Scattering parameters

Microwave Electronics

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Forward and backward voltages along a line

- In a homogeneous line, the forward and backward voltages in a point univoquely determine the voltage and currents on the whole line:

$$V(z) = V^+(z) + V^-(z) = V^+(0)\exp(-j\beta z) + V^-(0)\exp(j\beta z)$$

$$\frac{V^+(0)}{Z_0}\exp(-j\beta z) - \frac{V^-(0)}{Z_0}\exp(j\beta z) =$$
$$= I^+(0)\exp(-j\beta z) + I^-(0)\exp(j\beta z) = I^+(z) + I^-(z) = I(z)$$

$$Z_0 = \frac{V^+(z)}{I^+(z)} = -\frac{V^-(z)}{I^-(z)} \quad \text{characteristic impedance}$$

$$Z(z) = \frac{V(z)}{I(z)} \qquad \text{input impedance at section } z$$

The reflection coefficient

- Defined as the ratio between the backward and forward waves:

$$\boxed{\Gamma(z)=\frac{V^-(z)}{V^+(z)}=\frac{V^-(0)\exp(j\beta z)}{V^+(0)\exp(-j\beta z)}=\Gamma(0)\exp(j2\beta z)}$$

$$Z(z)=\frac{V(z)}{I(z)}=\frac{V^+(0)\exp(-j\beta z)+V^-(0)\exp(j\beta z)}{\frac{V^+(0)}{Z_0}\exp(-j\beta z)-\frac{V^-(0)}{Z_0}\exp(j\beta z)}=Z_0\frac{1+\frac{V^-(0)}{V^+(0)}\exp(j2\beta z)}{1-\frac{V^-(0)}{V^+(0)}\exp(j2\beta z)}=Z_0\frac{1+\Gamma(z)}{1-\Gamma(z)}$$

$$\boxed{\begin{aligned} z(z)&=\frac{Z(z)}{Z_0}=\frac{1+\Gamma(z)}{1-\Gamma(z)}\\ \Gamma(z)&=\frac{z(z)-1}{z(z)+1}=\frac{Z(z)-Z_0}{Z(z)+Z_0}\end{aligned}}$$

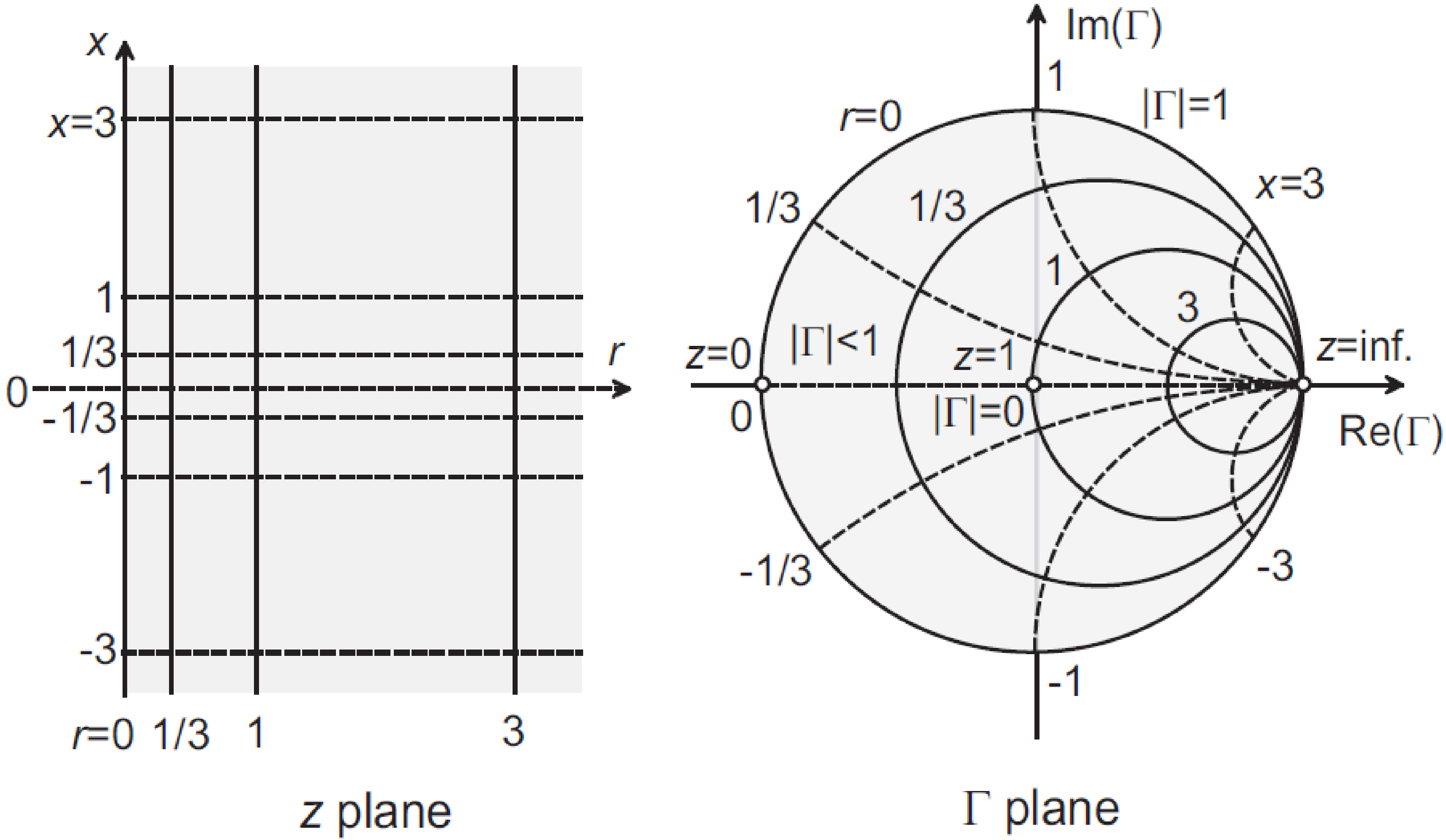
Mapping between gamma and zeta planes

- The trasformation gamma → zeta and viceversa defines the mapping between the complex planes zeta & gamma
- Straight lines in zeta plane become circles or straight lines in gamma plane, and viceversa
- In particular, constant resistance lines in zeta plane become circles in gamma plane, constant reactance lines in zeta plane become circles in gamma plane
- Particular points (origin, unit circle in gamma plane):

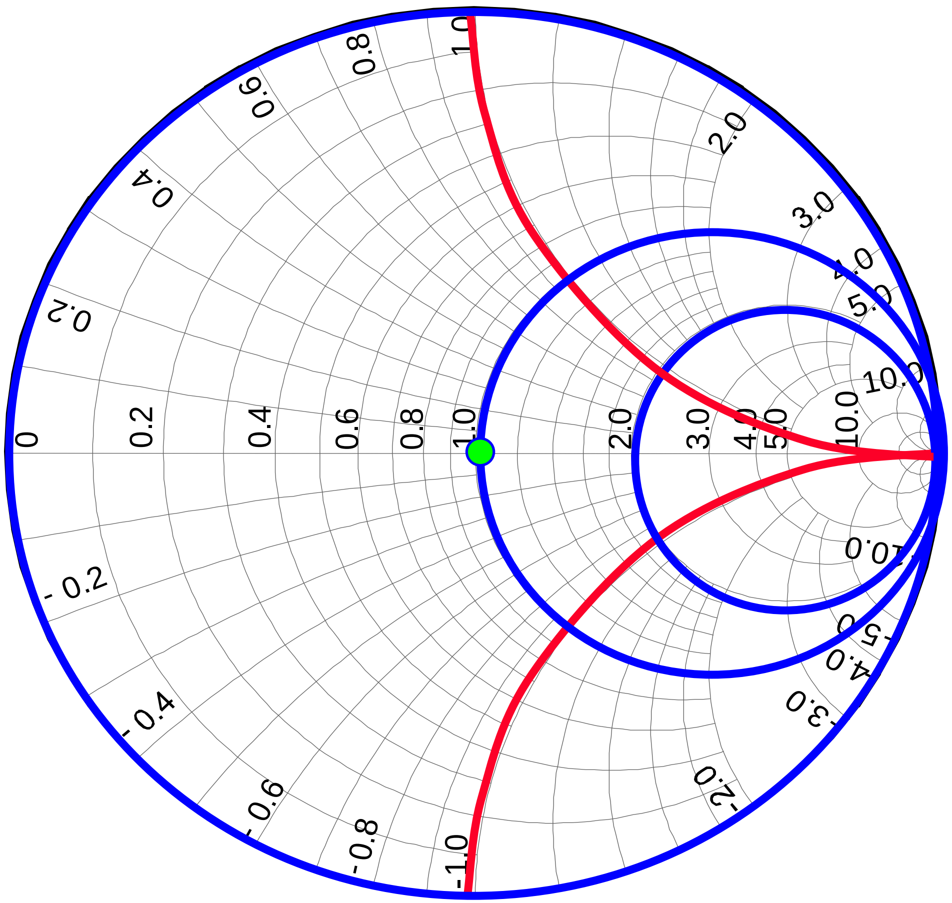
$$Z = Z_0 \to \Gamma(z) = \frac{Z_0 - Z_0}{Z_0 + Z_0} = 0$$

$$Z = jX \to \Gamma(z) = \frac{jX - Z_0}{jX + Z_0} \to \left|\Gamma(z)\right| = \frac{\sqrt{X^2 + Z_0^2}}{\sqrt{X^2 + Z_0^2}} = 1$$

