1+R)]\end{aligned}$$

The $V(t)$ can be considered as a security, hence its return follows from the formula

$$\begin{aligned}K_{\mathrm{lev}} &= \frac{V(1)-V(0)}{V(0)} \\ &= \frac{S(0)[1+K_S-(1-w)(1+R)-w]}{wS(0)} \\ &= \frac{K_S-(1-w)R}{w} \\ &= R + \frac{1}{w}(K_S - R).\end{aligned}$$

Since R is non-random and $w \geq 0$, the standard deviation of the leveraged return is $\sigma_{\mathrm{lev}} = \frac{1}{w}\sigma_S$.

Exercise 1.6. We will use the fact that $\mathbb{E}(K_3) = \frac{\mathbb{E}(S_3(1))-S_3(0)}{S_3(0)}$ and $\sigma_3 = \sqrt{\mathrm{Var}(K_3)} = \frac{1}{S_3(0)}\sqrt{\mathrm{Var}(S_3(1))}$. We have $\sigma_i \geq \sigma_3$ and $\mu_3 \geq \mu_i$ (for $i = 1, 2$) when

$$\begin{aligned}S_3(0) &\geq \frac{1}{\sigma_i}\sqrt{\mathrm{Var}(S_3(1))}, \qquad (9.1)\\ S_3(0) &\leq \frac{\mathbb{E}(S_3(1))}{1+\mu_i}.\end{aligned}$$

The solutions to the three cases are

(i) We have $\sigma_1 \geq \sigma_3$ and $\mu_3 \geq \mu_1$ when

$$S(0) \in [\frac{1}{\sigma_1}\sqrt{\mathrm{Var}(S(1))}, \frac{\mathbb{E}(S(1))}{1+\mu_1}] = [\frac{1}{0.25}20, \frac{100}{1+0.1}] = [80, 90, 909].$$

(ii) We have $\sigma_2 \geq \sigma_3$ and $\mu_2 \geq \mu_1$ when

$$S(0) \in [\frac{1}{\sigma_2}\sqrt{\mathrm{Var}(S(1))}, \frac{\mathbb{E}(S(1))}{1+\mu_2}] = [\frac{1}{0.3}20, \frac{100}{1+0.15}] = [66.666, 86.957].$$

(iii) No asset will be dominated by another asset for

$$\begin{aligned} S(0) &\in \{\mathbb{R}_+ \setminus [80, 90, 909]\} \cap \{\mathbb{R}_+ \setminus [66.666, 86.957]\} \\ &= (0, 66.666) \cup (90, 909, +\infty). \end{aligned}$$

Remark: This exercise demonstrates one weakness of considering variance as the risk measure. Condition (9.1) follows from the fact that when the price $S(0)$ is low then this creates large deviation of return from the expected return. Economically, having low prices is good for us, which means that in this case the constraint (9.1) is artificial.

9.2 Solutions to exercises from chapter 2

Exercise 2.1. From Example 2.1 we know that

$$\mu_1 = 10\%, \qquad \sigma_1 = 0.2,$$
$$\mu_2 = 5\%, \qquad \sigma_2 = 0.15.$$

Note that

$$K_{\mathbf{w}}(\omega_1) = \frac{V_{(x_1,x_2)}(1,\omega_1) - V_{(x_1,x_2)}(0)}{V_{(x_1,x_2)}(0)} = \frac{300 - 600}{600} = -50\%,$$
$$K_{\mathbf{w}}(\omega_2) = \frac{V_{(x_1,x_2)}(1,\omega_2) - V_{(x_1,x_2)}(0)}{V_{(x_1,x_2)}(0)} = \frac{900 - 600}{600} = 50\%,$$

which since $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$ gives

$$\mathbb{E}(K_{\mathbf{w}}) = 0$$
$$\sqrt{\mathrm{Var}(K_{\mathbf{w}})} = 0.5.$$

We see that we have lower expected return and higher risk on the strategy than on any of the two assets.

The reason behind this is that we take a short position on the asset S_1, which has high expected return. This, in our case, reduces the expected return on the strategy to zero.

Exercise 2.2. From the definition of variance we deduce that (for full derivation, follow mirror computations to the proof of Theorem 2.4, with $w_1 = w_2 = 1$)

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2Cov(X,Y)$$

so that

$$\begin{aligned}(\sigma_X + \sigma_Y)^2 - \sigma^2_{X+Y} &= 2[\sigma_X\sigma_Y - Cov(X,Y)] \\ &= 2[\mathbb{E}(X_c)\mathbb{E}(Y_c) - \mathbb{E}(X_cY_c)] \\ &\geq 0\end{aligned}$$

where we apply the Schwarz inequality to $X_c = X - \mathbb{E}(X)$ and $Y_c = Y - \mathbb{E}(Y)$ in the final step. Taking square roots we are done.

Exercise 2.3. Let $m = 30\%$. We need to find w so that

$$w\mu_1 + (1-w)\mu_2 = m.$$

We solve for w :

$$w = \frac{m - \mu_2}{\mu_1 - \mu_2},$$

and substitute the numbers

$$w = \frac{30\% - 20\%}{10\% - 20\%} = -1.$$

This means that the portfolio with the return 30% is $\mathbf{w} = (w_1, w_2) = (-1, 2)$.

Exercise 2.4. The attainable set is a horizontal half line that lies on the set $\{(\sigma, \mu) : \mu = \mu_1 = \mu_2\}$.

Exercise 2.5. The portfolio with the smallest risk is $(\mu_{\mathbf{w}_{\min}}, \sigma_{\mathbf{w}_{\min}})$ with $\sigma_{\mathbf{w}_{\min}} = 0$ and

$$\mu_{\mathbf{w}_{\min}} = \frac{\sigma_2}{\sigma_1 + \sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1 + \sigma_2}\mu_2.$$

The two half lines pass through the points $(\mu_{\mathbf{w}_{\min}}, \sigma_{\mathbf{w}_{\min}})$ and (μ_i, σ_i), for $i = 1, 2$. The formulae for the half lines are therefore

$$\mu = a_i\sigma + b,$$

with

$$b = \mu_{\mathbf{w}_{\min}} = \frac{\sigma_2}{\sigma_1 + \sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1 + \sigma_2}\mu_2$$

and

$$a_i = \frac{\mu_i - \mu_{\mathbf{w}_{\min}}}{\sigma_i} = \frac{1}{\sigma_i}\left(\mu_i - \frac{\sigma_2}{\sigma_1 + \sigma_2}\mu_1 - \frac{\sigma_1}{\sigma_1 + \sigma_2}\mu_2\right).$$

Simplifying gives:

$$\begin{aligned} a_1 &= \frac{1}{\sigma_1 + \sigma_2}\left(\mu_1 - \mu_2\right), \\ a_2 &= \frac{1}{\sigma_1 + \sigma_2}\left(\mu_2 - \mu_1\right). \end{aligned}$$

Exercise 2.6. The portfolio with the smallest risk is $(\mu_{\mathbf{w}_{\min}}, \sigma_{\mathbf{w}_{\min}})$ with $\sigma_{\mathbf{w}_{\min}} = 0$ and

$$\mu_{\mathbf{w}_{\min}} = \frac{-\sigma_2}{\sigma_1 - \sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1 - \sigma_2}\mu_2.$$

The two half lines pass through the points $(\mu_{\mathbf{w}_{\min}}, \sigma_{\mathbf{w}_{\min}})$ and (μ_i, σ_i), for $i = 1, 2$. The formulae for the half lines are therefore

$$\mu = a_i\sigma + b,$$

with

$$b = \mu_{\mathbf{w}_{\min}} = \frac{-\sigma_2}{\sigma_1 - \sigma_2}\mu_1 + \frac{\sigma_1}{\sigma_1 - \sigma_2}\mu_2,$$

and

$$a_i = \frac{\mu_i - \mu_{\mathbf{w}_{\min}}}{\sigma_i} = \frac{1}{\sigma_i}\left(\mu_i + \frac{\sigma_2}{\sigma_1 - \sigma_2}\mu_1 - \frac{\sigma_1}{\sigma_1 - \sigma_2}\mu_2\right).$$

Simplifying gives:

$$\begin{aligned} a_1 &= \frac{1}{\sigma_1 - \sigma_2}\left(\mu_1 - \mu_2\right), \\ a_2 &= \frac{1}{\sigma_1 - \sigma_2}\left(\mu_2 - \mu_1\right). \end{aligned}$$

Exercise 2.7. When $\rho_{12} = 1$ and $\sigma_1 = \sigma_2$ then

$$\begin{aligned} \sigma_{\mathbf{w}}^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12} \\ &= (w_1\sigma_1 + w_2\sigma_2)^2 \\ &= \sigma_1^2\left(w_1 + w_2\right)^2 \\ &= \sigma_1^2. \end{aligned}$$

This means that all portfolios have $\sigma_{\mathbf{w}} = \sigma_1 = \sigma_2$, regardless of the choice of $\mathbf{w}$. If $\mu_1 \neq \mu_2$, then the attainable set is a vertical line. If $\mu_1 = \mu_2$, then the attainable set is a single point $(\sigma_1, \mu_1) = (\sigma_2, \mu_2)$.

Exercise 2.8. First assume that $\rho_{12} > 1$. Let us take

$$w_1 = \frac{-\sigma_2}{\sigma_1 - \sigma_2} \qquad \text{and} \qquad w_2 = \frac{\sigma_1}{\sigma_1 - \sigma_2}.$$

Since exactly one of the weights is negative, $w_1w_2 < 0$. Thus

$$\begin{aligned} \sigma_{\mathbf{w}}^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12} \\ &= (w_1\sigma_1 + w_2\sigma_2)^2 + 2w_1w_2\sigma_1\sigma_2(\rho_{12} - 1) \\ &= 2w_1w_2\sigma_1\sigma_2(\rho_{12} - 1) \\ &< 0. \end{aligned}$$

For $\rho_{12} < -1$. Let us take

$$w_1 = \frac{\sigma_2}{\sigma_1 + \sigma_2} \qquad \text{and} \qquad w_2 = \frac{\sigma_1}{\sigma_1 + \sigma_2}.$$

Observe that since both w_1 and w_2 are positive and $1 + \rho_{12} < 0$

$$\begin{aligned} \sigma_{\mathbf{w}}^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12} \\ &= (w_1\sigma_1 - w_2\sigma_2)^2 + 2w_1w_2\sigma_1\sigma_2\left(1 + \rho_{12}\right) \\ &= 2w_1w_2\sigma_1\sigma_2\left(1 + \rho_{12}\right) \\ &< 0. \end{aligned}$$

This means that in both cases we can obtain negative variance of a portfolio. This stands against common sense and the mathematical properties of variance (which by definition is nonnegative). When using the formulas from portfolio theory with illegal initial data, we can thus arrive at misleading conclusions. When negative variance emerges from computations, illegal initial data are a possible source of such errors.

Exercise 2.9. We first investigate for which ρ_{12} we will have $w_1 < 0$. This happens when

$$w_1 = \frac{a}{a+b} = \frac{\sigma_2^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_2^2 + \sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2} < 0. \tag{9.2}$$

Since $|\rho_{12}| \leq 1$, we see that

$$\sigma_2^2 + \sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 \geq \sigma_2^2 + \sigma_1^2 - 2\sigma_1\sigma_2 = (\sigma_1 - \sigma_2)^2 \geq 0.$$

Thus (9.2) is equivalent to

$$\sigma_2^2 - \rho_{12}\sigma_1\sigma_2 < 0,$$

which gives the condition

$$\frac{\sigma_2}{\sigma_1} < \rho_{12}.$$

Similarly

$$w_1 = \frac{b}{a+b} = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2}{\sigma_2^2 + \sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2} < 0$$

is equivalent to

$$\frac{\sigma_1}{\sigma_2} < \rho_{12}.$$

This means that

$$\rho_{12} > \min\left\{\frac{\sigma_1}{\sigma_2}, \frac{\sigma_2}{\sigma_1}\right\},$$

implies that the minimum variance portfolio requires short selling.

Exercise 2.10. From the calculation leading to Corollary 2.9 we have $C^{-1} = \frac{1}{\det C}\begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix}$, so that, with $\mu = (\mu_1, \mu_2)$ we have

$$C^{-1}(\mu - R\mathbf{1}) = \frac{1}{\det C}\begin{bmatrix} \sigma_2^2(\mu_1 - R) - \sigma_{12}(\mu_2 - R) \\ -\sigma_{12}(\mu_1 - R) + \sigma_1^2(\mu_2 - R) \end{bmatrix} = \frac{1}{\det C}\begin{bmatrix} c \\ d \end{bmatrix}$$

where c, d are defined in Theorem 2.10. Similarly,

$$\mathbf{1}^T C^{-1}(\mu - R\mathbf{1}) = \frac{1}{\det C}(c + d),$$

which shows that the two expressions for the coefficients of the Market Portfolio are the same.

