Differential geometry, gauge theories, and gravity
M. Göckeler and T. Schücker

Solutions to the problems

1 Exterior algebra

Problem 1.1
Verify the transformation law (1.4) of the dual basis.

Solution 1.1
It is sufficient to show

$$\beta'^i(b'_k) = \delta^i{}_k$$

using

$$\beta'^i = \sum_{j=1}^{n} \gamma^i{}_j \beta^j \quad \text{and} \quad b'_k = \sum_{l=1}^{n} (\gamma^{-1})^l{}_k b_l :$$

$$\begin{aligned}\beta'^i(b'_k) &= \sum_j \gamma^i{}_j \beta^j \Big(\sum_l (\gamma^{-1})^l{}_k b_l \Big) = \sum_{j,l} \gamma^i{}_j (\gamma^{-1})^l{}_k \beta^j(b_l) \\ &= \sum_{j,l} \gamma^i{}_j (\gamma^{-1})^l{}_k \delta^j{}_l = \sum_j \gamma^i{}_j (\gamma^{-1})^j{}_k \qquad = \delta^i{}_k.\end{aligned}$$

Problem 1.2
Prove the Leibniz rule (1.28) for inner derivatives.

Solution 1.2
Let $\varphi \in \Lambda^p V$, $\psi \in \Lambda^q V$ and $v_0 = v$. Then we have:

$$\begin{aligned}[i_v(\varphi \wedge \psi)](v_1, v_2, \dots, v_{p+q-1}) &= [\varphi \wedge \psi](v, v_1, \dots, v_{p+q-1}) \\ &= \frac{1}{p!q!} \sum_{\pi \in \mathscr{S}_{p+q}} \varphi(v_{\pi(0)},\dots, v_{\pi(p-1)}) \psi(v_{\pi(p)},\dots, v_{\pi(p+q-1)}) \operatorname{sig}\pi =: \star.\end{aligned}$$

Define the permutation $\rho \in \mathscr{S}_{p+q-1}$ by

$$\pi(0) = \rho(1), \dots, \pi(i-1) = \rho(i), \pi(i) = 0, \pi(i+1) = \rho(i+1), \dots, \pi(p+q-1) = \rho(p+q-1).$$

Then $\operatorname{sig}\rho = (-1)^i \operatorname{sig}\pi$ and

$$\begin{aligned}
\star &= \frac{1}{p!q!}\sum_{i=0}^{p-1}\sum_{\rho\in\mathscr{S}_{p+q-1}} \varphi(v_{\rho(1)},\dots,v_{\rho(i)},v,v_{\rho(i+1)},\dots,v_{\rho(p-1)})\,\psi(v_{\rho(p)},\dots,v_{\rho(p+q-1)})\,(-1)^i \operatorname{sig}\rho \\
&\quad + \frac{1}{p!q!}\sum_{i=p}^{p+q-1}\sum_{\rho\in\mathscr{S}_{p+q-1}} \varphi(v_{\rho(1)},\dots,v_{\rho(p)})\,\psi(v_{\rho(p+1)},\dots,v_{\rho(i)},v,v_{\rho(i+1)},\dots,v_{\rho(p+q-1)})\,(-1)^i \operatorname{sig}\rho \\
&= \frac{p}{p!q!}\sum_{\rho\in\mathscr{S}_{p+q-1}} \varphi(v,v_{\rho(1)},\dots,v_{\rho(p-1)})\,\psi(v_{\rho(p)},\dots,v_{\rho(p+q-1)}) \operatorname{sig}\rho \\
&\quad + \frac{q}{p!q!}\sum_{\rho\in\mathscr{S}_{p+q-1}} \varphi(v_{\rho(1)},\dots,v_{\rho(p)})\,\psi(v,v_{\rho(p+1)},\dots,v_{\rho(p+q-1)})\,(-1)^p \operatorname{sig}\rho \\
&= \frac{1}{(p-1)!q!}\sum_{\rho\in\mathscr{S}_{p+q-1}} (i_v\varphi)(v_{\rho(1)},\dots,v_{\rho(p-1)})\,\psi(v_{\rho(p)},\dots,v_{\rho(p+q-1)}) \operatorname{sig}\rho \\
&\quad + \frac{1}{p!(q-1)!}\sum_{\rho\in\mathscr{S}_{p+q-1}} \varphi(v_{\rho(1)},\dots,v_{\rho(p)})\,(i_v\psi)(v_{\rho(p+1)},\dots,v_{\rho(p+q-1)})\,(-1)^p \operatorname{sig}\rho \\
&= \quad [i_v\varphi\wedge\psi](v_1,v_2,\dots,v_{p+q-1}) + (-1)^p\,[\varphi\wedge i_v\psi](v_1,v_2,\dots,v_{p+q-1}).
\end{aligned}$$

Problem 1.3

Derive the transformation property (1.44) of a density.

Solution 1.3

If $\omega = k\,\beta^1\wedge\cdots\wedge\beta^n = k'\,\beta'^1\wedge\cdots\wedge\beta'^n$ with $\beta'^i = \sum_{j=1}^n \gamma^i{}_j\beta^j$ we must show that $k' = \det\gamma^{-1}$. Indeed:

$$\begin{aligned}
\omega &= \frac{k'}{n!}\sum_{i_1,\dots,i_n=1}^{n} \varepsilon_{i_1,\dots,i_n}\,\beta'^{i_1}\wedge\cdots\wedge\beta'^{i_n} = \frac{k'}{n!}\sum_{\substack{i_1,\dots,i_n,\\ j_1,\dots,j_n}} \gamma^{i_1}{}_{j_1}\cdots\gamma^{i_n}{}_{j_n}\varepsilon_{i_1,\dots,i_n}\,\beta^{j_1}\wedge\cdots\wedge\beta^{j_n} \\
&= \frac{k'}{n!}\det\gamma\sum_{j_1,\dots,j_n} \varepsilon_{j_1,\dots,j_n}\,\beta^{j_1}\wedge\cdots\wedge\beta^{j_n} \quad = \quad (k'\det\gamma)\,\beta^1\wedge\cdots\wedge\beta^n.
\end{aligned}$$

2 Differential forms on open subsets of $\mathbb{R}^n$

Problem 2.1
Derive the formulas (2.14) and (2.29) for the transformation of frames associated with coordinate systems.

Solution 2.1
By definition (2.9) we have

$$\frac{\partial}{\partial y^j}(x) = \left(x, \frac{\partial x}{\partial y^j}\right) = \left(x, \sum_{i=1}^n \frac{\partial x^i}{\partial y^j}\frac{\partial x}{\partial x^i}\right) = \sum_{i=1}^n \frac{\partial x^i}{\partial y^j}\left(x, \frac{\partial x}{\partial x^i}\right) = \sum_{i=1}^n \frac{\partial x^i}{\partial y^j}\frac{\partial}{\partial x^i}(x)\,.$$

This proves (2.14).

According to (2.14) and (2.28) we calculate

$$\sum_{j=1}^n \frac{\partial y^i}{\partial x^j}\mathrm{d}x^j\left(\frac{\partial}{\partial y^l}\right) = \sum_{j,k=1}^n \frac{\partial y^i}{\partial x^j}\frac{\partial x^k}{\partial y^l}\mathrm{d}x^j\left(\frac{\partial}{\partial x^k}\right) = \sum_{j=1}^n \frac{\partial y^i}{\partial x^j}\frac{\partial x^j}{\partial y^l} = \frac{\partial y^i}{\partial y^l} = \delta^i{}_l\,.$$

This proves (2.29).

Problem 2.2
Prove that pullback and exterior derivative commute (equation (2.63)).

Solution 2.2
Since F^* and d are linear, it suffices to consider $\varphi = f(y)\mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_p}$. We find

$$\begin{aligned} F^*\mathrm{d}\varphi &= F^* \sum_i \frac{\partial f}{\partial y^i}\mathrm{d}y^i \wedge \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_p} \\ &= \sum_i \left(\frac{\partial f}{\partial y^i} \circ F\right) \sum_{j,j_1,\ldots,j_p} \frac{\partial F^i}{\partial x^j}\frac{\partial F^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial F^{i_p}}{\partial x^{j_p}}\mathrm{d}x^j \wedge \mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_p}\,. \end{aligned}$$

On the other hand we have

$$\begin{aligned} \mathrm{d}F^*\varphi &= \mathrm{d}\left(f \circ F \sum_{j_1,\ldots,j_p} \frac{\partial F^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial F^{i_p}}{\partial x^{j_p}}\mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_p}\right) \\ &= \mathrm{d}(f \circ F) \sum_{j_1,\ldots,j_p} \frac{\partial F^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial F^{i_p}}{\partial x^{j_p}}\mathrm{d}x^{j_1} \wedge \cdots \wedge \mathrm{d}x^{j_p}\,, \end{aligned}$$

because

$$\sum_{j,k} \frac{\partial^2 F^i}{\partial x^j \partial x^k}\mathrm{d}x^j \wedge \mathrm{d}x^k = 0\,.$$

Consequently

$$\mathrm{d}F^*\varphi = \sum_i \left(\frac{\partial f}{\partial y^i}\circ F\right) \sum_{j,j_1,\dots,j_p} \frac{\partial F^i}{\partial x^j}\frac{\partial F^{i_1}}{\partial x^{j_1}}\cdots\frac{\partial F^{i_p}}{\partial x^{j_p}}\mathrm{d}x^j\wedge\mathrm{d}x^{j_1}\wedge\cdots\wedge\mathrm{d}x^{j_p} = F^*\mathrm{d}\varphi\,.$$

Problem 2.3
Verify that the function (2.76) satisfies the equation $\mathrm{d}f=\varphi$.

Solution 2.3
For the function

$$f(x) = \int_0^1 \sum_{i=1}^n \varphi_i(tx)x^i\mathrm{d}t$$

we calculate

$$\frac{\partial f}{\partial x^j} = \int_0^1 \sum_{i=1}^n \left\{ \frac{\partial\varphi_i}{\partial y^j}\bigg|_{y=tx} tx^i + \varphi_i(tx)\delta^i{}_j \right\} \mathrm{d}t\,.$$

Since

$$0=\mathrm{d}\varphi=\sum_{i,j}\frac{\partial\varphi_i}{\partial y^j}\mathrm{d}y^j\wedge\mathrm{d}y^i$$

we have

$$\frac{\partial\varphi_i}{\partial y^j}=\frac{\partial\varphi_j}{\partial y^i}\,,$$

where the Cartesian coordinates have been temporarily denoted by y^i because of the substitution $y^k=tx^k$ needed above. Consequently

$$\begin{aligned}\frac{\partial f}{\partial x^j} &= \sum_{i=1}^n \int_0^1 \left\{ \frac{\partial\varphi_j}{\partial y^i}\bigg|_{y=tx} tx^i + \varphi_i(tx)\delta^i{}_j \right\} \mathrm{d}t = \int_0^1 t\frac{\partial}{\partial t}\varphi_j(tx)\,\mathrm{d}t + \int_0^1 \varphi_j(tx)\,\mathrm{d}t \\ &= t\varphi_j(tx)\bigg|_{t=0}^{t=1} - \int_0^1 \varphi_j(tx)\,\mathrm{d}t + \int_0^1 \varphi_j(tx)\,\mathrm{d}t = \varphi_j(x)\,.\end{aligned}$$

Therefore

$$\mathrm{d}f = \sum_{j=1}^n \frac{\partial f}{\partial x^j}\mathrm{d}x^j = \sum_{j=1}^n \varphi_j(x)\mathrm{d}x^j = \varphi\,.$$

Problem 2.4
Prove Stokes' theorem (2.118) with $p=n=2$ in the special case where K is a rectangle.

Solution 2.4
Write

$$K=\left\{x\in\mathbb{R}^2\middle| a^1\le x^1\le b^1, a^2\le x^2\le b^2\right\}$$

and

$$\varphi=\varphi_1\,\mathrm{d}x^1+\varphi_2\,\mathrm{d}x^2\,.$$

Then $\mathrm{d}\varphi$ is given by (2.119) and therefore

$$\int_K \mathrm{d}\varphi = \int_K \left(\frac{\partial\varphi_2}{\partial x^1} - \frac{\partial\varphi_1}{\partial x^2}\right) \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \int_{a^1}^{b^1} \mathrm{d}x^1 \int_{a^2}^{b^2} \mathrm{d}x^2 \left(\frac{\partial\varphi_2}{\partial x^1} - \frac{\partial\varphi_1}{\partial x^2}\right)$$
$$= \int_{a^2}^{b^2} \mathrm{d}x^2 \left(\varphi_2\big|_{x^1=b^1} - \varphi_2\big|_{x^1=a^1}\right) - \int_{a^1}^{b^1} \mathrm{d}x^1 \left(\varphi_1\big|_{x^2=b^2} - \varphi_1\big|_{x^2=a^2}\right) .$$

On the other hand, we have

$$\int_{\partial K} \varphi = \int_{a^1}^{b^1} \mathrm{d}x^1 \varphi_1\big|_{x^2=a^2} + \int_{a^2}^{b^2} \mathrm{d}x^2 \varphi_2\big|_{x^1=b^1} - \int_{a^1}^{b^1} \mathrm{d}x^1 \varphi_1\big|_{x^2=b^2} - \int_{a^2}^{b^2} \mathrm{d}x^2 \varphi_2\big|_{x^1=a^1} ,$$

whence (2.118) follows.

3 Metric structures

Problem 3.1
Verify the transformation law (3.5) of the matrix describing a metric under a change of basis.

Solution 3.1
Recall

$$b'_i = \sum_{j=1}^{n} (\gamma^{-1})^k{}_i b_k \quad \text{and} \quad g_{kl} = g(b_k, b_l).$$

Then

$$g'_{ij} = g(b'_i, b'_j) = \sum_{k,l} (\gamma^{-1})^k{}_i (\gamma^{-1})^l{}_j \, g(b_k, b_l) = \sum_{k,l} (\gamma^{-1\,T})_i{}^k \, g_{kl} \, (\gamma^{-1})^l{}_j = (\gamma^{-1\,T} \, g \, \gamma^{-1})_{ij}.$$

Problem 3.2
Calculate the square of the Hodge star.

Solution 3.2
We use the definition (3.16),

$$*(e^{i_1} \wedge \cdots \wedge e^{i_p}) := \varepsilon_{i_1 \ldots i_n} \, \eta^{i_1 i_1} \cdots \eta^{i_p i_p} \, e^{i_{p+1}} \wedge \cdots \wedge e^{i_n}, \quad \text{(no summations!)}$$

which has to be understood in the sense that $i_1, \dots, i_n$ is a permutation of $1, \dots, n$. By linearity of the Hodge star, it is sufficient to calculate the Hodge star of the above equation:

$$\begin{aligned}
**(e^{i_1} \wedge \cdots \wedge e^{i_p}) &= \varepsilon_{i_1 \dots i_n} \eta^{i_1 i_1} \cdots \eta^{i_p i_p} \, *(e^{i_{p+1}} \wedge \cdots \wedge e^{i_n}) \\
&= \varepsilon_{i_1 \dots i_n} \eta^{i_1 i_1} \cdots \eta^{i_p i_p} \, (\varepsilon_{i_{p+1} \dots i_n i_1 \dots i_p} \eta^{i_{p+1} i_{p+1}} \cdots \eta^{i_n i_n} \, e^{i_1} \wedge \cdots \wedge e^{i_p}) \\
&= \varepsilon_{i_1 \dots i_n} (-1)^{(n-p)p} \varepsilon_{i_1 \dots i_n} \eta^{i_1 i_1} \cdots \eta^{i_n i_n} \, e^{i_1} \wedge \cdots \wedge e^{i_p} \quad = (-1)^{np-p+s} \, e^{i_1} \wedge \cdots \wedge e^{i_p}.
\end{aligned}$$

Therefore $* * \varphi = (-1)^{p(n-1)+s} \varphi$ for any p-form φ.

Problem 3.3

Derive the expression (3.20) of the Hodge star with respect to an arbitrary basis.

Solution 3.3

Consider the p-form

$$\varphi = \frac{1}{p!} \sum_{j_1, \dots, j_p = 1}^{n} \varphi_{j_1 \dots j_p} \, \beta^{j_1} \wedge \cdots \wedge \beta^{j_p} \quad \text{with} \quad \beta^j = \sum_i \gamma^j{}_i \, e^i,$$

or equivalently $e^k = \sum_i (\gamma^{-1})^k{}_i \, \beta^i$. We know that

$$g^{-1} = (\gamma^{-1\,T} \eta \, \gamma^{-1})^{-1} = \gamma \, \eta \, \gamma^T.$$

Therefore

$$(g^{-1} \gamma^{T\,-1})^{jk} = \sum_i g^{ji} \, (\gamma^{-1})^k{}_i = \underline{\underline{\sum_i \gamma^j{}_i \, \eta^{ik}}} \quad \text{and} \quad \det \gamma^{-1} = |\det g_{..}|^{1/2}.$$

We also need the following expression for the determinant of an $n \times n$ matrix M:

$$\underline{\sum_{k_1, \dots, k_n} \varepsilon_{k_1 \dots k_n} \, M^{k_1}{}_{i_1} \cdots M^{k_n}{}_{i_n}} = \varepsilon_{i_1 \dots i_n} \det M \tag{$\star$}$$

and

$$\begin{aligned}
*(e^{i_1} \wedge \cdots \wedge e^{i_p}) &= \frac{1}{(n-p)!} \sum_{i_{p+1}, \dots, i_n} \varepsilon_{i_1 \dots i_n} \eta^{i_1 i_1} \cdots \eta^{i_p i_p} \, e^{i_{p+1}} \wedge \cdots \wedge e^{i_n} \\
&= \frac{1}{(n-p)!} \sum_{k_1, \dots, k_n} \varepsilon_{k_1 \dots k_n} \eta^{i_1 k_1} \cdots \eta^{i_p k_p} \, e^{k_{p+1}} \wedge \cdots \wedge e^{k_n}.
\end{aligned}$$

Now

$$
\begin{aligned}
*\,\varphi = &\ \frac{1}{p!}\sum_{\substack{j_1,\dots,j_p,\\ i_1,\dots,i_p}} \varphi_{j_1\dots j_p}\,\gamma^{j_1}{}_{i_1}\cdots\gamma^{j_p}{}_{i_p}\ *(e^{i_1}\wedge\cdots\wedge e^{i_p}) \\
= &\ \frac{1}{p!\,(n-p)!}\sum_{\substack{j_1,\dots,j_p,\\ i_1,\dots,i_p,\\ k_1,\dots,k_n}} \varphi_{j_1\dots j_p}\,\underline{\underline{\gamma^{j_1}{}_{i_1}}}\cdots\gamma^{j_p}{}_{i_p}\,\varepsilon_{k_1\dots k_n}\,\underline{\underline{\eta^{i_1k_1}}}\cdots\eta^{i_pk_p}\,e^{k_{p+1}}\wedge\cdots\wedge e^{k_n} \\
= &\ \frac{1}{p!\,(n-p)!}\sum_{\substack{j_1,\dots,j_p,\\ i_1,\dots,i_n,\\ k_1,\dots,k_n}} \varphi_{j_1\dots j_p}\,g^{j_1i_1}(\gamma^{-1})^{k_1}{}_{i_1}\cdots g^{j_pi_p}(\gamma^{-1})^{k_p}{}_{i_p}\,\varepsilon_{k_1\dots k_n} \\
&\qquad\cdot(\gamma^{-1})^{k_{p+1}}{}_{i_{p+1}}\cdots(\gamma^{-1})^{k_n}{}_{i_n}\,\beta^{i_{p+1}}\wedge\cdots\wedge\beta^{i_n} \\
= &\ \frac{1}{p!\,(n-p)!}\sum_{\substack{j_1,\dots,j_p,\\ i_1,\dots,i_n}} \varphi_{j_1\dots j_p}\,g^{j_1i_1}\cdots g^{j_pi_p}\,\varepsilon_{i_1\dots i_n}\det(\gamma^{-1})\beta^{i_{p+1}}\wedge\cdots\wedge\beta^{i_n} \\
= &\ \frac{|\det g_{..}|^{1/2}}{p!\,(n-p)!}\sum_{\substack{j_1,\dots,j_p,\\ i_1,\dots,i_n}} \varphi_{j_1\dots j_p}\,g^{j_1i_1}\cdots g^{j_pi_p}\,\varepsilon_{i_1\dots i_n}\,\beta^{i_{p+1}}\wedge\cdots\wedge\beta^{i_n}.
\end{aligned}
$$

Problem 3.4
Verify the expression (3.47) of the Laplace operator applied to a 0-form.