Constant resistance (-) and reactance (- -) circles



The Smith chart - reminder

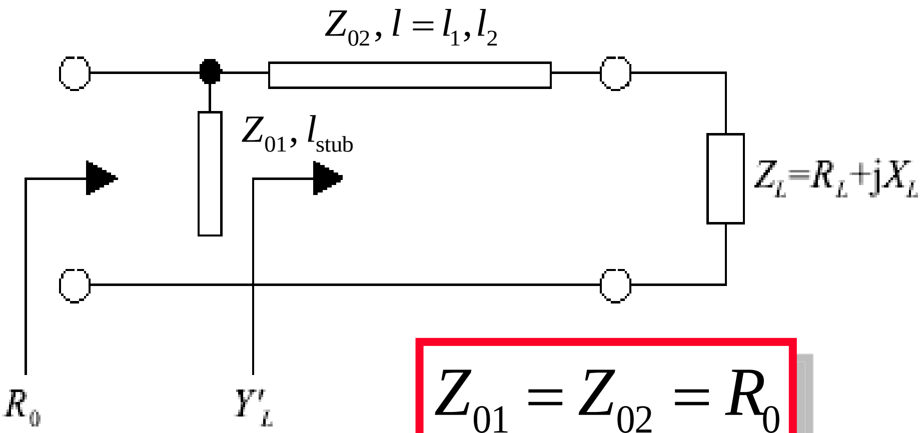


- Graphical representation of the relationship between the reflection coefficient and the normalized impedance (*R*=const. and *X*=const. lines in gamma plane):

$$\Gamma = \frac{z-1}{z+1}$$

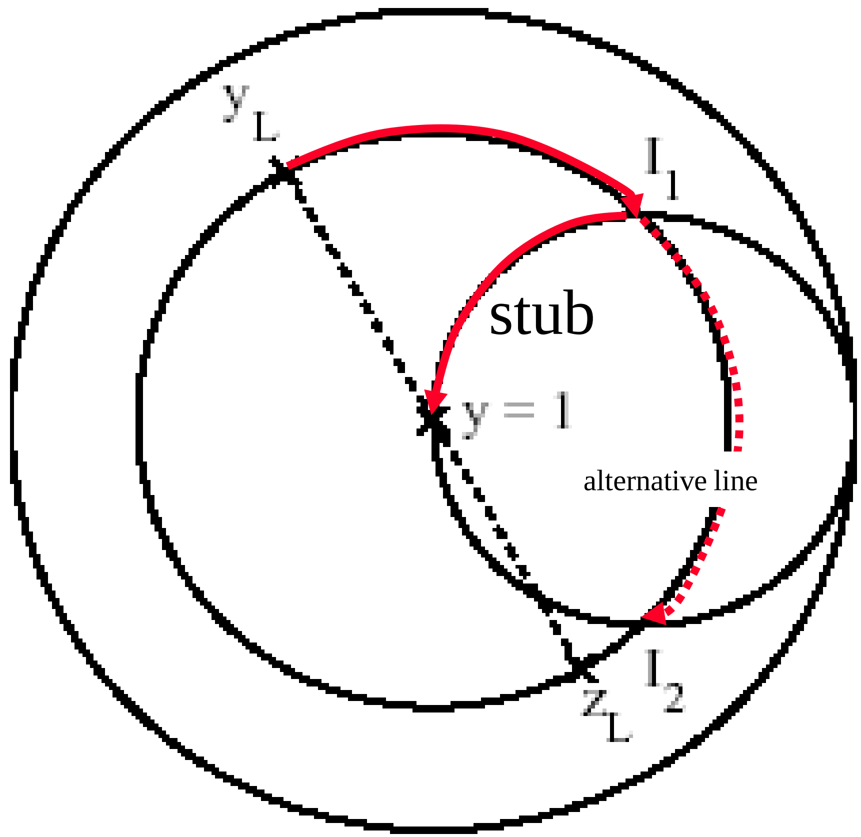
$$z = \frac{Z}{Z_0}$$

The reflection coefficient on the Smith chart & the use of the Smith chart



$$\boxed{Z_{01} = Z_{02} = R_0}$$

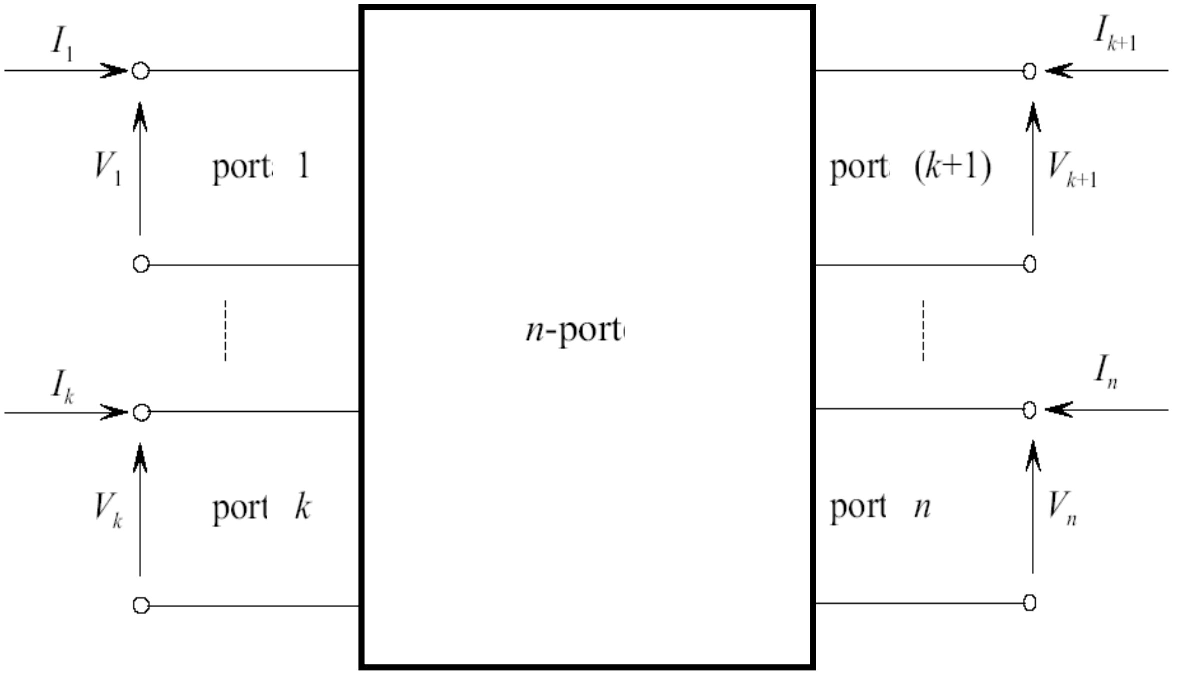
- Using the Smith chart as a graphical computer is today (almost) a lost art, but the chart still is ubiquitous as a representation tool



- Gamma rotates on the gamma plane along a line, thus allowing for a simple graphical way to evaluate the line impedance section by section
- Addition of series or parallel elements → the Smith chart as a design tool for matching sections (parallel stub here)

Representing a linear n-port

- A linear n-port without internal independent generators in sinusoidal steady-state can be described by a matrix relationship between current and voltage phasors



E.g. for a two-port:

Series (impedance)

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

Parallel (admittance)

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Hybrid

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$

Why S (scattering) parameters

- Total voltages and currents are difficult to measure at RF and microwaves, and even the definition of these quantities may be questionable in some cases (waveguide).
- In the measurement of conventional (Z,Y…) two-port parameters, short and open circuits are required. However, they are difficult to realise over a broad band of frequencies.
- Most active devices or circuits are not open- or short circuit stable
- Solution → measurement of progressive & regressive waves made on matched load → SCATTERING PARAMETERS

Power wave rationale

- All the black box parameters previously introduced are based on the total voltages and currents.
- Note that these total voltages can be considered to be composed out of positive and negative going (progressive and regressive, forward and backward) waves (as in a TX line).
- For higher frequencies it is often more convenient to use a two-port description in terms of forward and backward waves (called power waves)
- Power waves can be formally defined also in a lumped-parameter structure

What are power waves

- Voltage and current phasors in a port can be replaced by two proper linear combinations of them called **power waves a_k** and b_k
- Since V and I phasors have different dimensions the combination requires (for each port) a parameter (called *normalization impedance*) having the dimension of an impedance

The normalization impedance for port k, Z_{0k} is arbitrary but typically assumed as real → normalization resistance R_{0k}

$$\boxed{\Re\left(Z_{0k}\right) > 0}$$

The definition of power waves (real normalization impedance)

