Gains in a loaded two-port

Microwave Electronics

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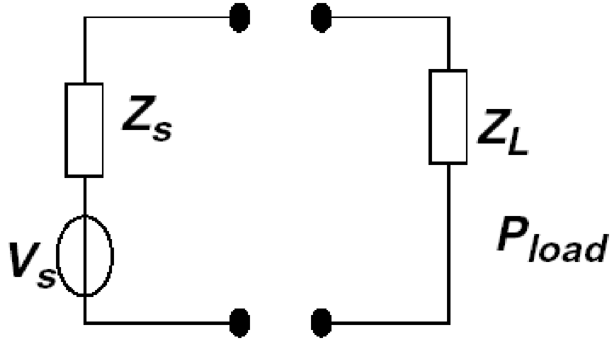
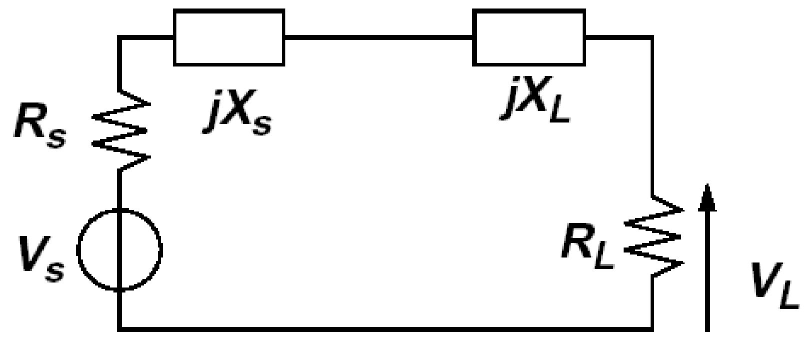
Power (conjugate) match →max. power transfer – usual approach

Consider a source with the complex impedance Z_S





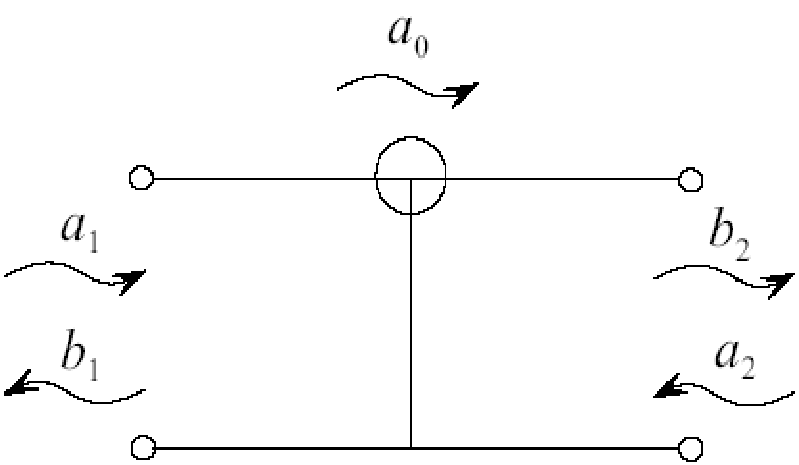
$$P_L = \frac{|V_L|^2}{R_L}, \quad V_L = V_S \frac{R_L}{(R_L + R_S) + j(X_L + X_S)}$$

$$P_L = \frac{|V_S|^2 R_L}{(R_L + R_S)^2 + (X_L + X_S)^2} \text{ max. vs. } X_L \to X_L = -X_S$$

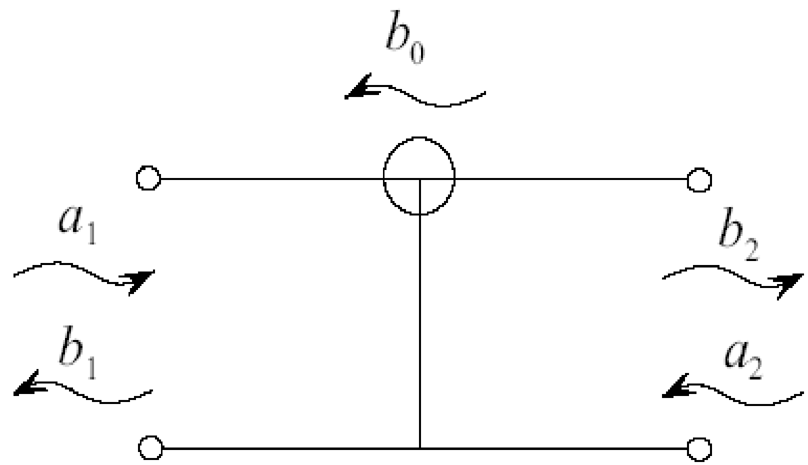
$$P_L = \frac{|V_S|^2 R_L}{(R_L + R_S)^2} \text{ max. vs. } R_L \to \frac{dP_L}{dR_L} = |V_S|^2 \left[\frac{R_S - R_L}{(R_L + R_S)^3} \right] = 0 \to R_L = R_S$$

Finally $Z_L = Z_S^*$ (conjugate or power match) $P_{L\max} = P_{Sav} = \dfrac{|V_S|^2}{4R_S}$

Forward and backward wave generators

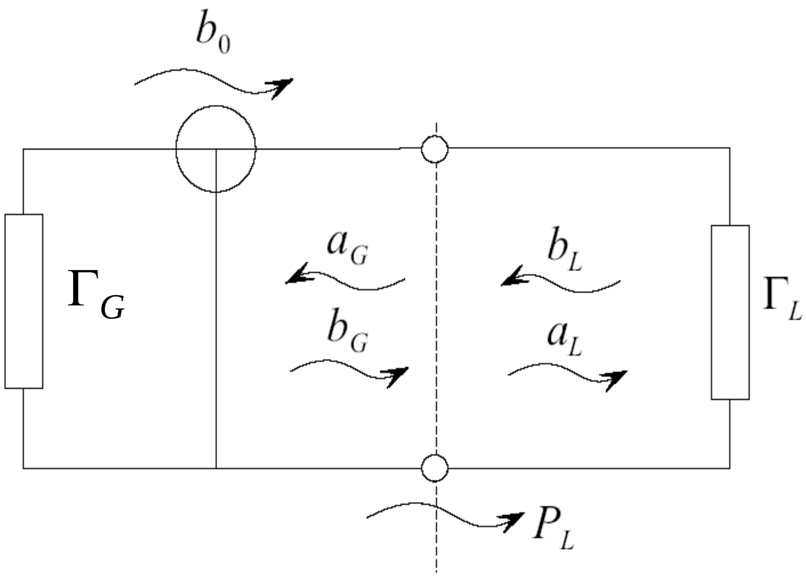


$$\begin{cases} b_1 = a_2 \\ b_2 = a_0 + a_1 \end{cases}$$



$$\begin{cases} b_1 = a_2 + b_0 \\ b_2 = a_1 \end{cases}$$

Power transfer in a loaded one-port



$$\begin{cases} a_L = b_G \\ a_G = b_L \\ b_G = b_0 + \Gamma_G a_G \\ b_L = \Gamma_L a_L \end{cases}$$