Solution 3.4
We know: $g^{-1} = \gamma\,\eta\,\gamma^T$ and $\det g^{-1} = (\det\gamma)^2\,(-1)^s$ where s is the number of minus signs in the metric. Therefore $\underline{\underline{(-1)^s\det g^{-1}}} = |\det g^{-1}| = 1/|g|$. We also need,

$$
\frac{1}{(n-1)!}\sum_{i_2,\dots,i_n=1}^{n}\varepsilon_{k\,i_2\dots i_n}\,\varepsilon_{i_1i_2\dots i_n} = \delta^k{}_{i_1}.
$$

Then:

$$\begin{aligned}
\Delta f &= -\delta \mathrm{d} f = -(-1)^{1+s} * \mathrm{d} * \mathrm{d} f = (-1)^s * \mathrm{d} * \sum_j \frac{\partial f}{\partial x^j}\, \mathrm{d}x^j \\
&= (-1)^s * \mathrm{d} \sum_j \frac{|\det g_{..}|^{1/2}}{(n-1)!} \sum_{i_1 \dots i_n} \frac{\partial f}{\partial x^j}\, g^{j i_1}\, \varepsilon_{i_1 \dots i_n}\, \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_n} \\
&= \frac{(-1)^s}{(n-1)!} * \sum_{\substack{j,k,\\ i_1,\dots,i_n}} \frac{\partial}{\partial x^k} \left(|g|^{1/2} \frac{\partial f}{\partial x^j}\, g^{j i_1} \right) \varepsilon_{i_1 \dots i_n}\, \mathrm{d}x^k \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_n} \\
&= \frac{\underline{\underline{(-1)^s}}}{\underset{\sim\sim\sim}{(n-1)!}} \sum_{\substack{j,k,\\ i_1,\dots,i_n,\\ j_1,\dots,j_n}} |g|^{1/2} \frac{\partial}{\partial x^k} \left(|g|^{1/2} \frac{\partial f}{\partial x^j}\, g^{j i_1} \right) \underset{\sim\sim\sim}{\varepsilon_{i_1 \dots i_n}}\, \underbrace{g^{k j_1} g^{i_2 j_2} \dots g^{i_n j_n}\, \varepsilon_{j_1 \dots j_n}} \\
&= \frac{1}{|g|^{1/2}} \sum_{j,k} \frac{\partial}{\partial x^k} \left(|g|^{1/2} \frac{\partial f}{\partial x^j}\, g^{jk} \right),
\end{aligned}$$

where we have used equation $(\star)$ of Solution 3.3 to replace the underbraced expression (together with the summations over $j_1, \dots, j_n$) by $\underline{\underline{(-1)^s |g|^{-1}}} \underset{\sim\sim\sim}{\varepsilon^{k\, i_2 \dots i_n}}$.

4 Gauge theories

Problem 4.1

Show that Maxwell's equations in terms of differential forms are given by (4.9) and (4.10).

Solution 4.1

The equation $\mathrm{d}F = 0$ means:

$$\begin{aligned}
0 &= \mathrm{d} \left(-\sum_{i=1}^{3} E_i \mathrm{d}x^0 \wedge \mathrm{d}x^i + \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} B_i \mathrm{d}x^j \wedge \mathrm{d}x^k \right) \\
&= -\sum_{i,j=1}^{3} \frac{\partial E_i}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^i \\
&\quad + \frac{1}{2} \sum_{i,j,k,l=1}^{3} \varepsilon_{ijk} \frac{\partial B_i}{\partial x^l} \mathrm{d}x^l \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k + \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \frac{\partial B_i}{\partial x^0} \mathrm{d}x^0 \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k .
\end{aligned}$$

This implies

$$-\sum_{j,k=1}^{3} \frac{\partial E_k}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^k + \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \frac{\partial B_i}{\partial x^0} \mathrm{d}x^0 \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k = 0$$

and

$$\sum_{i,j,k,l=1}^{3} \varepsilon_{ijk} \frac{\partial B_i}{\partial x^l} \mathrm{d}x^l \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k = 0\,.$$

From the first of these two equations we get successively

$$\frac{1}{2} \sum_{j,k=1}^{3} \left(\frac{\partial E_k}{\partial x^j} - \frac{\partial E_j}{\partial x^k} \right) \mathrm{d}x^0 \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k + \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \frac{\partial B_i}{\partial x^0} \mathrm{d}x^0 \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k = 0\,,$$

$$\frac{1}{2} \left(\frac{\partial E_k}{\partial x^j} - \frac{\partial E_j}{\partial x^k} \right) + \frac{1}{2} \sum_{i=1}^{3} \varepsilon_{ijk} \frac{\partial B_i}{\partial x^0} = 0\,,$$

$$\sum_{j,k=1}^{3} \varepsilon_{ljk} \left(\frac{\partial E_k}{\partial x^j} - \frac{\partial E_j}{\partial x^k} \right) + \sum_{i,j,k=1}^{3} \varepsilon_{ljk} \varepsilon_{ijk} \frac{\partial B_i}{\partial x^0} = 0\,,$$

$$2(\mathrm{curl} E)_l + 2 \sum_{i=1}^{3} \delta_{il} \frac{\partial B_i}{\partial x^0} = 0\,,$$

and finally

$$\mathrm{curl} E + \frac{\partial B}{\partial t} = 0\,.$$

Since

$$\mathrm{d}x^l \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k = \varepsilon_{ljk} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,,$$

the second equation implies

$$0 = \sum_{i,j,k,l=1}^{3} \varepsilon_{ijk} \varepsilon_{ljk} \frac{\partial B_i}{\partial x^l} = \sum_{i,l=1}^{3} 2\delta_{il} \frac{\partial B_i}{\partial x^l}$$

and consequently

$$\mathrm{div} B = 0\,.$$

Consider now $\mathrm{d}G = j$. We have

$$\begin{aligned} \mathrm{d}G = \sum_{i,j=1}^{3} \frac{\partial H_i}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^0 \wedge \mathrm{d}x^i + \frac{1}{2} \sum_{i,j,k,l=1}^{3} \varepsilon_{ijk} \frac{\partial D_i}{\partial x^l} \mathrm{d}x^l \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \\ + \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \frac{\partial D_i}{\partial x^0} \mathrm{d}x^0 \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \\ = -\sum_{j,k=1}^{3} \frac{\partial H_k}{\partial x^j} \mathrm{d}x^0 \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k + \frac{1}{2} \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \frac{\partial D_i}{\partial x^0} \mathrm{d}x^0 \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \\ + \frac{1}{2} \sum_{i,j,k,l=1}^{3} \varepsilon_{ijk} \varepsilon_{ljk} \frac{\partial D_i}{\partial x^l} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \end{aligned}$$

and

$$j = \frac{1}{3!}\sum_{i,j,k=1}^{3}\Big(j^0\varepsilon_{ijk}\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k - j^i\varepsilon_{ijk}\mathrm{d}x^0\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k + j^i\varepsilon_{ijk}\mathrm{d}x^j\wedge\mathrm{d}x^0\wedge\mathrm{d}x^k - j^i\varepsilon_{ijk}\mathrm{d}x^j\wedge\mathrm{d}x^k\wedge\mathrm{d}x^0\Big)$$

$$= \rho\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 - \frac{1}{2}\sum_{i,j,k=1}^{3} j^i\varepsilon_{ijk}\mathrm{d}x^0\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k\,.$$

Now the equation $\mathrm{d}G = j$ implies

$$\frac{1}{2}\sum_{i,l=1}^{3} 2\delta_{il}\frac{\partial D_i}{\partial x^l} = \rho\,,$$

i.e.,

$$\mathrm{div}D = \rho\,,$$

and

$$\frac{1}{2}\left(\frac{\partial H_j}{\partial x^k} - \frac{\partial H_k}{\partial x^j}\right) + \frac{1}{2}\sum_{i=1}^{3}\varepsilon_{ijk}\frac{\partial D_i}{\partial x^0} = -\frac{1}{2}\sum_{i=1}^{3}\varepsilon_{ijk}j^i\,,$$

$$\sum_{j,k=1}^{3}\varepsilon_{ljk}\left(\frac{\partial H_j}{\partial x^k} - \frac{\partial H_k}{\partial x^j}\right) + \sum_{i,j,k=1}^{3}\varepsilon_{ljk}\varepsilon_{ijk}\frac{\partial D_i}{\partial x^0} = -\sum_{i,j,k=1}^{3}\varepsilon_{ljk}\varepsilon_{ijk}j^i\,,$$

$$-2\,(\mathrm{curl}H)_l + 2\sum_{i=1}^{3}\delta_{il}\frac{\partial D_i}{\partial x^0} = -2\sum_{i=1}^{3}\delta_{il}j^i\,,$$

i.e.,

$$\mathrm{curl}H - \frac{\partial D}{\partial t} = j\,.$$

Problem 4.2
Verify the transformation laws (4.63) and (4.69).

Solution 4.2
For the field strength 2-form we write

$$\begin{aligned}
F' &= \mathrm{d}A' + A'\wedge A'\\
&= (\mathrm{d}\gamma)\wedge A\gamma^{-1} + \gamma(\mathrm{d}A)\gamma^{-1} - \gamma A\wedge\mathrm{d}\gamma^{-1} + (\mathrm{d}\gamma)\wedge\mathrm{d}\gamma^{-1} + \gamma A\gamma^{-1}\wedge\gamma A\gamma^{-1}\\
&\quad + \gamma A\gamma^{-1}\wedge\gamma\mathrm{d}\gamma^{-1} + \gamma(\mathrm{d}\gamma^{-1})\wedge\gamma A\gamma^{-1} + \gamma(\mathrm{d}\gamma^{-1})\wedge\gamma\mathrm{d}\gamma^{-1}\\
&= (\mathrm{d}\gamma)\wedge A\gamma^{-1} + \gamma(\mathrm{d}A)\gamma^{-1} - \gamma A\wedge\mathrm{d}\gamma^{-1} - \gamma(\mathrm{d}\gamma^{-1})\gamma\wedge\mathrm{d}\gamma^{-1} + \gamma A\wedge A\gamma^{-1}\\
&\quad + \gamma A\wedge\mathrm{d}\gamma^{-1} - (\mathrm{d}\gamma)\wedge A\gamma^{-1} + \gamma(\mathrm{d}\gamma^{-1})\gamma\wedge\mathrm{d}\gamma^{-1}\\
&= \gamma(\mathrm{d}A)\gamma^{-1} + \gamma A\wedge A\gamma^{-1} = \gamma F\gamma^{-1}\,.
\end{aligned}$$

For the V-valued 0-form Φ we have

$$\begin{aligned}(\mathrm{D}\Phi)' &= \mathrm{d}\Phi' + A' \wedge \Phi' = \mathrm{d}(\gamma\Phi) + \left(\gamma A\gamma^{-1} - (\mathrm{d}\gamma)\gamma^{-1}\right) \wedge \gamma\Phi \\ &= (\mathrm{d}\gamma) \wedge \Phi + \gamma \mathrm{d}\Phi + \gamma A \wedge \Phi - (\mathrm{d}\gamma) \wedge \Phi = \gamma\left(\mathrm{d}\Phi + A \wedge \Phi\right) = \gamma \mathrm{D}\Phi\,.\end{aligned}$$

<u>Problem 4.3</u>
Prove the Leibniz rule (4.70).

<u>Solution 4.3</u>

$$\begin{aligned}\mathrm{D}(\psi \wedge \Phi) &= \mathrm{d}(\psi \wedge \Phi) + A \wedge \psi \wedge \Phi = (\mathrm{d}\psi) \wedge \Phi + (-1)^q \psi \wedge \mathrm{d}\Phi + (-1)^q \psi \wedge A \wedge \Phi \\ &= (\mathrm{d}\psi) \wedge \Phi + (-1)^q \psi \wedge (\mathrm{d}\Phi + A \wedge \Phi) = (\mathrm{d}\psi) \wedge \Phi + (-1)^q \psi \wedge \mathrm{D}\Phi\,.\end{aligned}$$

<u>Problem 4.4</u>
Show that the action (4.75) is gauge invariant.

<u>Solution 4.4</u>
Consider the first term:

$$\mathrm{tr}\left(F' \wedge *F'\right) = \mathrm{tr}\left(\gamma F \gamma^{-1} \wedge *\gamma F \gamma^{-1}\right) = \mathrm{tr}\left(F \wedge *F\right)\,.$$

Recalling that $\gamma(x)$ is unitary, one finds in the case of the second term:

$$\left((\mathrm{D}\Phi)'\right)^{+} \wedge *(\mathrm{D}\Phi)' = (\gamma \mathrm{D}\Phi)^{+} \wedge *\gamma \mathrm{D}\Phi = (\mathrm{D}\Phi)^{+}\gamma^{+}\gamma \wedge *\mathrm{D}\Phi = (\mathrm{D}\Phi)^{+} \wedge *\mathrm{D}\Phi\,.$$

Consider finally the third term:

$$\Phi'^{+} \wedge *\Phi' = \Phi^{+}\gamma^{+} \wedge *\gamma\Phi = \Phi^{+}\gamma^{+}\gamma \wedge *\Phi = \Phi^{+} \wedge *\Phi\,.$$

<u>Problem 4.5</u>
Derive the field equations (4.84), (4.85) from the action (4.75).

<u>Solution 4.5</u>
Writing the variation of Φ as $\Phi \to \Phi + \varphi$ one gets

$$\begin{aligned}S &= \frac{1}{g^2}\int \mathrm{tr}(F \wedge *F) + \int (\mathrm{D}\Phi)^{+} \wedge *\mathrm{D}\Phi - m^2 \int \Phi^{+} \wedge *\Phi \\ &\to S + \int (\mathrm{D}\Phi)^{+} \wedge *\mathrm{D}\varphi + \int (\mathrm{D}\varphi)^{+} \wedge *\mathrm{D}\Phi - m^2 \int \varphi^{+} \wedge *\Phi - m^2 \int \Phi^{+} \wedge *\varphi\,.\end{aligned}$$

Ignoring boundary terms one finds with the help of Stokes' theorem the following formula, where φ is a 0-form:

$$\begin{aligned}\int (\mathrm{D}\varphi)^{+} \wedge \Phi &= \int \left(\mathrm{d}\varphi^{+}\right) \wedge \Phi + \int (A \wedge \varphi)^{+} \wedge \Phi = -\int \varphi^{+} \wedge \mathrm{d}\Phi + \int \varphi^{+} A^{+} \wedge \Phi \\ &= -\int \varphi^{+} \wedge \mathrm{d}\Phi - \int \varphi^{+} A \wedge \Phi = -\int \varphi^{+} \wedge \mathrm{D}\Phi\,.\end{aligned}$$

Consequently we have

$$\int (\mathrm{D}\varphi)^+ \wedge *\mathrm{D}\Phi = -\int \varphi^+ \wedge \mathrm{D} * \mathrm{D}\Phi \,.$$

Due to (3.45) we have

$$\int \Phi^+ \wedge *\varphi = \int \varphi^{\mathrm{t}} \wedge *\overline{\Phi} \,.$$

and

$$\begin{aligned}\int (\mathrm{D}\Phi)^+ \wedge *\mathrm{D}\varphi &= \int (\mathrm{D}\varphi)^{\mathrm{t}} \wedge *\overline{(\mathrm{D}\Phi)} = \overline{\int (\mathrm{D}\varphi)^+ \wedge *\mathrm{D}\Phi} = -\overline{\int \varphi^+ \wedge \mathrm{D} * \mathrm{D}\Phi} \\ &= -\int \varphi^{\mathrm{t}} \wedge \overline{\mathrm{D} * \mathrm{D}\Phi} \,.\end{aligned}$$

Therefore the variation of the action can be written as

$$\begin{aligned}S \to S &- \int \varphi^+ \wedge \big(\mathrm{D} * \mathrm{D}\Phi + m^2 * \Phi\big) - \int \varphi^{\mathrm{t}} \wedge \overline{(\mathrm{D} * \mathrm{D}\Phi + m^2 * \Phi)} \\ &= S + \int \varphi^+ \wedge * \big(*\mathrm{D} * \mathrm{D}\Phi - m^2\Phi\big) + \int \varphi^{\mathrm{t}} \wedge *\overline{(*\mathrm{D} * \mathrm{D}\Phi - m^2\Phi)} \,,\end{aligned}$$

and the field equation (4.85) follows.