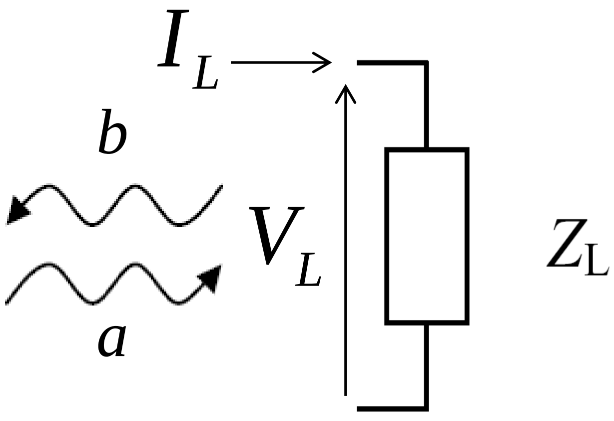
Z_{0k}=R_{0k} real and positive

$$\left\{\begin{array}{lcl} a_k & = & \dfrac{V_k + R_{0k} I_k}{2\sqrt{R_{0k}}} \\ b_k & = & \dfrac{V_k - R_{0k} I_k}{2\sqrt{R_{0k}}} \end{array}\right.$$

$\Longleftrightarrow$

$$\left\{\begin{array}{lcl} V_k & = & (a_k + b_k)\sqrt{R_{0k}} \\ I_k & = & \dfrac{a_k - b_k}{\sqrt{R_{0k}}}\,. \end{array}\right.$$

Closing a port on its normalization resistance



$$Z_L = R_0 \rightarrow V_L = R_0 I_L$$

$$a = \frac{V_L + R_0 I_L}{2\sqrt{R_0}} = \frac{V_L}{\sqrt{R_0}} = \sqrt{R_0}\, I_L$$

$$b = \frac{V_L - R_0 I_L}{2\sqrt{R_0}} = 0$$

- In a port closed on its normalization resistance the backward wave *b* is identically zero!
- This does not generally imply any “matching” in the meaning of maximum power transfer….

Why "power" waves?

- Power entering the *k*-th port:

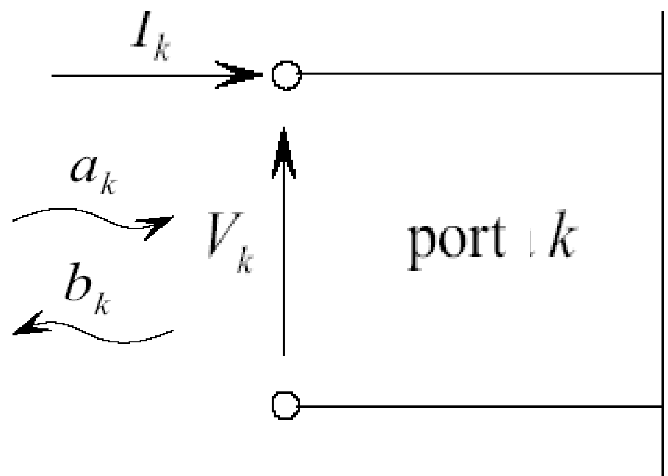
$$P_k = \Re\left(V_k I_k^*\right) = \Re\left(\left(a_k + b_k\right)\cdot\left(a_k^* - b_k^*\right)\right)$$

$$\vdots$$

$$P_k = \Re\left(V_k I_k^*\right) = \left|a_k\right|^2 - \left|b_k\right|^2$$

$\boldsymbol{a_k}$ forward power wave

$\boldsymbol{b_k}$ backward power wave



Power wave dimensions: $\sqrt{\mathrm{W}}$

Power waves and transmission lines I

- Given a section *z* of a T-line of impedance Z_0 power waves are connected to progressive and regressive waves:

$$V(z) = V^+ + V^- = a\sqrt{Z_0} + b\sqrt{Z_0}$$

$$I(z) = I^+ + I^- = \frac{V^+}{Z_0} - \frac{V^-}{Z_0} = \frac{a\sqrt{Z_0}}{Z_0} - \frac{b\sqrt{Z_0}}{Z_0} = \frac{a}{\sqrt{Z_0}} - \frac{b}{\sqrt{Z_0}}$$

$$\boxed{\begin{aligned} V^+ &= a\sqrt{Z_0} \\ V^- &= b\sqrt{Z_0} \end{aligned}}$$

- Power running on the line:

$$P(z) = \mathrm{Re}\{V(z)I^*(z)\} = \frac{\left|V^+\right|^2}{Z_0} - \frac{\left|V^-\right|^2}{Z_0} = |a|^2 - |b|^2$$

Power waves and transmission lines II

- While in a transmission line power waves exist from a physical standpoint as forward and backward waves, in a *n*-port their definition is “formal” → no propagation implied
- The normalization impedance does not have a *physical meaning* as the characteristic impedance does

The scattering matrix

In vector form

$$\underline{\mathrm{a}} = [a_1, a_2, a_3, \ldots]^T$$

$$\underline{\mathrm{b}} = [b_1, b_2, b_3, \ldots]^T$$

$$\mathbf{R_0} = diag\{R_{01}, R_{02}, R_{03}, \ldots\}$$

&

$$\underline{\boldsymbol{V}} = [V_1, V_2, V_3, \ldots]^T$$

$$\underline{\boldsymbol{I}} = [I_1, I_2, I_3, \ldots]^T$$

$$\underline{\boldsymbol{V}} = \mathbf{R_0}^{1/2}\left(\underline{\mathrm{a}} + \underline{\mathrm{b}}\right)$$

$$\underline{\boldsymbol{I}} = \mathbf{R_0}^{-1/2}\left(\underline{\mathrm{a}} - \underline{\mathrm{b}}\right)$$

Z-matrix representation

$$\underline{\boldsymbol{V}} = \mathbf{Z}\,\underline{\boldsymbol{I}}$$

Substituting V and I in terms of power waves a linear relationship is obtained between a and b

Scattering (S) matrix representation

$$\underline{\mathrm{b}} = \mathbf{S}\,\underline{\mathrm{a}}$$

S matrix in terms of Z matrix

$$\mathbf{S} \equiv (\mathbf{R}_0^{-1/2}\mathbf{Z}\mathbf{R}_0^{-1/2} + \mathbf{I})^{-1}(\mathbf{R}_0^{-1/2}\mathbf{Z}\mathbf{R}_0^{-1/2} - \mathbf{I}) =$$

$$\vdots$$

$$= \mathbf{R}_0^{-1/2}(\mathbf{Z} - \mathbf{R}_0)(\mathbf{Z} + \mathbf{R}_0)^{-1}\mathbf{R}_0^{1/2}.$$

- Take care: matrices <u>functions of the same matrix</u> commute (**AB=BA**, in general false)!
- If normalization impedances are the same for all ports (typically 50 Ohm):

$$\mathbf{R_0} = R_0\,\mathbf{I} \quad \left(G_0 = 1/R_0\right)$$

$$\boxed{\begin{aligned}\mathbf{S} &= (\mathbf{Z} - R_0\mathbf{I})(\mathbf{Z} + R_0\mathbf{I})^{-1} \\ &= (G_0\mathbf{I} - \mathbf{Y})(G_0\mathbf{I} + \mathbf{Y})^{-1}\end{aligned}}$$

Other conversion formulae

$$\boxed{\mathbf{S} = (\mathbf{I} + \mathbf{R}_0^{1/2}\mathbf{Y}\mathbf{R}_0^{1/2})^{-1}(\mathbf{I} - \mathbf{R}_0^{1/2}\mathbf{Y}\mathbf{R}_0^{1/2})}$$

$$\boxed{\mathbf{Z} = \mathbf{R}_0^{1/2}(\mathbf{I} + \mathbf{S})(\mathbf{I} - \mathbf{S})^{-1}\mathbf{R}_0^{1/2}}$$

$$\boxed{\mathbf{Y} = \mathbf{R}_0^{-1/2}(\mathbf{I} - \mathbf{S})(\mathbf{I} + \mathbf{S})^{-1}\mathbf{R}_0^{-1/2}}$$

Some S-matrix properties

- Total power dissipated by n-port:

$$\boxed{P_d = \sum_{i=1}^{N}\left(|a_i|^2 - |b_i|^2\right) = \left(\underline{a}^T \cdot \underline{a}^* - \underline{b}^T \cdot \underline{b}^*\right) \overset{(\underline{b}=\mathbf{S}\underline{a})}{=} \underline{a}^T\left(\mathbf{I} - \mathbf{S}^T\mathbf{S}^*\right)\underline{a}^*}$$

- Lossless (reactive) device:

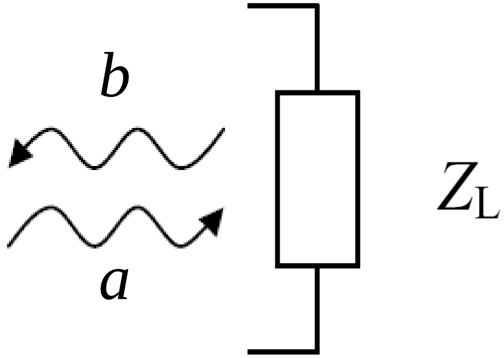
$$\boxed{\mathbf{S}^{T*}\mathbf{S} \equiv \mathbf{S}^{\dagger}\mathbf{S} = \mathbf{I} \quad \mathbf{S}\ \text{hermitian}}$$