$$\begin{cases} a_L = \dfrac{b_0}{1 - \Gamma_G \Gamma_L} \\ b_L = \dfrac{b_0 \Gamma_L}{1 - \Gamma_G \Gamma_L} \end{cases}$$

$$P_L = |a_L|^2 - |b_L|^2 = |a_L|^2 (1 - |\Gamma_L|^2) = |b_0|^2 \frac{1 - |\Gamma_L|^2}{|1 - \Gamma_G \Gamma_L|^2}$$

$$Z_L = Z_G^* \underset{R_0 \in \Re}{\Rightarrow} \Gamma_L = \Gamma_G^*$$

Power matching (maximum power transfer, $Z_L = Z_G^* \rightarrow \Gamma_L = \Gamma_G^*$)

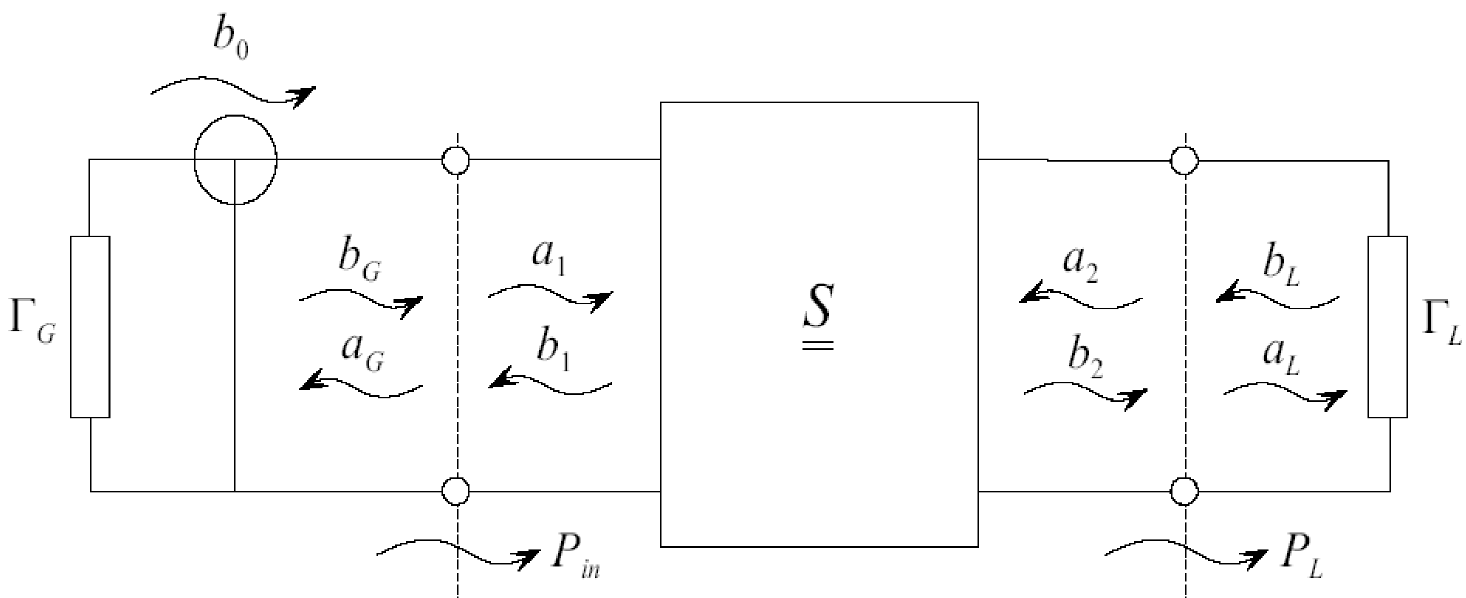
power to load $\rightarrow$ generator available power

$$P_{av} = \frac{|b_0|^2 \left(1 - |\Gamma_G^*|^2\right)}{|1 - \Gamma_G \Gamma_G^*|^2} = \frac{|b_0|^2 \left(1 - |\Gamma_G|^2\right)}{\left|1 - |\Gamma_G|^2\right|^2} = \frac{|b_0|^2 \left(1 - |\Gamma_G|^2\right)}{\left(1 - |\Gamma_G|^2\right)^2} =$$

$$= \frac{|b_0|^2}{1 - |\Gamma_G|^2} = |V_0|^2 \frac{R_0}{|Z_G + R_0|^2} \frac{|Z_G + R_0|^2}{|Z_G + R_0|^2 - |Z_G - R_0|^2} = \frac{|V_0|^2}{4 R_G}$$

$$P_L = P_{av} \frac{\left(1 - |\Gamma_G|^2\right)\left(1 - |\Gamma_L|^2\right)}{|1 - \Gamma_G \Gamma_L|^2}$$

Analysis of loaded two-port



$$
\begin{aligned}
b_G &= b_0 + \Gamma_G a_G \\
b_1 &= S_{11} a_1 + S_{12} a_2 \\
b_2 &= S_{21} a_1 + S_{22} a_2 \\
b_L &= \Gamma_L a_L \ .
\end{aligned}
\qquad
\begin{aligned}
a_1 &= b_G \\
b_1 &= a_G \\
b_2 &= a_L \\
a_2 &= b_L
\end{aligned}
$$

$$
\Rightarrow \quad
\begin{pmatrix}
1 & -\Gamma_G & 0 & 0 \\
-S_{11} & 1 & -S_{12} & 0 \\
-S_{21} & 0 & -S_{22} & 1 \\
0 & 0 & 1 & -\Gamma_L
\end{pmatrix}
\begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \end{pmatrix}
= b_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Solution in terms of power waves