Consider now the variation of A, written as $A \to A + a$ with $a = \omega T_b$, where ω is a real valued 1-form. We get

$$F = \mathrm{d}A + A \wedge A \to F + \mathrm{d}a + a \wedge A + A \wedge a \,,$$

$$\mathrm{D}\Phi = \mathrm{d}\Phi + A \wedge \Phi \to \mathrm{D}\Phi + a \wedge \Phi \,,$$

and consequently

$$\begin{aligned}S \to S &+ \frac{1}{g^2}\int \mathrm{tr}\big((\mathrm{d}a + a \wedge A + A \wedge a) \wedge *F\big) + \frac{1}{g^2}\int \mathrm{tr}\big(F \wedge *(\mathrm{d}a + a \wedge A + A \wedge a)\big) \\ &+ \int (a \wedge \Phi)^+ \wedge *\mathrm{D}\Phi + \int (\mathrm{D}\Phi)^+ \wedge *(a \wedge \Phi) \,.\end{aligned}$$

With the help of (3.45) and Stokes' theorem we find

$$
\begin{aligned}
S \to S &+ \frac{2}{g^2}\int \mathrm{tr}\big((\mathrm{d}a + a\wedge A + A\wedge a)\wedge *F\big) + \int \Phi^+ \wedge a^+ \wedge *\mathrm{D}\Phi + \int (\mathrm{D}\Phi)^+ \wedge *(a\wedge\Phi) \\
= S &+ \frac{2}{g^2}\int \mathrm{tr}\big((\mathrm{d}a)\wedge *F + a\wedge A\wedge *F - a\wedge (*F)\wedge A\big) \\
&\qquad - \int \Phi^+ \wedge a \wedge *\mathrm{D}\Phi + \int (\mathrm{D}\Phi)^+ \wedge *(a\wedge\Phi) \\
= S &+ \frac{2}{g^2}\int \mathrm{tr}\big(a\wedge \mathrm{d} * F + a\wedge A\wedge *F - a\wedge (*F)\wedge A\big) \\
&\qquad - \int \Phi^+ a \wedge *\mathrm{D}\Phi + \int (\mathrm{D}\Phi)^+ \wedge *(a\wedge\Phi) \\
= S &+ \frac{2}{g^2}\int \omega\wedge \mathrm{tr}\Big(T_b\big(\mathrm{d} * F + A\wedge *F - (*F)\wedge A\big)\Big) \\
&\qquad - \int \omega\wedge \Phi^+ T_b(*\mathrm{D}\Phi) + \int (\mathrm{D}\Phi)^+ \wedge (*\omega) T_b\Phi \\
= S &+ \frac{2}{g^2}\int \omega\wedge \mathrm{tr}\Big(T_b\big(\mathrm{d} * F + A\wedge *F - (*F)\wedge A\big)\Big) \\
&\qquad - \int \omega\wedge \Phi^+ T_b(*\mathrm{D}\Phi) + \int \omega \wedge \big(* (\mathrm{D}\Phi)^+\big) T_b\Phi \\
= S &+ \int \omega\wedge \left\{\frac{2}{g^2}\mathrm{tr}\big(T_b(\mathrm{d} * F + [A, *F])\big) - \Phi^+ T_b(*\mathrm{D}\Phi) + *(\mathrm{D}\Phi)^+ T_b\Phi\right\} \\
= S &+ \int \omega\wedge * * \left\{\frac{2}{g^2}\mathrm{tr}(T_b \mathrm{D} * F) - \Phi^+ T_b(*\mathrm{D}\Phi) + *(\mathrm{D}\Phi)^+ T_b\Phi\right\}\,.
\end{aligned}
$$

In this way we obtain the field equations

$$
\frac{2}{g^2}\mathrm{tr}(T_b \mathrm{D} * F) = \Phi^+ T_b(*\mathrm{D}\Phi) - *(\mathrm{D}\Phi)^+ T_b\Phi\,,
$$

from which (4.84) follows.

5 Einstein-Cartan theory

Problem 5.1
Derive the metric condition (5.23).

Solution 5.1
Let c be a curve in $\mathscr{U}$ with starting point x_0,

$$
\begin{aligned}
c : [0,1] &\to \mathscr{U} \\
\tau &\mapsto c(\tau)\,, \quad c(0) = x_0.
\end{aligned}
$$

The pull back $c^*\Gamma$ of the connection 1-form Γ reads in coordinates:

$$c^*(\Gamma_\mu \, \mathrm{d}x^\mu) = \Gamma_\mu \, \frac{\mathrm{d}c^\mu}{\mathrm{d}\tau} \, \mathrm{d}\tau.$$

Let v_0 be a tangent vector at x_0, $v_0 \in T_{x_0}\mathscr{U}$. We define the "parallel transport of v_0 along c by means of Γ" to be the one-parameter family of tangent vectors $v_\tau \in T_{c(\tau)}\mathscr{U}$ whose components with respect to the frame β^a,

$$v^a(\tau) := \beta^a|_{c(\tau)}(v_\tau),$$

solve the four linear first-order differential equations

$$\frac{\mathrm{d}v^a}{\mathrm{d}\tau} + (c^*\Gamma)^a{}_b v^b = 0 \quad \text{with initial conditions} \quad v^a(0) = \beta^a|_{x_0}(v_0)\,,$$

or in matrix-vector notation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} v(\tau) + (c^*\Gamma(\tau))v(\tau) = 0\,.$$

Let $w_0 \in T_{x_0}\mathscr{U}$ be a second tangent vector at x_0. The scalar product of the two parallel transports v_τ and w_τ is preserved if and only if

$$\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\, g_{c(\tau)}(v_\tau, w_\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\, \left[v^T(\tau)\, g(c(\tau))\, w(\tau)\right] \\
&= \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\, v^T(\tau)\right] g(c(\tau))\, w(\tau) + v^T(\tau) \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\, g(c(\tau))\right] w(\tau) + v^T(\tau)\, g(c(\tau))\, \frac{\mathrm{d}}{\mathrm{d}\tau}\, w(\tau) \\
&= v^T(\tau) \left[-(c^*\Gamma(\tau))^T g(c(\tau)) + \frac{\mathrm{d}}{\mathrm{d}\tau}\, g(c(\tau)) - g(c(\tau))\, c^*\Gamma(\tau)\right] w(\tau).
\end{aligned}$$

Since v_0, w_0 and c are arbitrary, this identity is equivalent to

$$-\Gamma^T g + \mathrm{d}g - g\Gamma =: \mathrm{D}g = 0.$$

<u>Problem 5.2</u>
Show that the connection (5.51) is metric and torsion-free.

<u>Solution 5.2</u>

$$\begin{aligned}
\Gamma^i{}_j = {}& \tfrac{1}{2} \left[C^i{}_{jk} - g^{ii'} g_{jj'} C^{j'}{}_{i'k} - g^{ii'} g_{kk'} C^{k'}{}_{i'j}\right] \beta^k \\
&+ \tfrac{1}{2} g^{ii'} \left[(\mathrm{d}g_{ji'})(b_k) + (\mathrm{d}g_{i'k})(b_j) - (\mathrm{d}g_{kj})(b_{i'})\right] \beta^k.
\end{aligned}$$

Note the last minus sign which was missing from equation (5.51) in the hard-cover edition.
– Metricity:

$$(\mathrm{D}g)_{ab} = \mathrm{d}g_{ab} - \Gamma^i{}_a g_{ib} - g_{ai}\Gamma^i{}_b =: \tfrac{1}{2}\mathrm{d}g_{ab} - \Gamma^i{}_a g_{ib} + ({}_a \leftrightarrow {}_b)$$

$$= \tfrac{1}{2}\mathrm{d}g_{ab} - \tfrac{1}{2}\left[C^i{}_{ak} - \underset{\sim\sim}{g^{ii'}}\, g_{aa'}C^{a'}{}_{i'k} - \underset{\sim\sim}{g^{ii'}}\, g_{kk'}C^{k'}{}_{i'a}\right] \underset{\sim}{g_{ib}}\,\beta^k$$

$$- \tfrac{1}{2}\underset{\sim\sim}{g^{ii'}}\left[(\mathrm{d}g_{ai'})(b_k) + (\mathrm{d}g_{i'k})(b_a) - (\mathrm{d}g_{ka})(b_{i'})\right] \underset{\sim}{g_{ib}}\,\beta^k + ({}_a \leftrightarrow {}_b)$$

$$= \cancel{\tfrac{1}{2}\mathrm{d}g_{ab}} - \tfrac{1}{2}\left(\left\{g_{bi}C^i{}_{ak} - g_{aa'}C^{a'}{}_{bk}\right\} - g_{kk'}C^{k'}{}_{ba}\right)\beta^k$$

$$\cancel{-\tfrac{1}{2}\mathrm{d}g_{ab}} - \tfrac{1}{2}\left[(\mathrm{d}g_{bk})(b_a) - (\mathrm{d}g_{ak})(b_b)\right]\beta^k + ({}_a \leftrightarrow {}_b) = 0,$$

because the three remaining terms, $\{\ldots\}$, $C^{k'}{}_{ba}$, $[\ldots]$ are antisymmetric under exchange of the indices a and b.
– Torsion vanishes:

$$T^i = \mathrm{d}\beta^i + \Gamma^i{}_{jk}\beta^k \wedge \beta^j$$

$$= \cancel{\tfrac{1}{2}C^i{}_{jk}\beta^j \wedge \beta^k} + \tfrac{1}{2}\left[\cancel{C^i{}_{jk}} - \left\{g^{ii'}g_{jj'}C^{j'}{}_{i'k} + g^{ii'}g_{kk'}C^{k'}{}_{i'j}\right\}\right]\beta^k \wedge \beta^j$$

$$+ \tfrac{1}{2}g^{ii'}\left[\{(\mathrm{d}g_{ji'})(b_k) + (\mathrm{d}g_{i'k})(b_j)\} - (\mathrm{d}g_{kj})(b_{i'})\right]\beta^k \wedge \beta^j = 0,$$

because the three remaining terms, $\{\ldots\}$, $\{\ldots\}$, $\mathrm{d}g_{kj}$ are symmetric under exchange of the indices k and j.

Problem 5.3
Consider a piece of the pseudosphere, that is an open subset of $\mathbb{R}^2$ endowed with the metric whose matrix with respect to a holonomic frame $\mathrm{d}u,\ \mathrm{d}v$ is

$$(g^{ij}) = \begin{pmatrix} \mathrm{e}^v & 0 \\ 0 & 1 \end{pmatrix}.$$

Calculate the Riemannian connection, curvature and curvature scalar.

Solution 5.3

$$
\begin{aligned}
e^1 &= e^{-v/2}\mathrm{d}u, \quad e^2 = \mathrm{d}v,\\
\mathrm{d}e^1 &= -\tfrac{1}{2}e^{-v/2}\mathrm{d}v\wedge\mathrm{d}u = \tfrac{1}{2}e^1\wedge e^2, \quad \mathrm{d}e^2 = 0,\\
C^1{}_{12} &= -C^1{}_{21} = \tfrac{1}{2}, \quad \text{(all other } C\text{'s vanish)}\\
\omega^i{}_{jk} &= \tfrac{1}{2}\left[C^i{}_{jk} - C^j{}_{ik} - C^k{}_{ij}\right],\\
\omega^1{}_{21} &= -\tfrac{1}{2}, \quad \omega^1{}_{22} = 0,\\
\omega^1{}_2 &= -\omega^2{}_1 = -\tfrac{1}{2}e^1 = -\tfrac{1}{2}e^{-v/2}\mathrm{d}u,\\
R &= \mathrm{d}\omega + \cancel{\tfrac{1}{2}[\omega,\omega]}, \quad R^1{}_2 = \tfrac{1}{4}e^{-v/2}\mathrm{d}v\wedge\mathrm{d}u = -\tfrac{1}{4}e^1\wedge e^2,\\
R^1{}_{212} &= R^2{}_{121} = -\tfrac{1}{4}, \qquad R^{ab}{}_{ab} = -\tfrac{1}{2}.
\end{aligned}
$$

Problem 5.4
Prove (5.78).

Solution 5.4
To show:

$$\varepsilon_{adrs}\varepsilon^{bkrs} = -2\left(\delta^b{}_a\delta^k{}_d - \delta^k{}_a\delta^b{}_d\right) \quad \text{with} \quad \varepsilon_{0123} = 1.$$

Our metric has signature $+---$, therefore $\varepsilon^{0123} = -1$. For a and d given, the left-hand side is antisymmetric in b and k, and different from zero if and only if $a \neq d$ and (b,k) is a permutation of (a,d). Hence it is sufficient to consider the case $b = a$ and $k = d$ where

$$\sum_{r,s}\varepsilon_{adrs}\varepsilon^{bkrs} = 2\sum_{r<s}\varepsilon_{adrs}\varepsilon^{bkrs} = 2\varepsilon_{0123}\varepsilon^{0123} = -2.$$

Problem 5.5
Give explicit expressions of the metric and connection for Cartan's example, fig. 5.2. Calculate the curvature and the torsion.

Solution 5.5
For notational ease we denote the Cartesian coordinates by both (x^1, x^2, x^3) and (x, y, z). With respect to the frame $\beta^a = \mathrm{d}x^a$ the metric tensor is

$$(g_{ab}) = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

The equation of parallel transport along a curve $c(\tau)$ is (cf. (5.21))

$$\frac{\mathrm{d}v^a}{\mathrm{d}\tau} + \Gamma^a{}_{b\mu}\frac{\mathrm{d}c^\mu}{\mathrm{d}\tau}v^b = 0.$$

In the x- and y-directions,

$$c(\tau)=\begin{pmatrix}x_0+\tau\\ y_0\\ z_0\end{pmatrix}\quad\text{and}\quad c(\tau)=\begin{pmatrix}x_0\\ y_0+\tau\\ z_0\end{pmatrix},$$

the parallel transports are ordinary translations: $\mathrm{d}v^a/\mathrm{d}\tau=0$ and therefore $\Gamma_1=0$ and $\Gamma_2=0$. When transported in the z-direction,

$$c(\tau)=\begin{pmatrix}x_0\\ y_0\\ z_0+\tau\end{pmatrix},$$

any initial vector v_0 turns clockwise around the z-direction:

$$v(\tau)=\begin{pmatrix}\cos(\omega\tau) & \sin(\omega\tau) & 0\\ -\sin(\omega\tau) & \cos(\omega\tau) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}v_{0x}\\ v_{0y}\\ v_{0z}\end{pmatrix},\quad \omega>0,$$

Inserting these expressions for $c(\tau)$ and $v(\tau)$ into the equation of parallel transport above we obtain a vector equation of the form $M(\tau)v_0+N(\tau)v_0=0$ with two explicit matrices M and N. Since the initial vector v_0 is arbitrary we have the matrix equation $M(\tau)+N(\tau)=0$ or

$$\Gamma_3=\omega\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad\text{and altogether}\quad \Gamma=\omega\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\mathrm{d}z.$$

This connection is metric,

$$\mathrm{D}g=\mathrm{d}g-\Gamma^Tg-g\Gamma=0,\quad\text{and flat,}\quad R=\mathrm{d}\Gamma+\tfrac{1}{2}[\Gamma,\Gamma]=0,$$

and has non-vanishing torsion:

$$\begin{aligned}T^a=&\mathrm{d}\mathrm{d}x^a+\Gamma^a{}_b\wedge\mathrm{d}x^b=\Gamma^a{}_{b3}\,\mathrm{d}z\wedge\mathrm{d}x^b,\\ T=&\omega\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\mathrm{d}z\wedge\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\\ \mathrm{d}z\end{pmatrix}=\omega\begin{pmatrix}\mathrm{d}y\wedge\mathrm{d}z\\ \mathrm{d}z\wedge\mathrm{d}x\\ 0\end{pmatrix}.\end{aligned}$$

As a bonus we indicate the “geodesic coordinates (X,Y,Z) centered at (x_0,y_0,z_0)”. In general they are defined by the unique solution of the geodesic equation

$$\frac{\mathrm{d}^2Q^\mu}{\mathrm{d}\tau^2}+\Gamma^\mu{}_{\nu\alpha}\frac{\mathrm{d}Q^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}Q^\alpha}{\mathrm{d}\tau}=0,\quad\text{with initial conditions}\quad Q^\mu(0)=\begin{pmatrix}x_0\\ y_0\\ z_0\end{pmatrix},\ \frac{\mathrm{d}Q^\mu}{\mathrm{d}\tau}(0)=\begin{pmatrix}X\\ Y\\ Z\end{pmatrix},$$

and then by setting $x^\mu(X,Y,Z):=Q^\mu(1)$. In our case we have

$$Q^\mu(\tau)=\begin{pmatrix}\{X\sin(\omega Z\tau)\ -Y\,[1-\cos(\omega Z\tau)]\}\,/(\omega Z)\ +x_0\\ \{Y\sin(\omega Z\tau)\ +X\,[1-\cos(\omega Z\tau)]\}\,/(\omega Z)\ +y_0\\ Z\tau+z_0\end{pmatrix}$$

and

$$x^\mu(X,Y,Z) = \begin{pmatrix} \{X\sin(\omega Z)\ - Y\,[1-\cos(\omega Z)]\}\}\,/(\omega Z)\ + x_0 \\ \{Y\sin(\omega Z)\ + X\,[1-\cos(\omega Z)]\}\}\,/(\omega Z)\ + y_0 \\ Z + z_0 \end{pmatrix}.$$

6 The Lie derivative

Problem 6.1
Verify the Jacobi identity (6.20) for the Lie bracket.

Solution 6.1
To show:

$$[v,[u,w]] = [[v,u],w] + [u,[v,w]].$$

We use the linearity and the Leibniz rule of partial derivatives as well as the short hand $\partial_i := \partial/\partial x^i$ and the summation convention. Let us start with the left-hand side of the Jacobi identity:

$$\begin{aligned} \text{lhs} =& [v, (u^i\partial_i w^j - w^i\partial_i u^j)\partial_j] \\ =& [v^k\partial_k(u^i\partial_i w^j) - v^k\partial_k(w^i\partial_i u^j) - u^i(\partial_i w^k)\partial_k v^j + w^i(\partial_i u^k)\partial_k v^j]\partial_j \\ =& [\underline{v^k(\partial_k u^i)\partial_i w^j} + \underline{\underline{v^k u^i\partial_k\partial_i w^j}} - \underset{\sim\sim\sim\sim\sim}{v^k(\partial_k w^i)\partial_i u^j} - \underset{-\,-\,-\,-}{v^k w^i\partial_k\partial_i u^j} \\ & - \underset{\cdots\cdots}{u^i(\partial_i w^k)\partial_k v^j} + w^i(\partial_i u^k)\partial_k v^j]\partial_j, \end{aligned}$$

$$\begin{aligned} \text{rhs} =& [(v^k\partial_k u^i - u^k\partial_k v^i)\partial_i, w] + [u, (v^k\partial_k w^j - w^k\partial_k v^j)\partial_j] \\ =& [v^k(\partial_k u^i)\partial_i w^j - u^k(\partial_k v^i)\partial_i w^j - w^i\partial_i(v^k\partial_k u^j) + w^i\partial_i(u^k\partial_k v^j) \\ & + u^i\partial_i(v^k\partial_k w^j) - u^i\partial_i(w^k\partial_k v^j) - v^k(\partial_k w^i)\partial_i u^j + w^k(\partial_k v^i)\partial_i u^j]\partial_j \\ =& [\underline{v^k(\partial_k u^i)\partial_i w^j} - \cancel{u^k(\partial_k v^i)\partial_i w^j} - \cancel{w^i(\partial_i v^k)\partial_k u^j} - \underset{-\,-\,-\,-}{w^i v^k\partial_i\partial_k u^j} \\ & + w^i(\partial_i u^k)\partial_k v^j + w^i\cancel{u^k}\cancel{\partial_i}\cancel{\partial_k}v^j + \cancel{u^i(\partial_i v^k)\partial_k w^j} + \underline{\underline{u^i v^k\partial_i\partial_k w^j}} \\ & - \underset{\cdots\cdots}{u^i(\partial_i w^k)\partial_k v^j} - u^i\cancel{w^k}\cancel{\partial_i}\cancel{\partial_k}v^j - \underset{\sim\sim\sim\sim\sim}{v^k(\partial_k w^i)\partial_i u^j} + \cancel{w^k(\partial_k v^i)\partial_i u^j}]\partial_j. \end{aligned}$$

Problem 6.2
Prove Cartan's identity (6.34).