Exercise 2.11. First we compute **m** using (2.16). We can then compute the variance of the return of the market portfolio using (2.10)

$$\sigma_{\mathbf{m}}^2 = \mathbf{m}^{\mathrm{T}} C \mathbf{m}.$$

Optimal investments lie on the capital market line. The investor needs to hold a combination of the market portfolio and the risk-free security. We assume that he spends λV on the market portfolio and invests $(1-\lambda)V$ risk-free. The desired λ can be computed from the standard deviation of the return of the position

$$\lambda^2 \sigma_{\mathbf{m}}^2 + (1-\lambda)^2 \sigma_R^2 + 2\mathrm{cov}(K_{\mathbf{m}}, R) = \sigma^2.$$

Since $\mathrm{cov}(K_{\mathbf{m}}, R) = 0$ and $\sigma_R = 0$, above gives

$$\lambda = \frac{\sigma_{\mathbf{m}}}{\sigma}.$$

Since the investor spends λV on the market portfolio, the vector

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda V \mathbf{m},$$

gives us the amount v_1 invested in the first asset, and v_2 invested in the second asset. As mentioned above, $(1-\lambda)V$ is invested risk-free.

Exercise 2.12. Since $\sigma_{12} = \sigma_{21} = \rho_{12}\sigma_1\sigma_2$, $\sigma_{11} = \sigma_1^2$ and $\sigma_{22} = \sigma_2^2$,

$$C = \begin{bmatrix} 0.01 & -0.015 \\ -0.015 & 0.09 \end{bmatrix}.$$

$$\begin{aligned} \mu_1 &= 10\%, & \sigma_1 &= 0.1, & \rho_{12} &= -0.5, \\ \mu_2 &= 20\%, & \sigma_2 &= 0.3, & R &= 5\%. \end{aligned}$$

We first find **m**

$$\mathbf{m} = \frac{C^{-1}(\boldsymbol{\mu} - R\mathbf{1})}{\mathbf{1}^{\mathrm{T}} C^{-1}(\boldsymbol{\mu} - R\mathbf{1})} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix},$$

and compute

$$\begin{aligned} \mu_{\mathbf{m}} &= \mathbf{m}^{\mathrm{T}} \boldsymbol{\mu} = 0.125, \\ \sigma_{\mathbf{m}} &= \sqrt{\mathbf{m}^{\mathrm{T}} C \mathbf{m}} = 0.075. \end{aligned}$$

From Example 2.13 we know that the optimal investment is a portfolio on the capital market line with expected return equal to

$$m = R + \frac{1}{a}\left(\frac{\mu_{\mathbf{m}} - R}{\sigma_{\mathbf{m}}}\right)^2 = 0.25.$$

We assume that we spend λV on the market portfolio and invests $(1-\lambda)\,V$ risk-free. As in Example 2.12, the desired λ is

$$\lambda = \frac{m - R}{\mu_{\mathbf{m}} - R} = \frac{8}{3}.$$

Since the investor spends λV on the market portfolio, the vector

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda V \mathbf{m} = \begin{bmatrix} 6000 \\ 2000 \end{bmatrix},$$

gives us the amount v_1 invested in the first asset, and v_2 invested in the second asset. As mentioned above, $(1-\lambda)\,V = -5000$ is invested risk-free. In other words, we borrow 5000 at the risk free interest rate and invest this together with $V = 3000$, spending 6000 on the first asset, and 2000 on the second asset.

Exercise 2.13. We first show that

$$\frac{x-h}{a} - \frac{y-k}{b} = 0$$

is an asymptote. To show this, consider a point (x, y_a) with

$$y_a = y_a(x) = b\frac{x-h}{a} + k,$$

on the asymptote. We need to show that the distance between such points and the hyperbola converges to zero as $|x|$ goes to infinity. Let (x, y_h) be a point on the hyperbola, meaning that

$$\left(\frac{x-h}{a}\right)^2 - \left(\frac{y_h - k}{b}\right)^2 = 1.$$

We will show that $\lim_{|x|\to+\infty} |y_h(x) - y_a(x)| = 0$ (which implies that $\lim_{|x|\to+\infty} \|(x, y_h(x)) - (x, y_a(x))\| = 0$).

Let

$$X = X(x) = \frac{x-h}{a},$$

$$Y_a = Y_a(x) = \frac{y_a(x) - k}{b},$$

$$Y_h = Y_h(x) = \frac{y_h(x) - k}{b}.$$

Using the facts that

$$X^2 - {Y_h}^2 = 1,$$
$$X - Y_a = 0,$$

we compute

$$\begin{aligned}
|Y_h - Y_a| &= \left| \sqrt{X^2 - 1} - X \right| \\
&= \left| \frac{\left(\sqrt{X^2 - 1} - X\right)\left(\sqrt{X^2 - 1} + X\right)}{\sqrt{X^2 - 1} + X} \right| \\
&= \frac{1}{\left| \sqrt{X^2 - 1} + X \right|} \\
&\overset{|X| \to +\infty}{\to} 0.
\end{aligned}$$

Since $|X(x)|$ converges to infinity as $|x|$ converges to infinity, above implies that

$$\lim_{|x| \to +\infty} |y_h(x) - y_a(x)| = \lim_{|x| \to +\infty} |b|\,|Y_h(x) - Y_a(x)| = 0,$$

as required.

Showing that

$$\frac{x - h}{a} + \frac{y - k}{b} = 0$$

is also an asymptote follows from mirror computations.

9.3 Solutions to exercises from chapter 3

Exercise 3.1. The system

$$\begin{aligned} 2x - \frac{1}{2}\lambda &= 0, \\ 2y - \frac{1}{2}\lambda &= 0, \\ \frac{1}{2}x + \frac{1}{2}y - \frac{1}{2} &= 0, \end{aligned}$$

is equivalent to

$$A\mathbf{p} = \mathbf{b}, \tag{9.3}$$

for

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ \lambda \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \qquad A = \begin{bmatrix} 2 & 0 & -\frac{1}{2} \\ 0 & 2 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

Let

$$A_x = \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 2 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \quad A_y = \begin{bmatrix} 2 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \quad A_\lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

We can compute

$$\det(A) = 1, \qquad \det(A_x) = \frac{1}{2},$$
$$\det\left(A_y\right) = \frac{1}{2}, \qquad \det(A_\lambda) = 2.$$

By the Cramer's rule, the solution to (9.3) is

$$x = \frac{\det(A_x)}{\det(A)} = \frac{1}{2}, \qquad y = \frac{\det\left(A_y\right)}{\det(A)} = \frac{1}{2}, \qquad \lambda = \frac{\det(A_\lambda)}{\det(A)} = 2.$$

Exercise 3.2. Inserting (3.7) into (3.6) gives:

$$\begin{aligned} &2\sigma_1^2 x^* + 2\sigma_{12} y^* - \lambda^* \\ &= 2\sigma_1^2 \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} + 2\sigma_{12} \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} - \frac{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \\ &= 2\frac{\sigma_1^2\left(\sigma_2^2 - \sigma_{12}\right) + \sigma_{12}\left(\sigma_1^2 - \sigma_{12}\right) - \left(\sigma_1^2\sigma_2^2 - \sigma_{12}^2\right)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \\ &= 0, \end{aligned}$$

$$
\begin{aligned}
&2\sigma_{12}x^* + 2\sigma_2^2 y^* - \lambda^* \\
&= 2\sigma_{12}\frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} + 2\sigma_2^2\frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} - 2\frac{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \\
&= 2\frac{\sigma_{12}\left(\sigma_2^2 - \sigma_{12}\right) + \sigma_2^2\left(\sigma_1^2 - \sigma_{12}\right) - \left(\sigma_1^2\sigma_2^2 - \sigma_{12}^2\right)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
x^* + y^* - 1 &= \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} + \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} - 1 \\
&= \frac{\sigma_2^2 - \sigma_{12} + \sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} - 1 \\
&= 0,
\end{aligned}
$$

as required.

Exercise 3.3. To plot $g(x, y) = 1$, it is enough to plot the function $y = 1 - x$. To plot $f(x, y) = \sigma^2$ we need to solve

$$x^2\sigma_1^2 + y^2\sigma_2^2 + 2xy\rho_{12}\sigma_1\sigma_2 - \sigma^2 = 0.$$

The solution for x, given y, is

$$x(y) = \frac{-2y\rho_{12}\sigma_1\sigma_2 \pm \sqrt{(2y\rho_{12}\sigma_1\sigma_2)^2 - 4\sigma_1^2\left(y^2\sigma_2^2 - \sigma^2\right)}}{2\sigma_1^2}.$$

The solution for y, given x, is

$$y(x) = \frac{-2x\rho_{12}\sigma_1\sigma_2 \pm \sqrt{(2x\rho_{12}\sigma_1\sigma_2)^2 - 4\sigma_2^2\left(x^2\sigma_1^2 - \sigma^2\right)}}{2\sigma_2^2}.$$

We can use the above equations for $x(y), y(x)$ to plot $f(x, y) = \sigma$. This is done in the file `Exercise_3_3.xlsx`.

Exercise 3.4. The system of equations

$$
\begin{aligned}
\nabla f(x, y, z) - \lambda\nabla g(x, y, z) &= 0 \\
g(x, y, z) &= 0
\end{aligned}
$$

leads to

$$
\begin{aligned}
-\lambda 2x &= 0, \\
\lambda 2y &= 0, \\
1 - \lambda 2z &= 0, \\
x^2 - y^2 + z^2 - 1 &= 0,
\end{aligned}
$$

which implies

$$x = y = 0,$$
$$z = \pm 1,$$
$$\lambda = \pm\frac{1}{2}.$$

Hence $\left(x^*, y^*, z_1^*\right) = (0, 0, 1)$ and $\left(x^*, y^*, z_2^*\right) = (0, 0, -1)$ are the candidates for a minimum. For these two points we see that

$$f(x^*, y^*, z_1^*) = 1,$$
$$f(x^*, y^*, z_2^*) = -1,$$

hence (x^*, y^*, z_2^*) remains the only candidate. If we take

$$(x^{**}, y^{**}, z^{**}) = (0, 2, -2)\,,$$

then (x^{**}, y^{**}, z^{**}) satisfies the constraint $g(x^{**}, y^{**}, z^{**}) = 0$ and

$$f\,(x^{**}, y^{**}, z^{**}) = -2 < -1 = f\,(x^*, y^*, z^*)\,,$$

and we therefore see that (x^*, y^*, z^*) is not a solution to the problem:

$$\begin{aligned}&\min f\,(x, y, z)\,,\\ &\text{under the constraints: } g(x, y, z) = 0.\end{aligned}$$

Exercise 3.5. The system of equations

$$\begin{aligned}\nabla f(x, y, z) - \lambda \nabla g(x, y, z) &= 0,\\ g(x, y, z) &= 0,\end{aligned}$$

leads to

$$\begin{aligned}1 - \lambda 2x &= 0,\\ 1 - \lambda 2y &= 0,\\ 1 - \lambda 2z &= 0,\\ x^2 + y^2 + z^2 - 1 &= 0,\end{aligned}$$

which implies that

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1,$$

hence $\lambda = \pm\frac{\sqrt{3}}{2}$. Substituting λ into the system gives two solutions

$$(x_1, y_1, z_1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),$$
$$(x_2, y_2, z_2) = \left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right).$$

Since $f(x_1, y_1, z_1) = \sqrt{3} > -\sqrt{3} = f(x_2, y_2, z_2)$, we see that (x_2, y_2, z_2) is the only candidate for a solution to the problem.

Let $A = \left\{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\right\}$. With this notation our problem can be written as: find

$$\min_{(x,y,z)\in A} f(x, y, z).$$

Since A is compact we know that a solution to this problem exists. The point (x_2, y_2, z_2) is the only candidate hence the solution must be (x_2, y_2, z_2).

Exercise 3.6. Let us consider

$$f(x, y, z) = xyz$$
$$g(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$$

We can compute

$$\nabla f = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}, \qquad \nabla g = \begin{bmatrix} 2\frac{x}{a^2} \\ 2\frac{y}{b^2} \\ 2\frac{z}{c^2} \end{bmatrix},$$

which by the method of Lagrange multipliers leads to the system of equations

$$yz = \lambda 2\frac{x}{a^2},$$
$$xz = \lambda 2\frac{y}{b^2},$$
$$xy = \lambda 2\frac{z}{c^2},$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

From the first equation we get $2\lambda = a^2\frac{yz}{x}$, which substituted into the second and third gives

$$xz = \lambda 2\frac{y}{b^2} = a^2\frac{yz}{x}\frac{y}{b^2},$$
$$xy = \lambda 2\frac{z}{c^2} = a^2\frac{yz}{x}\frac{z}{c^2},$$

hence

$$\frac{x^2}{a^2} = \frac{y^2}{b^2},$$
$$\frac{x^2}{a^2} = \frac{z^2}{c^2},$$

meaning that

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Using the constraint we see that

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3\frac{x^2}{a^2} = 3\frac{y^2}{b^2} = 3\frac{z^2}{c^2},$$

hence

$$x = \pm\frac{a}{\sqrt{3}}, \qquad y = \pm\frac{b}{\sqrt{3}}, \qquad z = \pm\frac{c}{\sqrt{3}}.$$

These parameters describe the eight vertices of the maximal box. Since it must be centred at the origin, the volume of the maximal box is $8|xyz|$, where x, y, z are as above, hence the maximal volume is $\frac{8}{3\sqrt{3}}abc$.