- Reciprocal device:

$$\boxed{\mathbf{S} = \mathbf{S}^T \quad \mathbf{S}\ \text{symmetric}}$$

- For a passive device the eigenvalues of $\mathbf{S}^{\dagger}\mathbf{S}$ have magnitude <1

The S-matrix of a one-port



$$\boxed{S_{11} = \Gamma_L = \frac{b}{a}}$$

Γ_L reflection coefficient

$$\boxed{\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

$$\boxed{Z_L = Z_0 \frac{1+\Gamma_L}{1-\Gamma_L}}$$

$$\boxed{Y_L = \frac{1}{Z_0} \frac{1-\Gamma_L}{1+\Gamma_L}}$$

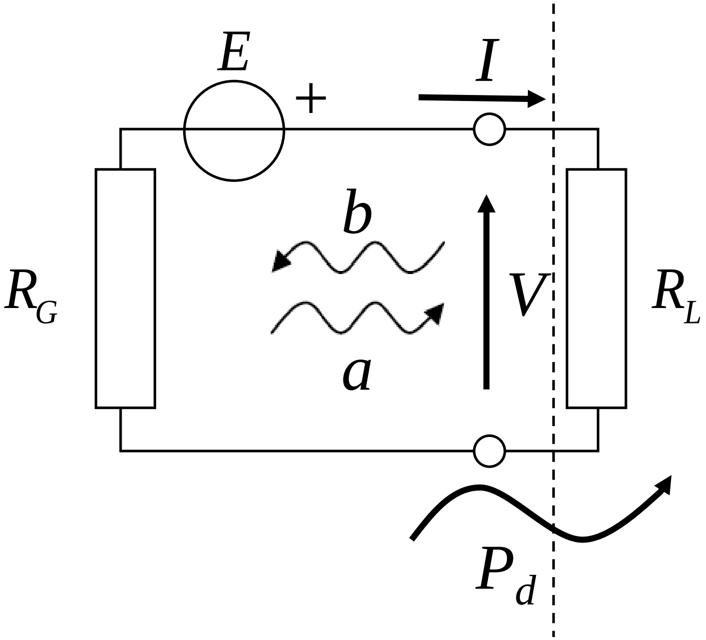
Example – Does the choice of Z_0 change the circuit solution? - I

$$\text{choose } Z_0 = R_L \text{ (real)} \Rightarrow \Gamma_L = 0$$

$$b = \Gamma_L a = 0$$

$$a = \frac{V + Z_0 I}{2\sqrt{Z_0}} = \frac{V + R_L I}{2\sqrt{R_L}} = \frac{V}{\sqrt{R_L}}$$

$$P_d = |a|^2 - |b|^2 = \boxed{\frac{|V|^2}{R_L}}$$



- The normalization impedance “matches” the load

Example – Does the choice of Z_0 change the circuit solution? - II

$$\text{real } Z_0 \neq R_L \Rightarrow \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0};$$

$$a = \frac{V + Z_0 I}{2\sqrt{Z_0}} = \frac{V(1 + Z_0 / R_L)}{2\sqrt{Z_0}}$$

- The normalization impedance is "mismatched"

$$P_d = |a|^2 - |b|^2 = |a|^2\left(1 - |\Gamma_L|^2\right) = \frac{|V|^2 |1 + Z_0 / R_L|^2}{4Z_0}\left(1 - \frac{|1 - Z_0 / R_L|^2}{|1 + Z_0 / R_L|^2}\right) =$$

$$= \frac{|V|^2}{4Z_0}\left(|1 + Z_0 / R_L|^2 - |1 - Z_0 / R_L|^2\right) =$$

$$= \frac{|V|^2}{4Z_0}\left(1 + |Z_0 / R_L|^2 + 2\Re(Z_0 / R_L) - 1 - |Z_0 / R_L|^2 + 2\Re(Z_0 / R_L)\right) =$$

$$= \frac{|V|^2}{4Z_0} 4\left(Z_0 / R_L\right) = \boxed{\frac{|V|^2}{R_L}}$$

n-port with independent sources

Open circuit voltages $\neq 0$

$$\underline{V} = \mathbf{Z}\,\underline{I} + \underline{V}_0$$

Short-circuit currents $\neq 0$

$$\underline{I} = \mathbf{Y}\,\underline{V} + \underline{I}_0$$

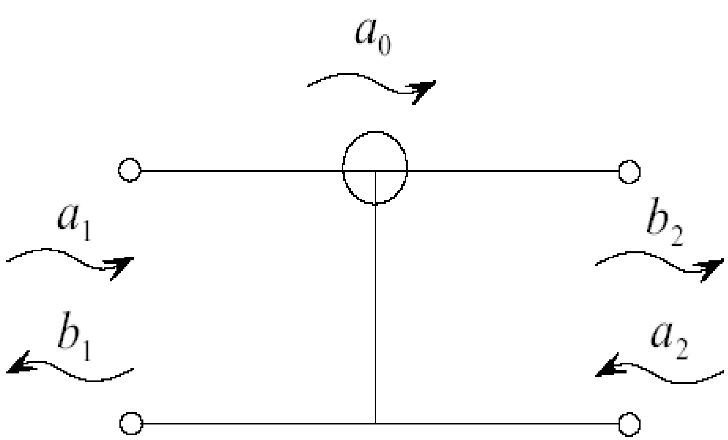
b **power waves $\neq 0$ when all ports are closed on the normalization impedances (→ all *a*=0, check!)**

$$\underline{b} = \mathbf{S}\underline{a} + \underline{b}_0$$

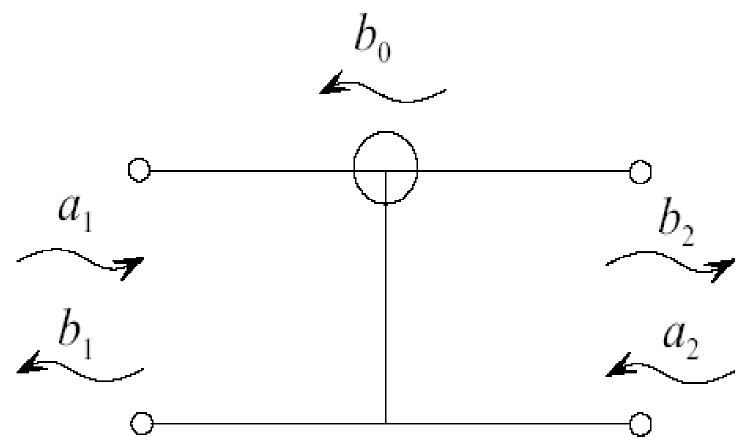
$$\underline{b}_0 \equiv (\mathbf{R}_0^{-1/2}\,\mathbf{Z}\mathbf{R}_0^{-1/2} + \mathbf{I}\,)^{-1}\mathbf{R}_0^{-1/2}\,\underline{V}_0 = \mathbf{R}_0^{1/2}\,(\mathbf{Z} + \mathbf{R}_0\,)^{-1}\underline{V}_0\;.$$

Forward & backward power wave generators

- While in series / parallel representations ideal voltage and current sources are introduced, in the power wave representation of non-autonomous *n*-ports we introduce forward and backward power wave generators



$$\begin{cases} b_1 = a_2 \\ b_2 = a_0 + a_1 \end{cases}$$



$$\begin{cases} b_1 = a_2 + b_0 \\ b_2 = a_1 \end{cases}$$

Why the symbol: forward wave generator = voltage + current source

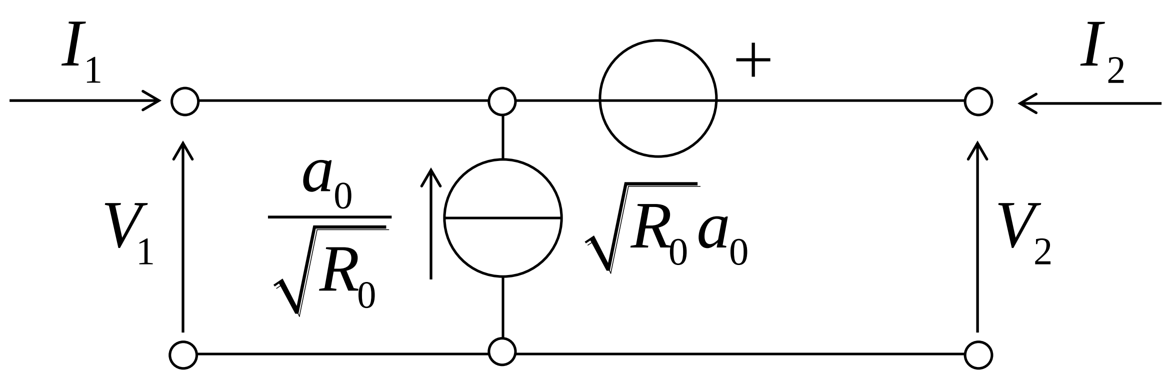
$$b_1 = a_2 \Rightarrow \frac{V_1 - R_0 I_1}{2\sqrt{R_0}} = \frac{V_2 + R_0 I_2}{2\sqrt{R_0}}$$

$$\Rightarrow V_1 - R_0 I_1 = V_2 + R_0 I_2 \Rightarrow V_1 - V_2 = R_0 I_1 + R_0 I_2$$