Solution 6.2
Our task is to prove

$$i_v\mathrm{d} + \mathrm{d}i_v = L_v.$$

Applied to a 0-form f this equation reduces to $i_v \mathrm{d}f = L_v f$, which we have proved in (6.15). Applied to an exact 1-form $\mathrm{d}f$ our equation becomes

$$i_v \mathrm{d}\mathrm{d}f + \mathrm{d}i_v \mathrm{d}f = \mathrm{d}L_v f = L_v \mathrm{d}f.$$

The last equality holds because Lie derivative and exterior derivative commute, (6.14). According to the general rule (6.27), $i_v \mathrm{d} + \mathrm{d} i_v$ is a derivation because interior and exterior derivatives are derivations. By (6.13) the Lie derivative itself is a derivation. Therefore Cartan's identity is correct for any differential form, since any form can be written as a sum of products of 0-forms and exact 1-forms.

Problem 6.3

Consider a piece of the 2-sphere with its standard metric (5.55). Show that the following three vector fields

$$\frac{\partial}{\partial\varphi}, \quad \cos\varphi\,\frac{\cos\theta}{\sin\theta}\,\frac{\partial}{\partial\varphi} + \sin\varphi\frac{\partial}{\partial\theta}, \quad -\sin\varphi\,\frac{\cos\theta}{\sin\theta}\,\frac{\partial}{\partial\varphi} + \cos\varphi\frac{\partial}{\partial\theta}$$

are Killing vectors. What Lie algebra do they generate?

Solution 6.3

A vector field $v = v^\alpha\, \partial/\partial x^\alpha =: v^\alpha \partial_\alpha$ is a Killing vector if and only if

$$L_v g(u, w) = g(L_v u, w) + g(u, L_v w)$$

for any two vector fields u and w. With $u = \partial_\mu$ and $w = \partial_\nu$ we obtain:

$$L_v g_{\mu\nu} = v^\alpha \partial_\alpha g_{\mu\nu} = g(-(\partial_\mu v^\alpha)\partial_\alpha, \partial_\nu) + g(\partial_\mu, -(\partial_\nu v^\alpha)\partial_\alpha) = -(\partial_\mu v^\alpha) g_{\alpha\nu} - (\partial_\nu v^\alpha) g_{\mu\alpha}\,.$$

Therefore v is a Killing vector if and only if for all μ and ν

$$v^\alpha \partial_\alpha g_{\mu\nu} + (\partial_\mu v^\alpha) g_{\alpha\nu} + (\partial_\nu v^\alpha) g_{\mu\alpha} = 0. \qquad (\star)$$

holds. In our case with $x^1 = \varphi$, $x^2 = \theta$, the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} \sin^2\theta & 0 \\ 0 & 1 \end{pmatrix}.$$

The Killing equation $(\star)$ then reads for

$$\begin{aligned} \mu = \nu = 1: &\qquad 2\sin\theta\,\cos\theta\, v^2 + 2\sin^2\theta\,\partial_\varphi v^1 = 0, \\ \mu = \nu = 2: &\qquad 2\,\partial_\theta v^2 = 0, \\ \mu = 1,\ \nu = 2: &\qquad \partial_\varphi v^2 + \sin^2\theta\,\partial_\theta v^1 = 0, \end{aligned}$$

with the following three linearly independent solutions:

$$v_{(1)} = \begin{pmatrix} v^1_{(1)} \\ v^2_{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \partial_\varphi, \quad v_{(2)} = \begin{pmatrix} +\cos\varphi\,\frac{\cos\theta}{\sin\theta} \\ \sin\varphi \end{pmatrix} = +\cos\varphi\,\frac{\cos\theta}{\sin\theta}\,\partial_\varphi + \sin\varphi\,\partial_\theta,$$

$$v_{(3)} = \begin{pmatrix} -\sin\varphi\,\frac{\cos\theta}{\sin\theta} \\ \cos\varphi \end{pmatrix} = -\sin\varphi\,\frac{\cos\theta}{\sin\theta}\,\partial_\varphi + \cos\varphi\,\partial_\theta.$$

They are a basis of the Lie algebra $so(3)$, cf. (8.54):

$$[v_{(a)}, v_{(b)}] = \sum_{c=1}^{3} \varepsilon_{abc} v_{(c)} .$$

Two of these commutation relations are easy and the third is:

$$[v_{(2)}, v_{(3)}] = \left[\cos\varphi \, \frac{\cos\theta}{\sin\theta} \, \partial_\varphi + \sin\varphi \, \partial_\theta , - \sin\varphi \, \frac{\cos\theta}{\sin\theta} \, \partial_\varphi + \cos\varphi \, \partial_\theta \right]$$

$$= \cos\varphi \, \frac{\cos\theta}{\sin\theta} \left(- \cos\varphi \, \frac{\cos\theta}{\sin\theta} \right) \partial_\varphi + \cancel{\cos\varphi \, \frac{\cos\theta}{\sin\theta} (- \sin\varphi) \, \partial_\theta} - \sin\varphi \, \sin\varphi \, \frac{-1}{\sin^2\theta} \, \partial_\varphi$$

$$+ \sin\varphi \, \frac{\cos\theta}{\sin\theta} \left(- \sin\varphi \, \frac{\cos\theta}{\sin\theta} \right) \partial_\varphi + \cancel{\sin\varphi \, \cos\varphi \, \frac{\cos\theta}{\sin\theta}} \, \partial_\theta - \cos\varphi \, \cos\varphi \, \frac{-1}{\sin^2\theta} \, \partial_\varphi$$

$$= \frac{1}{\sin^2\theta} \left(- \cos^2\varphi \, \cos^2\theta - \sin^2\varphi \, \cos^2\theta + \sin^2\varphi + \cos^2\varphi \right) \partial_\varphi = \frac{1 - \cos^2\theta}{\sin^2\theta} \, \partial_\varphi = v_{(1)} ,$$

where we have used

$$\partial_\theta \, \frac{\cos\theta}{\sin\theta} = \frac{-1}{\sin^2\theta} .$$

7 Manifolds

Problem 7.1
Show that the definition of differentiability of maps between two manifolds is independent of the coordinates chosen.

Solution 7.1
Consider a map $F : M \to N$ where M and N are manifolds. Let $(\mathscr{U}', \alpha')$, $(\mathscr{U}, \alpha)$ be charts for M with $x \in \mathscr{U} \cap \mathscr{U}'$, and let $(\mathscr{V}', \beta')$, $(\mathscr{V}, \beta)$ be charts for N with $F(x) \in \mathscr{V} \cap \mathscr{V}'$. The representation of F with the help of the primed coordinates, $\beta' \circ F \circ \alpha'^{-1}$, is related to the representation of F in terms of the unprimed coordinates, $\beta \circ F \circ \alpha^{-1}$, by

$$\beta' \circ F \circ \alpha'^{-1} = \beta' \circ \beta^{-1} \circ \beta \circ F \circ \alpha^{-1} \circ \alpha \circ \alpha'^{-1} .$$

As $\beta' \circ \beta^{-1}$ and $\alpha \circ \alpha'^{-1}$ are C^∞, we see that $\beta \circ F \circ \alpha^{-1}$ is C^∞ at $\alpha(x)$ if and only if $\beta' \circ F \circ \alpha'^{-1}$ is C^∞ at $\alpha'(x)$.

Problem 7.2
Prove the theorem in section 7.4.

Solution 7.2 (see Choquet-Bruhat et al., p.240)
Let $x_0 \in N$ and $(\mathscr{U}, \alpha)$ be a chart for M with $x_0 \in \mathscr{U}$. Denote the corresponding coordinates

by $x^1, \dots, x^n$. Since the mapping $x \mapsto (f^1(x), \dots, f^p(x))$ is of rank p at x_0, it is possible to label the coordinates such that the $p \times p$-matrix

$$\left(\frac{\partial f^A}{\partial x^i}\right)_{\substack{A=1,\dots,p \\ i=n-p+1,\dots,n}}$$

has nonvanishing determinant at x_0. Then there exists a neighbourhood $\bar{\mathscr{U}} \subset \mathscr{U}$ with $x_0 \in \bar{\mathscr{U}}$ where the following change of coordinates is admissible:

$$\begin{aligned} y^i &= x^i \quad &, \quad i &= 1, \dots, n-p\,, \\ y^i &= f^{i-(n-p)}(x) \quad &, \quad i &= n-p+1, \dots, n\,. \end{aligned}$$

Therefore $(\bar{\mathscr{U}}, \bar{\alpha})$ with $\bar{\alpha}(x) = (y^1, \dots, y^n)$ is a chart for M. If $x \in \bar{\mathscr{U}} \cap N$, one has

$$\bar{\alpha}(x) = (y^1, \dots, y^{n-p}, 0, \dots, 0)\,.$$

Such a chart can be found for each $x_0 \in N$. Hence N is an $(n-p)$-dimensional submanifold of M.

Problem 7.3

Show that the differentiable structure of S^2 as a submanifold of $\mathbb{R}^3$ coincides with the differentiable structure given in section 7.1. Hint: Use polar coordinates in $\mathbb{R}^3$.

Solution 7.3

Let

$$M = \mathbb{R}^3 \qquad \text{and} \qquad N = S^2 = \left\{ x \in M \,\middle|\, \sum_{i=1}^{3} (x^i)^2 = 1 \right\}.$$

Define a chart $(\mathscr{U}, \alpha)$ for M by

$$\mathscr{U} = \mathbb{R}^3 - \left\{ x \in \mathbb{R}^3 \middle| x^1 \geq 0\,,\, x^2 = 0 \right\}$$

and $\alpha(x) = (\vartheta, \varphi, r)$, where

$$\begin{aligned} x^1 &= r \sin\vartheta \cos\varphi\,, \\ x^2 &= r \sin\vartheta \sin\varphi\,, \\ x^3 &= r \cos\vartheta\,. \end{aligned}$$

For $x \in \mathscr{U}$ we have $r > 0$, $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$. Define

$$\bar{\mathscr{U}} = \mathscr{U} \cap S^2 = S^2 - \left\{ x \in S^2 \middle| x^1 \geq 0\,,\, x^2 = 0 \right\}.$$

Since $x \in \mathscr{U} \cap S^2$ implies $\alpha(x) = (\vartheta, \varphi, 1)$, we see that $(\bar{\mathscr{U}}, \bar{\alpha})$ with $\bar{\alpha}(x) = (\vartheta, \varphi)$ is a chart for S^2 as a submanifold of $\mathbb{R}^3$, and we have to show that $(\bar{\mathscr{U}}, \bar{\alpha})$ is compatible with the charts $(\mathscr{U}_i, \alpha_i)$ introduced in section 7.1.

The change of coordinates is given by

$$\left(\alpha_{\frac{1}{2}} \circ \bar{\alpha}^{-1}\right)(\vartheta, \varphi) = \alpha_{\frac{1}{2}}(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) = \frac{1}{1 \pm \cos\vartheta}(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi)$$

for $(\vartheta, \varphi) \in \bar{\alpha}(\mathscr{U}_{\frac{1}{2}} \cap \bar{\mathscr{U}})$. On $\bar{\mathscr{U}}$ we have $0 < \vartheta < \pi$. Hence the above change of coordinates is C^∞. The inverse map is

$$\left(\bar{\alpha} \circ \alpha_{\frac{1}{2}}^{-1}\right)(y^1, y^2) = (\vartheta, \varphi) \quad \text{for} \quad (y^1, y^2) \in \alpha_{\frac{1}{2}}(\mathscr{U}_{\frac{1}{2}} \cap \bar{\mathscr{U}}),$$

where

$$\begin{aligned} \vartheta &= \arccos\left(\pm\frac{1-(y^1)^2-(y^2)^2}{1+(y^1)^2+(y^2)^2}\right), \\ \varphi &= -\mathrm{i}\,\log\frac{y^1+\mathrm{i}y^2}{\sqrt{(y^1)^2+(y^2)^2}} \end{aligned}$$

with the branch of the logarithm chosen such that $0 < \varphi < 2\pi$. Since $\alpha_{\frac{1}{2}}(x) \neq (0,0)$ for $x \in \bar{\mathscr{U}}$, we see that $\left(\bar{\alpha} \circ \alpha_{\frac{1}{2}}^{-1}\right)$ is C^∞ as well. Therefore $(\bar{\mathscr{U}}, \bar{\alpha})$ is compatible with $(\mathscr{U}_i, \alpha_i)$.

In order to cover all of S^2 we need one more chart. For instance, one can take

$$\mathscr{U} = \mathbb{R}^3 - \left\{x \in \mathbb{R}^3 \middle| x^1 \leq 0\,,\, x^3 = 0\right\}$$

and $\alpha(x) = (\vartheta, \varphi, r)$, where

$$\begin{aligned} x^1 &= -r\sin\vartheta\cos\varphi\,, \\ x^2 &= r\cos\vartheta\,, \\ x^3 &= -r\sin\vartheta\sin\varphi\,. \end{aligned}$$

As above one sees that the resulting chart $(\bar{\mathscr{U}}, \bar{\alpha})$ for S^2 is compatible with $(\mathscr{U}_i, \alpha_i)$. Since the two $\bar{\mathscr{U}}$s cover S^2, we have obtained the desired result.

Problem 7.4
Verify that the Lie bracket of two vector fields on a manifold is represented in terms of coordinates by (6.18).

Solution 7.4
Represent the vector fields v, w in terms of coordinates:

$$v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i} \quad , \quad w = \sum_{i=1}^n w^i \frac{\partial}{\partial x^i}\,.$$

According to (7.37) their Lie bracket is given by

$$
\begin{aligned}
[v,w](f) = v(w(f)) - w(v(f)) &= v\left(\sum_j w^j \frac{\partial}{\partial x^j} f\right) - w\left(\sum_i v^i \frac{\partial}{\partial x^i} f\right) \\
&= \sum_{i,j} v^i \frac{\partial}{\partial x^i}\left(w^j \frac{\partial}{\partial x^j} f\right) - \sum_{i,j} w^j \frac{\partial}{\partial x^j}\left(v^i \frac{\partial}{\partial x^i} f\right) \\
&= \sum_{i,j}\left(v^i \frac{\partial w^j}{\partial x^i}\frac{\partial f}{\partial x^j} + v^i w^j \frac{\partial^2 f}{\partial x^i \partial x^j} - w^j \frac{\partial v^i}{\partial x^j}\frac{\partial f}{\partial x^i} - w^j v^i \frac{\partial^2 f}{\partial x^i \partial x^j}\right) \\
&= \sum_{i,j}\left(v^i \frac{\partial w^j}{\partial x^i} - w^i \frac{\partial v^j}{\partial x^i}\right)\frac{\partial}{\partial x^j} f\,.
\end{aligned}
$$

This proves (6.18).

Problem 7.5

Prove the coordinate independence of the definition (7.46) of the tangent mapping explicitly.

Solution 7.5

Consider coordinates $x^1, \dots, x^n$ and $\bar{x}^1, \dots, \bar{x}^n$ on a neighbourhood of $x \in M$ and coordinates $y^1, \dots, y^p$ and $\bar{y}^1, \dots, \bar{y}^p$ on a neighbourhood of $F(x) \in N$. In terms of these coordinates, the map $F : M \to N$ is represented by

$$y^j = F^j(x^1, \dots, x^n)$$

and by

$$\bar{y}^j = \bar{F}^j(\bar{x}^1, \dots, \bar{x}^n)\,,$$

respectively ($j = 1, 2, \dots, p$). For $v \in T_x M$ we have

$$v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}(x) = \sum_{i=1}^n \bar{v}^i \frac{\partial}{\partial \bar{x}^i}(x)\,,$$

where according to (7.29)

$$\bar{v}^i = \sum_{m=1}^n \frac{\partial \bar{x}^i}{\partial x^m} v^m\,.$$

Due to (7.28) we have

$$\frac{\partial}{\partial \bar{y}^j} = \sum_{l=1}^p \frac{\partial y^l}{\partial \bar{y}^j}\frac{\partial}{\partial y^l}\,.$$

Therefore we get

$$
\begin{aligned}
\sum_{j=1}^p \sum_{i=1}^n \bar{v}^i \frac{\partial \bar{F}^j}{\partial \bar{x}^i}\frac{\partial}{\partial \bar{y}^j} &= \sum_{l,j=1}^p \sum_{i,m=1}^n \frac{\partial \bar{x}^i}{\partial x^m} v^m \frac{\partial \bar{F}^j}{\partial \bar{x}^i}\frac{\partial y^l}{\partial \bar{y}^j}\frac{\partial}{\partial y^l} \\
&= \sum_{l=1}^p \sum_{m=1}^n v^m \left(\sum_{j=1}^p \sum_{i=1}^n \frac{\partial y^l}{\partial \bar{y}^j}\frac{\partial \bar{F}^j}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^m}\right)\frac{\partial}{\partial y^l}\,.
\end{aligned}
$$

The expression in parentheses equals $\dfrac{\partial F^l}{\partial x^m}$, because

$$F^j(x^1,\ldots) = y^j\left(\bar{F}^1(\bar{x}^1(x^1,\ldots),\ldots),\ldots\right) .$$

Therefore

$$T_xF(v) = \sum_{j=1}^{p}\sum_{i=1}^{n} \bar{v}^i \frac{\partial \bar{F}^j}{\partial \bar{x}^i}\frac{\partial}{\partial \bar{y}^j} = \sum_{l=1}^{p}\sum_{m=1}^{n} v^m \frac{\partial F^l}{\partial x^m}\frac{\partial}{\partial y^l} .$$

8 Lie groups

Problem 8.1

Show that $\mathbb{R}^3$ with the multiplication law

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + x_1y_2 - x_2y_1 \end{pmatrix}$$

is a Lie group and that its left action is an affine representation. Calculate the structure constants of the Lie algebra with respect to a convenient basis.

Solution 8.1

For notational convenience, we put all indices upstairs.
First show that we have a group:

- neutral element: $e = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- inverse: $\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}^{-1} = \begin{pmatrix} -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$
- associativity: $\left[\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}\right]\begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} = \begin{pmatrix} x^1 + y^1 \\ x^2 + y^2 \\ x^3 + y^3 + x^1y^2 - x^2y^1 \end{pmatrix}\begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix}$

$$= \begin{pmatrix} x^1 + y^1 + z^1 \\ x^2 + y^2 + z^2 \\ x^3 + y^3 + x^1y^2 - x^2y^1 + z^3 + (x^1 + y^1)z^2 - (x^2 + y^2)z^1 \end{pmatrix}$$

$$= \begin{pmatrix} x^1 + y^1 + z^1 \\ x^2 + y^2 + z^2 \\ x^3 + y^3 + z^3 + x^1y^2 + x^1z^2 + y^1z^2 - x^2y^1 - x^2z^1 - y^2z^1 \end{pmatrix},$$

which is indeed equal to

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \left[\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix} \right] = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \begin{pmatrix} y^1 + z^1 \\ y^2 + z^2 \\ y^3 + z^3 + y^1 z^2 - y^2 z^1 \end{pmatrix}$$
$$= \begin{pmatrix} x^1 + y^1 + z^1 \\ x^2 + y^2 + z^2 \\ x^3 + y^3 + z^3 + y^1 z^2 - y^2 z^1 + x^1 (y^2 + z^2) - x^2 (y^1 + z^1) \end{pmatrix}$$
$$= \begin{pmatrix} x^1 + y^1 + z^1 \\ x^2 + y^2 + z^2 \\ x^3 + y^3 + z^3 + x^1 y^2 + x^1 z^2 + y^1 z^2 - x^2 y^1 - x^2 z^1 - y^2 z^1 \end{pmatrix}.$$

Second, our group is a Lie group, because multiplication and inverse are differentiable.