- Solving the system one has:

$$a_1 = b_G = b_0 \frac{1 - S_{22}\Gamma_L}{(1 - S_{11}\Gamma_G)(1 - S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_G\Gamma_L}$$

$$b_1 = a_G = b_0 \frac{S_{12}S_{21}\Gamma_L + S_{11}(1 - S_{22}\Gamma_L)}{(1 - S_{11}\Gamma_G)(1 - S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_G\Gamma_L} =$$

$$= b_0 \frac{S_{11} - \Delta_S\Gamma_L}{(1 - S_{11}\Gamma_G)(1 - S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_G\Gamma_L}$$

$$a_2 = b_L = b_0 \frac{\Gamma_L S_{21}}{(1 - S_{11}\Gamma_G)(1 - S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_G\Gamma_L}$$

$$b_2 = a_L = b_0 \frac{S_{21}}{(1 - S_{11}\Gamma_G)(1 - S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_G\Gamma_L}$$

Δ_S is the determinant of the S matrix

$$\Delta_S = S_{11}S_{22} - S_{12}S_{21}$$

Some input port parameters

- Input reflection coefficient:

$$\Gamma_{in} = \frac{b_1}{a_1} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} = \frac{S_{11} - \Delta_S \Gamma_L}{1 - S_{22}\Gamma_L}.$$

- Input power:

$$P_{in} = |a_1|^2 - |b_1|^2 = |b_0|^2 \frac{|1 - S_{22}\Gamma_L|^2 - |S_{11} - \Delta_S\Gamma_L|^2}{|(1 - S_{11}\Gamma_G)(1 - S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_G\Gamma_L|^2},$$

alternatively:

$$P_{in} = |a_1|^2(1 - |\Gamma_{in}|^2) = |b_0|^2 \frac{1 - |\Gamma_{in}|^2}{|1 - \Gamma_G\Gamma_{in}|^2}.$$

Input reflection coefficient - proof

$$\boxed{\begin{aligned} b_1 &= S_{11}a_1 + S_{12}a_2 \\ b_2 &= S_{21}a_1 + S_{22}a_2 \end{aligned}}$$

$$a_2 = \Gamma_L b_2 \Rightarrow$$

$$b_1 = S_{11}a_1 + S_{12}\Gamma_L b_2$$

$$b_2 = S_{21}a_1 + S_{22}\Gamma_L b_2$$

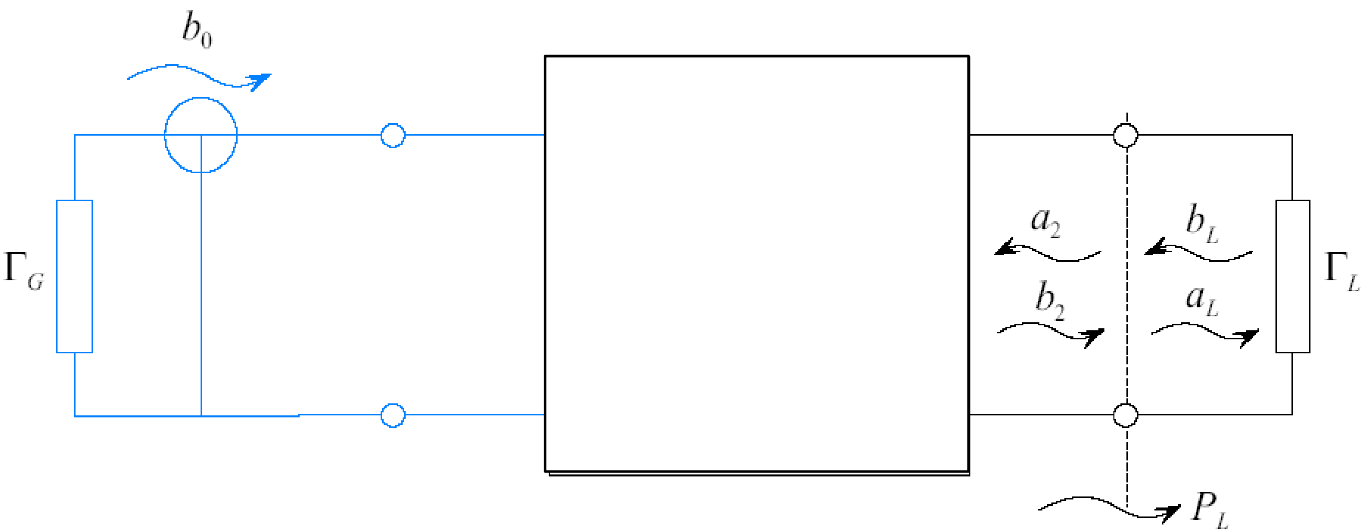
$$\Rightarrow b_2 = \frac{S_{21}a_1}{1 - S_{22}\Gamma_L}$$

$$\Rightarrow b_1 = S_{11}a_1 + \frac{S_{12}S_{21}\Gamma_L a_1}{1 - S_{22}\Gamma_L} \Rightarrow$$

$$\boxed{\frac{b_1}{a_1} = \Gamma_{in} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} = \frac{S_{11} - S_{11}S_{22}\Gamma_L + S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} = \frac{S_{11} - \Delta_S\Gamma_L}{1 - S_{22}\Gamma_L}}$$

Output equivalent circuit

- It is useful to evaluate the equivalent circuit of the two-port as seen from the output port:

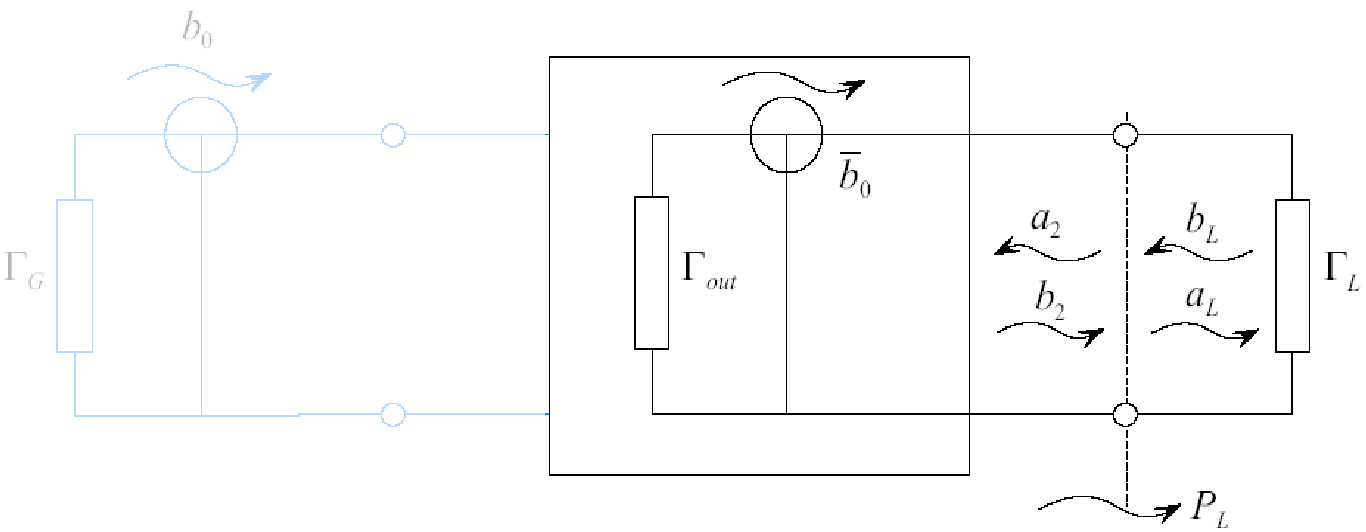


$$b_2 = \bar{b}_0 + \Gamma_{out} a_2.$$

$$\bar{b}_0 = b_0 \frac{S_{21}}{1 - S_{11}\Gamma_G} \qquad \Gamma_{out} = S_{22} + \frac{S_{12}S_{21}\Gamma_G}{1 - S_{11}\Gamma_G} = \frac{S_{22} - \Delta_S \Gamma_G}{1 - S_{11}\Gamma_G}$$

Output equivalent circuit

- It is useful to evaluate the equivalent circuit of the two-port as seen from the output port:



$$b_2 = \overline{b}_0 + \Gamma_{out} a_2.$$

$$\overline{b}_0 = b_0 \frac{S_{21}}{1 - S_{11}\Gamma_G} \qquad \Gamma_{out} = S_{22} + \frac{S_{12}S_{21}\Gamma_G}{1 - S_{11}\Gamma_G} = \frac{S_{22} - \Delta_S \Gamma_G}{1 - S_{11}\Gamma_G}$$

Proof

- Load the output with the normalization resistance, the input gamma will be by definition S_{11} thus:

$$\Gamma_L = 0 \Rightarrow a_2 = 0 \Rightarrow b_2 = \overline{b_0} = S_{21} a_1$$

$$\text{since } b_2 = \overline{b_0} + \Gamma_{out} a_2 = \overline{b} \text{ with } a_2 = 0$$

$$a_1 = \frac{b_0}{1 - S_{11}\Gamma_G} \text{ from the input port circuit}$$

$$\boxed{\overline{b_0} = \frac{S_{21} b_0}{1 - S_{11}\Gamma_G}}$$

- The output gamma is obtained by superposition setting b_0 =0 and using the input gamma result.

Power on the load

- Equivalent expressions:

$$P_L = |a_L|^2 - |b_L|^2 = |b_0|^2 \frac{|S_{21}|^2(1-|\Gamma_L|^2)}{|(1-S_{11}\Gamma_G)(1-S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_G\Gamma_L|^2}$$

$$P_L = |\bar{b}_0|^2 \frac{1-|\Gamma_L|^2}{|1-\Gamma_{out}\Gamma_L|^2} = |b_0|^2 \frac{|S_{21}|^2(1-|\Gamma_L|^2)}{|1-\Gamma_L\boxed{\Gamma_{out}}|^2|1-S_{11}\Gamma_G|^2} .$$

Two-port gains

- Having defined:
  - The **input power** at port 1 P_{in}
  - The **generator (input) available power** $P_{av,in}$
  - The **power on the load** P_L
  - The **load (output) available power** $P_{av,L}$

  we define the following power ratios:

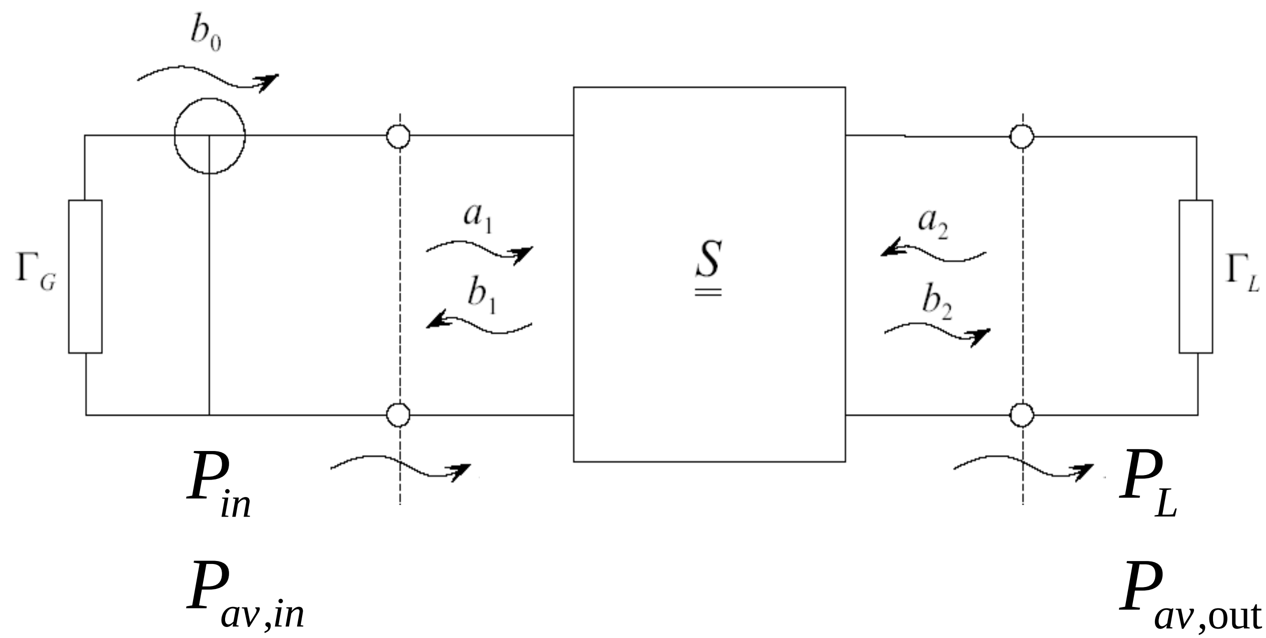
  - **Operational gain →** $$G_{op} = \frac{P_L}{P_{in}}$$
  - **Available power gain →** $$G_{av} = \frac{P_{av,L}}{P_{av,in}}$$
  - **Transducer gain →** $$G_t = \frac{P_L}{P_{av,in}}$$