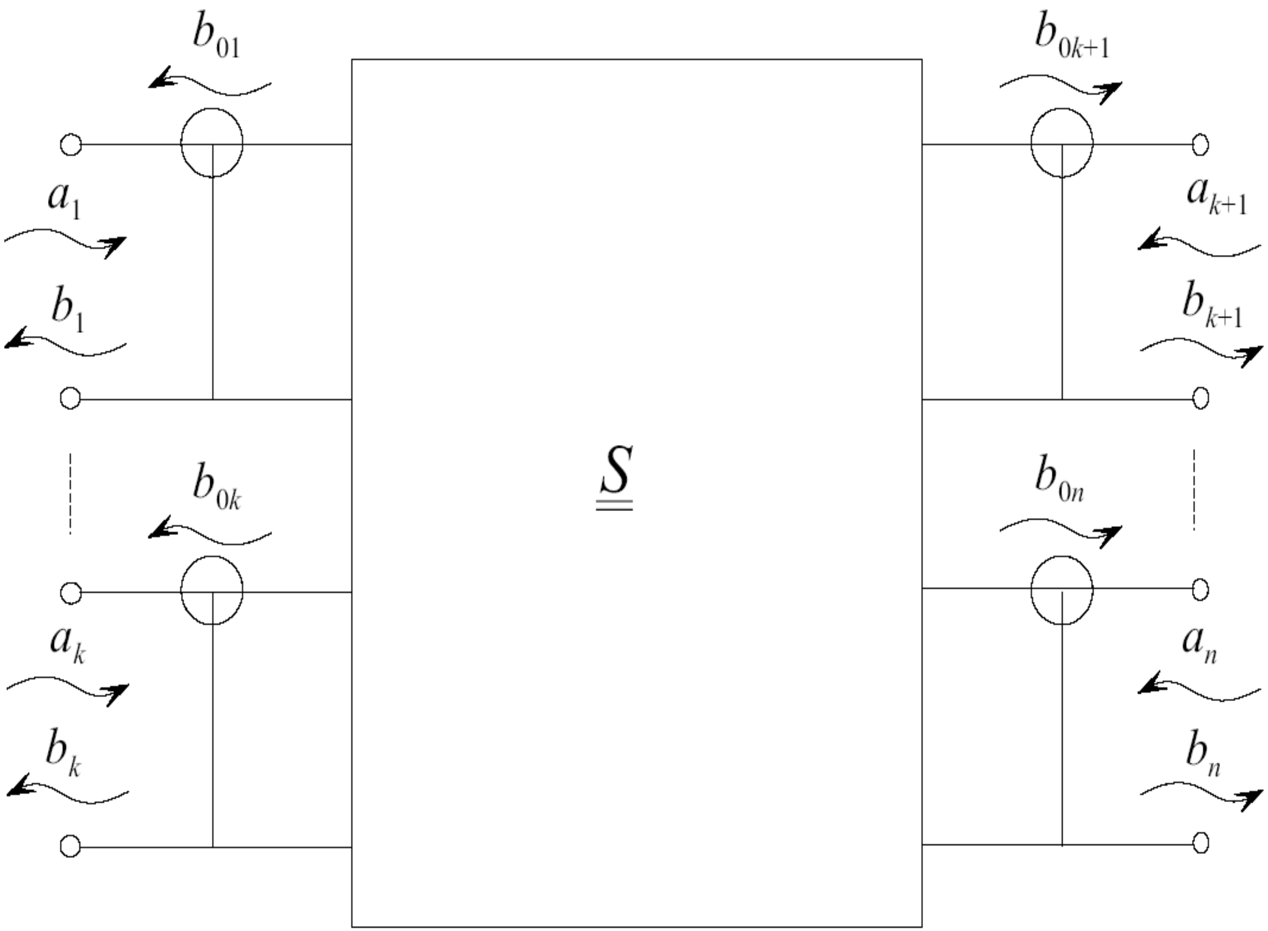
$$b_2 = a_1 + a_0 \Rightarrow \frac{V_2 - R_0 I_2}{2\sqrt{R_0}} = \frac{V_1 + R_0 I_1}{2\sqrt{R_0}} + a_0$$

$$\Rightarrow V_2 - R_0 I_2 = V_1 + R_0 I_1 + 2\sqrt{R_0}\,a_0 \Rightarrow V_1 - V_2 = -R_0 I_2 - R_0 I_1 - 2\sqrt{R_0}\,a_0$$

$$\Rightarrow \boxed{V_2 = V_1 + \sqrt{R_0}\,a_0, \quad I_2 = -I_1 - \frac{a_0}{\sqrt{R_0}}}$$

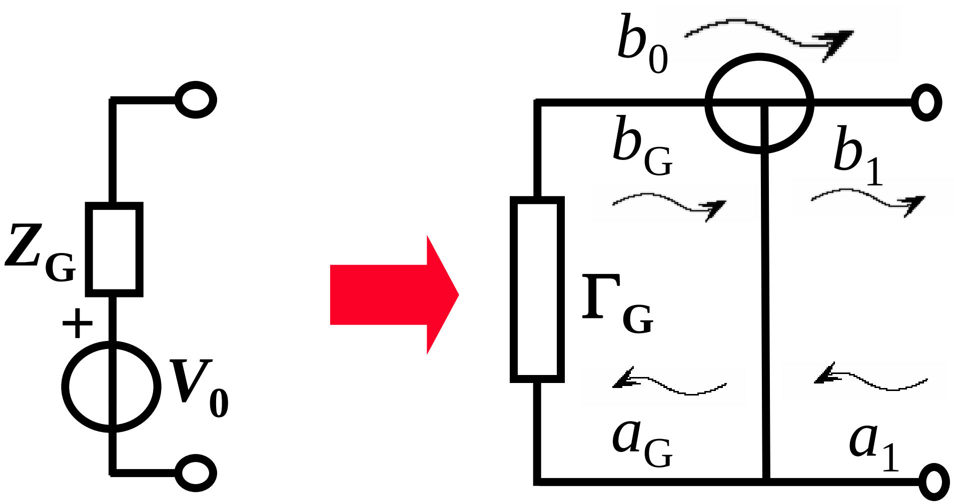


Equivalent circuit of autonomous *n*-port



$\underline{b}_0$ can be derived from open-circuit voltages

Example: real voltage generator I



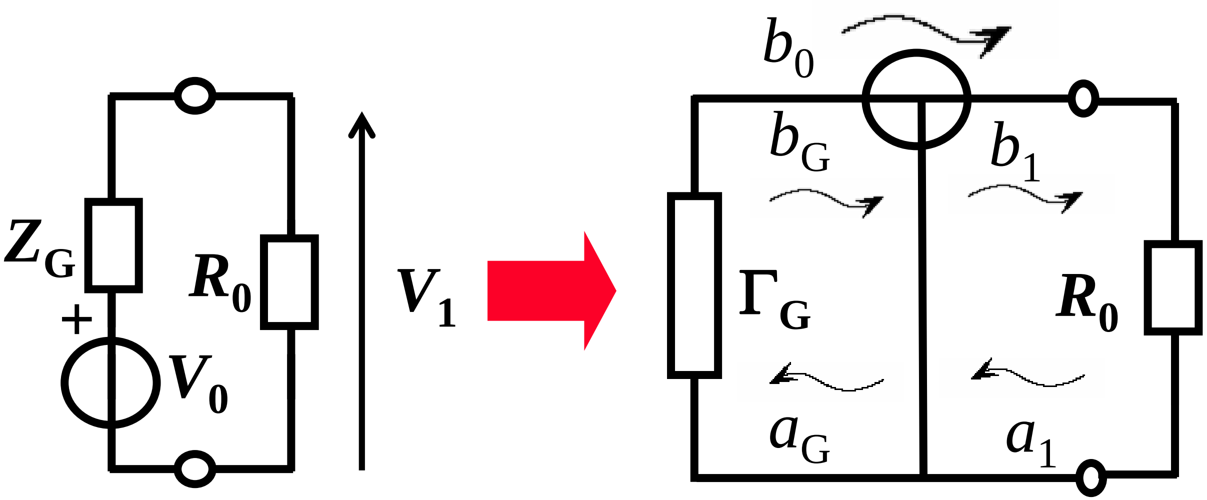
$$a_G = a_1$$
$$b_1 = b_G + b_0$$
$$b_G = \Gamma_G a_G$$
$$b_1 = \Gamma_G a_1 + b_0$$

$$\Gamma_G = \frac{Z_G - R_0}{Z_G + R_0}$$

$$b_0 = \frac{\sqrt{R_0}}{Z_G + R_0} V_0$$

Why?