The left translation or action is affine:

$$L_g \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} g^1 \\ g^2 \\ g^3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} g^1 \\ g^2 \\ g^3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -g^2 & g^1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}.$$

To construct the (left) invariant vector fields we write the left translation in the form:

$$L_g \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} g^1 + x^1 \\ g^2 + x_2 \\ g^3 + x^3 + g^1 x^2 - g^2 x^1 \end{pmatrix} =: \begin{pmatrix} F^1(x) \\ F^2(x) \\ F^3(x) \end{pmatrix},$$

and its tangent map (using the shorthand $\partial_i := \partial / \partial x^i$ and the summation convention),

$$\begin{aligned} TL_g \; : \; T_x\mathbb{R}^3 &\longrightarrow T_{L_g x}\mathbb{R}^3 \\ v^i \partial_i &\longmapsto v^i (\partial_i F^j) \partial_j = v^1 \partial_1 + v^2 \partial_2 + (v^3 + g^1 v^2 - g^2 v^1) \partial_3 \,. \end{aligned}$$

With $(a, b, c)^T \in T_0\mathbb{R}^3$ we obtain the generic invariant vector field

$$v(x) = a\,\partial_1 + b\,\partial_2 + (c + b\,x^1 - a\,x^2)\,\partial_3 \,.$$

As basis of the Lie algebra $\mathfrak{g}$ we choose

$$A := \partial_1 - x^2 \partial_3, \quad B := \partial_2 + x^1 \partial_3, \quad C := \partial_3.$$

There is only one non-vanishing commutator:

$$[A, B] = [\partial_1 - x^2 \partial_3,\, \partial_2 + x^1 \partial_3] = \partial_3 + \partial_3 = 2\,C.$$

Problem 8.2

Show that the Lie algebra of $SO(n)$ is isomorphic to the Lie algebra of antisymmetric matrices.

Solution 8.2

$SO(n)$ is the submanifold of $\mathbb{R}^{n^2}$ defined by the system of equations

$$MM^T = \mathbb{1}, \quad \det M = 1.$$

In the neighbourhood of $\mathbb{1}$ the group $SO(n)$ is a surface in $\mathbb{R}^{n^2}$ in the sense discussed on pages 9 and 10. To calculate $T_{\mathbb{1}}SO(n)$ we consider a curve $M(\tau)$ in $SO(n)$ starting at $\mathbb{1} =: M(0)$. The tangent vectors $\mathrm{d}/\mathrm{d}\tau M(0)$ for all such curves constitute $T_{\mathbb{1}}SO(n)$. From $MM^T = \mathbb{1}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau} M(0) + \left(\frac{\mathrm{d}}{\mathrm{d}\tau} M(0)\right)^T = 0.$$

Therefore

$$T_{\mathbb{1}}SO(n) = \left\{ \sum_{i_1,i_2} a^{i_1 i_2} \frac{\partial}{\partial x^{i_1 i_2}}(\mathbb{1}) \quad \text{with} \quad a^{i_1 i_2} = -a^{i_2 i_1} \right\}.$$

The invariant vector fields are calculated as in (8.34). Their commutators are again given by (8.35).

Problem 8.3

Define $C \in gl_n$ by

$$e^C = e^{\lambda A} e^{\lambda B}$$

with $A, B \in gl_n$, and $\lambda \in \mathbb{R}$. Consider the expansion of C in powers of λ and calculate as many terms as you wish (Campbell-Hausdorff formula).

Solution 8.3

$$\begin{aligned} e^{\lambda A} e^{\lambda B} &= \left(\mathbb{1} + \lambda A + \frac{\lambda^2}{2!} A^2 + \frac{\lambda^3}{3!} A^3 + \cdots\right)\left(\mathbb{1} + \lambda B + \frac{\lambda^2}{2!} B^2 + \frac{\lambda^3}{3!} B^3 + \cdots\right) \\ &= \mathbb{1} + \lambda(A+B) + \frac{\lambda^2}{2!}(A^2 + 2AB + B^2) + \frac{\lambda^3}{3!}(A^3 + 3A^2B + 3AB^2 + B^3) + \cdots \end{aligned}$$

Write

$$C = \lambda C_1 + \frac{\lambda^2}{2!} C_2 + \frac{\lambda^3}{3!} C_3 + \cdots$$

Then

$$\begin{aligned} e^C &= \mathbb{1} + \lambda C_1 + \frac{\lambda^2}{2!} C_2 + \frac{\lambda^3}{3!} C_3 + \cdots + \frac{1}{2!}\left(\lambda C_1 + \frac{\lambda^2}{2!} C_2 + \cdots\right)^2 + \frac{1}{3!}(\lambda C_1 + \cdots)^3 + \cdots \\ &= \mathbb{1} + \lambda C_1 + \frac{\lambda^2}{2!}\left[C_2 + C_1^2\right] + \frac{\lambda^3}{3!}\left[C_3 + \tfrac{3}{2} C_1 C_2 + \tfrac{3}{2} C_2 C_1 + C_1^3\right] + \cdots \end{aligned}$$

whence

$$
\begin{aligned}
C_1 &= A + B,\\
C_2 &= A^2 + 2AB + B^2 - A^2 - AB - BA - B^2 = [A, B],\\
C_3 &= \cancel{A^3} + \underline{3A^2B} + \underline{\underline{3AB^2}} + \cancel{B^3}\\
&\quad - \cancel{A^3} - \underline{A^2B} - ABA - \underset{\sim\sim}{BA^2} - \underline{\underline{AB^2}} - BAB - \underset{---}{B^2A} - \cancel{B^3}\\
&\quad - \underline{\tfrac{3}{2}A^2B} + \cancel{\tfrac{3}{2}ABA} - \cancel{\tfrac{3}{2}ABA} + \underset{\sim\sim}{\tfrac{3}{2}BA^2} - \tfrac{3}{2}\cancel{B}A\cancel{B} + \underset{---}{\tfrac{3}{2}B^2A} - \underline{\underline{\tfrac{3}{2}AB^2}} + \tfrac{3}{2}\cancel{B}A\cancel{B}\\
&= \underline{\tfrac{1}{2}A^2B} - ABA + \underset{\sim\sim}{\tfrac{1}{2}BA^2} + \underline{\underline{\tfrac{1}{2}AB^2}} - BAB + \underset{---}{\tfrac{1}{2}B^2A} \;=\; \tfrac{1}{2}[A - B, [A, B]].
\end{aligned}
$$

In conclusion:

$$e^A e^B = e^{A + B + (1/2)[A,B] + (1/12)[A-B,[A,B]] + \cdots}.$$

Problem 8.4

Use the Maurer-Cartan structure equation (8.67) to prove that (8.70) is equivalent to the Jacobi identity.

Solution 8.4

The Jacobi identity,

$$[v, [u, w]] = [[v, u], w] + [u, [v, w]],$$

is often written as a sum of three cyclic permutations,

$$[[v, u], w] + [[w, v], u] + [[u, w], v] = 0,$$

or in a basis $\{A_i\}$ of our Lie algebra and with $v = A_{i_1}$, $u = A_{i_2}$ and $w = A_{i_3}$,

$$\sum_k \left(f_{i_1 i_2}{}^k f_{k i_3}{}^l + f_{i_3 i_1}{}^k f_{k i_2}{}^l + f_{i_2 i_3}{}^k f_{k i_1}{}^l \right) = 0,$$

or using the antisymmetry of the structure constants, $f_{i_1 i_2}{}^k = -f_{i_2 i_1}{}^k$,

$$\sum_k \sum_{\pi \in \mathscr{S}_3} \mathrm{sig}\pi \, f_{i_{\pi(1)} i_{\pi(2)}}{}^k f_{k i_{\pi(3)}}{}^l.$$

We write the Maurer-Cartan structure equation (8.67),

$$\mathrm{d}\zeta^l = -\tfrac{1}{2} \sum_{k, i_3} f_{k i_3}{}^l \zeta^k \wedge \zeta^{i_3},$$

and take its exterior derivative:

$$
\begin{aligned}
\mathrm{d}^2\zeta^l &= -\tfrac{1}{2} \sum_{k, i_3} f_{k i_3}{}^l \mathrm{d}\zeta^k \wedge \zeta^{i_3} + \tfrac{1}{2} \sum_{k, i_3} f_{k i_3}{}^l \zeta^k \wedge \mathrm{d}\zeta^{i_3} = -\sum_{k, i_3} f_{k i_3}{}^l \mathrm{d}\zeta^k \wedge \zeta^{i_3}\\
&= \tfrac{1}{2} \sum_{k, i_1, i_2, i_3} f_{i_1 i_2}{}^k f_{k i_3}{}^l \zeta^{i_1} \wedge \zeta^{i_2} \wedge \zeta^{i_3} = \tfrac{1}{2} \sum_{i_1 < i_2 < i_3} \left(\sum_k \sum_{\pi \in \mathscr{S}_3} \mathrm{sig}\pi \, f_{i_{\pi(1)} i_{\pi(2)}}{}^k f_{k i_{\pi(3)}}{}^l \right) \zeta^{i_1} \wedge \zeta^{i_2} \wedge \zeta^{i_3}.
\end{aligned}
$$

The desired equivalence between equation (8.70), $\mathrm{d}^2\zeta^l = 0$, and the Jacobi identity follows because the $\zeta^{i_1} \wedge \zeta^{i_2} \wedge \zeta^{i_3}$ with $i_1 < i_2 < i_3$ are a basis of the invariant 3-forms.

9 Fibre bundles

Problem 9.1
Give a differentiable atlas for the frame bundle $F(M)$.

Solution 9.1
Let $\{(\mathscr{U}_r, \alpha_r)\}$ be a C^∞-atlas for M and denote the coordinates on $\mathscr{U}_r$ by $x_r^1, \ldots, x_r^n$. Then $\{(\pi^{-1}(\mathscr{U}_r), \bar{\alpha}_r)\}$ is a C^∞-atlas for $F(M)$, where $\pi : F(M) \to M$ is the projection and $\bar{\alpha}_r : \pi^{-1}(\mathscr{U}_r) \to \mathbb{R}^{n+n^2}$ is given by

$$\bar{\alpha}_r(x, b_1, \ldots, b_n) = \left(x_r^1, \ldots, x_r^n, \mathrm{d}x_r^1(b_1), \mathrm{d}x_r^2(b_1), \ldots, \mathrm{d}x_r^n(b_n)\right) \equiv \left(x_r^1, \ldots, x_r^n, (\mathrm{d}x_r^j(b_i))\right)$$

for $(x, b_1, \ldots, b_n) \in \pi^{-1}(\mathscr{U}_r)$. It remains to be shown that $\bar{\alpha}_r \circ \bar{\alpha}_s^{-1}$ is C^∞. We have

$$\bar{\alpha}_r \circ \bar{\alpha}_s^{-1}\left(x_s^1, \ldots, x_s^n, (\mathrm{d}x_s^j(b_i))\right) = \left(x_r^1, \ldots, x_r^n, (\mathrm{d}x_r^j(b_i))\right) ,$$

where

$$(x_r^1, \ldots, x_r^n) = \alpha_r \circ \alpha_s^{-1}(x_s^1, \ldots, x_s^n)$$

and

$$\mathrm{d}x_r^j(b_i) = \sum_l \frac{\partial x_r^j}{\partial x_s^l} \mathrm{d}x_s^l(b_i) \, .$$

This means

$$\bar{\alpha}_r \circ \bar{\alpha}_s^{-1}\left(x_s^1, \ldots, x_s^n, (g^j{}_i)\right) = \left(x_r^1, \ldots, x_r^n, \left(\sum_l \frac{\partial x_r^j}{\partial x_s^l} g^l{}_i\right)\right) .$$

The coordinates x_r^i are infinitely often differentiable functions of $x_s^1, \ldots, x_s^n$, consequently the same holds for the derivatives $\dfrac{\partial x_r^j}{\partial x_s^l}$. The dependence upon $g^j{}_i$ is linear and therefore C^∞ as well.

Problem 9.2
Consider $SU(2)$ as the subgroup of $SU(3)$ consisting of all $SU(3)$-matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} .$$

Then, according to the theorem mentioned at the end of section 9.5, $SU(3)$ may be regarded as an $SU(2)$-bundle over S^5. Find local trivializations of this bundle over S^5 – {north pole} and S^5 – {south pole}. Calculate the corresponding transition function.

Solution 9.2
Considering S^5 as a subset of $\mathbb{C}^3$,

$$S^5 = \left\{(a,b,c) \in \mathbb{C}^3 \Big| |a|^2 + |b|^2 + |c|^2 = 1\right\} ,$$

the projection $\pi : SU(3) \to S^5$ is given by

$$\pi(U) = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{31} \end{pmatrix} .$$

To prove this one has to show that

$$\pi(U) = \pi(V) \Leftrightarrow U = V \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$. It is trivial to prove "$\Leftarrow$". To show "$\Rightarrow$" note that $(V^+U)_{11} = 1$, because $U_{i1} = V_{i1}$ for $i = 1,2,3$ and $U, V \in SU(3)$. Since $V^+U \in SU(3)$, one can conclude from this:

$$(V^+U)_{12} = (V^+U)_{13} = (V^+U)_{21} = (V^+U)_{31} = 0 .$$

Consequently

$$V^+U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{A} \end{pmatrix} \quad \text{with} \quad A \in SU(2) .$$

A section of $(SU(3), S^5, \pi)$ over $S^5 - \{(0,0,+\mathrm{i})\}$ is given by

$$\begin{aligned} S^5 - \{(0,0,+\mathrm{i})\} &\to SU(3) \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\mapsto U_+(a,b,c) \end{aligned}$$

and a section over $S^5 - \{(0,0,-\mathrm{i})\}$ is given by

$$\begin{aligned} S^5 - \{(0,0,-\mathrm{i})\} &\to SU(3) \\ \begin{pmatrix} a \\ b \\ c \end{pmatrix} &\mapsto U_-(a,b,c) \end{aligned}$$

where

$$U_+(a,b,c) = \begin{pmatrix} a & \dfrac{\mathrm{i}}{1+\mathrm{i}c}(1 - \mathrm{i}\bar{c} - |a|^2) & -\dfrac{a\bar{b}}{1-\mathrm{i}\bar{c}} \\ b & -\dfrac{\mathrm{i}\bar{a}b}{1+\mathrm{i}c} & \dfrac{1}{1-\mathrm{i}\bar{c}}(1 - \mathrm{i}\bar{c} - |b|^2) \\ c & -\bar{a} & \mathrm{i}\bar{b}\dfrac{1+\mathrm{i}c}{1-\mathrm{i}\bar{c}} \end{pmatrix} , \quad c \neq \mathrm{i} \quad ,$$

$$U_-(a,b,c) = \begin{pmatrix} a & \dfrac{-\mathrm{i}}{1-\mathrm{i}c}(1+\mathrm{i}\bar{c}-|a|^2) & -\dfrac{a\bar{b}}{1+\mathrm{i}\bar{c}} \\ b & \dfrac{\mathrm{i}\bar{a}b}{1-\mathrm{i}c} & \dfrac{1}{1+\mathrm{i}\bar{c}}(1+\mathrm{i}\bar{c}-|b|^2) \\ c & -\bar{a} & -\mathrm{i}\bar{b}\dfrac{1-\mathrm{i}c}{1+\mathrm{i}\bar{c}} \end{pmatrix} \quad , \quad c \neq -\mathrm{i}$$

with $|a|^2+|b|^2+|c|^2 = 1$. So we get local trivializations over $S^5 - \{(0,0,+\mathrm{i})\}$ and over $S^5 - \{(0,0,-\mathrm{i})\}$:

$$\begin{aligned} \pi^{-1}\big(S^5 - \{(0,0,\pm\mathrm{i})\}\big) &\to \big(S^5 - \{(0,0,\pm\mathrm{i})\}\big) \times SU(2) \\ U \quad &\mapsto \big(\pi(U), V_\pm(U)\big)\,, \end{aligned}$$

where $V_\pm(U) \in SU(2)$ is such that

$$U = U_\pm\big(U_{11}, U_{21}, U_{31}\big) \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{V_\pm(U)} \end{pmatrix}.$$

According to (9.8) the transition function

$$g_{-+} \;:\; S^5 - \{(0,0,+\mathrm{i}),(0,0,-\mathrm{i})\} \to SU(2)$$

is determined by

$$g_{-+}\big(U_{11},U_{21},U_{31}\big)V_+(U) = V_-(U)\,.$$

Therefore

$$U = U_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{V_-} \end{pmatrix} = U_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{g_{-+}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{V_+} \end{pmatrix}.$$

Consequently,

$$U_- \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \multicolumn{2}{c}{g_{-+}} \end{pmatrix} = U_+\,.$$

This yields

$$g_{-+}(a,b,c) = \begin{pmatrix} 1 - \dfrac{2|b|^2}{(1+\mathrm{i}\bar{c})(1+\mathrm{i}c)} & -\dfrac{2\mathrm{i}a\bar{b}}{1+\bar{c}^2} \\ -\dfrac{2\mathrm{i}\bar{a}b}{1+c^2} & 1 - \dfrac{2|b|^2}{(1-\mathrm{i}c)(1-\mathrm{i}\bar{c})} \end{pmatrix}.$$

Problem 9.3
Show that a vector bundle with an m-dimensional fibre is trivial if and only if it admits m sections linearly independent at each point.

Solution 9.3
If E is a trivial vector bundle over M, we may assume that it is of the form $E = M \times \mathbb{R}^m$. Then m sections that are linearly independent at each point are given by

$$\begin{aligned} M \ni x \mapsto & \big(x, (1, 0, 0, \dots, 0)\big) \\ & \big(x, (0, 1, 0, \dots, 0)\big) \\ & \qquad \vdots \\ & \big(x, (0, 0, 0, \dots, 1)\big) . \end{aligned}$$

Conversely, if $\sigma_i : M \to E$ $(i = 1, 2, \dots, m)$ are m sections that are everywhere linearly independent and $\pi : E \to M$ denotes the projection, we can express any $y \in E$ as

$$y = \sum_{i=1}^{m} a_i \sigma_i(x) ,$$

where $x = \pi(y)$. Consequently, the map

$$y \mapsto (x, a_1, \dots, a_m) \in M \times \mathbb{R}^m$$

is a global trivialization of E.

10 Monopoles, instantons, and related fibre bundles

Problem 10.1
Apply the gauge transformation (10.24) to the 't Hooft–Polyakov ansatz (10.17).