A note on gain measurements

- The **transducer gain** is easy to measure because it only require a power measurement on the load → the input available power is derived from the generator setup (e.g. I set 100 mW available power on a 50 Ohm generator)
- The measurement of the **available power gain** further requires to match the output load (a so-called load-pull measurement)
- The **operational power gain** requires input and output power measurement → setup similar to the S-matrix measurement, a network analyzer is required (also scalar, only magnitudes of power waves, not phases)

What do gains depend on?

- Apart from the S parameters, gains depend on the load and generator reflection coefficients as follows:
  - **Operational gain →** on Γ_L
  - **Available power gain →** Γ_G
  - **Transducer gain →** on Γ_L and Γ_G .
- Why?



Why?

- Operational gain → on Γ_L
  - In fact if the *generator* impedance is changed, this changes the input matching and the input power, but the power on the load changes <u>of the same amount</u>
- Available power gain →on Γ_G
  - By definition the available power at the output is the power on the load when power transfer on the load is maximum (conjugate matching), thus the load impedance **is related to the generator impedance**
- Transducer gain → on Γ_L and Γ_G
  - In this case the input available power changes with generator gamma while the power on the load changes with the generator and load gamma (both independently)

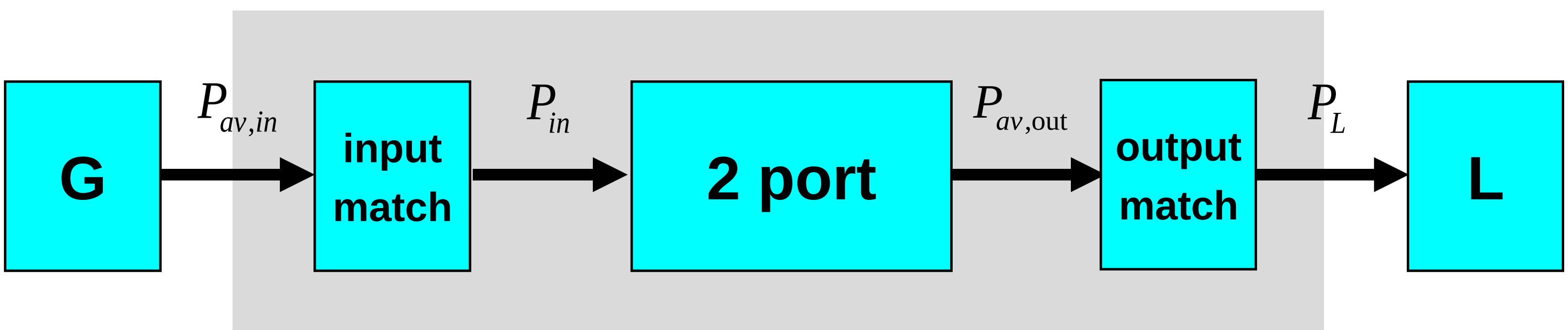
Maximum gain and maximum power transfer: same thing?

- The **maximum power transfer** occurs when the generator yields the maximum power, i.e.
  - the input power is <u>the generator (input) available power</u>
  - the power on the load is <u>the output available power</u>
- The maximum power transfer implies power impedance matching **simultaneously** at the input and output (*if this is possible*)
- The maximum power transfer implies maximum gain (for all gains!), but *maximum gain alone* implies *maximum power transfer* only for **<u>transducer gain</u>**, see later.

Power transfer and gains - I

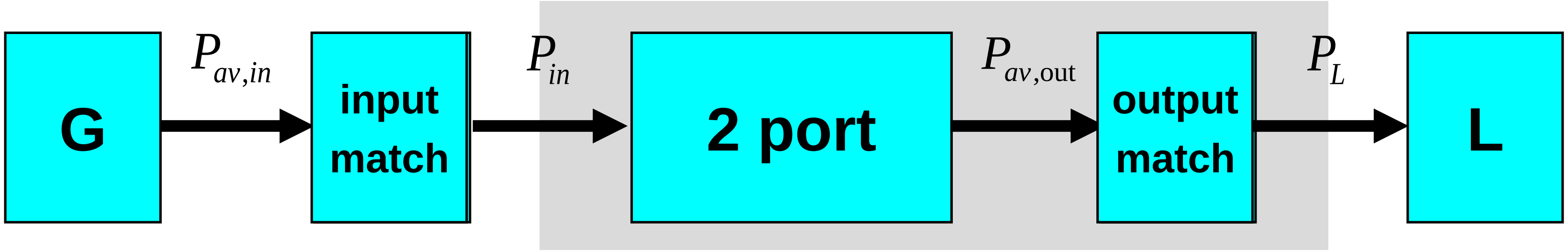
$$P_{in} = P_{av,in} \frac{\left(1-\left|\Gamma_G\right|^2\right)\left(1-\left|\Gamma_{in}(\Gamma_L)\right|^2\right)}{\left|1-\Gamma_G\Gamma_{in}(\Gamma_L)\right|^2}$$

$$P_L = P_{av,\mathrm{out}} \frac{\left(1-\left|\Gamma_{out}(\Gamma_G)\right|^2\right)\left(1-\left|\Gamma_L\right|^2\right)}{\left|1-\Gamma_{out}(\Gamma_G)\Gamma_L\right|^2}$$



$G_t(\Gamma_G,\Gamma_L) = \dfrac{P_L}{P_{av,in}} \Rightarrow$ max. transducer gain implies max power transfer

Power transfer and gains - II



$$P_{in} = P_{av,in} \frac{\left(1-\left|\Gamma_G\right|^2\right)\left(1-\left|\Gamma_{in}(\Gamma_L)\right|^2\right)}{\left|1-\Gamma_G \Gamma_{in}(\Gamma_L)\right|^2}$$