Example: real voltage generator II



$$a_1 = 0,\ \ b_1 = b_0$$

$$V_1 = \frac{V_0 R_0}{Z_G + R_0} = \sqrt{R_0}\,(a_1 + b_1) =$$

$$= \sqrt{R_0}\,b_0 \rightarrow b_0 = \frac{V_0}{Z_G + R_0}\sqrt{R_0}$$

if $Z_G = R_0$

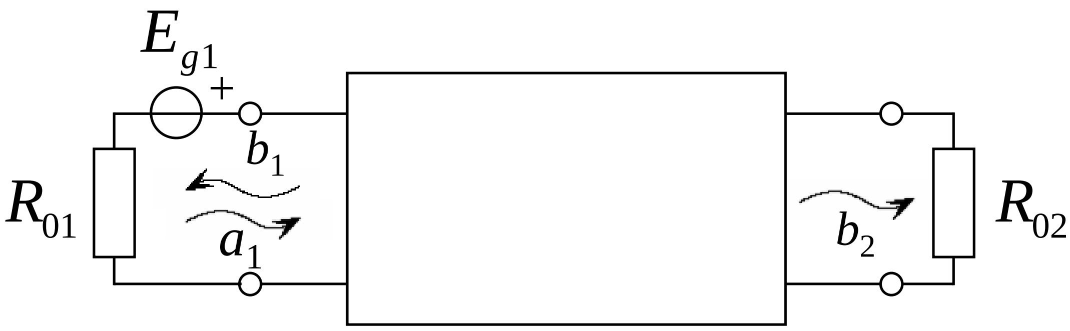
$$b_0 = \frac{1}{2\sqrt{R_0}} V_0$$

$$a_G = a_1$$

$$b_G = 0$$

$$b_1 = b_0$$

Measuring the S matrix – two port I



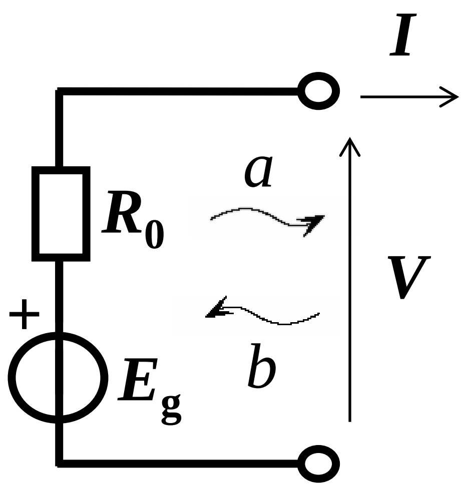
$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \qquad S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$

- We need to cancel a_2 and measure the response at port 1
- Port 2 is closed on the normalization resistance, in this case we have (check!):

$$a_2 = 0 \qquad b_2 = \frac{V_2}{\sqrt{R_{02}}} \qquad a_1 = \frac{E_{g1}}{2\sqrt{R_{01}}}$$

- Therefore scattering parameters are measured on an *n*-port closed on resistive loads (potentially wideband)

Check!



$$V = (a+b)\sqrt{R_0}$$

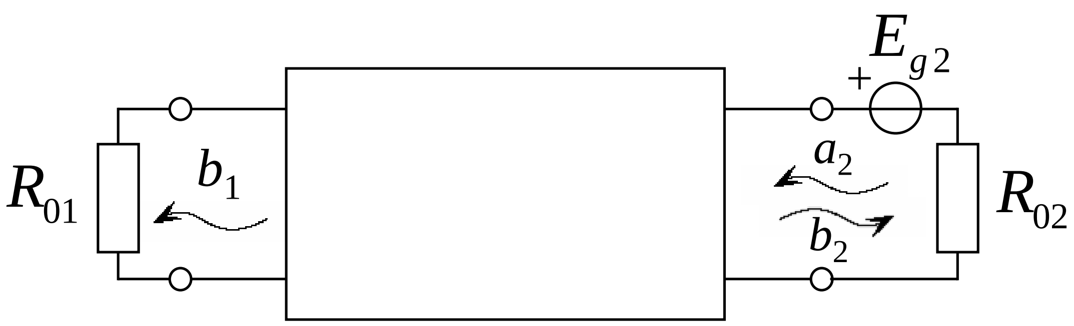
$$I = \frac{a-b}{\sqrt{R_0}}$$

$$V = -R_0 I + E_g$$

$$(a+b)\sqrt{R_0} = -R_0\frac{a-b}{\sqrt{R_0}} + E_g$$

$$(a+b+a-b)\sqrt{R_0} = E_g \Rightarrow a = \frac{E_g}{2\sqrt{R_0}}$$

Measuring the S matrix – two port II



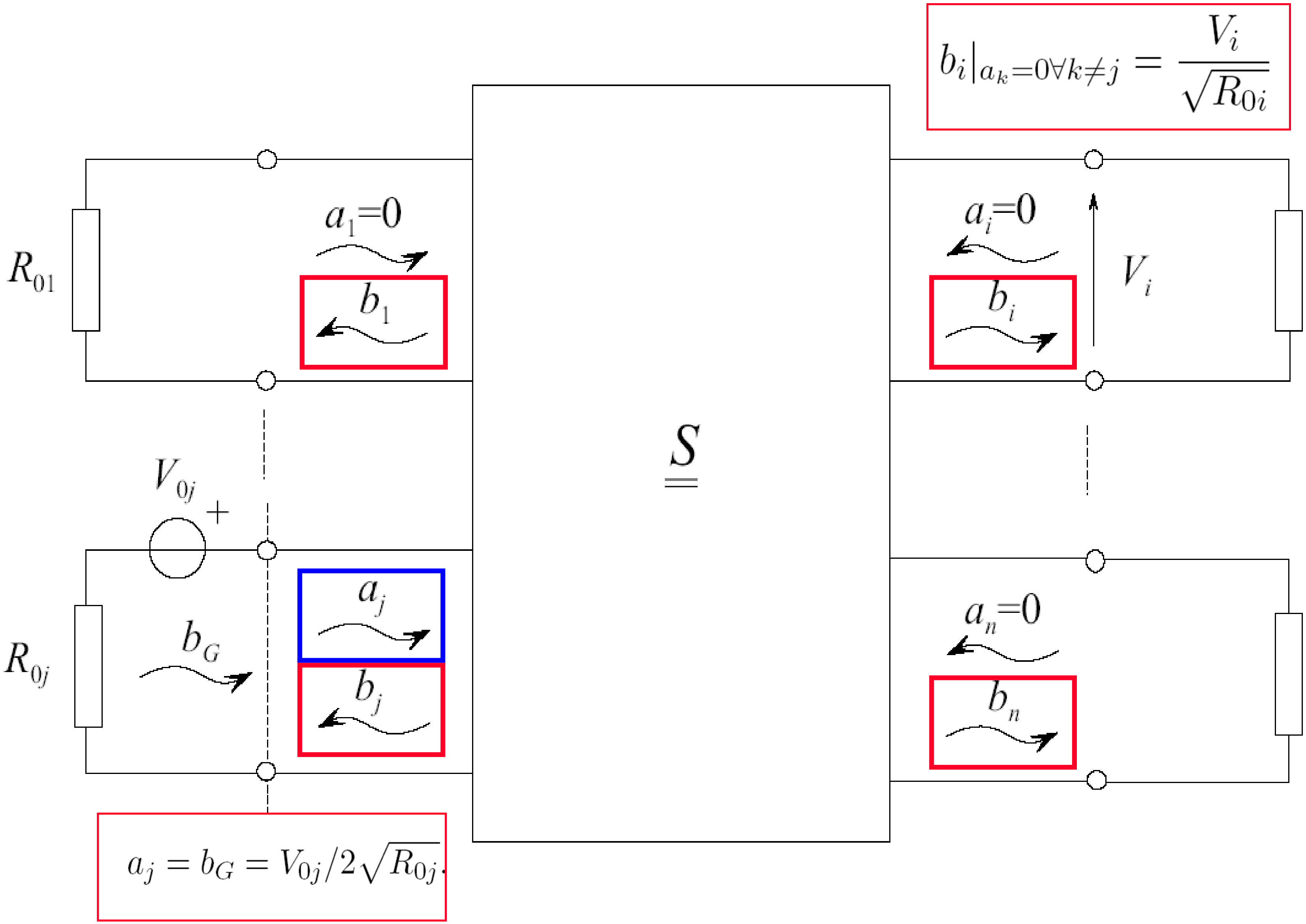
$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} \qquad S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0}$$

- We need to cancel a_1 and measure the response at port 2
- Port 1 is closed on the normalization resistance, in this case we have (check!):

$$a_1 = 0 \qquad b_1 = \frac{V_1}{\sqrt{R_{01}}} \qquad a_2 = \frac{E_{g2}}{2\sqrt{R_{02}}}$$

- Therefore scattering parameters are measured on an *n*-port closed on resistive loads (potentially wideband)

Setup to measure the j-th column of S (n-port)