Solution 10.1
We have $\gamma(x) = R(\hat{\gamma}(x))$ with

$$\hat{\gamma}(x) = \begin{pmatrix} \cos(\vartheta/2) & \mathrm{e}^{-\mathrm{i}\varphi} \sin(\vartheta/2) \\ -\mathrm{e}^{\mathrm{i}\varphi} \sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} .$$

Here, $R : SU(2) \to SO(3)$ is the map defined in (8.58) where it was called φ. Therefore

$$\Phi' = \gamma \Phi = R(\hat{\gamma}) \Phi = \hat{f}^{-1} \left(\hat{\gamma} \hat{f}(\Phi) \hat{\gamma}^{-1} \right) ,$$

where

$$\hat{f}(\Phi) = -\frac{\mathrm{i}}{2}\begin{pmatrix} \Phi^3 & \Phi^1 - \mathrm{i}\Phi^2 \\ \Phi^1 + \mathrm{i}\Phi^2 & -\Phi^3 \end{pmatrix} = -\frac{\mathrm{i}}{2}\frac{f(r)}{gr^2}\begin{pmatrix} x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & -x^3 \end{pmatrix}.$$

One finds

$$\hat{\gamma}\hat{f}(\Phi)\hat{\gamma}^{-1} = -\frac{\mathrm{i}}{2}\frac{f(r)}{gr}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and consequently

$$\Phi'^i = \delta_{i3}\frac{f(r)}{gr}.$$

In order to compute the gauge transformed vector potential we make use of the fact that the Lie algebras of $SU(2)$ and $SO(3)$ are isomorphic. So we work in $su(2)$ and consider the vector potential

$$\hat{A} = \frac{1-h(r)}{r^2}\sum_{a,i,j=1}^{3}\tau_a\varepsilon_{aij}x^j\mathrm{d}x^i$$

with $\tau_a = -(\mathrm{i}/2)\sigma_a$ (cf.(8.55)). We want to calculate

$$\hat{A}' = \hat{\gamma}\hat{A}\hat{\gamma}^{-1} + \hat{\gamma}\mathrm{d}\hat{\gamma}^{-1}.$$

One finds

$$\hat{\gamma}\mathrm{d}\hat{\gamma}^{-1} = \frac{1}{2}\begin{pmatrix} \mathrm{i}(1-\cos\vartheta)\mathrm{d}\varphi & \mathrm{e}^{-\mathrm{i}\varphi}(-\mathrm{d}\vartheta + \mathrm{i}\sin\vartheta\,\mathrm{d}\varphi) \\ \mathrm{e}^{\mathrm{i}\varphi}(\mathrm{d}\vartheta + \mathrm{i}\sin\vartheta\,\mathrm{d}\varphi) & -\mathrm{i}(1-\cos\vartheta)\mathrm{d}\varphi \end{pmatrix}$$

and

$$\frac{1}{r^2}\sum_{a,i,j=1}^{3}\tau_a\varepsilon_{aij}x^j\mathrm{d}x^i = \frac{1}{r^2}\left(\tau_1(x^3\mathrm{d}x^2 - x^2\mathrm{d}x^3) + \tau_2(x^1\mathrm{d}x^3 - x^3\mathrm{d}x^1) + \tau_3(x^2\mathrm{d}x^1 - x^1\mathrm{d}x^2)\right)$$

with

$$\begin{aligned} \frac{1}{r^2}(x^3\mathrm{d}x^2 - x^2\mathrm{d}x^3) &= \sin\varphi\,\mathrm{d}\vartheta + \sin\vartheta\cos\vartheta\cos\varphi\,\mathrm{d}\varphi, \\ \frac{1}{r^2}(x^1\mathrm{d}x^3 - x^3\mathrm{d}x^1) &= -\cos\varphi\,\mathrm{d}\vartheta + \sin\vartheta\cos\vartheta\sin\varphi\,\mathrm{d}\varphi, \\ \frac{1}{r^2}(x^2\mathrm{d}x^1 - x^1\mathrm{d}x^2) &= -\sin^2\vartheta\,\mathrm{d}\varphi. \end{aligned}$$

Consequently,

$$\frac{1}{r^2}\sum_{a,i,j=1}^{3}\tau_a\varepsilon_{aij}x^j\mathrm{d}x^i = -\frac{\mathrm{i}}{2}\begin{pmatrix} -\sin^2\vartheta\,\mathrm{d}\varphi & \mathrm{e}^{-\mathrm{i}\varphi}(\mathrm{i}\,\mathrm{d}\vartheta + \sin\vartheta\cos\vartheta\,\mathrm{d}\varphi) \\ \mathrm{e}^{\mathrm{i}\varphi}(-\mathrm{i}\,\mathrm{d}\vartheta + \sin\vartheta\cos\vartheta\,\mathrm{d}\varphi) & \sin^2\vartheta\,\mathrm{d}\varphi \end{pmatrix}$$

and therefore

$$\hat{\gamma}\hat{A}\hat{\gamma}^{-1} = -\frac{\mathrm{i}}{2}(1-h(r))\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi}(\mathrm{i}\,\mathrm{d}\vartheta + \sin\vartheta\,\mathrm{d}\varphi) \\ \mathrm{e}^{\mathrm{i}\varphi}(-\mathrm{i}\,\mathrm{d}\vartheta + \sin\vartheta\,\mathrm{d}\varphi) & 0 \end{pmatrix}.$$

Thus we obtain

$$\hat{A}' = \tau_1\big(- h(r)\sin\vartheta\cos\varphi\,\mathrm{d}\varphi - h(r)\sin\varphi\,\mathrm{d}\vartheta\big) + \tau_2\big(- h(r)\sin\vartheta\sin\varphi\,\mathrm{d}\varphi + h(r)\cos\varphi\,\mathrm{d}\vartheta\big)$$
$$+ \tau_3\big(- (1-\cos\vartheta)\mathrm{d}\varphi\big) = \sum_{a=1}^{3} \tau_a A'^a$$

with A'^a as given in (10.25).

Problem 10.2
Verify (10.31).

Solution 10.2
Since

$$\mathrm{d}\frac{1}{r} = -\frac{1}{r^2}\mathrm{d}r = -\frac{1}{r^3}\sum_{i=1}^{4} x^i \mathrm{d}x^i\,,$$

one finds

$$\gamma^{-1}\mathrm{d}\gamma = \frac{1}{r}\begin{pmatrix} x^4 + \mathrm{i}x^3 & x^2 + \mathrm{i}x^1 \\ -x^2 + \mathrm{i}x^1 & x^4 - \mathrm{i}x^3 \end{pmatrix} \mathrm{d}\frac{1}{r}\begin{pmatrix} x^4 - \mathrm{i}x^3 & -x^2 - \mathrm{i}x^1 \\ x^2 - \mathrm{i}x^1 & x^4 + \mathrm{i}x^3 \end{pmatrix}$$
$$= -\frac{1}{r^2}\sum_{j=1}^{4} x^j \mathrm{d}x^j + \frac{1}{r^2}\left(x^4 - 2\sum_{l=1}^{3} x^l \tau_l\right)\left(\mathrm{d}x^4 + 2\sum_{k=1}^{3}\tau_k \mathrm{d}x^k\right)\,.$$

Using

$$\tau_j\tau_l = -\frac{1}{4}\delta_{jl} + \frac{1}{2}\sum_{m=1}^{3}\varepsilon_{jlm}\tau_m$$

one arrives at the desired result:

$$\gamma^{-1}\mathrm{d}\gamma = \sum_{m=1}^{3}\tau_m\left[-\frac{2}{r^2}\left(x^m\mathrm{d}x^4 - x^4\mathrm{d}x^m + \sum_{l,k=1}^{3}\varepsilon_{lkm}x^l\mathrm{d}x^k\right)\right]\,.$$

Problem 10.3
Calculate the field strength belonging to the instanton ansatz (10.29).

Solution 10.3
With

$$\tilde{\zeta} = \gamma^*\zeta = \sum_{j=1}^{3}\tau_j\tilde{\zeta}^j$$

the ansatz (10.29) reads $A = f(r)\tilde{\zeta}$. From (8.73) it follows $\mathrm{d}\tilde{\zeta} = -\frac{1}{2}[\tilde{\zeta},\tilde{\zeta}]$. Consequently:

$$F = \mathrm{d}A + \frac{1}{2}[A,A] = f'(r)\mathrm{d}r\wedge\tilde{\zeta} + f(r)\mathrm{d}\tilde{\zeta} + \frac{1}{2}f^2(r)[\tilde{\zeta},\tilde{\zeta}] = f'(r)\mathrm{d}r\wedge\tilde{\zeta} + \frac{1}{2}(f^2 - f)[\tilde{\zeta},\tilde{\zeta}]\,.$$

Due to

$$[\tau_l, \tau_j] = \sum_{m=1}^{3} \varepsilon_{ljm} \tau_m$$

one finds

$$[\tilde{\zeta}, \tilde{\zeta}] = \sum_{l,j,m=1}^{3} \varepsilon_{ljm} \tau_m \tilde{\zeta}^l \wedge \tilde{\zeta}^j$$

and therefore

$$F = \sum_{m=1}^{3} \tau_m \left[\frac{\mathrm{d}f}{\mathrm{d}r} \mathrm{d}r \wedge \tilde{\zeta}^m + \frac{1}{2}(f^2 - f) \sum_{l,j=1}^{3} \varepsilon_{ljm} \tilde{\zeta}^l \wedge \tilde{\zeta}^j \right] .$$

Problem 10.4
Show that (10.48) is the Riemannian connection belonging to the metric described by the ansatz (10.47).

Solution 10.4
Since $\eta_{ij} = \delta_{ij}$, the spin connection ω must satisfy $\omega^{ij} = -\omega^{ji}$. This is trivially fulfilled, because (10.48) is meant to be completed such that ω becomes antisymmetric. Hence ω is metric. It remains to be shown that the torsion vanishes, i.e.,

$$\mathrm{d}e^i + \sum_{j=1}^{4} \omega^{ij} \wedge e^j = 0 .$$

In the case $i = 1$ we have

$$\mathrm{d}e^1 = 0$$

and

$$\sum_{j=1}^{4} \omega^{1j} \wedge e^j = 0 ,$$

because $\omega^{1j} \propto e^j$.

For the following note that $\mathrm{d}\tilde{\zeta} = -\frac{1}{2}[\tilde{\zeta}, \tilde{\zeta}]$ (see solution 10.3) implies

$$\begin{aligned} \mathrm{d}\tilde{\zeta}^1 &= -\tilde{\zeta}^2 \wedge \tilde{\zeta}^3 , \\ \mathrm{d}\tilde{\zeta}^2 &= -\tilde{\zeta}^3 \wedge \tilde{\zeta}^1 , \\ \mathrm{d}\tilde{\zeta}^3 &= -\tilde{\zeta}^1 \wedge \tilde{\zeta}^2 . \end{aligned}$$

Then we get for $i = 2$

$$\mathrm{d}e^2 = -\frac{1}{2}\mathrm{d}r \wedge \tilde{\zeta}^1 - \frac{1}{2} r \left(-\tilde{\zeta}^2 \wedge \tilde{\zeta}^3 \right) = \frac{1}{rf} e^1 \wedge e^2 + \frac{2}{rg} e^3 \wedge e^4$$

and

$$\sum_{j=1}^{4}\omega^{2j}\wedge e^j = \frac{1}{rf}e^2\wedge e^1 + \frac{2-g^2}{rg}e^4\wedge e^3 - \frac{g}{r}e^3\wedge e^4 = -\frac{1}{rf}e^1\wedge e^2 - \frac{2}{rg}e^3\wedge e^4\,.$$

For $i = 3$ we find

$$\mathrm{d}e^3 = -\frac{1}{2}\mathrm{d}r\wedge\tilde{\zeta}^2 - \frac{1}{2}r\left(-\tilde{\zeta}^3\wedge\tilde{\zeta}^1\right) = \frac{1}{rf}e^1\wedge e^3 + \frac{2}{rg}e^4\wedge e^2$$

and

$$\sum_{j=1}^{4}\omega^{3j}\wedge e^j = \frac{1}{rf}e^3\wedge e^1 - \frac{2-g^2}{rg}e^4\wedge e^2 + \frac{g}{r}e^2\wedge e^4 = -\frac{1}{rf}e^1\wedge e^3 - \frac{2}{rg}e^4\wedge e^2\,.$$

Finally, we obtain in the case $i = 4$

$$\mathrm{d}e^4 = -\frac{1}{2}g\mathrm{d}r\wedge\tilde{\zeta}^3 - \frac{r}{2}g'\mathrm{d}r\wedge\tilde{\zeta}^3 - \frac{1}{2}rg\left(-\tilde{\zeta}^1\wedge\tilde{\zeta}^2\right) = \left(\frac{1}{rf}+\frac{g'}{gf}\right)e^1\wedge e^4 + 2\frac{g}{r}e^2\wedge e^3$$

and

$$\sum_{j=1}^{4}\omega^{4j}\wedge e^j = \left(\frac{1}{rf}+\frac{g'}{gf}\right)e^4\wedge e^1 + \frac{g}{r}e^3\wedge e^2 - \frac{g}{r}e^2\wedge e^3 = -\left(\frac{1}{rf}+\frac{g'}{gf}\right)e^1\wedge e^4 - 2\frac{g}{r}e^2\wedge e^3\,.$$

Problem 10.5

Verify the expression (10.59) for the fundamental vector field λB.

Solution 10.5

The fundamental vector field $\lambda B \in \mathrm{vect}(S^3)$ is defined by (cf.(10.56))

$$(\lambda B)(z_1,z_2) = T_1\mathrm{bit}_{(z_1,z_2)}(B(1)) = T_1\mathrm{bit}_{(z_1,z_2)}\left(b\frac{\partial}{\partial\alpha}(1)\right)\,.$$

In terms of coordinates, $\mathrm{bit}_{(z_1,z_2)}$ is given by

$$\alpha\mapsto\left(z_1\mathrm{e}^{\mathrm{i}\alpha}, z_2\mathrm{e}^{\mathrm{i}\alpha}\right)\,.$$

Introducing real coordinates y^j according to (10.58) we get for this map the representation

$$\alpha\mapsto\left(y^1\cos\alpha - y^2\sin\alpha\,,\; y^1\sin\alpha + y^2\cos\alpha\,,\; y^3\cos\alpha - y^4\sin\alpha\,,\; y^3\sin\alpha + y^4\cos\alpha\right)\,.$$

Now (7.46) yields

$$(\lambda B)(z_1,z_2) = b\left(-y^2\frac{\partial}{\partial y^1} + y^1\frac{\partial}{\partial y^2} - y^4\frac{\partial}{\partial y^3} + y^3\frac{\partial}{\partial y^4}\right)\,.$$

Problem 10.6
Repeat the calculations of section 10.6 with the symmetry group $SU(2)$ instead of $SO(3)$.

Solution 10.6
Upon replacing the symmetry group $SO(3)$ by $SU(2)$, the 't Hooft–Polyakov ansatz for the scalar fields becomes invariant under gauge transformations of the form

$$\gamma(x) = \exp\left(-\beta(x)\sum_{j=1}^{3} x^j \tau_j\right) \in SU(2) \quad , \quad x \in S^2 .$$

As indicated in (10.89) we have to take the following embedding of $U(1)$ in $SU(2)$:

$$\begin{aligned} U(1) &\to SU(2) \\ \mathrm{e}^{\mathrm{i}\alpha} &\mapsto \exp\left(-2\alpha\sum_{j=1}^{3} x^j \tau_j\right) . \end{aligned}$$

On one side, we have a trivial $SU(2)$ principal bundle over S^2,

$$\tilde{P} = S^2 \times SU(2) ,$$

where gauge transformations are of the form

$$\begin{aligned} S^2 \times SU(2) &\to S^2 \times SU(2) \\ (x, w) &\mapsto (x, \gamma(x)w) \end{aligned}$$

with $\gamma : S^2 \to SU(2)$. On the other side, we look for a $U(1)$ principal bundle (P, S^2, π). In this case, gauge transformations are given by maps

$$\gamma_r : \mathscr{U}_r \to U(1) \quad , \quad r = 1, 2 \quad ,$$

with $\gamma_1 = \gamma_2$ on $\mathscr{U}_1 \cap \mathscr{U}_2$, i.e., we have one function $\tilde{\gamma} : S^2 \to U(1)$, from which we can construct a bundle automorphism $(f, \mathrm{id}_{U(1)}, \mathrm{id}_{S^2})$:

$$\begin{aligned} f : P &\to P \\ p &\mapsto f(p) = p\,\tilde{\gamma}(\pi(p)) . \end{aligned}$$

We require the existence of a bundle map $(f_P, \varphi, \mathrm{id}_{S^2})$ with

$$\begin{aligned} f_P &: P \to \tilde{P} , \\ \varphi &: U(1) \to SU(2) \end{aligned}$$

such that

$$\tilde{\gamma}(x) = \mathrm{e}^{\mathrm{i}\beta(x)/2}$$

(gauge transformation in P) corresponds to

$$\gamma(x) = \exp\left(-\beta(x)\sum_{j=1}^{3} x^j \tau_j\right)$$

(gauge transformation in $\tilde{P}$). The factor 1/2 in the expression for $\tilde{\gamma}$ takes care of the fact that

$$\exp\left(-2\pi\sum_{j=1}^{3}x^j\tau_j\right)=-1\,.$$

The properties (b), (c) in the definition of a bundle map of principal bundles enable us to write

$$f_P(p)=(\pi(p),G(p))\quad,\quad p\in P\,,$$

where $G:P\to SU(2)$ satisfies

$$G\left(p\,\mathrm{e}^{\mathrm{i}\alpha}\right)=G(p)\,\varphi\left(\mathrm{e}^{\mathrm{i}\alpha}\right)\,.$$

The required correspondence of gauge transformations in P and $\tilde{P}$ means:

$$G(p)\,\varphi\left(\mathrm{e}^{\mathrm{i}\beta(x)/2}\right)=\exp\left(-\beta(x)\sum_{j=1}^{3}x^j\tau_j\right)G(p)\quad,\quad x=\pi(p)\,,$$

for all $p\in P$. Introducing local trivializations of P we can write for $p\in\pi^{-1}(\mathscr{U}_r)$

$$G(p)=\tilde{G}_r(\pi(p),f_r(p))\,,$$

where $\tilde{G}_r$ satisfies

$$\tilde{G}_r\left(x,\mathrm{e}^{\mathrm{i}\alpha'+\mathrm{i}\alpha}\right)=\tilde{G}_r\left(x,\mathrm{e}^{\mathrm{i}\alpha'}\right)\varphi\left(\mathrm{e}^{\mathrm{i}\alpha}\right)\,,$$

and we have to fulfill the equation

$$\tilde{G}_r\left(x,\mathrm{e}^{\mathrm{i}\alpha'}\right)\varphi\left(e^{\mathrm{i}\beta/2}\right)=\exp\left(-\beta\sum_{j=1}^{3}x^j\tau_j\right)\tilde{G}_r\left(x,\mathrm{e}^{\mathrm{i}\alpha'}\right)$$

for $r=1,2$. This is possible only if

$$\varphi\left(e^{\mathrm{i}\beta/2}\right)=\exp\left(-\beta\sum_{j=1}^{3}n^j\tau_j\right)$$

with $(n^1)^2+(n^2)^2+(n^3)^2=1$. Choose $n^3=1$, i.e., $\varphi\left(e^{\mathrm{i}\beta/2}\right)=\mathrm{e}^{-\beta\tau_3}$. Then

$$\tilde{G}_1\left(x,\mathrm{e}^{\mathrm{i}\alpha}\right)=\begin{pmatrix}\cos(\vartheta/2) & -\mathrm{e}^{-\mathrm{i}\varphi}\sin(\vartheta/2)\\ \mathrm{e}^{\mathrm{i}\varphi}\sin(\vartheta/2) & \cos(\vartheta/2)\end{pmatrix}\mathrm{e}^{-2\alpha\tau_3}\quad\text{for }\vartheta<\pi$$

and

$$\tilde{G}_2\left(x,\mathrm{e}^{\mathrm{i}\alpha}\right)=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\varphi}\cos(\vartheta/2) & -\sin(\vartheta/2)\\ \sin(\vartheta/2) & \mathrm{e}^{\mathrm{i}\varphi}\cos(\vartheta/2)\end{pmatrix}\mathrm{e}^{-2\alpha\tau_3}\quad\text{for }0<\vartheta$$

do the job. Introducing the abbreviation

$$A(\vartheta,\varphi)=\begin{pmatrix}\cos(\vartheta/2) & -\mathrm{e}^{-\mathrm{i}\varphi}\sin(\vartheta/2)\\ \mathrm{e}^{\mathrm{i}\varphi}\sin(\vartheta/2) & \cos(\vartheta/2)\end{pmatrix}$$

one has to show for $\vartheta < \pi$:

$$\mathrm{e}^{-\beta\tau_3} = A^{-1}(\vartheta,\varphi)\exp\big(-\beta(\sin\vartheta\cos\varphi\,\tau_1 + \sin\vartheta\sin\varphi\,\tau_2 + \cos\vartheta\,\tau_3)\big)A(\vartheta,\varphi)\,.$$

So it suffices to check that

$$A^{-1}(\vartheta,\varphi)(\sin\vartheta\cos\varphi\,\tau_1 + \sin\vartheta\sin\varphi\,\tau_2 + \cos\vartheta\,\tau_3)A(\vartheta,\varphi) = \tau_3\,.$$

This is easily done. The case $0 < \vartheta$ is treated analogously.