Exercise 3.7. Consider the following constrained maximisation problem:

$$\begin{aligned} &\max f(\mathbf{v}), \\ &\text{under the constraints: } \mathbf{g}(\mathbf{v}) = \mathbf{0}. \end{aligned} \tag{9.4}$$

Assume that $\mathbf{g}(\mathbf{v}) = A\mathbf{v} - \mathbf{c}$ and that

$$\mathbf{w}^{\mathrm{T}} H(f, \mathbf{v})\mathbf{w} \leq 0. \tag{9.5}$$

Let $h(\mathbf{v}) = -f(\mathbf{v})$. We see that when we solve

$$\begin{aligned} &\min h(\mathbf{v}), \\ &\text{under the constraints: } \mathbf{g}(\mathbf{v}) = \mathbf{0}, \end{aligned} \tag{9.6}$$

then $\max f(\mathbf{v}) = -\min h(\mathbf{v})$. Since $H(h, \mathbf{v}) = -H(f, \mathbf{v})$, by condition (9.5) we see that

$$\mathbf{w}^{\mathrm{T}} H(h, \mathbf{v})\mathbf{w} \geq 0,$$

which by Theorem 3.4 means that the solution $\mathbf{v}^*$ of (3.9) is a solution of the problem (9.6). Since $\max f(\mathbf{v}) = -\min h(\mathbf{v})$ we thus see that $\mathbf{v}^*$ is also the solution of problem (9.4).

9.4 Solutions to exercises from chapter 4

Exercise 4.1. Let Id denote the identity matrix. The following condition holds:

$$CC^{-1} = \mathrm{Id}.$$

We can compute

$$\mathrm{Id} = \mathrm{Id}^{\mathrm{T}} = \left(CC^{-1}\right)^{\mathrm{T}} = \left(C^{-1}\right)^{\mathrm{T}} C^{\mathrm{T}}.$$

Since $C^{\mathrm{T}} = C$, this means that

$$\left(C^{-1}\right)^{\mathrm{T}} C = \mathrm{Id},$$

hence $\left(C^{-1}\right)^{\mathrm{T}}$ is an inverse matrix of C:

$$C^{-1} = \left(C^{-1}\right)^{\mathrm{T}},$$

as required.

Exercise 4.2. Since

$$\sigma_{ij} = \operatorname{cov}\left(K_i, K_j\right) = \mathbb{E}\left((K_i - \mu_i)\left(K_j - \mu_j\right)\right)$$
$$= \mathbb{E}\left(\left(K_j - \mu_j\right)(K_i - \mu_i)\right) = \operatorname{cov}\left(K_j, K_i\right) = \sigma_{ji},$$

we see that C is symmetric.

To prove that C is positive semidefinite we need to prove that for any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^2$

$$\mathbf{x}^{\mathrm{T}} C \mathbf{x} \geq 0.$$

Let $X = \sum_{i=1}^{n} x_i K_i$. Using the same argument as for the proof of Theorem 4.1 we have

$$\operatorname{Var}(X) = \mathbf{x}^{\mathrm{T}} C \mathbf{x}.$$

Since variance is nonnegative, we obtain $\mathbf{x}^{\mathrm{T}} C \mathbf{x} \geq 0$.

The covariance matrix does not have to be invertible. As an example, consider two securities, first of which is the risk free asset. Assume that the second has the variance of the return equal to σ^2. Then

$$C = \begin{bmatrix} 0 & 0 \\ 0 & \sigma^2 \end{bmatrix},$$

which is not invertible.

Exercise 4.3. We saw in the previous Exercise that any covariance matrix C is symmetric. If C is also invertible, there is an orthogonal matrix P

such that $P^{\mathrm{T}}CP = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ where the eigenvalues λ_i need not be distinct. Since

$$0 \neq \det C = \lambda_1 \lambda_2 ... \lambda_n,$$

the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T C \mathbf{x} =$ is non-negative for each $\mathbf{x} \neq \mathbf{0}$. The columns of the orthogonal matrix P form an orthogonal basis $\{\mathbf{v}_i : i \leq n\}$ of $\mathbb{R}^n$ and, relative to this basis, $Q(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i^2$ for any $\mathbf{x}$. Applying this with $\mathbf{x} = \mathbf{v}_i$ for each $i \leq n$ we see that each $\lambda_i \geq 0$. Since their product is non-zero, each must be strictly positive. This shows that $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

Exercise 4.4. Since $w_j = \frac{1}{n}$ we have

$$\sigma_{\mathbf{w}}(n) = \left(\frac{1}{n}\right)^2 \sum_{j,k=1}^{n} \sigma_{jk}.$$

Mathematically, such a limit might not be convergent as n tends to zero. As an example we can consider

$$\begin{aligned} \sigma_{kk} &= 2^k \qquad \text{for } k = 1, 2 \ldots \\ \sigma_{jk} &= 0 \qquad \text{for } j \neq k. \end{aligned}$$

This example though is artificial and one would not expect this to happen in real life.

In a situation where assets are independent (which implies $\sigma_{jk} = 0$ for $j \neq k$) and when $\sigma_{kk} \leq c$ for $k = 1, 2, \ldots$ then

$$\sigma_{\mathbf{w}}(n) = \left(\frac{1}{n}\right)^2 \sum_{k=1}^{n} \sigma_{kk} \leq \left(\frac{1}{n}\right)^2 nc \to 0.$$

Exercise 4.5. We can compute

$$C^{-1} = \begin{bmatrix} 100.0 & 0 & 0 \\ 0 & 50.0 & 0 \\ 0 & 0 & 25.0 \end{bmatrix}$$

The matrix M is equal to

$$M = \begin{bmatrix} \boldsymbol{\mu}^{\mathrm{T}} C^{-1} \boldsymbol{\mu} & \boldsymbol{\mu}^{\mathrm{T}} C^{-1} \mathbf{1} \\ \boldsymbol{\mu}^{\mathrm{T}} C^{-1} \mathbf{1} & \mathbf{1}^{\mathrm{T}} C^{-1} \mathbf{1} \end{bmatrix} = \begin{bmatrix} 5.25 & 27.5 \\ 27.5 & 175 \end{bmatrix},$$

and $\det(M) = 162.5$. For $m = 0.25$ we have

$$\begin{aligned} \det(M_1) &= m \mathbf{1}^{\mathrm{T}} C^{-1} \mathbf{1} - \boldsymbol{\mu}^{\mathrm{T}} C^{-1} \mathbf{1} = 16.25 \\ \det(M_2) &= \boldsymbol{\mu}^{\mathrm{T}} C^{-1} \boldsymbol{\mu} - m \boldsymbol{\mu}^{\mathrm{T}} C^{-1} \mathbf{1} = -1.625, \end{aligned}$$

giving

$$\mathbf{w} = \frac{1}{\det(M)} C^{-1} \left(\det(M_1)\boldsymbol{\mu} + \det(M_2)\mathbf{1}\right) = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}.$$

Exercise 4.6. We can compute

$$C^{-1} = \begin{bmatrix} 100.0 & 0 & 0 \\ 0 & 100.0 & -50.0 \\ 0 & -50.0 & 50.0 \end{bmatrix}.$$

The matrix M is equal to

$$M = \begin{bmatrix} \boldsymbol{\mu}^{\mathrm{T}}C^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}^{\mathrm{T}}C^{-1}\mathbf{1} \\ \boldsymbol{\mu}^{\mathrm{T}}C^{-1}\mathbf{1} & \mathbf{1}^{\mathrm{T}}C^{-1}\mathbf{1} \end{bmatrix} = \begin{bmatrix} 3.5 & 20 \\ 20 & 150 \end{bmatrix},$$

giving $\det(M) = 125.0$ and

$$\mathbf{a} = \frac{1}{\det(M)} C^{-1} \left(\left(\mathbf{1}^{\mathrm{T}}C^{-1}\mathbf{1}\right)\boldsymbol{\mu} - \left(\boldsymbol{\mu}^{\mathrm{T}}C^{-1}\mathbf{1}\right)\mathbf{1}\right) = \begin{bmatrix} -4.0 \\ -2.0 \\ 6.0 \end{bmatrix},$$

$$\mathbf{b} = \frac{1}{\det(M)} C^{-1} \left(\left(\boldsymbol{\mu}^{\mathrm{T}}C^{-1}\boldsymbol{\mu}\right)\mathbf{1} - \left(\boldsymbol{\mu}^{\mathrm{T}}C^{-1}\mathbf{1}\right)\boldsymbol{\mu}\right) = \begin{bmatrix} 1.2 \\ 0.6 \\ -0.8 \end{bmatrix}.$$

The vector of weights corresponding to $m = 0.2$ is

$$\mathbf{w}_{m=0.2} = m\mathbf{a} + \mathbf{b} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix}.$$

Exercise 4.7. The minimum variance line on the (w_1, w_2) plane is a straight line resulting from a projection of $m\mathbf{a} + \mathbf{b}$ onto the first two co-ordinates

$$\begin{aligned} \begin{bmatrix} w_1(m) \\ w_2(m) \end{bmatrix} &= m \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ &= m \begin{bmatrix} -4.0 \\ -2.0 \end{bmatrix} + \begin{bmatrix} 1.2 \\ 0.6 \end{bmatrix} \end{aligned}$$

Its plot is given in Figure 9.1.

For $m = 0.1$ the portfolio is

$$\mathbf{w}_{m=0.1} = 0.1 \begin{bmatrix} -4.0 \\ -2.0 \\ 6.0 \end{bmatrix} + \begin{bmatrix} 1.2 \\ 0.6 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 0.4 \\ -0.2 \end{bmatrix}$$

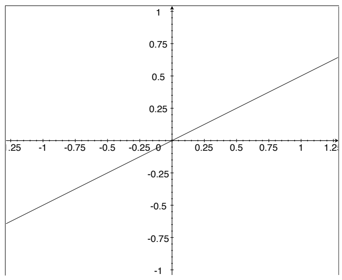

Figure 9.1 The minimum variance line on the (w_1, w_2) plane.

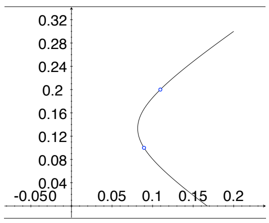

Figure 9.2 The minimum variance line on the (σ, μ) plane.

The portfolio for $m = 0.2$ is computed in the solution to Exercise 4.6. Their variances are

$$\sigma^2_{m=0.1} = 0.008$$
$$\sigma^2_{m=0.2} = 0.012.$$

The covariance can be computed using Proposition 4.2, giving

$$\text{Cov} = 0.004.$$

By Corollary 4.8, we can compute the risk and expected return of portfolios on the minimum variance line using

$$\sigma^2_{\mathbf{w}} = \alpha^2\sigma^2_{m=0.1} + (1-\alpha)^2\sigma^2_{m=0.2} + 2\alpha(1-\alpha)\,\text{Cov}$$
$$\mu_{\mathbf{w}} = \alpha 0.1 + (1-\alpha)\,0.2.$$

The plot of $(\sigma_{\mathbf{w}}, \mu_{\mathbf{w}})$ is given in Figure 9.2.