Further remarks

- To evaluate the S parameters we can conveniently make use of a generator with internal impedance equal to the normalization resistance, to change column we only need to change the generator
- The diagonal of ***S*** corresponds to the reflection coefficients seen from each port when all others are closed on the normalization resistances
- For the out-of-diagonal coefficients (transmission coefficients) we have (check!):

$$a_j = b_G = V_{0j}/2\sqrt{R_{0j}}. \qquad b_i|_{a_k=0\forall k\neq j} = \frac{V_i}{\sqrt{R_{0i}}}$$

$$\boxed{S_{ij}|_{i\neq j} = \left.\frac{b_i}{a_j}\right|_{a_k=0\forall k\neq j} = 2\frac{V_i}{V_{0j}}\sqrt{\frac{R_{0j}}{R_{0i}}}\ .}$$

Solving a network made of connected n-ports

- A network deriving from the connection of several n-ports can be solved by exploiting as unknowns the power waves and as constitutive equations the scattering matrices of each n-port
- For example, in a network with m 2-ports we have 4m unknowns but also 2m constitutive relationships (from scattering parameters) and 2m continuity relationships derived from the Kirchhoff voltage and current laws
- Power waves are not necessarily continuous across two connected ports → only if the normalization impedance is the same on both sides

Connecting *n*-ports → power wave continuity

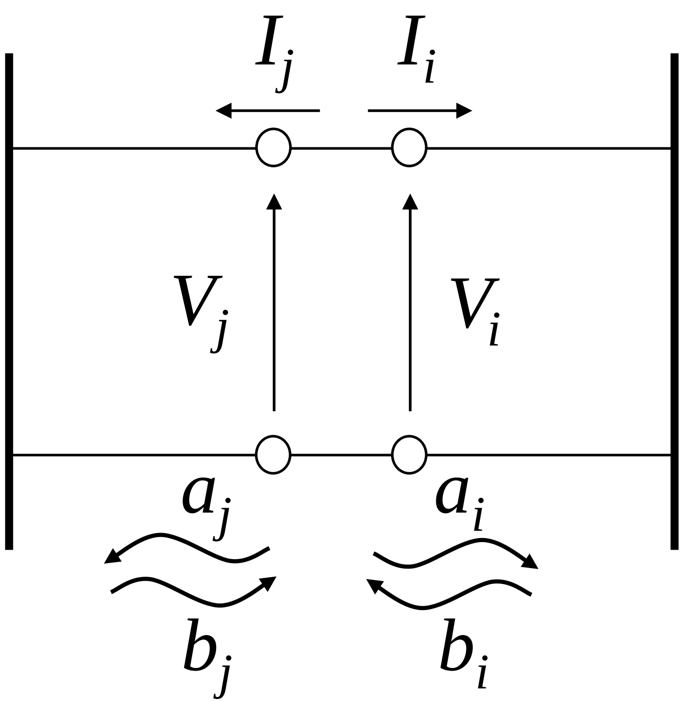
V and *I* continuity

$$V_j = V_i$$

$$I_j = -I_i$$

$$\sqrt{R_{0j}}(a_j + b_j) = \sqrt{R_{0i}}(a_i + b_i)$$

$$(a_j - b_j)/\sqrt{R_{0j}} = -(a_i - b_i)/\sqrt{R_{0i}}$$



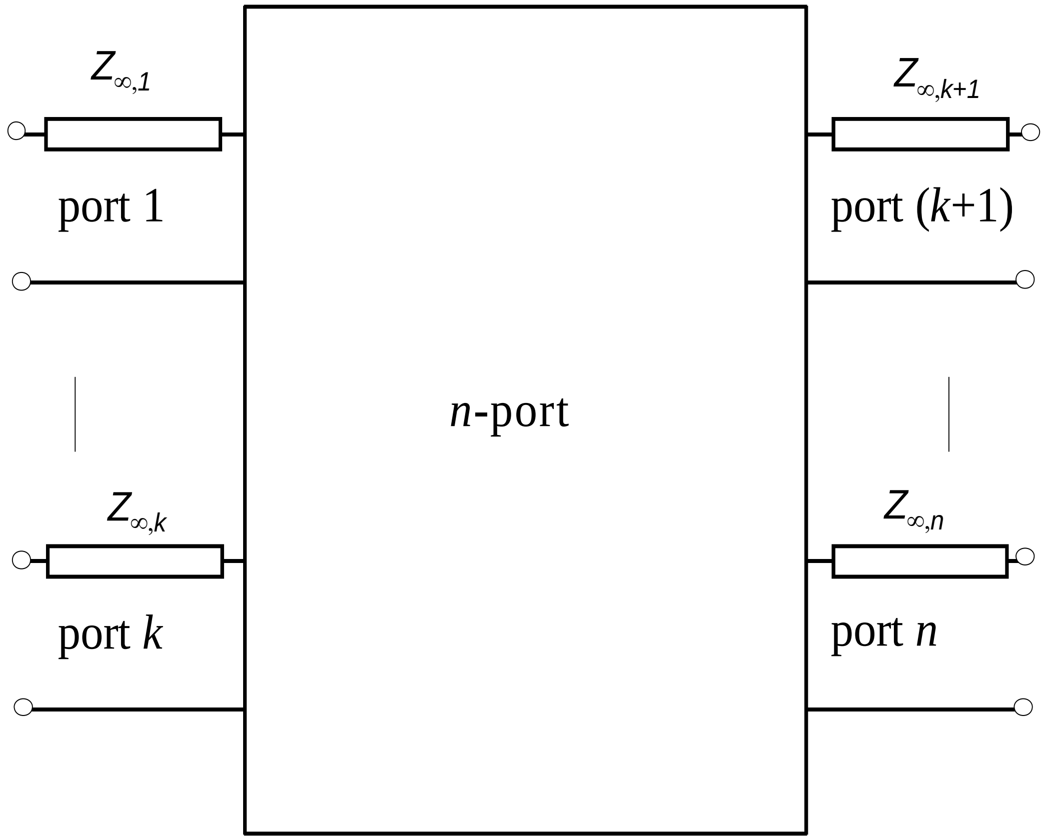
- If the normalization impedances are different the linear relationship between power waves is non-diagonal
- If they are equal the simple continuity holds:

$$a_j = b_i$$

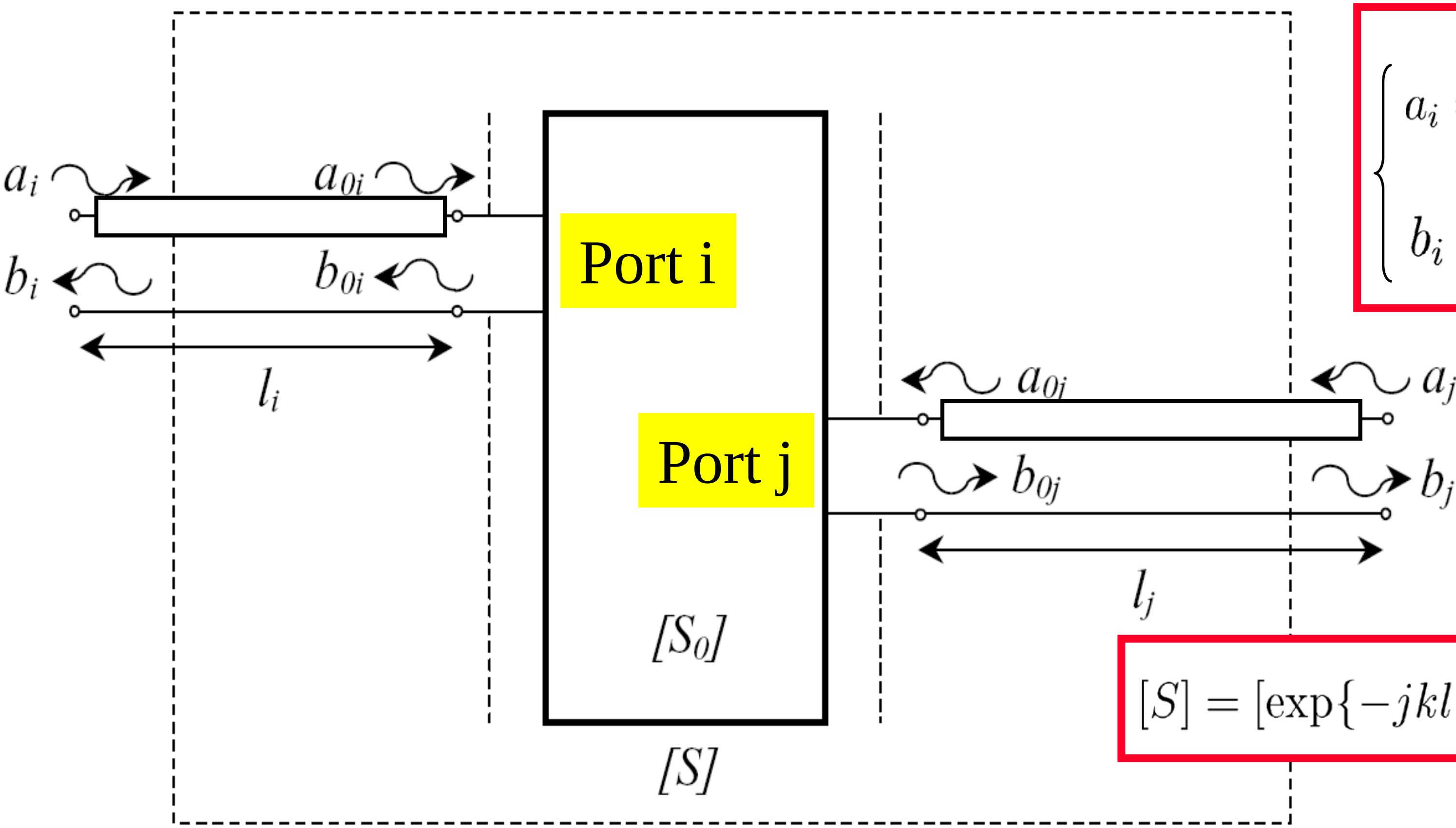
$$a_i = b_j$$

Reference plane shift I

- Suppose we connect to all ports transmission line with characteristic impedance equal to the port nomalization impedance, how is the S matrix transformed?