For $p \in \pi^{-1}(\mathscr{U}_1) \cap \pi^{-1}(\mathscr{U}_2)$ we must have

$$\tilde{G}_1(\pi(p), f_1(p)) = \tilde{G}_2(\pi(p), f_2(p))\,.$$

With $f_r(p) = \mathrm{e}^{\mathrm{i}\alpha_r}$ this means

$$\begin{pmatrix} \cos(\vartheta/2) & -\mathrm{e}^{-\mathrm{i}\varphi}\sin(\vartheta/2) \\ \mathrm{e}^{\mathrm{i}\varphi}\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} e^{-2\alpha_1\tau_3} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi}\cos(\vartheta/2) & -\sin(\vartheta/2) \\ \sin(\vartheta/2) & \mathrm{e}^{\mathrm{i}\varphi}\cos(\vartheta/2) \end{pmatrix} \mathrm{e}^{-2\alpha_2\tau_3}$$

for $0 < \vartheta < \pi$. One finds: $f_2(p) = \mathrm{e}^{\mathrm{i}\varphi} f_1(p)$. Hence P is the bundle of the Dirac monopole with $m = 1$.

Turning to the connections, we look for the uniquely determined connection $\tilde{\mathscr{A}}$ on $\tilde{P}$ with

$$f_P^*\tilde{\mathscr{A}} = (T_1\varphi)\mathscr{A}\,,$$

where $\mathscr{A}$ is the connection on P represented locally by (10.10), (10.11) for $m = 1$ and

$$\begin{aligned} T_1\varphi : T_1U(1) &\to T_eSU(2) \\ u(1) \ni \mathrm{i}b &\mapsto -2b\,\tau_3 \in su(2)\,. \end{aligned}$$

As in section 10.6 we find for $x \in \mathscr{U}_r$

$$(T_1\varphi)A_{r,x} = \tilde{G}_r(x,1)^{-1}\tilde{A}_x\tilde{G}_r(x,1) + \tilde{G}_r(x,1)^{-1}\mathrm{d}\,\tilde{G}_r(x,1)\,. \qquad (\star)$$

Since $\tilde{G}_1(x,1) = \hat{\gamma}(x)^{-1}$ with $\hat{\gamma}$ as in solution 10.1, we can conclude from that exercise:

$$\begin{aligned} &\tilde{G}_1(x,1)^{-1}\left(\frac{1-h(r)}{r^2}\sum_{a,i,j=1}^{3}\tau_a\varepsilon_{aij}x^j\mathrm{d}x^i\right)\tilde{G}_1(x,1) + \tilde{G}_1(x,1)^{-1}\mathrm{d}\,\tilde{G}_1(x,1) \\ &= \tau_1\big(-h(r)\sin\vartheta\cos\varphi\,\mathrm{d}\varphi - h(r)\sin\varphi\,\mathrm{d}\vartheta\big) + \tau_2\big(-h(r)\sin\vartheta\sin\varphi\,\mathrm{d}\varphi + h(r)\cos\varphi\,\mathrm{d}\vartheta\big) \\ &\quad + \tau_3\big(-(1-\cos\vartheta)\,\mathrm{d}\varphi\big) \xrightarrow[r\to\infty]{} \tau_3\big(-(1-\cos\vartheta)\,\mathrm{d}\varphi\big)\,. \end{aligned}$$

On the other hand, from (10.10) with $m = 1$ we get

$$(T_1\varphi)A_{1,x} = (T_1\varphi)\left(\frac{\mathrm{i}}{2}(1-\cos\vartheta)\,\mathrm{d}\varphi\right) = \tau_3\big(-(1-\cos\vartheta)\,\mathrm{d}\varphi\big)\,.$$

Hence, $(\star)$ with $r = 1$ is satisfied if $\tilde{A}$ is the potential of the 't Hooft–Polyakov ansatz on the sphere at infinity. As is easily checked, the same holds in the case $r = 2$.

Problem 10.7

Prove (10.98) in the special case where G is a matrix group and $I(F^2) = \mathrm{tr}(F \wedge F)$.

Solution 10.7

With the help of the Bianchi identity

$$\mathrm{d}F = -[A, F] = F \wedge A - A \wedge F$$

we get

$$\begin{aligned} \mathrm{d}I(F^2) &= \mathrm{d}\,\mathrm{tr}(F \wedge F) = \mathrm{tr}\big(\mathrm{d}(F \wedge F)\big) = \mathrm{tr}\big((\mathrm{d}F) \wedge F + F \wedge \mathrm{d}F\big) = 2\,\mathrm{tr}\big(F \wedge \mathrm{d}F\big) \\ &= 2\,\mathrm{tr}\big(F \wedge (F \wedge A - A \wedge F)\big) = 2\,\mathrm{tr}\big(F \wedge F \wedge A - F \wedge A \wedge F\big) \\ &= 2\,\mathrm{tr}\big(F \wedge F \wedge A - F \wedge F \wedge A\big) = 0\,. \end{aligned}$$

Problem 10.8

Derive (10.107) from the Chern–Simons formula.

Solution 10.8

The invariant symmetric bilinear form corresponding to

$$I(F^2) = \mathrm{tr}(F \wedge F)$$

is given by

$$I(A_1, A_2) = \mathrm{tr}(A_1 A_2) \quad , \quad A_1, A_2 \in \mathfrak{g}\,.$$

We want to calculate $Q\big(A, \mathring{A}\big)$ for two local potentials ($\mathfrak{g}$-valued 1-forms) A and $\mathring{A}$ according to (10.102). Defining

$$A_\tau = \mathring{A} + \tau\alpha \quad \text{with} \quad \alpha = A - \mathring{A}\,,\ 0 \le \tau \le 1$$

(see (10.99)), we have the corresponding field strength

$$\begin{aligned} F_\tau &= \mathrm{d}A_\tau + A_\tau \wedge A_\tau = \mathrm{d}\mathring{A} + \tau\mathrm{d}\alpha + \mathring{A} \wedge \mathring{A} + \tau\alpha \wedge \mathring{A} + \tau\mathring{A} \wedge \alpha + \tau^2\alpha \wedge \alpha \\ &= \mathring{F} + \tau\big(\mathrm{d}\alpha + \alpha \wedge \mathring{A} + \mathring{A} \wedge \alpha\big) + \tau^2\alpha \wedge \alpha\,. \end{aligned}$$

Now we get

$$\begin{aligned} Q\big(A, \mathring{A}\big) &= 2\int_0^1 \mathrm{d}\tau\,\mathrm{tr}\Big(\big(A - \mathring{A}\big) \wedge F_\tau\Big) \\ &= 2\int_0^1 \mathrm{d}\tau\,\mathrm{tr}\Big(\alpha \wedge \mathring{F} + \tau\big(\alpha \wedge \mathrm{d}\alpha + \alpha \wedge \alpha \wedge \mathring{A} + \alpha \wedge \mathring{A} \wedge \alpha\big) + \tau^2\alpha \wedge \alpha \wedge \alpha\Big) \\ &= \mathrm{tr}\Big(2\alpha \wedge \mathring{F} + \alpha \wedge \mathrm{d}\alpha + \alpha \wedge \alpha \wedge \mathring{A} + \alpha \wedge \mathring{A} \wedge \alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha\Big) \\ &= \mathrm{tr}\Big(2\alpha \wedge \mathring{F} + \alpha \wedge \mathrm{d}\alpha + 2\alpha \wedge \mathring{A} \wedge \alpha + \frac{2}{3}\alpha \wedge \alpha \wedge \alpha\Big)\,. \end{aligned}$$

Inserting this expression in the Chern-Simons formula (10.101) one arrives at (10.107).

Problem 10.9
Verify (10.114).

Solution 10.9
Since we want to apply Stokes' theorem, we first have to think about orientations. The boundaries of $S^n_\pm$ can be identified with S^{n-1}:

$$\partial S^n_+ = \partial S^n_- = \big\{(x^1,\dots,x^{n+1}) \in S^n \,|\, x^{n+1}=0\big\} = S^{n-1} \subset S^n\,.$$

S^n and S^{n-1} are oriented according to the rule given on p. 111. S^n_+ and S^n_- inherit their orientations from S^n, and the orientations of $\partial S^n_\pm$ are as explained in section 7.10. So the orientations of ∂S^n_+ and ∂S^n_- are opposite to each other and we have to find out which one agrees with the orientation of S^{n-1}. To answer this question it suffices to consider one point in ∂S^n_+. Take, for instance, $x_0 = (1,0,\dots,0) \in \partial S^n_+ \subset \mathbb{R}^{n+1}$. $T_{x_0}S^n$ is spanned by $\dfrac{\partial}{\partial x^2}(x_0),\dots,\dfrac{\partial}{\partial x^{n+1}}(x_0)$ and an outward normal vector is $n(x_0) = \dfrac{\partial}{\partial x^1}(x_0)$. Since $\dfrac{\partial}{\partial x^1}(x_0),\dots,\dfrac{\partial}{\partial x^{n+1}}(x_0)$ is an oriented basis of $T_{x_0}\mathbb{R}^{n+1}$, an oriented basis of $T_{x_0}S^n$ is given by $\dfrac{\partial}{\partial x^2}(x_0),\dots,\dfrac{\partial}{\partial x^{n+1}}(x_0)$. An outward normal vector of $S^n_+ \subset S^n$ at x_0 is $n_+(x_0) := -\dfrac{\partial}{\partial x^{n+1}}(x_0)$. So

$$\frac{\partial}{\partial x^2}(x_0),\dots,\frac{\partial}{\partial x^n}(x_0), -n_+(x_0)$$

is an oriented basis of $T_{x_0}S^n_+$, whence

$$(-1)^n n_+(x_0), \frac{\partial}{\partial x^2}(x_0),\dots,\frac{\partial}{\partial x^n}(x_0)$$

is also an oriented basis of $T_{x_0}S^n_+$. Consequently, $\dfrac{\partial}{\partial x^2}(x_0),\dots,\dfrac{\partial}{\partial x^n}(x_0)$ is an oriented basis of $T_{x_0}\partial S^n_+$ if n is even. On the other hand, an oriented basis of $T_{x_0}S^{n-1}$ is given by $\dfrac{\partial}{\partial x^2}(x_0),\dots,\dfrac{\partial}{\partial x^n}(x_0)$ as the above considerations show. Therefore the orientations of ∂S^n_+ and S^{n-1} agree for even n. (If n is odd, ∂S^n_- and S^{n-1} have the same orientation.)

Starting from (10.113) and using (10.108) we get with the help of Stokes' theorem:

$$\begin{aligned}\int_{S^4} c_2 &= \frac{1}{8\pi^2}\int_{S^4_+} \mathrm{d}\Big(\mathrm{tr}\big((\mathrm{d}A_1)\wedge A_1 + \tfrac{2}{3}A_1\wedge A_1\wedge A_1\big)\Big)\\ &\quad+\frac{1}{8\pi^2}\int_{S^4_-} \mathrm{d}\Big(\mathrm{tr}\big((\mathrm{d}A_2)\wedge A_2 + \tfrac{2}{3}A_2\wedge A_2\wedge A_2\big)\Big)\\ &= \frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}\big((\mathrm{d}A_1)\wedge A_1 + \tfrac{2}{3}A_1\wedge A_1\wedge A_1\big)\\ &\quad-\frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}\big((\mathrm{d}A_2)\wedge A_2 + \tfrac{2}{3}A_2\wedge A_2\wedge A_2\big)\,.\end{aligned}$$

On S^3 we have

$$A_2 = \gamma A_1 \gamma^{-1} + \gamma \mathrm{d}\gamma^{-1}$$

(cf. (10.91)) with γ given by (10.30) for $r = 1$. Writing A instead of A_1 we obtain

$$\begin{aligned}
8\pi^2 \int_{S^4} c_2 = \int_{S^3} \mathrm{tr}\Big\{ &(\mathrm{d}A) \wedge A + \tfrac{2}{3} A \wedge A \wedge A - (\mathrm{d}\gamma) \wedge A \wedge A\gamma^{-1} - (\mathrm{d}A) \wedge A \\
&+ A \wedge (\mathrm{d}\gamma^{-1}) \wedge \gamma A - (\mathrm{d}\gamma) \wedge (\mathrm{d}\gamma^{-1}) \wedge \gamma A \gamma^{-1} - (\mathrm{d}\gamma) \wedge A \wedge \mathrm{d}\gamma^{-1} \\
&- \gamma(\mathrm{d}A) \wedge \mathrm{d}\gamma^{-1} + \gamma A \wedge (\mathrm{d}\gamma^{-1}) \wedge \gamma \mathrm{d}\gamma^{-1} - (\mathrm{d}\gamma) \wedge (\mathrm{d}\gamma^{-1}) \wedge \gamma \mathrm{d}\gamma^{-1} \\
-\tfrac{2}{3}\big[&A \wedge A \wedge A + \gamma A \wedge A \wedge \mathrm{d}\gamma^{-1} + A \wedge (\mathrm{d}\gamma^{-1}) \wedge \gamma A \\
&+ \gamma A \wedge (\mathrm{d}\gamma^{-1}) \wedge \gamma \mathrm{d}\gamma^{-1} + (\mathrm{d}\gamma^{-1}) \wedge \gamma A \wedge A + \gamma(\mathrm{d}\gamma^{-1}) \wedge \gamma A \wedge \mathrm{d}\gamma^{-1} \\
&+ (\mathrm{d}\gamma^{-1}) \wedge \gamma(\mathrm{d}\gamma^{-1}) \wedge \gamma A + \gamma(\mathrm{d}\gamma^{-1}) \wedge \gamma(\mathrm{d}\gamma^{-1}) \wedge \gamma(\mathrm{d}\gamma^{-1}) \big] \Big\} \\
= \int_{S^3} \mathrm{tr}\Big[&(\gamma^{-1}\mathrm{d}\gamma) \wedge (\gamma^{-1}\mathrm{d}\gamma) \wedge A + (\gamma^{-1}\mathrm{d}\gamma) \wedge \mathrm{d}A - \tfrac{1}{3}(\gamma^{-1}\mathrm{d}\gamma) \wedge (\gamma^{-1}\mathrm{d}\gamma) \wedge (\gamma^{-1}\mathrm{d}\gamma) \Big] .
\end{aligned}$$

(Note that $\gamma \mathrm{d}\gamma^{-1} = -(\mathrm{d}\gamma)\gamma^{-1}$.) Since

$$\mathrm{d}\big((\gamma^{-1}\mathrm{d}\gamma) \wedge A\big) = -(\gamma^{-1}\mathrm{d}\gamma) \wedge (\gamma^{-1}\mathrm{d}\gamma) \wedge A - (\gamma^{-1}\mathrm{d}\gamma) \wedge \mathrm{d}A$$

and $\partial S^3 = \emptyset$, we find

$$\begin{aligned}
8\pi^2 \int_{S^4} c_2 &= \int_{S^3} \mathrm{tr}\Big[- \mathrm{d}\big((\gamma^{-1}\mathrm{d}\gamma) \wedge A\big) - \tfrac{1}{3}(\gamma^{-1}\mathrm{d}\gamma) \wedge (\gamma^{-1}\mathrm{d}\gamma) \wedge (\gamma^{-1}\mathrm{d}\gamma) \Big] \\
&= -\tfrac{1}{3} \int_{S^3} \mathrm{tr}\Big((\gamma^{-1}\mathrm{d}\gamma) \wedge (\gamma^{-1}\mathrm{d}\gamma) \wedge (\gamma^{-1}\mathrm{d}\gamma) \Big) .
\end{aligned}$$

11 Spin

Problem 11.1

Construct an isomorphism between $Spin(0,3)$ and $SU(2)$.

Solution 11.1

$$\begin{aligned}
&\text{Let} \quad v := v_1 e^1 + v_2 e^2 + v_3 e^3, \qquad \text{with} \quad \sum_{a=1}^{3} v_a^2 = 1, \\
&\text{and} \quad w := w_1 e^1 + w_2 e^2 + w_3 e^3, \quad \text{with} \quad \sum_{a=1}^{3} w_a^2 = 1.
\end{aligned}$$

Then the Clifford product vw is

$$\begin{aligned}
vw = &(-v_1 w_1 - v_2 w_2 - v_3 w_3)\mathbb{1} + (v_1 w_2 - v_2 w_1) e^1 e^2 \\
&+ (v_2 w_3 - v_3 w_2) e^2 e^3 + (v_3 w_1 - v_1 w_3) e^3 e^1.
\end{aligned}$$

Since for any unit vectors $\vec{v}$, $\vec{w}$ in $\mathbb{R}^3$, we have

$$(\vec{v} \cdot \vec{w})^2 + (\vec{v} \times \vec{w})^2 = 1,$$

and on the other hand any unit vector in $\mathbb{R}^3$ can be written as vector product of two unit vectors, any element of the form

$$x_1 \mathbb{1} - x_2 e^1 e^2 - x_3 e^2 e^3 - x_4 e^3 e^1 \quad \text{with} \quad \sum_{i=1}^{4} x_i^2 = 1$$

is element of $Spin(0,3)$. Furthermore any $Spin(0,3)$ element can be written in this form, because

$$\begin{aligned}(x_1 \mathbb{1} - x_2 e^1 e^2 - x_3 e^2 e^3 - x_4 e^3 e^1)&(y_1 \mathbb{1} - y_2 e^1 e^2 - y_3 e^2 e^3 - y_4 e^3 e^1) \\ &= (z_1 \mathbb{1} - z_2 e^1 e^2 - z_3 e^2 e^3 - z_4 e^3 e^1)\end{aligned}$$

with

$$\begin{aligned} z_1 &= x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4, \\ z_2 &= x_1 y_2 + x_2 y_1 - x_3 y_4 + x_4 y_3, \\ z_3 &= x_1 y_3 + x_3 y_1 + x_2 y_4 - x_4 y_2, \\ z_4 &= x_1 y_4 + x_4 y_1 + x_3 y_2 - x_2 y_3. \end{aligned} \tag{$\star$}$$

The fact that $\sum_{i=1}^{4} z_i^2 = 1$, will follow from the group isomorphism to $SU(2)$, where the $SU(2)$ elements are written in the form of (8.11):

$$\begin{pmatrix} x_1 + \mathrm{i}\, x_2 & -x_3 + \mathrm{i}\, x_4 \\ x_3 + \mathrm{i}\, x_4 & x_1 - \mathrm{i}\, x_2 \end{pmatrix} \quad \text{with} \quad \sum_{i=1}^{4} x_i^2 = 1.$$

Indeed,

$$\begin{pmatrix} x_1 + \mathrm{i}\, x_2 & -x_3 + \mathrm{i}\, x_4 \\ x_3 + \mathrm{i}\, x_4 & x_1 - \mathrm{i}\, x_2 \end{pmatrix} \begin{pmatrix} y_1 + \mathrm{i}\, y_2 & -y_3 + \mathrm{i}\, y_4 \\ y_3 + \mathrm{i}\, y_4 & y_1 - \mathrm{i}\, y_2 \end{pmatrix} = \begin{pmatrix} z_1 + \mathrm{i}\, z_2 & -z_3 + \mathrm{i}\, z_4 \\ z_3 + \mathrm{i}\, z_4 & z_1 - \mathrm{i}\, z_2 \end{pmatrix}$$

with the z_i satisfying the equations ($\star$).