Exercise 4.8. Let $\sigma_1 = \sigma_{m=0.1}$ and $\sigma_2 = \sigma_{m=0.2}$ and $\sigma_{12} = \text{Cov}$ for parameters computed in the solution of Exercise 4.7. We look for α for which

$$\alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\sigma_{12} = \sigma^2 = 0.007.$$

This is a quadratic equation

$$\alpha^2\left(\sigma_1^2+\sigma_2^2-2\sigma_{12}\right)+\alpha 2\left(\sigma_{12}-\sigma_2^2\right)+\sigma_2^2-\sigma^2=0$$

with solutions

$$\alpha=\frac{-2\left(\sigma_{12}-\sigma_2^2\right)\pm\sqrt{\left[2\left(\sigma_{12}-\sigma_2^2\right)\right]^2-4\left(\sigma_1^2+\sigma_2^2-2\sigma_{12}\right)\left(\sigma_2^2-\sigma^2\right)}}{2\left(\sigma_1^2+\sigma_2^2-2\sigma_{12}\right)}.$$

giving

$$\begin{aligned}\alpha_1&=\frac{25}{30},\\ \alpha_2&=\frac{1}{2}.\end{aligned}$$

The corresponding expected returns are

$$\begin{aligned}\mu_1&=\alpha_1 0.1+(1-\alpha_1)\,0.2=\frac{7}{60},\\ \mu_2&=\alpha_2 0.1+(1-\alpha_2)\,0.2=0.15.\end{aligned}$$

Since $\mu_2>\mu_1$, the efficient portfolio with $\sigma^2=0.007$ is

$$\begin{aligned}&\alpha_2\mathbf{w}_{m=0.1}+(1-\alpha_2)\,\mathbf{w}_{m=0.2}\\ &=\frac{1}{2}\begin{bmatrix}0.8\\0.4\\-0.2\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0.4\\0.2\\0.4\end{bmatrix}=\begin{bmatrix}0.6\\0.3\\0.1\end{bmatrix}.\end{aligned}$$

Exercise 4.9. Since

$$\mu_{\mathbf{w}_{\min}}=\boldsymbol{\mu}^{\mathrm{T}}\mathbf{w}_{\mathbf{min}}=\boldsymbol{\mu}^{\mathrm{T}}\frac{C^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}C^{-1}\mathbf{1}},$$

for $r=\mu_{\mathbf{w}_{\min}}$ we have

$$\begin{aligned}\mathbf{1}^{\mathrm{T}}C^{-1}(\boldsymbol{\mu}-r\mathbf{1})&=\mathbf{1}^{\mathrm{T}}C^{-1}\left(\boldsymbol{\mu}-\boldsymbol{\mu}^{\mathrm{T}}\frac{C^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}C^{-1}\mathbf{1}}\mathbf{1}\right)\\ &=\mathbf{1}^{\mathrm{T}}C^{-1}\boldsymbol{\mu}-\frac{\boldsymbol{\mu}^{\mathrm{T}}C^{-1}\mathbf{1}}{\mathbf{1}^{\mathrm{T}}C^{-1}\mathbf{1}}\mathbf{1}^{\mathrm{T}}C^{-1}\mathbf{1}\\ &=\mathbf{1}^{\mathrm{T}}C^{-1}\boldsymbol{\mu}-\boldsymbol{\mu}^{\mathrm{T}}C^{-1}\mathbf{1}\\ &=0\end{aligned}$$

hence we have a division by zero in

$$\mathbf{m}=\frac{C^{-1}(\boldsymbol{\mu}-r\mathbf{1})}{\mathbf{1}^{\mathrm{T}}C^{-1}(\boldsymbol{\mu}-r\mathbf{1})}.$$

The geometric reason behind this is that the point $(0, R)$ is the focus point of the MVL hyperbola. Thus there is no tangency point on MVL with a straight line emanating from $(0, R)$.

Exercise 4.10. We start by computing $\mathbf{m}_1$ and $\mathbf{m}_2$

$$\mathbf{m}_1 = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.4 \end{bmatrix} \qquad \mathbf{m}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The expected returns and variances are

$$\begin{aligned} \mu_{\mathbf{m}_1} &= 0.2, & \mu_{\mathbf{m}_2} &= 0.3, \\ \sigma^2_{\mathbf{m}_1} &= 0.012, & \sigma^2_{\mathbf{m}_2} &= \sigma_3^2 = 0.04. \end{aligned}$$

(i) Since for $\sigma^2 = 0.003$ we have $\sigma < \sigma_{\mathbf{m}_1}$, the efficient portfolio is a combination of a risk–free investment at r_1 and an investment in the tangency portfolio $\mathbf{m}_1$. The corresponding variance of such an investment is $\alpha^2\sigma^2_{\mathbf{m}_1}$.

We need to find an α for which

$$\alpha^2\sigma^2_{\mathbf{m}_1} = \sigma^2,$$

which gives

$$\alpha = \sqrt{\frac{\sigma^2}{\sigma^2_{\mathbf{m}_1}}} = \sqrt{\frac{0.003}{0.012}} = 0.5.$$

This means that we need to invest $(1-\alpha)V = 500$ risk free at the rate r_1 and invest $\alpha V \mathbf{m}_1$ amongst the remaining assets. Thus we need to invest 200 in the first asset, 100 in the second, and 200 in the third asset.

(ii) Since for $\sigma^2 = 0.023$ we have $\sigma^2_{\mathbf{m}_1} < \sigma^2 < \sigma^2_{\mathbf{m}_2}$, the efficient investment with variance equal to σ^2 lies on the minimum variance line. Any portfolio on the MVL is a linear combination of $\mathbf{m}_1$ and $\mathbf{m}_2$. We need to find an α for which

$$\alpha^2\sigma^2_{\mathbf{m}_1} + (1-\alpha)^2\sigma^2_{\mathbf{m}_2} + 2\alpha(1-\alpha)\operatorname{Cov}(K_{\mathbf{m}_1}, K_{\mathbf{m}_2}) = \sigma^2. \tag{9.7}$$

Since

$$\operatorname{Cov}(K_{\mathbf{m}_1}, K_{\mathbf{m}_2}) = \mathbf{m}_1^{\mathrm{T}} C \mathbf{m}_2 = 0.02,$$

all the parameters in (9.7) are known, and we can solve the quadratic equation just as we have done in the solution of Exercise 4.8. The solution is

$$\alpha = \frac{1}{2},$$

meaning that we need to invest

$$\alpha V\mathbf{m}_1 + (1-\alpha)\, V\mathbf{m}_2$$

amongst the risky securities. Thus we invest 200 in the first asset, 100 in the second, and 700 in the third asset.

(iii) Since for $\sigma^2 = 0.16$ we have $\sigma^2 > \sigma^2_{\mathbf{m}_2}$, the efficient portfolio is a combination of a risk–free loan at r_2 and an investment in the tangency portfolio $\mathbf{m}_2$. The corresponding variance of such an investment is $\alpha^2\sigma^2_{\mathbf{m}_2}$.

We need to find an α for which

$$\alpha^2\sigma^2_{\mathbf{m}_2} = \sigma^2,$$

which gives

$$\alpha = \sqrt{\frac{\sigma^2}{\sigma^2_{\mathbf{m}_2}}} = \sqrt{\frac{0.16}{0.04}} = 2.$$

This means that we need to invest $(1-\alpha)V = -1000$ at a rate r_2 (meaning that we borrow 1000) and invest $\alpha V\mathbf{m}_2$ amongst the remaining assets. This means that we need to invest 2000 in the third asset.

9.5 Solutions to exercises from chapter 5

Exercise 5.1. From the bilinearity of covariance we know that

$$\begin{aligned}\beta_{\mathbf{w}} &= \frac{\operatorname{Cov}(K_{\mathbf{w}}, K_{\mathbf{m}})}{\sigma_{\mathbf{m}}^2} \\ &= \frac{\operatorname{Cov}(w_1 K_1 + \ldots + w_n K_n, K_{\mathbf{m}})}{\sigma_{\mathbf{m}}^2} \\ &= w_1 \frac{\operatorname{Cov}(K_1, K_{\mathbf{m}})}{\sigma_{\mathbf{m}}^2} + \ldots + w_n \frac{\operatorname{Cov}(K_n, K_{\mathbf{m}})}{\sigma_{\mathbf{m}}^2} \\ &= w_1\beta_1 + \ldots + w_n\beta_n.\end{aligned}$$

From (5.1) we know that

$$\mu_i = R + \beta_i(\mu_{\mathbf{m}} - R).$$

We can now compute

$$\begin{aligned}\mu_{\mathbf{w}} &= w_1\mu_1 + \ldots + w_n\mu_n \\ &= w_1\left(R + \beta_1(\mu_{\mathbf{m}} - R)\right) + \ldots + w_n\left(R + \beta_n(\mu_{\mathbf{m}} - R)\right) \\ &= R + (w_1\beta_1 + \ldots + w_n\beta_n)(\mu_{\mathbf{m}} - R) \\ &= R + \beta_{\mathbf{w}}(\mu_{\mathbf{m}} - R),\end{aligned}$$

as required.

Exercise 5.2. From Chapter 4 we know that the weights of the tangency portfolios for R_k are

$$\mathbf{m}_k = \frac{1}{\gamma_k} C^{-1}(\boldsymbol{\mu} - R_k\mathbf{1})$$

for $\gamma_k = \mathbf{1}^{\mathrm{T}} C^{-1}(\boldsymbol{\mu} - R_k\mathbf{1})$ and $k = 1, 2$. Let $\mathbf{w}$ be any portfolio. Applying Proposition 4.2,

$$\frac{\operatorname{Cov}(K_{\mathbf{w}}, K_{\mathbf{m}_k})}{\sigma_{\mathbf{m}_k}^2} = \frac{\mathbf{w}^{\mathrm{T}} C\mathbf{m}_k}{\mathbf{m}_k^{\mathrm{T}} C\mathbf{m}_k} = \frac{\frac{1}{\gamma}\mathbf{w}^{\mathrm{T}}(\boldsymbol{\mu} - R_k\mathbf{1})}{\frac{1}{\gamma}\mathbf{m}_k^{\mathrm{T}}(\boldsymbol{\mu} - R_k\mathbf{1})}.$$

Since $\mathbf{w}^{\mathrm{T}}\boldsymbol{\mu} = \mu_{\mathbf{w}}$, $\mathbf{m}_k^{\mathrm{T}}\boldsymbol{\mu} = \mu_{\mathbf{m}_k}$ and $\mathbf{w}^{\mathrm{T}}\mathbf{1} = \mathbf{m}_k^{\mathrm{T}}\mathbf{1} = 1$, this gives

$$\frac{\operatorname{Cov}(K_{\mathbf{w}}, K_{\mathbf{m}_k})}{\sigma_{\mathbf{m}_k}^2} = \frac{\mu_{\mathbf{w}} - R_k}{\mu_{\mathbf{m}_k} - R_k},$$

and by rearranging

$$\mu_{\mathbf{w}} = R_k + \frac{\operatorname{Cov}(K_{\mathbf{w}}, K_{\mathbf{m}_k})}{\sigma_{\mathbf{m}_k}^2}\left(\mu_{\mathbf{m}_k} - R_k\right).$$

Taking a portfolio where everything is invested in the i-th asset gives

$$\mu_i = R_k + \frac{\mathrm{Cov}(K_i, K_{\mathbf{m}_k})}{\sigma^2_{\mathbf{m}_k}} (\mu_{\mathbf{m}_k} - R_k),$$

for $k = 1, 2$, as required.

Exercise 5.3. For

$$f(\alpha, \beta) = \sum_{j=1}^{d} \left(y_j - \alpha - \beta x_j\right)^2,$$

we have

$$\frac{\partial f}{\partial \alpha} = -2 \sum_{j=1}^{d} \left(y_j - \alpha - \beta x_j\right) = -2d\,(\bar{y} - \alpha - \beta \bar{x}),$$

$$\frac{\partial f}{\partial \beta} = -2 \sum_{j=1}^{d} \left(y_j - \alpha - \beta x_j\right) x_j = -2d\,(\overline{xy} - \bar{x}\alpha - \beta \overline{xx}),$$

The system

$$\frac{\partial f}{\partial \alpha} = 0,$$
$$\frac{\partial f}{\partial \beta} = 0,$$

is therefore equivalent to

$$\bar{y} - \alpha - \beta \bar{x} = 0,$$
$$\overline{xy} - \bar{x}\alpha - \beta \overline{xx} = 0.$$

Multiplying the first equation by $\bar{x}$ and subtracting the two gives

$$\overline{xy} - \bar{x}\bar{y} - \bar{x}\alpha - \beta \overline{xx} + \beta \bar{x}\bar{x} = 0,$$

which leads to

$$\beta = \frac{\bar{x}\bar{y} - \overline{xy}}{\bar{x}\bar{x} - \overline{xx}}.$$

Since $\bar{y} - \alpha - \beta \bar{x} = 0$, it follows that

$$\alpha = \bar{y} - \beta \bar{x}.$$

9.6 Solutions to exercises from chapter 6

Exercise 6.1. If $X \preceq_U Y$ and $Y \preceq_U Z$ then by the definition of $\preceq_U$,

$$U(X) \le U(Y) \qquad \text{and} \qquad U(Y) \le U(Z),$$

hence $U(X) \le U(Z)$, which again from the definition of $\preceq_U$ implies that

$$X \preceq_U Z.$$

This finishes the proof of transitivity of $\preceq_U$.

For any random variables X, Y we either have $U(X) \le U(Y)$ or $U(X) \ge U(Y)$, which by definition of $\preceq_U$ means that either $X \preceq_U Y$ or $Y \preceq_U X$. This proves completeness of $\preceq_U$.

Exercise 6.2. Suppose that there exists a utility U that represents the lexicographic order. Since a utility is differentiable, it has to be continuous. Let $(x_0, y_0) \in \mathbb{R}^2$ and $y_1 > y_0$. Since U is strictly increasing

$$U(x_0, y_0) < U(x_0, y_1). \tag{9.8}$$

Let $x_1 > x_0$. Since U represents $\preceq_{\text{lex}}$, and $(x_0, y_1) \preceq_{\text{lex}} (x_1, y_0)$, we must have

$$U(x_0, y_1) \le U(x_1, y_0),$$

hence

$$U(x_1, y_0) \le \lim_{x_1 \to x_0} U(x_1, y_0) \tag{9.9}$$

On the other hand, since U is continuous

$$\lim_{x_1 \to x_0} U(x_1, y_1) = U(x_0, y_0). \tag{9.10}$$

Combining (9.8)–(9.10) gives

$$\lim_{x_1 \to x_0} U(x_1, y_1) = U(x_0, y_0) < U(x_0, y_1) < \lim_{x_1 \to x_0} U(x_1, y_1),$$

a contradiction. Thus, such U can not exist.