Reference plane shift II



$$\begin{cases} a_i = a_{0i} e^{jk_i l_i} \\ b_i = b_{0i} e^{-jk_i l_i} \end{cases}$$

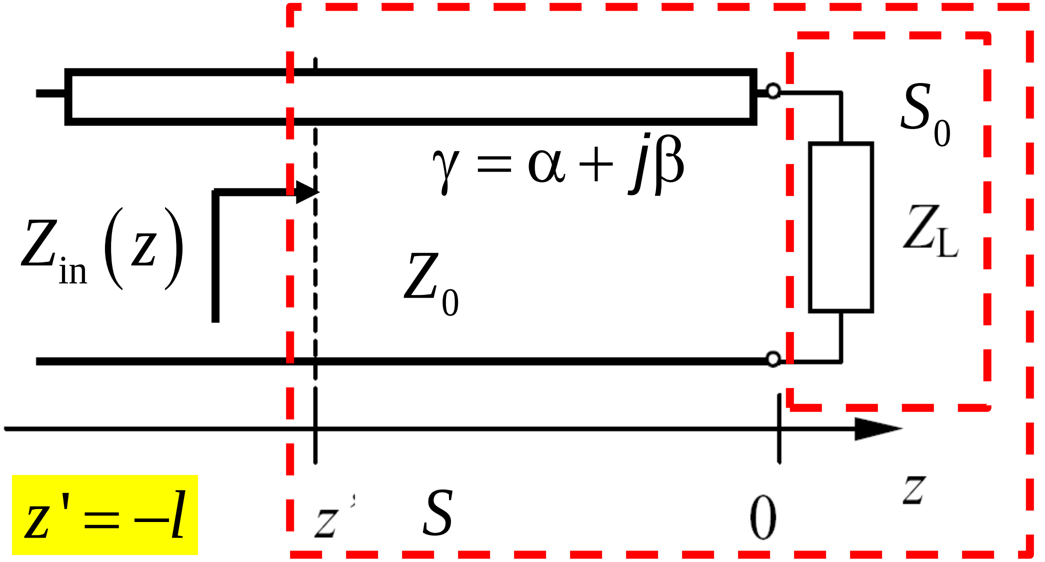
$$[S] = [\exp\{-jkl\}][S_0][\exp\{-jkl\}]$$

$$S_{ii} = S_{0ii} e^{-2jk_i l_i}$$

$$S_{ij} = S_{0ij} e^{-j(k_i l_i + k_j l_j)}$$

$$\exp(\pm jkl) = \mathrm{diag}\{\ \exp\{\pm jk_1 l_1\}\ \ \ldots\ \exp\{\pm jk_N l_N\}\ \}$$

Example: loaded line input impedance



Assume as the reference impedance Z_0 of the line

$$S_0 = \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

From line theory

$$\Gamma(z') = \Gamma_L \exp\left(j2\beta z' + 2\alpha z'\right)$$

$$S = \Gamma(-l) = \Gamma_L \exp\left(-j2\beta l - 2\alpha l\right)$$

Reference plane shift

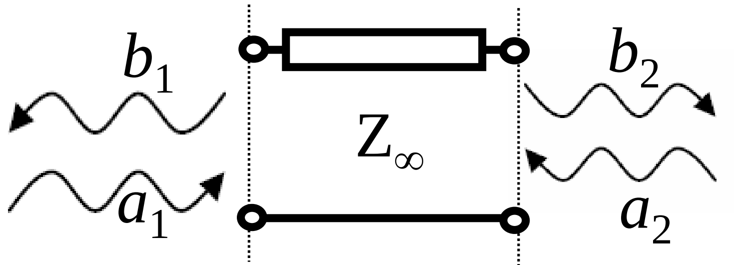
$$S = \exp\left(-j\beta l - \alpha l\right) S_0 \exp\left(-j\beta l - \alpha l\right)$$

$$Z_{in}(z) = Z_0 \frac{1+S}{1-S} = Z_0 \frac{1+S_0 \exp(-2\gamma l)}{1-S_0 \exp(-2\gamma l)} = Z_0 \frac{Z_L + Z_0 \tanh(\gamma l)}{Z_0 + Z_L \tanh(\gamma l)}$$

Scattering matrix of a line section

- We choose $R_{01}=R_{02}=Z_\infty$

$$S=\begin{pmatrix} 0 & \exp(-j\beta L) \\ \exp(-j\beta L) & 0 \end{pmatrix}$$



- If the normalization resistance is different from Z_∞

$$S_{11}^{(R_0)}=S_{22}^{(R_0)}\overset{def}{=}\Gamma_{in}^{(R_0)}=\frac{Z_\infty\left(1+\Gamma_{in}^{(Z_\infty)}\right)-R_0\left(1-\Gamma_{in}^{(Z_\infty)}\right)}{Z_\infty\left(1+\Gamma_{in}^{(Z_\infty)}\right)+R_0\left(1-\Gamma_{in}^{(Z_\infty)}\right)}$$

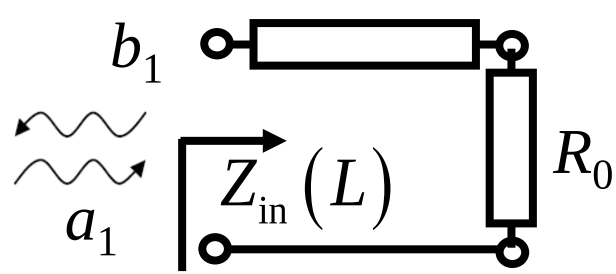
$$S_{21}^{(R_0)}=S_{12}^{(R_0)}=\left(1+S_{11}^{(R_0)}\right)\exp(-j\beta L)\frac{1+\Gamma_L^{(Z_\infty)}}{1+\Gamma_{in}^{(Z_\infty)}}$$

$$\Gamma_L^{(Z_\infty)}=\frac{R_0-Z_\infty}{R_0+Z_\infty}\quad(\text{ref. impedance}=Z_\infty)$$

$$\Gamma_{in}^{(Z_\infty)}=\Gamma_L^{(Z_\infty)}\exp(-2j\beta L)\quad(\text{ref. impedance}=Z_\infty)$$

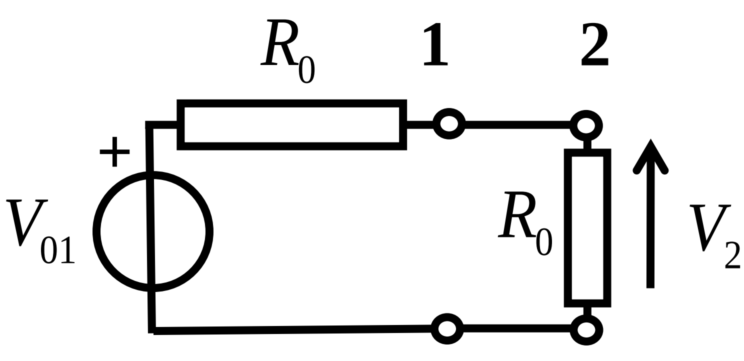
To evaluate S_{11} close the line on R_0 and evaluate Z_{in}

Then: $\mathbf{S_{11}\overset{def}{=}\Gamma_{in}^{(R_0)}=(Z_{in}-R_0)/(Z_{in}+R_0)}$



Proof with reference plane shift

- We choose R_{01}=R_{02}=Z_∞
- Start with a two-port made of two short circuits.
- Close on the normalization resistance at port 2 and excite at port 1 with a real generator having the normalization resistance as the internal resistance
- The reflection coefficient is 0 because of matching and furthermore (the structure is symmetrical and reciprocal):



$$S_{21} = S_{12} = 2\frac{V_2}{V_{01}} = 2\times\frac{1}{2}$$

$$\bar{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ with plane shift at port 1:}$$

$$S = \begin{pmatrix} e^{-j\beta L} & 0 \\ 0 & e^{-j\beta L} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{-j\beta L} \\ e^{-j\beta L} & 0 \end{pmatrix}$$