$G_{op}(\Gamma_L) = \dfrac{P_L}{P_{in}} \Rightarrow$ max. operational gain implies max. power transfer only if input power matching is also achieved

Power transfer and gains – III

$$P_L = P_{av,\text{out}} \frac{\left(1-\left|\Gamma_{out}(\Gamma_G)\right|^2\right)\left(1-\left|\Gamma_L\right|^2\right)}{\left|1-\Gamma_{out}(\Gamma_G)\Gamma_L\right|^2}$$

$$G_{av}(\Gamma_G) = \frac{P_{av,\text{out}}}{P_{av,in}} \Rightarrow$$ max. av. power gain $\rightarrow$ max. power transfer only with output power matching

Operational gain

- We start with operational gain – easier to optimize since it only depends on the (complex) load gamma [We could have started from available power gain as well]
- Doing all the math we obtain:

$$G_{op} = \frac{P_L}{P_{in}} = |S_{21}|^2 \frac{1 - |\Gamma_L|^2}{|1 - S_{22}\Gamma_L|^2 - |S_{11} - \Delta_S \Gamma_L|^2}$$

- or:

$$\boxed{G_{op} = |S_{21}|^2 \frac{1 - |\Gamma_L|^2}{1 - |S_{11}|^2 + |\Gamma_L|^2(|S_{22}|^2 - |\Delta_S|^2) + 2\Re(\Gamma_L(S_{11}^* \Delta_S - S_{22}))} \; .}$$

- The operational gain is a real function of the complex variable “gamma of the load” which is in turn defined in the Smith chart → let us have a look at **constant gain curves**

Operational gain - proof

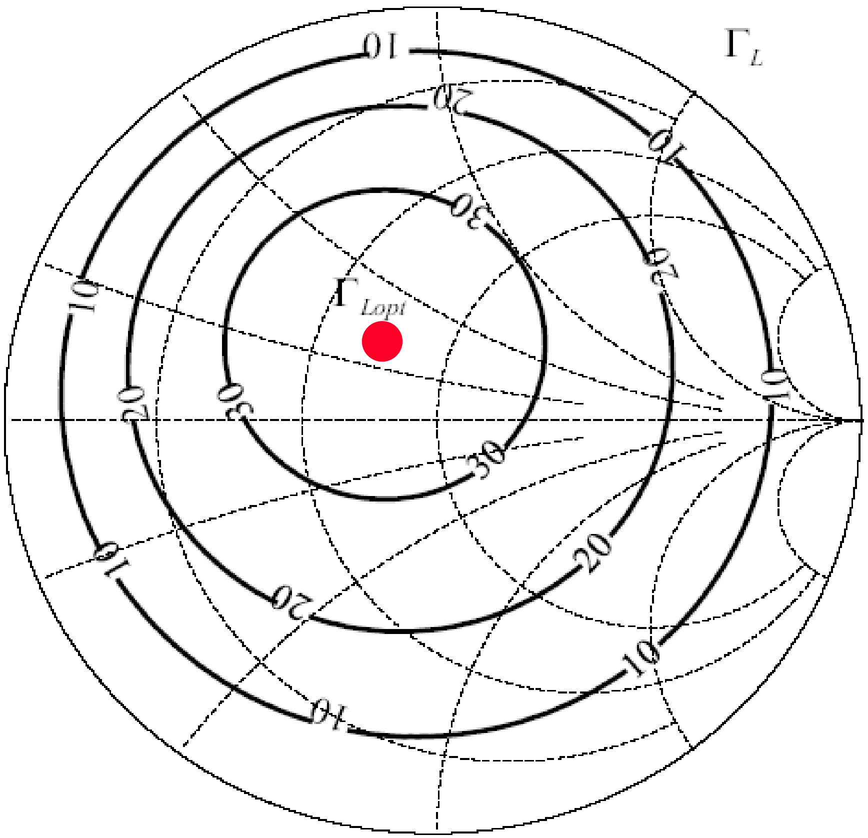
$$P_{in} = |b_0|^2 \frac{|1 - S_{22}\Gamma_L|^2 - |S_{11} - \Delta_S \Gamma_L|^2}{|(1 - S_{11}\Gamma_G)(1 - S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_G\Gamma_L|^2}$$

$$P_L = |b_0|^2 \frac{|S_{21}|^2 \left(1 - |\Gamma_L|^2\right)}{|(1 - S_{11}\Gamma_G)(1 - S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_G\Gamma_L|^2}$$

$$G_{op} = \frac{P_L}{P_{in}} = \frac{|S_{21}|^2 \left(1 - |\Gamma_L|^2\right)}{|1 - S_{22}\Gamma_L|^2 - |S_{11} - \Delta_S \Gamma_L|^2}$$

Notice that the input and load power depend on the generator gamma **in the same way** → their ratio is **independent on the generator gamma**

Constant operational gain circles



- The constant level curves of the operational gain in the complex plane Γ_L are *circles*
- The gain is maximum in a point → optimum Γ_L; the maximum gain is:

$$G_{op_{MAX}} = \left|\frac{S_{21}}{S_{12}}\right| \left(K - \sqrt{K^2 - 1}\right)$$

- However this maximum exists only if $K \geq 1$ where:

Linville (Rollet) stability factor

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta_S|^2}{2|S_{21}||S_{12}|}$$

The Unilateral Case (S_{12}=0)

- Simple if considered as a particular case

$$\Gamma_{in}=S_{11},\ \Gamma_{out}=S_{22},\ P_L=\left|S_{21}\right|^2\frac{1-\left|\Gamma_L\right|^2}{\left|1-S_{22}\Gamma_L\right|^2}\left|a_1\right|^2,\ P_{in}=\left(1-\left|S_{11}\right|^2\right)\left|a_1\right|^2$$

$$\Rightarrow \quad G_{OP}=\frac{P_L}{P_{in}}=\left|S_{21}\right|^2\frac{1-\left|\Gamma_L\right|^2}{\left|1-S_{22}\Gamma_L\right|^2\left(1-\left|S_{11}\right|^2\right)} \quad \Rightarrow \quad G_{OPMAX}=G_{OP}\Big|_{\Gamma_L=S_{22}^*}=\frac{\left|S_{21}\right|^2}{\left(1-\left|S_{22}\right|^2\right)\left(1-\left|S_{11}\right|^2\right)}$$

- To be dealt with care if derived from the general previous seen expressions since K→∞ when S_{12}→0 ......