Exercise 6.3. To show that a function is strictly increasing and concave it is enough to show that the derivative is strictly positive and that the second derivative is negative.

(i) For $a > 0$,

$$(-\mathrm{e}^{-ax})' = a\mathrm{e}^{-ax} > 0,$$
$$(-\mathrm{e}^{-ax})'' = -a^2\mathrm{e}^{-ax} < 0.$$

(ii) For $x > 0$,

$$(\ln x)' = \frac{1}{x} > 0,$$
$$(\ln x)'' = \frac{-1}{x^2} < 0.$$

(iii) For $a < 1$, holds $a - 1 < 0$ hence for $x > 0$

$$(ax^a)' = a^2 x^{a-1} > 0,$$
$$(ax^a)'' = a^2 (a-1) x^{a-2} < 0.$$

(iv) For $b > 0$ and $x < \frac{1}{b}$

$$\left(x - \frac{1}{2}bx^2\right)' = 1 - bx > 1 - b\frac{1}{b} = 0,$$
$$\left(x - \frac{1}{2}bx^2\right)'' = -b < 0.$$

Exercise 6.4. Substituting

$$u = 0.1, \quad m = 0, \quad d = -0.1,$$

into the formulae for π_i from Example 6.15, we obtain

$$\pi_1(x) = x,$$
$$\pi_2(x) = 1 - 2x,$$
$$\pi_3(x) = x.$$

Since in Example 6.15 we have shown that $X^*(\omega_i) = \frac{\lambda p_i}{\pi_i} = \frac{V p_i}{\pi_i}$,

$$X^*(\omega) = \begin{cases} \frac{V}{4x} & \text{for } \omega = \omega_1, \\ \frac{V}{2-4x} & \text{for } \omega = \omega_2, \\ \frac{V}{4x} & \text{for } \omega = \omega_3. \end{cases}$$

For $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, where x_1 is invested in the risk free asset and x_2 is invested in the risky security, we have

$$V_{\mathbf{x}}(1)(\omega) = \begin{cases} x_1 + x_2 S(0)(1+u) & \text{for } \omega = \omega_1, \\ x_1 + x_2 S(0)(1+m) & \text{for } \omega = \omega_2, \\ x_1 + x_2 S(0)(1+d) & \text{for } \omega = \omega_3. \end{cases}$$

Since we need to have $V_{\mathbf{x}}(1) = X^*$

$$x_1 + x_2 S(0)(1+u) = \frac{V}{4x} = x_1 + x_2 S(0)(1+d)$$

hence

$$x_2 u = x_2 d.$$

Since $u = -d$, this means that $x_2 = 0$. Since $V_{\mathbf{x}}(1) = X^*$ and $x_2 = 0$,

$$x_1 = \frac{V}{4x},$$
$$x_1 = \frac{V}{2-4x},$$

therefore

$$4x = 2 - 4x,$$

thus $x = 0.25$.

Exercise 6.5. For

$$f(X_1, \ldots, X_N) = \sum_{i=1}^{N} p_i u(X_i)$$
$$g(X_1, \ldots, X_N) = \sum_{i=1}^{N} \pi_i X_i - V$$

equation (3.9) implies that

$$\begin{bmatrix} p_1 u'(X_1) \\ \vdots \\ p_N u'(X_N) \end{bmatrix} - \lambda \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_N \end{bmatrix} = 0.$$

From the above we see that for $i = 1, \ldots, N$

$$X_i = (u')^{-1}\left(\frac{\pi_i}{p_i}\right). \tag{9.11}$$

In addition, we also have the constraint

$$\sum_{i=1}^{N} \pi_i X_i - V = 0.$$

Inserting X_i from (9.11) and rearranging gives

$$V = \sum_{i=1}^{N} \pi_i (u')^{-1}\left(\frac{\pi_i}{p_i}\right). \tag{9.12}$$

Equations (9.11), (9.12) combined give the claim.

Exercise 6.6. Let x denote the number of shares of the first asset. Then

$$V_{\mathbf{x}}(1) = xS_1(1) + \frac{V - xS_1(0)}{S_2(0)} S_2(1).$$

The problem of finding the maximal expected utility is reduced to maximising a function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = u(V_{\mathbf{x}}(1)) = \sum_{i=1}^{N} p_i \mathrm{e}^{xS_1(1)(\omega_i) - a\frac{V - xS_1(0)}{S_2(0)} S_2(1)(\omega_i)}.$$

This can easily be done numerically. An example of a solution for $V = 100$ is given in the file `Exercise_6_6.xlsx`.

Exercise 6.7. Since the model is complete, using (6.17) we can compute

$$\boldsymbol{\pi}^{\mathrm{T}} = \mathbf{S}(0)\,(\mathbf{S}(1))^{-1}. \tag{9.13}$$

For our utility

$$u'(x) = a\mathrm{e}^{-ax},$$
$$(u')^{-1}(x) = \frac{-1}{a}\ln\left(\frac{x}{a}\right).$$

Since

$$(u')^{-1}\left(\frac{\pi_i}{\lambda p_i}\right) = -\frac{1}{a}\ln\left(\frac{\pi_i}{a\lambda p_i}\right) = -\frac{1}{a}\ln\left(\frac{\pi_i}{ap_i}\right) + \frac{1}{a}\ln(\lambda),$$
$$\sum_{i=1}^{N}\pi_i = \frac{1}{1+R},$$

and from (6.13) we can compute

$$\begin{aligned} V &= \sum_{i=1}^{N}\pi_i (u')^{-1}\left(\frac{\pi_i}{\lambda p_i}\right) \\ &= -\sum_{i=1}^{N}\pi_i\frac{1}{a}\ln\left(\frac{\pi_i}{ap_i}\right) + \sum_{i=1}^{N}\pi_i\frac{1}{a}\ln(\lambda) \\ &= -\sum_{i=1}^{N}\pi_i\frac{1}{a}\ln\left(\frac{\pi_i}{ap_i}\right) + \frac{1}{1+R}\frac{1}{a}\ln(\lambda), \end{aligned}$$

hence

$$\frac{1}{a}\ln(\lambda) = (1+R)\left[V + \sum_{i=1}^{N}\pi_i\frac{1}{a}\ln\left(\frac{\pi_i}{ap_i}\right)\right]. \tag{9.14}$$

We can substitute the above into (6.12) to compute

$$X_i^* = (u')^{-1}\left(\frac{\pi_i}{\lambda p_i}\right) = -\frac{1}{a}\ln\left(\frac{\pi_i}{ap_i}\right) + \frac{1}{a}\ln(\lambda). \tag{9.15}$$

Finally, the $\mathbf{x}$ can be computed from (6.18),

$$\mathbf{x}^* = (\mathbf{S}(1))^{-1} X^*. \tag{9.16}$$

We now carry out the numerical computations, taking $V = 100$ as an example. Using (9.13)

$$\boldsymbol{\pi}^{\mathrm{T}} = \begin{bmatrix} 1 & 100 & 200 \end{bmatrix} \begin{bmatrix} 1.02 & 120 & 180 \\ 1.02 & 110 & 220 \\ 1.02 & 90 & 200 \end{bmatrix}^{-1} = \begin{bmatrix} 0.1569 & 0.3529 & 0.4706 \end{bmatrix}.$$

We now use (9.14)

$$\begin{aligned} \frac{1}{a} \ln(\lambda) &= 1.02 \Bigg[100 + 0.1569 \frac{1}{0.01} \ln\left(\frac{0.1569}{0.01 \cdot 0.25}\right) \\ &\quad + 0.3529 \frac{1}{0.01} \ln\left(\frac{0.3529}{0.01 \cdot 0.5}\right) \\ &\quad + 0.4706 \frac{1}{0.01} \ln\left(\frac{0.4706}{0.01 \cdot 0.25}\right) \Bigg] \\ &= 572.88. \end{aligned}$$

From (9.15)

$$\begin{aligned} X_1^* &= -\frac{1}{0.01} \ln\left(\frac{0.1569}{0.01 \cdot 0.25}\right) + 572.88 = 158.95, \\ X_2^* &= -\frac{1}{0.01} \ln\left(\frac{0.3529}{0.01 \cdot 0.5}\right) + 572.88 = 147.21, \\ X_3^* &= -\frac{1}{0.01} \ln\left(\frac{0.4706}{0.01 \cdot 0.25}\right) + 572.88 = 49.108. \end{aligned}$$

Finally, using (9.16) we obtain the strategy

$$\mathbf{x} = \begin{bmatrix} 1.02 & 120 & 180 \\ 1.02 & 110 & 220 \\ 1.02 & 90 & 200 \end{bmatrix}^{-1} \begin{bmatrix} 158.95 \\ 147.21 \\ 49.108 \end{bmatrix} = \begin{bmatrix} -465.13 \\ 4.1589 \\ 0.74622 \end{bmatrix}.$$

Exercise 6.8. If u is concave, then $\varphi = -u$ is convex. By Jensen's inequality $\varphi(\mathbb{E}(X)) \leq \mathbb{E}(\varphi(X))$, which gives

$$-u(\mathbb{E}(X)) = \varphi(\mathbb{E}(X)) \leq \mathbb{E}(\varphi(X)) = -\mathbb{E}(u(X)),$$

hence

$$u(\mathbb{E}(X)) \geq \mathbb{E}(u(X)).$$

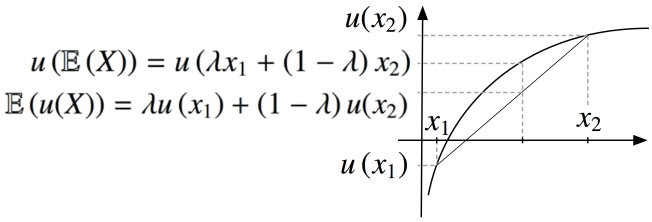

Figure 9.3 Concavity of u and risk aversion.

Conversely, let us assume that an investor is risk averse and consider a random variable with binomial distribution

$$X = \begin{cases} x_1 & \text{with probability } \lambda \\ x_2 & \text{with probability } 1-\lambda, \end{cases}$$

for $\lambda \in [0,1]$. Then since $u(\mathbb{E}(X)) \geq \mathbb{E}(u(X))$ we see that

$$u(\lambda x_1 + (1-\lambda)x_2) = u(\mathbb{E}(X)) \geq \mathbb{E}(u(X)) = \lambda u(x_1) + (1-\lambda)u(x_2),$$

which means that u is concave. A graphical representation for this random variable is given in Figure 9.3

Exercise 6.9. From the initial data for $\mathbf{S}(0)$ and $\mathbf{S}(1)$ it follows that

$$K_1 = 2\%, \qquad K_2(\omega) = \begin{cases} 20\% & \text{for } \omega = \omega_1 \\ 10\% & \text{for } \omega = \omega_2 \\ -10\% & \text{for } \omega = \omega_3 \end{cases}$$

$$K_3 = \begin{cases} -10\% & \text{for } \omega = \omega_1 \\ 10\% & \text{for } \omega = \omega_2 \\ 0\% & \text{for } \omega = \omega_3 \end{cases}$$

and simple computation leads to

$$R = 2\%, \qquad \boldsymbol{\mu} = \begin{bmatrix} 7.5\% \\ 2.5\% \end{bmatrix},$$

$$C = \begin{bmatrix} 0.011875 & -0.001875 \\ -0.001875 & 0.006875 \end{bmatrix}.$$

We can compute

$$\mathbf{m} = \frac{C^{-1}(\boldsymbol{\mu} - R\mathbf{1})}{\mathbf{1}^{\mathrm{T}}C^{-1}(\boldsymbol{\mu} - R\mathbf{1})} = \begin{bmatrix} 0.704\,55 \\ 0.295\,45 \end{bmatrix},$$

and

$$\mu_{\mathbf{m}} = \mathbf{m}^{\mathrm{T}}\boldsymbol{\mu} = 0.060228,$$
$$\sigma_{\mathbf{m}} = \sqrt{\mathbf{m}^{\mathrm{T}}C\mathbf{m}} = 0.075592.$$