The Unilateral case from the general approach 1

From the Rollet formula

$$2K\left|S_{12}\right| = \frac{1-\left|S_{11}\right|^2-\left|S_{22}\right|^2+\left|S_{11}\right|^2\left|S_{22}\right|^2}{\left|S_{21}\right|} = \frac{\left(1-\left|S_{11}\right|^2\right)\left(1-\left|S_{22}\right|^2\right)}{\left|S_{21}\right|}$$

From the max op gain formula

$$G_{OPMAX}\left|S_{12}\right| = \left|S_{21}\right|\left(K-\sqrt{K^2-1}\right)$$

When K→∞ (K real!)

$$G_{OPMAX} = \left|\frac{S_{21}}{S_{12}}\right|\left(K-\left|K\right|\sqrt{1-{}^{1}\!/_{K^2}}\right) \simeq \left|\frac{S_{21}}{S_{12}}\right|\left[K-\left|K\right|\left(1-\frac{1}{2K^2}\right)\right] = \left|\frac{S_{21}}{S_{12}}\right|\left(K-\left|K\right|+\frac{1}{2\left|K\right|}\right)$$

Hence, when S_{12}→0 K→+∞

$G_{OPMAX} \to \frac{\left|S_{21}\right|}{2\left|K\right|\left|S_{12}\right|} = \frac{\left|S_{21}\right|}{2K\left|S_{12}\right|}$ and from top page expr $G_{OPMAX} \to \frac{\left|S_{21}\right|^2}{\left(1-\left|S_{11}\right|^2\right)\left(1-\left|S_{22}\right|^2\right)}$

The Unilateral case from the general approach 2

$$G_{OPMAX}=\left|\frac{S_{21}}{S_{12}}\right|\left(K-|K|\sqrt{1-\frac{1}{K^2}}\right)\simeq\left|\frac{S_{21}}{S_{12}}\right|\left[K-|K|\left(1-\frac{1}{2K^2}\right)\right]=\left|\frac{S_{21}}{S_{12}}\right|\left(K-|K|+\frac{1}{2|K|}\right)$$

Instead, when $S_{12}\rightarrow 0$ but $K\rightarrow -\infty$ G_{OPMAX} is going to $-\infty$ too.

That means instability!

The negative sign of G_{OPMAX} means also that one of the two ports has the power flowing in reverted direction.

From the expression $K|S_{12}|=\dfrac{\left(1-|S_{11}|^2\right)\left(1-|S_{22}|^2\right)}{2|S_{21}|}$

We can see that in a unilateral device condition $K\rightarrow+\infty$ is verified when $|S_{11}|<1$ and $|S_{22}|<1$ while $K\rightarrow-\infty$ if $|S_{11}|>1$ or $|S_{22}|>1$

[In the case $|S_{11}|>1$ and $|S_{22}|>1$ both port powers have reverted sign so that the power flow is not going from generator to load, but in the opposite direction, and virtually the gain is positive.]

Max. operational gain & max. power transfer

- If K>=1 the operational gain has a maximum; the condition on K corresponds to an *unconditionally stable* two-port → i.e. a two port stable for any choice of load and generator impedances
- When the operational gain is maximum (condition on optimum load gamma) the **maximum power transfer** between generator and load **occurs when the generator is power matched to the two-port input** (condition on optimum generator gamma)
- The set of two optimum load and generator gammas corresponds to **simultaneous power matching** at the two ports (for analytic expressions of optimum gammas see text)

Available power gain

- One has:

$$G_{av} = \frac{P_{av,\text{out}}}{P_{av,in}} = \frac{|S_{21}|^2 \left(1 - |\Gamma_G|^2\right)}{|1 - S_{11}\Gamma_G|^2 - |S_{22} - \Delta_S \Gamma_G|^2}$$

or:

$$\boxed{G_{av} = |S_{21}|^2 \frac{1 - |\Gamma_G|^2}{1 - |S_{22}|^2 + |\Gamma_G|^2(|S_{11}|^2 - |\Delta_S|^2) + 2\Re(\Gamma_G(S_{22}^*\Delta_S - S_{11}))}\ .}$$

- The available power gain is defined by its constant level curves in the complex plane Γ_G (*circles* again!). The gain is zero on the Smith chart unit circle (reactive source impedance → *infinite input available power*) → dual situation vs. operational gain

Available power gain - proof

$$P_{in} = |b_0|^2 \frac{1-|\Gamma_{in}|^2}{|1-\Gamma_G \Gamma_{in}|^2} \Rightarrow P_{av,in} \underset{\Gamma_{in}=\Gamma_G^*}{=} \frac{|b_0|^2}{1-|\Gamma_G|^2}$$

$$P_L = \frac{|b_0|^2 |S_{21}|^2 \left(1-|\Gamma_L|^2\right)}{|1-\Gamma_L \Gamma_{out}|^2 |1-S_{11}\Gamma_G|^2} \Rightarrow P_{av,\mathrm{out}} \underset{\Gamma_L=\Gamma_{out}^*}{=} \frac{|b_0|^2 |S_{21}|^2}{\left(1-|\Gamma_{out}|^2\right)|1-S_{11}\Gamma_G|^2}$$

$$G_{av} = \frac{P_{av,\mathrm{out}}}{P_{av,in}} = \frac{|b_0|^2 |S_{21}|^2}{\left(1-|\Gamma_{out}|^2\right)|1-S_{11}\Gamma_G|^2} \frac{1-|\Gamma_G|^2}{|b_0|^2} =$$

$$\underset{\Gamma_{out}=\frac{S_{22}-\Delta_S\Gamma_G}{1-S_{11}\Gamma_G}}{=} \frac{|S_{21}|^2 \left(1-|\Gamma_G|^2\right)}{|1-S_{11}\Gamma_G|^2 - |S_{22}-\Delta_S \Gamma_G|^2}$$

The input and output available power **do not depend** on the load gamma → their ratio is **independent on the load gamma**

G_{op} – G_{av} duality and maximum values

- Taking into account that the two following formulae are obtained by exchanging 1→2, L→G (the determinant ad *K* are invariant with respect to this change):

$$\frac{G_{av}}{|S_{21}|^2} = \frac{1-|\Gamma_G|^2}{|1-S_{11}\Gamma_G|^2 - |S_{22}-\Delta_S\Gamma_G|^2} \qquad \frac{G_{op}}{|S_{21}|^2} = \frac{1-|\Gamma_L|^2}{|1-S_{22}\Gamma_L|^2 - |S_{11}-\Delta_S\Gamma_L|^2}$$

- we immediately get the two maxima:

$$\left(\frac{G_{op}}{|S_{21}|^2}\right)_{\max} = \frac{1}{|S_{12}S_{21}|}\left(K-\sqrt{K^2-1}\right) \Rightarrow \left(\frac{G_{av}}{|S_{21}|^2}\right)_{\max} = \frac{1}{|S_{21}S_{12}|}\left(K-\sqrt{K^2-1}\right)$$

- i.e.:

$$\boxed{\left(G_{op}\right)_{\max} = \left(G_{av}\right)_{\max}}$$

Maximum available gain (MAG)

- MAG coincides with maximum operational gain, but again a maximum exists only for $K >= 1$ (unconditionally stable two-port)
- MAG corresponds to an optimum choice of generator gamma, to achieve *maximum power transfer* we also have to power match the load to the two-port output (optimum load gamma)
- We obtain again that maximum power transfer corresponds to conjugate matching at the two ports simultaneously → **same condition** already discussed

Transducer gain

- One has the following espression, depending on both generator and load gammas:

$$G_t = \frac{P_L}{P_{av,in}} = \left|S_{21}\right|^2 \frac{\left(1-\left|\Gamma_L\right|^2\right)\left(1-\left|\Gamma_G\right|^2\right)}{\left|\left(1-S_{11}\Gamma_G\right)\left(1-S_{22}\Gamma_L\right)-S_{12}S_{21}\Gamma_L\Gamma_G\right|^2}$$

the optimum condition directly corresponds to maximum power transfer, i.e. again to simultaneous conjugate matching at ports 1 and 2:

$$\begin{cases} \Gamma_G = \Gamma^*_{in}(\Gamma_L) \\ \Gamma_L = \Gamma^*_{out}(\Gamma_G) \end{cases}$$

- The optimum transducer gain (obviously) is **the same** as MAG and maximum operational gain.

Transducer gain - proof

$$P_{in} = |b_0|^2 \frac{1-|\Gamma_{in}|^2}{|1-\Gamma_G\Gamma_{in}|^2} \Rightarrow P_{av,in} \underset{\Gamma_{in}=\Gamma_G^*}{=} \frac{|b_0|^2}{1-|\Gamma_G|^2} \text{ depends on } \Gamma_G$$

$$P_L = \frac{|b_0|^2 |S_{21}|^2 \left(1-|\Gamma_L|^2\right)}{\left|(1-S_{11}\Gamma_G)(1-S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_L\Gamma_G\right|^2} \text{ depends on } \Gamma_G, \Gamma_L$$

$$G_t = \frac{P_L}{P_{av,in}} = \frac{|b_0|^2 |S_{21}|^2 \left(1-|\Gamma_L|^2\right)}{\left|(1-S_{11}\Gamma_G)(1-S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_L\Gamma_G\right|^2} \frac{1-|\Gamma_G|^2}{|b_0|^2} =$$

$$= \frac{|S_{21}|^2 \left(1-|\Gamma_L|^2\right)\left(1-|\Gamma_G|^2\right)}{\left|(1-S_{11}\Gamma_G)(1-S_{22}\Gamma_L) - S_{12}S_{21}\Gamma_L\Gamma_G\right|^2} \text{ depends on } \Gamma_G, \Gamma_L$$

Unilateral transducer gain

- Transducer gain of a *unilateral device* where S_{12}=0 (is that meaningful and why? "always" unconditionally stable):

$$G_u = G_t|_{S_{12}=0} = |S_{21}|^2 \frac{(1-|\Gamma_L|^2)(1-|\Gamma_G|^2)}{|1-\Gamma_L S_{22}|^2 |1-\Gamma_G S_{11}|^2}$$

the **maximum unilateral gain (MUG)** is obtained when the two ports are power matched, a condition which can now be obtained separately at port 1 and 2:

$$\begin{cases} \Gamma_G = S_{11}^* \\ \Gamma_L = S_{22}^* \end{cases} \Longrightarrow G_{u_{max}} = \frac{|S_{21}|^2}{(1-|S_{11}|^2)(1-|S_{22}|^2)}$$

Summary and MSG (Maximum Stable Gain)

$$G_{op} = \frac{P_L}{P_{in}} = |S_{21}|^2 \frac{1-|\Gamma_L|^2}{1-|S_{11}|^2+|\Gamma_L|^2(|S_{22}|^2-|\Delta_S|^2)+2\Re(\Gamma_L(S_{11}^*\Delta_S-S_{22}))}$$

$$G_{av} = \frac{P_{av,out}}{P_{av,in}} = |S_{21}|^2 \frac{1-|\Gamma_G|^2}{1-|S_{22}|^2+|\Gamma_G|^2(|S_{11}|^2-|\Delta_S|^2)+2\Re(\Gamma_G(S_{22}^*\Delta_S-S_{11}))}$$

$$G_t = \frac{P_L}{P_{av,in}} = |S_{21}|^2 \frac{(1-|\Gamma_L|^2)(1-|\Gamma_G|^2)}{|(1-\Gamma_L S_{22})(1-\Gamma_G S_{11})-S_{12}S_{21}\Gamma_G\Gamma_L|^2}$$

$$\mathrm{MAG} = \frac{|S_{21}|}{|S_{12}|}\left(K-\sqrt{K^2-1}\right) \xrightarrow{K=1} \mathrm{MSG} = \frac{|S_{21}|}{|S_{12}|}$$

MSG=MAG for K=1→*figure of merit* if K<=1

$$\mathrm{MUG} = \frac{|S_{21}|^2}{(1-|S_{11}|^2)(1-|S_{22}|^2)}$$

Conclusion on simultaneous power matching

- **Simultaneous power matching** leads to **maximum gain(s)** and **maximum power transfer** → basic design of maximum gain amplifier
- Can be implemented only if the stability factor is equal to or larger than one → this happens if the device is *unconditionally stable* (unconditional stability implies K>=1)
- Names to be remembered:
  - **MAG** (Maximum Available Gain, the maximum gain);
  - **MSG** (Maximum Stable Gain, often used as a figure of merit as a function of frequency if the device is potentially unstable);
  - **MUG** (Maximum Unilateral Gain, less important)