To find a portfolio with the highest certainty equivalent we need to maximise

$$\mathbb{E}(X) - \gamma(X) = V\left[\mu - \frac{aV}{2}\sigma^2\right] + V.$$

This is equivalent to maximising

$$u(\sigma, \mu) = \mu - \frac{aV}{2}\sigma^2.$$

From Example 2.13 we know that the solution is a portfolio with the return equal to

$$\mu = R + \frac{1}{aV}\left(\frac{\mu_{\mathbf{m}} - R}{\sigma_{\mathbf{m}}}\right)^2,$$

and

$$\sigma = \frac{1}{aV}\frac{\mu_{\mathbf{m}} - R}{\sigma_{\mathbf{m}}}.$$

Taking $a = 0.01$, $V = 100$, as in the solution to Exercise 6.7, we get

$$\mu = 0.02 + \frac{1}{0.01 \cdot 100}\left(\frac{0.060228 - 0.02}{0.07559\,2}\right)^2 = 0.30321,$$
$$\sigma = \frac{1}{0.01 \cdot 100}\frac{0.060228 - 0.02}{0.07559\,2} = 0.532\,17.$$

Exercise 6.10. We continue the solution of Exercise 6.9. We wish to find a portfolio on the CML with the return

$$m = 0.30321.$$

As in Example 2.12 we compute

$$\lambda = \frac{m - R}{\mu_{\mathbf{m}} - R} = \frac{0.30321 - 0.02}{0.060228 - 0.02} = 7.040\,1.$$

We spend $(1 - \lambda)V = -6.040\,1 \cdot 100 = -604.01$ risk free (since the number is negative we take a short position; in other words, borrow 604.01) and spend

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda V \mathbf{m} = 7.0401 \cdot 100 \begin{bmatrix} 0.70455 \\ 0.29545 \end{bmatrix} = \begin{bmatrix} 496.01 \\ 208.00 \end{bmatrix}$$

on the two risky assets. We thus obtain the strategy

$$\mathbf{x} = \begin{bmatrix} (1-\lambda)\, V/S_1(0) \\ v_1/S_2(0) \\ v_2/S_3(0) \end{bmatrix} = \begin{bmatrix} -604.01 \\ 4.9601 \\ 1.04 \end{bmatrix}.$$

The solution differs from the solution to Exercise 6.7. The reason for this is that in Exercise 6.7 we find the exact solution which maximises the expected utility. In this exercise we maximise the certainty equivalent. This in principle is equivalent to maximising expected utility, but we use

$$\gamma(X) \approx -\frac{V^2}{2}\frac{u''(V(1+\mu))}{u'(V(1+\mu))}\sigma^2, \tag{9.17}$$

for our computation of $\mathbb{E}(X) - \gamma(X)$. Since (9.17) is only an approximation of $\gamma(X)$, the solution we have found is also an approximate solution. It will usually be different from the true one (found in the solution to Exercise 6.7).

9.7 Solutions to exercises from chapter 7

Exercise 7.1

Proposition (reformulation of Proposition 7.4 for lower quantiles)
Let X, Y be random variables. Then for any $\beta \in (0, 1)$

(i) $X \geq Y$ *implies* $q_\beta(X) \geq q_\beta(Y)$
(ii) *for any* $b \in \mathbb{R}$, $q_\beta(X + b) = q_\beta(X) + b$
(iii) *for* $b > 0$, $q_\beta(bX) = bq_\beta(X)$
(iv) $q_\beta(-X) = -q^{1-\beta}(X)$

Proof We begin by proving (iv). By Proposition 7.4 we know that for any X, $q^\alpha(-X) = -q_{1-\alpha}(X)$ when $0 < \alpha < 1$, hence also $-q^\alpha(X) = q_{1-\alpha}(-X)$. Set $\beta = 1 - \alpha$, then (iv) follows: $q_\beta(-X) = -q^{1-\beta}(X)$.

For (i), we have $-X \leq -Y$, so $q^\alpha(-X) \leq q^\alpha(-Y)$, i.e., $-q^\alpha(-X) \geq -q^\alpha(-Y)$. By (iv), with $\beta = 1 - \alpha$, with $-X, -Y$ instead of X, Y, we then obtain

$$q_\beta(X) = -q^\alpha(-X) \geq -q^\alpha(-Y) = q_\beta(Y).$$

For (ii), let $b \in \mathbb{R}$, so, applying Proposition 7.4 (ii),

$$-q^\alpha(-(X + b)) = -q^\alpha(-X) + b = q_\beta(X) + b,$$

using (iv) with X instead of $-X$.

For (iii), with $b > 0$ we apply Proposition 7.4 (iii) to $b(-X)$ to obtain similarly that

$$q_\beta(bX) = -q^\alpha(b(-X)) = -bq^\alpha(-X) = bq_\beta(X).$$

□

Lemma (reformulation of Lemma 7.5 for lower quantiles)
If $F_X(x)$ is continuous and strictly increasing then $q_\alpha(X) = F_X^{-1}(\alpha)$.

Proof We need only replace $<$ by $\leq$ and the upper by the lower quantile throughout the proof of Lemma 7.5. □

Lemma (reformulation of Lemma 7.6 for lower quantiles)
Let X be a random variable. If $f : \mathbb{R} \to \mathbb{R}$ is left-continuous and non-decreasing then

$$q_\alpha(f(X)) = f(q_\alpha(X)).$$

Proof Since f is left-continuous and non decreasing, for any $y \in \mathbb{R}$ there exists an $x \in \mathbb{R}$ such that

$$f^{-1}((-\infty, y]) = (-\infty, x].$$

We need to show two facts to obtain our result:
1. for any $y > f(q_\alpha(X))$ we have

$$F_{f(X)}(y) \geq \alpha,$$

2. for any $y < f(q_\alpha(X))$ we have

$$F_{f(X)}(y) < \alpha.$$

For the proof of the first fact we take $y > f(q_\alpha(X))$ and an x such that $f^{-1}((-\infty, y]) = (-\infty, x]$. Note that since $y > f(q_\alpha(X))$ and since f is non-decreasing we have $x \geq q_\alpha(X)$. Therefore

$$\begin{aligned}
F_{f(X)}(y) &= P(f(X) \leq y) \\
&= P(X \leq x) \\
&= F_X(x) \\
&\geq \alpha \qquad \text{(since } x \geq q_\alpha(X) \text{ and by definition of } q_\alpha(X)\text{)}.
\end{aligned}$$

We take any $y < f(q_\alpha(X))$ and x such that $f^{-1}((-\infty, y]) = (-\infty, x]$. Since f is non-decreasing and $f(x) \leq y < f(q_\alpha(X))$, we see that $x < q_\alpha(X)$. We can now compute

$$\begin{aligned}
F_{f(X)}(y) &= P(f(X) \leq y) \\
&= P(X \leq x) \\
&= F_X(x) \\
&< \alpha \qquad \text{(since } x < q_\alpha(X) \text{ and by definition of } q_\alpha(X)\text{)}.
\end{aligned}$$

□

Exercise 7.2
All three properties follow immediately from their counterparts in Propostion 7.4, since $VaR^\alpha(X) = -q^\alpha(X)$.
Exercise 7.3
We invest $wS(0)$ of our own cash, hence

$$V(0) = wS(0).$$

At time one

$$V(1) = S(1) - (1 + R)(1 - w)S(0).$$

The discounted levraged gain G_{lev} is therefore

$$\begin{aligned} G_{\mathrm{lev}} &= \frac{V(1)}{1+R} - V(0) \\ &= \frac{S(1) - (1+R)(1-w)S(0)}{1+R} - wS(0) \\ &= \frac{S(1)}{1+R} - S(0) \\ &= G_S, \end{aligned}$$

where G_S is the discounted gain from an investment in stock. Hence,

$$VaR(G_{\mathrm{lev}}) = VaR(G_S).$$

We need to keep in mind that we invest only $wS(0)$ of our own funds. If w is small, then the resulting VaR can be very large in comparison to the size of the investment. This agrees with the intuition that leveraged positions are more risky.

Exercise 7.4

By Lemma 7.17 we know that

$$q^{\alpha}(S(T))) = S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}N^{-1}(\alpha)}. \tag{9.18}$$

Using the fact that

$$N^{-1}(0.05) = -1.644853627,$$

and substituting the numbers into (9.18) we obtain

$$q^{5\%}(S(T))) = S(0)e^{\left(0.1 - \frac{0.2^2}{2}\right)T + 0.2\sqrt{1}(-1.644853627)} = 77.96.$$

Therefore by (7.16)

$$\begin{aligned} \mathrm{VaR}^{5\%}(X) &= S(0) - e^{-rT}q^{5\%}(S(T)) \\ &= 100 - e^{-0.03}77.96 \\ &= 24.344. \end{aligned}$$

Exercise 7.5

To compute $\mathbb{E}\left(X_{(x,y)}\right)$ we use the fact that

$$\mathbb{E}(S(T)) = \mathbb{E}\left(S(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z}\right) = S(0)e^{\mu T}.$$

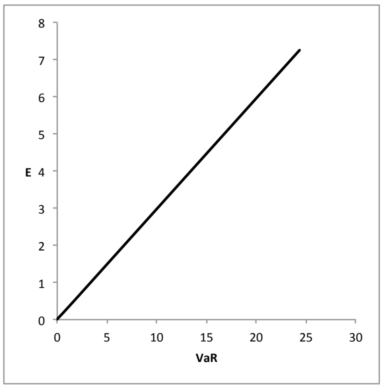

Figure 9.4 The plot of $VaR^{\alpha}\left(X_{(x,y)}\right)$ against $\mathbb{E}\left(X_{(x,y)}\right)$.

This gives

$$\begin{aligned}
\mathbb{E}\left(X_{(x,y)}\right) &= \mathbb{E}\left(\mathrm{e}^{-rT}V_{(x,y)}(T) - V_{(x,y)}(0)\right)\\
&= \mathrm{e}^{-rT}\mathbb{E}\left(V_{(x,y)}(T)\right) - V_{(x,y)}(0)\\
&= \mathrm{e}^{-rT}\mathbb{E}\left(xS(T) + yA(T)\right) - V_{(x,y)}(0)\\
&= \mathrm{e}^{-rT}x\mathbb{E}\left(S(T)\right) + y\mathrm{e}^{-rT}A(T) - \left[xS(0) + y\right]\\
&= \mathrm{e}^{-rT}xS(0)\,\mathrm{e}^{\mu T} + y - \left[xS(0) + y\right]\\
&= xS(0)\left[e^{(\mu-r)T} - 1\right].
\end{aligned}$$

The graph in Figure 9.4 is produced in the file `Exercise_7_5.xlsx`.

Exercise 7.6

Using the properties of VaR^{α} from Proposition 7.9 we obtain

$$\begin{aligned}
VaR^{\alpha}\left(X_{(x,\theta)}\right) &= VaR^{\alpha}\left(x\mathrm{e}^{-rT}S(T) + \theta\mathrm{e}^{-rT}(F - S(T)) - xS(0)\right)\\
&= VaR^{\alpha}\left(x\mathrm{e}^{-rT}S(T) + \theta\mathrm{e}^{-rT}(S(0)\mathrm{e}^{rT} - S(T)) - xS(0)\right)\\
&= VaR^{\alpha}\left((x-\theta)\,\mathrm{e}^{-rT}S(T) + (\theta - x)\,S(0)\right)\\
&= (x-\theta)\,\mathrm{e}^{-rT}VaR^{\alpha}\left(S(T)\right) + (x-\theta)\,S(0)\\
&= (x-\theta)\left[S(0) - \mathrm{e}^{-rT}q^{\alpha}(S(T))\right]
\end{aligned}$$

Similarly, from the properties of mathematical expectation

$$\begin{aligned}\mathbb{E}\left(X_{(x,\theta)}\right) &= \mathbb{E}\left(x\mathrm{e}^{-rT}S(T) + \theta\mathrm{e}^{-rT}(F - S(T)) - xS(0)\right)\\ &= \mathbb{E}\left((x-\theta)\,\mathrm{e}^{-rT}S(T) + (\theta - x)\,S(0)\right)\\ &= (x-\theta)\left(\mathrm{e}^{-rT}\mathbb{E}\left(S(T)\right) - S(0)\right)\\ &= (x-\theta)\,S(0)\left(\mathrm{e}^{(\mu-r)T} - 1\right).\end{aligned}$$

The plot is identical to the one from Exercise 7.5. The computations and the plot are made in the file `Exercise_7_6.xlsx`.

Exercise 7.7

The smallest VaR^{α} is 20.77455399. This is attainable for $K = 87.19709189$. The problem is solved using Excel solver in file `Exercise_7_7.xlsx`

Exercise 7.8

The problem is solved using Excel in the file `Exercise_7_8.xlsx`

9.8 Solutions to exercises from chapter 8

Exercise 8.1

We use the properties of VaR proved in Proposition 7.9: for $X \le Y, \lambda \ge 0$ and m real, we see that

(i) by Proposition 7.9 (i)

$$\mathrm{AVaR}^\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}^\beta(X)d\beta \ge \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}^\beta(Y)d\beta = \mathrm{AVaR}^\alpha(Y);$$

(ii) by Proposition 7.9 (ii)

$$\begin{aligned}\mathrm{AVaR}^\alpha(X+m) &= \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}^\beta(X+m)d\beta \\ &= \frac{1}{\alpha}\int_0^\alpha [\mathrm{VaR}^\beta(X)-m]d\beta \\ &= \mathrm{AVaR}^\alpha(X) - m;\end{aligned}$$

(iii) by Proposition 7.9 (iii)

$$\mathrm{AVaR}^\alpha(\lambda X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}^\beta(\lambda X)d\beta = \lambda\frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}^\beta(X)d\beta = \lambda\mathrm{AVaR}^\alpha(X).$$

Exercise 8.2

The proof of Lemma 8.4 extends without change to any function $q : (0,1) \to \mathbb{R}$ satisfying $q(F_X(x)-) \le x \le q(F_X(x))$ in place of $q^\alpha(X)$ (such functions are known as quantile functions for X; q^α being the largest and q_α the smallest), since such a q can differ from q^α in at most countably many points, i.e., a null set for the uniform distribution (Lebesgue measure on $[0,1]$).

Exercise 8.3

Same comment as for Exercise 8.2.

Exercise 8.4

We need only check the claims made for X : if $X(\omega) > q_\alpha(X)$ then $\mathbf{1}_X^\alpha(\omega) = 0$, so $\mathbf{1}_Z^\alpha - \mathbf{1}_X^\alpha \ge 0$ since $\mathbf{1}_Z^\alpha(\omega) \in [0,1]$. If $X(\omega) < q_\alpha(X)$ we have $\mathbf{1}_X^\alpha(\omega) = 1$, while, again, $\mathbf{1}_Z^\alpha \in [0,1]$, so $\mathbf{1}_Z^\alpha - \mathbf{1}_X^\alpha \le 0$.

Set $\overline{W}(\alpha) = \frac{1}{\alpha}\mathbb{E}(W\mathbf{1}_W^\alpha) = -AVaR^\alpha(W)$ for any W. Applying this to $W = X, Y$ and Z respectively,

$$\begin{aligned}&\alpha[\overline{Z}(\alpha) - \overline{X}(\alpha) - \overline{Y}(\alpha)] \\ &= \alpha\mathbb{E}[(X+Y)\mathbf{1}_Z^\alpha - X\mathbf{1}_X^\alpha - Y\mathbf{1}_Y^\alpha] \\ &= \mathbb{E}[X(\mathbf{1}_Z^\alpha - \mathbf{1}_X^\alpha) + Y(\mathbf{1}_Z^\alpha - \mathbf{1}_Y^\alpha)] \\ &= \mathbb{E}[X(\mathbf{1}_Z^\alpha - \mathbf{1}_X^\alpha)] + \mathbb{E}[Y(\mathbf{1}_Z^\alpha - \mathbf{1}_Y^\alpha)\end{aligned}$$

The first expectation is decreased if we replace X by $q_\alpha(X)$: to see this split Ω into the three disjoint events $\{X = q_\alpha(X)\}$,$\{X > q_\alpha(X)\}$ and $\{X < q_\alpha(X)\}$. The integral over $\{X = q_\alpha(X)\}$ obviously stays unchanged; on $\{X > q_\alpha(X)\}$ we have $(X - q_\alpha(X)(\mathbf{1}_Z^\alpha - \mathbf{1}_X^\alpha) \geq 0$, since both factors are non-negative, while on $\{X < q_\alpha(X)\}$, the factors $(X - q_\alpha(X))$ and $(\mathbf{1}_Z^\alpha - \mathbf{1}_X^\alpha)$ are both non-positive, so their product is non-negative. The same arguments apply with X replaced by Y, so we have shown that

$$\begin{aligned}
&\alpha[\overline{Z}(\alpha) - \overline{X}(\alpha) - \overline{Y}(\alpha)]\\
&\geq q_\alpha(X)\mathbb{E}[(\mathbf{1}_Z^\alpha - \mathbf{1}_X^\alpha)] + q_\alpha(Y)\mathbb{E}[(\mathbf{1}_Z^\alpha - \mathbf{1}_Y^\alpha]\\
&= q_\alpha(X)(\alpha - \alpha) + q_\alpha(Y)(\alpha - \alpha)\\
&= 0.
\end{aligned}$$

So $\overline{Z}(\alpha) \geq \overline{X}(\alpha) + \overline{Y}(\alpha)$, which implies that $AVaR^\alpha(X + Y) = -\overline{Z}(\alpha) \leq -\overline{X}(\alpha) - \overline{Y}(\alpha) = AVaR^\alpha(X) + AVaR^\alpha(Y)$.

Exercise 8.5

(i) To show that, if $X \leq Y$ then $TCE^\alpha(X) \geq TCE^\alpha(Y)$ it is enough to show that

$$\mathbb{E}[X|\{X \leq q^\alpha(X)\}] \leq \mathbb{E}[Y|Y \leq q^\alpha(Y)]$$

By Lemma 8.4, given a uniformly distributed random variable $U : (0, 1) \mapsto \mathbb{R}$, the real random variable $g_X(x) = q^{U(x)}(X)$ has the same distribution as X and, similarly, the real random variable $g_Y(x) = q^{U(x)}(Y)$ has the same distribution as Y.

The upper quantile $q^\alpha(X)$ is right-continuous: given $\varepsilon > 0$ we can find N such that

$$\begin{aligned}
0 &\leq q^{\alpha+\frac{1}{n}}(X) - q^\alpha(X)\\
&= \inf\{x : F_X(x) > \alpha + \frac{1}{n}\} - \inf\{x : F_X(x) > \alpha\}\\
&< \varepsilon
\end{aligned}$$

whenever $n > N$, as $\{x : F_X(x) > \alpha\} = \cup_{n\geq 1}\{x : F_X(x) > \alpha + \frac{1}{n}\}$.

Also recall that $X \leq Y$ implies $q^s(X) \leq q^s(Y)$ for any $s \in (0, 1)$ (Proposition 7.4).

So, by construction of g_X, g_Y and the right-continuity of q^α we have

$$\begin{aligned}
\mathbb{E}[X|\{X \le q^\alpha(X)\}] &= \mathbb{E}[g_X(U)|\{g_X(U) \le q^\alpha(g_X(U))\}] \\
&= \mathbb{E}[g_X(U)|\{g_X(U) \le g_X(q^\alpha(U))\}] \\
&= \mathbb{E}[g_X(U)|\{U \le q^\alpha(U)\}] \\
&\le \mathbb{E}[g_Y(U)|\{U \le q^\alpha(U)\}] \\
&= \mathbb{E}[g_Y(U)|\{g_Y(U) \le g_Y(q^\alpha(U))\}] \\
&= \mathbb{E}[g_Y(U)|\{g_Y(U) \le q^\alpha(g_Y(U))\}] \\
&= \mathbb{E}[Y|\{Y \le q^\alpha(Y)\}],
\end{aligned}$$

which completes the proof.

(ii) $\mathrm{TCE}^\alpha(X+m) = -\mathbb{E}[X+m|\{X+m \le q^\alpha(X+m)\}]$ for any real m.

By Proposition 7.4 we know that $q^\alpha(X+m) = q^\alpha(X) + m\}$ so that

$$\{X + m \le q^\alpha(X+m\} = \{X \le q^\alpha(X)\}$$

and so $-\mathbb{E}[X+m|\{X+m \le q^\alpha(X+m)\}] = -\mathbb{E}[X|\{X \le q^\alpha(X)\}] - m = \mathrm{TCE}^\alpha(X) - m$, as required.

(iii) For $\lambda > 0$ we have $q^\alpha(\lambda X) = \lambda q^\alpha(X)$, so that $\{\lambda X \le q^\alpha(\lambda X)\} = \{X \le q^\alpha(X)\}$.Therefore

$$\mathrm{TCE}^\alpha(\lambda X) = -\lambda\mathbb{E}[X|\{X \le q^\alpha(X)\}] = \lambda\mathrm{TCE}^\alpha(X).$$

For $\lambda = 0$ both sides are 0, since $\mathbb{E}[0|A) = 0$ for any A.

Exercise 8.6

We use the formula for AVaR given in Proposition 8.5:

$$\mathrm{AVaR}^\alpha(X) = -\frac{1}{\alpha}[\mathbb{E}(X\mathbf{1}_X^\alpha) + q^\alpha(X)(\alpha - P(X < q_X^\alpha))].$$

In Example 8.13 we have $q^\alpha(X) = 0 = q^\alpha(Y)$,so that, with $\alpha = 0.05$,

$$\mathrm{AVaR}^\alpha(X) = -\frac{1}{0.05}\{X(\omega_1)P(\omega_1) = 60 = \mathrm{AVaR}^\alpha(Y).$$

For $Z = X + Y$ we obtain $q^\alpha(Z) = -100$, so that $\{Z < q^\alpha(Z)\}$ is empty, and hence

$$\mathrm{AVaR}^\alpha(Z) = -\frac{1}{\alpha}(-100\alpha) = 100.$$

This verifies that in this example we have subadditivity:

$$\mathrm{AVaR}^\alpha(X+Y) \le \mathrm{AVaR}^\alpha(X) + \mathrm{AVaR}^\alpha(Y).$$

Exercise 8.7

Using the formula for $V_{(x,y)}(t)$ and property (iii) from Proposition 8.2 we compute

$$\begin{aligned}\mathrm{AVaR}^{\alpha}\left(X_{(x,y)}\right) &= \mathrm{AVaR}^{\alpha}\left(\mathrm{e}^{-rT}V_{(x,y)}(T) - V_{(x,y)}(0)\right)\\ &= \mathrm{AVaR}^{\alpha}\left(\mathrm{e}^{-rT}\left[xS(T) + y\mathrm{e}^{rT}\right] - [xS(0) + y]\right)\\ &= x\mathrm{AVaR}^{\alpha}\left(\mathrm{e}^{-rT}S(T) - S(0)\right).\end{aligned}$$

From Lemma 8.15 we see that

$$\mathrm{AVaR}^{\alpha}\left(X_{(x,y)}\right) = x\left(S(0) - \frac{1}{\alpha}S(0)\mathrm{e}^{(\mu-r)T}N\left(q^{\alpha}(Z) - \sigma\sqrt{T}\right)\right).$$

Suppose now that we invest $V(0)$ in our strategy. For simplicity, let us consider $V(0) = S(0)$ (to generalise to different $V(0)$ it is enough to rescale the argument below by a constant). If we consider a strategy with $y > 0$, then since $V(0) = S(0) = xS(0) + y$,

$$x = \frac{S(0) - y}{S(0)} < 1.$$

So,

$$\begin{aligned}\mathrm{AVaR}^{\alpha}\left(X_{(x,y)}\right) &= x\left(S(0) - \frac{1}{\alpha}S(0)\mathrm{e}^{(\mu-r)T}N\left(q^{\alpha}(Z) - \sigma\sqrt{T}\right)\right)\\ &< S(0) - \frac{1}{\alpha}S(0)\mathrm{e}^{(\mu-r)T}N\left(q^{\alpha}(Z) - \sigma\sqrt{T}\right)\\ &= \mathrm{AVaR}^{\alpha}\left(\mathrm{e}^{-rT}S(T) - S(0)\right).\end{aligned}$$

We see that $\mathrm{AVaR}^{\alpha}\left(X_{(x,y)}\right)$ is smaller than the one for investing $V(0) = S(0)$ in stock.

Exercise 8.8

We compute

$$\begin{aligned}\mathrm{AVaR}^{\alpha}\left(X_{(x,\theta)}\right) &= \mathrm{AVaR}^{\alpha}\left(\mathrm{e}^{-rT}V_{(x,\theta)}(T) - V_{(x,\theta)}(0)\right)\\ &= \mathrm{AVaR}^{\alpha}\left(\mathrm{e}^{-rT}\left[S(T) + \theta(F - S(T))\right] - S(0)\right)\\ &= \mathrm{AVaR}^{\alpha}\left(\mathrm{e}^{-rT}\left[S(T) + \theta(S(0)\mathrm{e}^{rT} - S(T))\right] - S(0)\right)\\ &= \mathrm{AVaR}^{\alpha}\left((1-\theta)\left(\mathrm{e}^{-rT}S(T) - S(0)\right)\right)\\ &= (1-\theta)\,\mathrm{AVaR}^{\alpha}\left(\mathrm{e}^{-rT}S(T) - S(0)\right).\end{aligned}$$

From Lemma 8.15 we see that

$$\mathrm{AVaR}^{\alpha}\left(X_{(x,\theta)}\right) = (1-\theta)\left(S(0) - \frac{1}{\alpha}S(0)\mathrm{e}^{(\mu-r)T}N\left(q^{\alpha}(Z) - \sigma\sqrt{T}\right)\right).$$

Exercise 8.9

If $K \geq q^\alpha(S(T))$ then since

$$q^\alpha(S(T)) = S(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}q^\alpha(Z)\right),$$

we see that

$$= -\frac{\ln\frac{S(0)}{K} + \left(\mu - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \geq q^\alpha(Z)$$

hence

$$d_-^{\mu,\alpha} = \max\left(d_-^\mu, -q^\alpha(Z)\right) = -q^\alpha(Z).$$

If $x = z$ and $K \geq q^\alpha(S(T))$, then

$$\begin{aligned}
\mathrm{AVaR}^\alpha\left(X_{(x,z)}\right) &= V_{(x,x)}(0) - \frac{1}{\alpha}\mathrm{e}^{(\mu-r)T}\left[xS(0)N\left(q^\alpha(Z) - \sigma\sqrt{T}\right) + zP^\alpha(K)\right] \\
&= V_{(x,x)}(0) - \frac{1}{\alpha}x\mathrm{e}^{(\mu-r)T}[S(0)N\left(-d_-^{\mu,\alpha} - \sigma\sqrt{T}\right) \\
&\qquad + K\mathrm{e}^{-\mu T}N(-d_-^{\mu,\alpha}) - S(0)N\left(-d_+^{\mu,\alpha}\right)] \\
&= V_{(x,x)}(0) - \frac{1}{\alpha}x\mathrm{e}^{(\mu-r)T}[S(0)N\left(-d_-^{\mu,\alpha} - \sigma\sqrt{T}\right) \\
&\qquad K\mathrm{e}^{-\mu T}N(-d_-^{\mu,\alpha}) - S(0)N\left(-d_-^{\mu,\alpha} - \sigma\sqrt{T}\right)] \\
&= V_{(x,x)}(0) - \frac{1}{\alpha}x\mathrm{e}^{(\mu-r)T}K\mathrm{e}^{-\mu T}N(-d_-^{\mu,\alpha}) \\
&= V_{(x,x)}(0) - \frac{1}{\alpha}x\mathrm{e}^{(\mu-r)T}K\mathrm{e}^{-\mu T}N(q^\alpha(Z)) \\
&= V_{(x,x)}(0) - \frac{1}{\alpha}\mathrm{e}^{(\mu-r)T}xK\mathrm{e}^{-\mu T}\alpha \\
&= V_{(x,x)}(0) - \mathrm{e}^{-rT}xK.
\end{aligned}$$

On the other hand, by (7.18) we know that

$$\mathrm{VaR}^\alpha\left(X_{(x,z)}\right) = V_{(x,z)}(0) - \mathrm{e}^{-rT}\left(xq^\alpha(S(T)) + z\left(K - q^\alpha(S(T))\right)^+\right).$$

If $x = z$ and $K \geq q^\alpha(S(T))$, then

$$\begin{aligned}
\mathrm{VaR}^\alpha\left(X_{(x,z)}\right) &= V_{(x,x)}(0) - \mathrm{e}^{-rT}\left(xq^\alpha(S(T)) + x\left(K - q^\alpha(S(T))\right)\right) \\
&= V_{(x,x)}(0) - \mathrm{e}^{-rT}xK,
\end{aligned}$$

which concludes the solution.

Exercise 8.10

$$\begin{aligned}\mathbb{E}(X_{(x,z)}) &= \mathbb{E}\left(e^{-rT}V_{(x,z)}(T) - V_{(x,z)}(0)\right)\\ &= \mathbb{E}\left(e^{-rT}\left(xS(T) + zH(T)\right) - V_{(x,z)}(0)\right)\\ &= e^{-rT}\left(x\mathbb{E}\left(S(T)\right) + z\mathbb{E}\left(H(T)\right)\right) - V_{(x,z)}(0).\end{aligned} \tag{9.19}$$

Direct computation of

$$\begin{aligned}\mathbb{E}\left(S(T)\right) &= \int_{-\infty}^{+\infty} S(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx,\\ \mathbb{E}\left(H(T)\right) &= \int_{-\infty}^{+\infty}\left(K - S(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\right)^+\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx,\end{aligned}$$

leads to

$$\mathbb{E}\left(S(T)\right) = S(0)e^{\mu T}, \tag{9.20}$$

and

$$\mathbb{E}\left(H(T)\right) = e^{\mu T}P(\mu, T, K, S(0), \sigma). \tag{9.21}$$

Substituting (9.20) and (9.21) into (9.19) gives the result.

Exercise 8.11

From Exercise 8.7 we know that

$$\mathrm{AVaR}^{\alpha}\left(X_{(x,y)}\right) = x\left(S(0) - \frac{1}{\alpha}S(0)e^{(\mu-r)T}N\left(q^{\alpha}(Z) - \sigma\sqrt{T}\right)\right).$$

From Exercise 7.5,

$$\mathbb{E}\left(X_{(x,y)}\right) = xS(0)\left[e^{(\mu-r)T} - 1\right].$$

We use these to produce the plot from Figure 9.5. From the plot we see that hedging with puts gives better results.

The computations are done in the file `Exercise_8_11.xlsx`

Exercise 8.12

We first compute AVaR^{α} for a position in x stocks and θ forward contracts. The payoff on a forward is $F - S(T)$, where $F = e^{rT}S(0)$. We define

$$\begin{aligned}V_{(x,\theta)}(0) &= xS(0),\\ V_{(x,\theta)}(T) &= xS(T) + \theta\left(F - S(T)\right),\end{aligned}$$

and take

$$X_{(x,\theta)} = e^{-rT}V_{(x,\theta)}(T) - V_{(x,\theta)}(0).$$

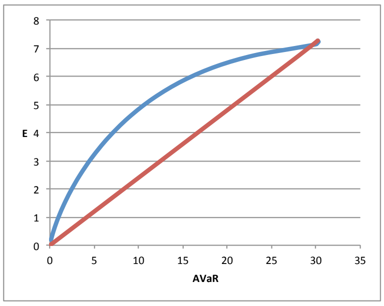

Figure 9.5 Comparison of hedging with puts (blue) with hedging by investing in the risk free asset (red)

Using properties of AVaR$^\alpha$ together with Lemma 8.15

$$\begin{aligned}\mathrm{AVaR}^\alpha\left(X_{(x,\theta)}\right) &= \mathrm{AVaR}^\alpha\left(\mathrm{e}^{-rT}xS(T)+\mathrm{e}^{-rT}\theta(F-S(T))-xS(0)\right)\\ &= \mathrm{AVaR}^\alpha\left(\mathrm{e}^{-rT}xS(T)+\mathrm{e}^{-rT}\theta\left(\mathrm{e}^{rT}S(0)-S(T)\right)-xS(0)\right)\\ &= \mathrm{AVaR}^\alpha\left((x-\theta)\left[\mathrm{e}^{-rT}S(T)-S(0)\right]\right)\\ &= (x-\theta)\,\mathrm{AVaR}^\alpha\left(\mathrm{e}^{-rT}S(T)-S(0)\right)\\ &= (x-\theta)\left[S(0)-\frac{1}{\alpha}S(0)\mathrm{e}^{(\mu-r)T}N\left(q^\alpha(Z)-\sigma\sqrt{T}\right)\right].\end{aligned}$$

The expectation

$$\mathbb{E}\left(X_{(x,\theta)}\right)=(x-\theta)\,S(0)\left(\mathrm{e}^{(\mu-r)T}-1\right)$$

was computed in Exercise 7.6. We can take $x = 1$, and $\theta \in [0,1]$ and compare the plot of

$$\{(\mathrm{AVaR}^\alpha(X_{(x,\theta)}),\mathbb{E}(X_{(x,\theta)})) : x=1,\theta\in[0,1]\} \tag{9.22}$$

with the plot

$$\left\{\left(\mathrm{AVaR}^\alpha\left(X_{(x,y)}\right),\mathbb{E}\left(X_{(x,y)}\right)\right) : y\geq 0, xS(0)+y=S(0)\right\} \tag{9.23}$$

from Exercise 8.11. The graphs (9.22) and (9.23) are identical. The computations and the graphs can be found in the file `Exercise_8_12.xlsx`.

Exercise 8.13

The proof follows from identical arguments. The only difference is that instead of taking $z(K-S(T))^+$ use $\sum_{i=1}^n z_i(K_i-S(T))^+$. All the computations follow along identical lines to the proof of Proposition 8.19.

Exercise 8.14

The simulation can be found in the file `Exercise_8_14.xlsm`.

Exercise 8.15

If ρ is monotone, then $X \geq Y$ implies $\rho(X) \leq \rho(Y)$. With $Y = 0$ this becomes $\rho(X) \leq \rho(0)$ when $X \geq 0$. If ρ is also positive homogeueous $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda = 0$ shows that $\rho(0) = 0$. Therefore $X \geq 0$ implies $\rho(X) \leq 0$.

Exercise 8.16

Suppose ρ is coherent. Suppose $0 \leq \alpha \leq 1$, and X, Y are bounded. Using subadditivity and then positive homogeneity gives

$$\rho(\alpha X + (1-\alpha)Y) \leq \rho(\alpha X) + \rho((1-\alpha)Y) = \alpha\rho(X) + (1-\alpha)\rho(Y)$$

so that ρ is convex. If ρ is convex and positive homogeneous, then for any bounded X, Y, using (iii) with $Z = \frac{1}{2}(X+Y)$ and $\lambda = 2$, and the convexity of ρ,

$$\begin{aligned} \rho(X+Y) &= 2\rho(\frac{1}{2}(X+Y)) \\ &\leq 2[\frac{1}{2}\rho(X) + \frac{1}{2}\rho(Y)] \\ &= \rho(X) + \rho(Y), \end{aligned}$$

so ρ is subadditive, hence coherent.

Exercise 8.17

To prove that $\rho_\mu(X+m) = \rho_\mu(X) - m$ we need the fact that μ is a probability measure on $(0,1)$, since

$$\int_0^1 \rho_\alpha(X+m)d\mu(\alpha) = \int_0^1 (\rho_\alpha(X)-m)d\mu(\alpha) = \rho_\mu(X)-m\int_0^1 d\mu(\alpha) = \rho_\mu(X)-m.$$

The other three requirements for coherence are immediate from the linearity and positivity of the integral.

Exercise 8.18

We use the fact that $\rho_{\max}(X) = \inf\{r \in \mathbb{R} : X + r \geq 0\ P\text{-a.s.}\}$ to check the four axioms for coherence.

(i) If $X \leq Y$ then $X + r \geq 0$ implies that $Y + r \geq 0$, hence

$$\inf\{r : X + r \geq 0\ P - a.s\} \geq \inf\{r : Y + r \geq 0\ P - a.s\}.$$

(ii) For any real m, r we have $(X+m) + r = X + (m+r)$, so that

$$\begin{aligned} &\inf\{r : (X+m) + r \geq 0\ P - a.s\} = \inf\{r : X + (m+r) \geq 0\ P - a.s\} \\ &\quad = \inf\{r : X + r \geq 0\ P - a.s\} - m. \end{aligned}$$

(iii) For $\lambda \geq 0$ we have

$$\inf\{r : \lambda X + r \geq 0 \ P - a.s\} = \inf\{\lambda s : \lambda(X + s) \geq 0 \ P - a.s\}$$
$$= \lambda \inf\{s : X + s \geq 0 \ P - a.s\}.$$

(iv) Write $z = X + Y$. For any real r we can write $Z + r = (X + s) + (Y + t)$ where $r = s + t$. Now

$$\inf\{r : Z + r \geq 0 \ P - a.s\} = \inf\{s + t : (X + s) + (Y + t) \geq 0 \ P - a.s.\}$$
$$\leq \inf\{s : X + s \geq 0 \ P - a.s\} + \inf\{t : Y + t \geq 0 \ P - a.s\}.$$

Exercise 8.19

In this example there are no non-empty P-null sets and $\Omega = \{\omega_1, \omega_2, \omega_3\}$ is finite, so that we can write

$$\mathrm{WCE}^{\alpha}(X) = -\inf\{\mathbb{E}[X|A] : P(A) > \alpha\}.$$

The sets satisfying the condition $P(A) > \alpha$ for X are $A_1 = \{\omega_1, \omega_2\}$ and the four sets involving ω_3. Since $X(\omega_1) = -100$ and its other values are 0, it is clear that the infimum is a minimum and is attained when we take A_1. We have

$$\mathbb{E}[X|A_1] = \frac{1}{P(A_1)}\{-100P(\omega_1) + 0\} = -50$$

so that $\mathrm{WCE}^{\alpha}(X) = 50$. The same argument holds for Y, with the roles of ω_1 and ω_2 interchanged.

With $Z = X + Y$ we have $Z = -100$ on A_1 and so $\mathrm{WCE}^{\alpha}(Z) = 100$, by a similar argument.

Therefore WCE^{α} is additive in this case.

In Exercise 8.6 we verified that $\mathrm{AVaR}^{\alpha}(X) = 60 = \mathrm{AVaR}^{\alpha}(Y)$.

Comparing the results we have obtained for X (here, in Exercise 8.6 and Example 8.13) we see that

$$\mathrm{AVaR}^{\alpha}(X) = 60$$
$$\mathrm{WCE}^{\alpha}(X) = 50$$
$$\mathrm{TCE}^{\alpha}(X) = 3.$$

Moreover, $\mathrm{VaR}^{\alpha}(X) = 0$, since $P(\{\omega_1\} = 0.03 < 0.05 = \alpha$.

Together, these results verify that the identities proved in Proposition 8.29 hold in this simple example.