Problem 11.2
Calculate the square of γ_5.

Solution 11.2

$$\begin{aligned} \gamma_5 :&= \rho(e^1 e^2 \cdots e^n), \\ \gamma_5^2 &= \rho(e^1 e^2 \cdots e^n)\rho(e^1 e^2 \cdots e^n) = \rho(e^1 e^2 \cdots e^n \, e^1 e^2 \cdots e^n) \\ &= (-1)^{n-1} \rho(e^1 e^2 \cdots e^n \, e^n e^1 e^2 \cdots e^{n-1}) = (-1)^{n-1} \eta^{nn} \rho(e^1 e^2 \cdots e^{n-1} \, e^1 e^2 \cdots e^{n-1}) \\ &= (-1)^{\sum_{i=1}^{n-1} i} \prod_{i=1}^{n} \eta^{ii} \rho(\mathbb{1}) = (-1)^{(n-1)n/2+s} \mathbb{1}. \end{aligned}$$

Problem 11.3

Calculate the square of the Dirac operator in a flat, torsion-free space.

Solution 11.3

$$\begin{aligned} \not{D}\psi &= \gamma^\mu \partial_\mu \psi, \quad \partial_\mu := \partial/\partial x^\mu \text{ and summation convention,} \\ \not{D}\not{D}\psi &= \gamma^\nu \partial_\nu \, \gamma^\mu \partial_\mu \psi = \gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \psi = \tfrac{1}{2}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)\partial_\nu\partial_\mu\psi = \eta^{\mu\nu}\partial_\nu\partial_\mu\psi = \Box\psi. \end{aligned}$$

Problem 11.4

Show that (11.72) defines an invariant sesquilinear form.

Solution 11.4

First let us show that for any sesquilinear form $(\cdot,\cdot)$

$$(\gamma^i\psi,\chi) = (\psi,\gamma^i\chi) \quad \text{for all} \quad i = 1,2,\ldots,n \qquad (\star)$$

implies

$$(\rho_g\psi,\rho_g\chi) = (\psi,\chi) \quad \text{for all} \quad g \in Spin(r,s)^e.$$

It is sufficient to prove this last equation for any generator g of $Spin(r,s)^e$. The most general such generator is

$$g = (v_i e^i)(w_j e^j),$$

using the summation convention with real coefficients v_i, w_j satisfying

$$v_i \eta^{ii} v_i = w_j \eta^{jj} w_j = \pm 1.$$

Then

$$\begin{aligned} (\rho_g\psi,\rho_g\chi) &= v_i w_j v_k w_l \, (\gamma^i\gamma^j\psi, \gamma^k\gamma^l\chi) = v_i v_k w_j w_l \, (\psi, \gamma^j\gamma^i\gamma^k\gamma^l\chi) \\ &= v_i v_k \eta^{ik} w_j w_l \, (\psi, \gamma^j\gamma^l\chi) = (v_i\eta^{ii}v_i)(w_j\eta^{jj}w_j)\,(\psi,\chi) \quad = (\psi,\chi). \end{aligned}$$

Now we consider the special case $r = 1$, $s = n-1$ and unitary γ^i:

$$(\gamma^i)^+ = (\gamma^i)^{-1}, \quad i = 0,1,\ldots,n-1.$$

Then we have:

$$\left(\gamma^0\right)^2 = \mathbb{1} \quad \text{and} \quad \left(\gamma^i\right)^2 = -\mathbb{1} \text{ for } i = 1,2,\ldots,n.$$

Therefore

$$\gamma^{0+} = \gamma^0 \quad \text{and} \quad \gamma^{i+} = -\gamma^i \text{ for } i = 1,2,\ldots,n,$$

or

$$\gamma^{i+}\gamma^0 = \gamma^0\gamma^i \text{ for } i = 0,1,2,\ldots,n.$$

Now we consider also the special sesquilinear form $(\psi,\chi) := \psi^+\gamma^0\chi$. Then we show:

$$(\gamma^i\psi,\chi) = (\gamma^i\psi)^+\gamma^0\chi = \psi^+\gamma^{i+}\gamma^0\chi = \psi^+\gamma^0\gamma^i\chi = (\psi,\gamma^i\chi),$$

and our first implication proves the invariance of the last sesquilinear form.

<u>Problem 11.5</u>
Verify the second equation of (11.74).

<u>Solution 11.5</u>
We use again the summation convention. We suppose that the field $\psi(x)$ decreases sufficiently fast for large x and that we can neglect integrals over an exact form of maximal degree ('surface terms'). We also suppose that the sesquilinear form satisfies the property $(\star)$ in the Solution 11.4.

$$
\begin{aligned}
&\overline{\mathrm{i}\int *(\psi, \not{D}\psi)} = -\mathrm{i}\int *(\not{D}\psi, \psi) \\
&= -\,\mathrm{i}\int (\gamma^{-1})^{\mu}{}_{i}(\gamma^{i}[\partial_\mu + \tfrac{1}{4}\omega_{ab\mu}\gamma^a\gamma^b]\psi, \psi)\ \det\gamma\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n \\
&= -\,\mathrm{i}\int \cancel{\partial_\mu[\det\gamma\,(\gamma^{-1})^{\mu}{}_{i}(\gamma^i\psi, \psi)]\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n} \\
&\quad +\mathrm{i}\int \left\{\partial_\mu[\det\gamma\,(\gamma^{-1})^{\mu}{}_{i}]\right\}(\gamma^i\psi, \psi)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n \\
&\quad +\mathrm{i}\int \det\gamma\,(\gamma^{-1})^{\mu}{}_{i}\,(\gamma^i\psi, \partial_\mu\psi)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n \\
&\quad -\mathrm{i}\int \det\gamma\,(\gamma^{-1})^{\mu}{}_{i}\,\tfrac{1}{4}\omega_{ab\mu}(\gamma^i\gamma^a\gamma^b\psi, \psi)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n \\
&= +\mathrm{i}\int \left\{\partial_\mu[\det\gamma\,(\gamma^{-1})^{\mu}{}_{i}]\right\}(\psi, \gamma^i\psi)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n \\
&\quad +\mathrm{i}\int \det\gamma\,(\gamma^{-1})^{\mu}{}_{i}\,(\psi, \gamma^i\partial_\mu\psi)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n \\
&\quad +\mathrm{i}\int \det\gamma\,(\gamma^{-1})^{\mu}{}_{i}\,\tfrac{1}{4}\omega_{ab\mu}(\psi, \gamma^a\gamma^b\gamma^i\psi)\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^n =: \star.
\end{aligned}
$$

In the last summand we substitute

$$\gamma^a\gamma^b\gamma^i = \gamma^i\gamma^a\gamma^b - \gamma^b 2\eta^{ia} + \gamma^a 2\eta^{ib}$$

and re-arrange $\star$ to take the form

$$\star = \mathrm{i}\int *F_i(\psi, \gamma^i\psi) + \mathrm{i}\int *(\psi, \not{D}\psi)$$

with

$$F_i := \det\gamma^{-1}\,\partial_\mu[\det\gamma\,(\gamma^{-1})^{\mu}{}_{i}] - \omega^{k}{}_{i\mu}(\gamma^{-1})^{\mu}{}_{k}.$$

We use successively for the square matrix $\gamma(x)$

$$\partial_\mu \det\gamma = \det\gamma\,\mathrm{tr}(\gamma^{-1}\partial_\mu\gamma), \quad \partial_\mu\gamma^{-1} = -\gamma^{-1}(\partial_\mu\gamma)\,\gamma^{-1},$$

and the gauge transformation (5.84),

$$\partial_\mu \gamma^a{}_\nu = \gamma^a{}_\alpha \, \Gamma^\alpha{}_{\nu\mu} - \omega^a{}_{b\mu} \, \gamma^b{}_\nu,$$

to compute

$$\begin{aligned}
F_i &= \det\gamma^{-1} (\partial_\mu \det\gamma)\, (\gamma^{-1})^\mu{}_i + \partial_\mu (\gamma^{-1})^\mu{}_i - \omega^k{}_{i\mu} (\gamma^{-1})^\mu{}_k \\
&= (\gamma^{-1})^\nu{}_a (\partial_\mu \gamma^a{}_\nu)(\gamma^{-1})^\mu{}_i - (\gamma^{-1})^\mu{}_a (\partial_\mu \gamma^a{}_\nu)(\gamma^{-1})^\nu{}_i - \omega^k{}_{i\mu} (\gamma^{-1})^\mu{}_k \\
&= (\gamma^{-1})^\nu{}_a \, \gamma^a{}_\alpha \, \Gamma^\alpha{}_{\nu\mu} \, (\gamma^{-1})^\mu{}_i - \cancel{(\gamma^{-1})^\nu{}_a \, \omega^a{}_{b\mu}} \, \cancel{\gamma^b{}_\nu \, (\gamma^{-1})^\mu{}_i} \\
&\quad - (\gamma^{-1})^\mu{}_a \, \gamma^a{}_\alpha \, \Gamma^\alpha{}_{\nu\mu} \, (\gamma^{-1})^\nu{}_i + (\gamma^{-1})^\mu{}_a \, \cancel{\omega^a{}_{b\mu} \, \gamma^b{}_\nu} (\gamma^{-1})^\nu{}_i \\
&\quad - \cancel{\omega^k{}_{i\mu} (\gamma^{-1})^\mu{}_k} \\
&= (\Gamma^\nu{}_{\nu\mu} - \Gamma^\nu{}_{\mu\nu})\, (\gamma^{-1})^\mu{}_i .
\end{aligned}$$

Finally we develop the torsion:

$$\begin{aligned}
T^a &= \tfrac{1}{2} \, T^a{}_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = \mathrm{d}(\gamma^a{}_\nu \, \mathrm{d}x^\nu) + \omega^a{}_{b\mu} \, \mathrm{d}x^\mu \wedge \gamma^b{}_\nu \, \mathrm{d}x^\nu \\
&= \tfrac{1}{2} \left[\partial_\mu \gamma^a{}_\nu - \partial_\nu \gamma^a{}_\mu + \omega^a{}_{b\mu} \, \gamma^b{}_\nu - \omega^a{}_{b\nu} \, \gamma^b{}_\mu \right] \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu .
\end{aligned}$$

Therefore we have

$$\begin{aligned}
T^a{}_{\mu\nu} &= \partial_\mu \gamma^a{}_\nu - \partial_\nu \gamma^a{}_\mu + \omega^a{}_{b\mu} \, \gamma^b{}_\nu - \omega^a{}_{b\nu} \, \gamma^b{}_\mu \\
&= \gamma^a{}_\alpha \, \Gamma^\alpha{}_{\nu\mu} - \cancel{\omega^a{}_{b\mu} \, \gamma^b{}_\nu} - \gamma^a{}_\alpha \, \Gamma^\alpha{}_{\mu\nu} + \cancel{\omega^a}{}_{b\nu} \, \gamma^b{}_\mu + \cancel{\omega^a{}_{b\mu} \, \gamma^b{}_\nu} - \cancel{\omega^a}{}_{b\nu} \, \gamma^b{}_\mu ,
\end{aligned}$$

and

$$F_i = (\Gamma^\nu{}_{\nu\mu} - \Gamma^\nu{}_{\mu\nu})\, (\gamma^{-1})^\mu{}_i = T^a{}_{\mu\nu} \, (\gamma^{-1})^\nu{}_a \, (\gamma^{-1})^\mu{}_i = T^a{}_{ia},$$

with

$$T^a = \tfrac{1}{2} \, T^a{}_{rs} \, e^r \wedge e^s .$$

The same computation in orthonormal rather then holonomic frames is more compact and it is instructive to compare the two.

From equation (11.62) we know that $\mathrm{D}\gamma^i = 0$. We will use the Leibniz rule for the exterior covariant derivative:

$$
\begin{aligned}
\overline{\mathrm{i}\int *(\psi, \not{D}\psi)} &= \overline{\frac{\mathrm{i}}{(n-1)!}\int (\psi, \gamma^{i_1}\mathrm{D}\psi)\wedge e^{i_2}\wedge\cdots\wedge e^{i_n}\,\varepsilon_{i_1\ldots i_n}} \\
&= \frac{-\mathrm{i}}{(n-1)!}\int (\gamma^{i_1}\mathrm{D}\psi, \psi)\wedge e^{i_2}\wedge\cdots\wedge e^{i_n}\,\varepsilon_{i_1\ldots i_n} \\
&= \frac{-\mathrm{i}}{(n-1)!}\int \mathrm{D}[(\gamma^{i_1}\psi, \psi)\wedge e^{i_2}\wedge\cdots\wedge e^{i_n}\,\varepsilon_{i_1\ldots i_n}] \\
&\quad + \frac{\mathrm{i}}{(n-1)!}\int (\gamma^{i_1}\psi, \mathrm{D}\psi)\wedge e^{i_2}\wedge\cdots\wedge e^{i_n}\,\varepsilon_{i_1\ldots i_n} \\
&\quad + \frac{\mathrm{i}}{(n-2)!}\int (\gamma^{i_1}\psi, \psi)\wedge T^{i_2}\wedge e^{i_3}\wedge\cdots\wedge e^{i_n}\,\varepsilon_{i_1\ldots i_n} \\
&= \frac{-\mathrm{i}}{(n-1)!}\int \cancel{\mathrm{d}[(\gamma^{i_1}\psi, \psi)\wedge e^{i_2}\wedge\cdots\wedge e^{i_n}\,\varepsilon_{i_1\ldots i_n}]} + \mathrm{i}\int *(\psi, \not{D}\psi) \\
&\quad + \frac{\mathrm{i}}{(n-2)!}\int (\gamma^{i_1}\psi, \psi)\wedge \tfrac{1}{2}\,T^{i_2}{}_{rs}\,e^r\wedge e^s\wedge e^{i_3}\wedge\cdots\wedge e^{i_n}\,\varepsilon_{i_1\ldots i_n} \\
&= \mathrm{i}\int *(\psi, \not{D}\psi) + \mathrm{i}\int *(\psi, \gamma^a\psi)T^b{}_{ab},
\end{aligned}
$$

where we have used the identity:

$$
e^r\wedge e^s\wedge e^{i_3}\wedge\cdots\wedge e^{i_n}\,\varepsilon_{i_1\ldots i_n} = e^1\wedge\cdots\wedge e^n\,(n-2)!\,(\delta^r{}_{i_1}\,\delta^s{}_{i_2} - \delta^s{}_{i_1}\,\delta^r{}_{i_2}).
$$

12 An algebraic approach to anomalies

Problem 12.1

Prove the properties (12.12) and (12.13) of the Ward operator.

Solution 12.1

$$
\begin{aligned}
W(E)(p\wedge p') &= [p(\varphi^i + f^i)\wedge p'(\varphi^i + f^i) - p(\varphi^i)\wedge p'(\varphi^i)]_{\mathrm{lin}}|_{f^i=-R_E\varphi^i} \\
&= [\cancel{p(\varphi^i)\wedge p'(\varphi^i)} + p(\varphi^i)\wedge[p'(\varphi^i+f^i) - p'(\varphi^i)] + [p(\varphi^i+f^i) - p(\varphi^i)]\wedge p'(\varphi^i) \\
&\quad + [p(\varphi^i+f^i) - p(\varphi^i)]\wedge[p'(\varphi^i+f^i) - p'(\varphi^i)] - \cancel{p(\varphi^i)\wedge p'(\varphi^i)}]_{\mathrm{lin}}|_{f^i=-R_E\varphi^i} \\
&= \left\{[p(\varphi^i+f^i) - p(\varphi^i)]_{\mathrm{lin}}\wedge p'(\varphi^i) + p(\varphi^i)\wedge[p'(\varphi^i+f^i) - p'(\varphi^i)]_{\mathrm{lin}}\right\}|_{f^i=-R_E\varphi^i} \\
&= (W(E)p)\wedge p' + p\wedge W(E)p'
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}W(E)p &= \mathrm{d}\left\{[p(\varphi^i+f^i) - p(\varphi^i)]_{\mathrm{lin}}|_{f^i=-R_E\varphi^i}\right\} = [\mathrm{d}p(\varphi^i+f^i) - \mathrm{d}p(\varphi^i)]_{\mathrm{lin}}|_{f^i=-R_E\varphi^i} \\
&= W(E)\mathrm{d}p
\end{aligned}
$$

Problem 12.2

Show that the Adler-Bardeen anomaly (12.33) is of Stora's type.

Solution 12.2

Let us rewrite equation (12.29) using $[\Omega, A] = -[A, \Omega]$ and the symmetry of I:

$$
\begin{aligned}
\mathfrak{a}(\Omega) = {} & 3\int_0^1 \mathrm{d}\tau I(\Omega, F_\tau, F_\tau) - 6\int_0^1 \mathrm{d}\tau(\tau^2-\tau) I([A,\Omega], F_\tau, A) \\
& \hspace{6cm} \text{(now use the invariance of } I\text{)} \\
= {} & 3\int_0^1 \mathrm{d}\tau I(\Omega, F_\tau, F_\tau) + 6\int_0^1 \mathrm{d}\tau(\tau^2-\tau) I(\Omega, [A, F_\tau], A) \\
& + 6\int_0^1 \mathrm{d}\tau(\tau^2-\tau) I(\Omega, F_\tau, [A,A]) \qquad \text{(now use the definition of } F_\tau\text{)} \\
= {} & 3\int_0^1 \mathrm{d}\tau\, \tau^2 I(\Omega, \mathrm{d}A, \mathrm{d}A) + 3\int_0^1 \mathrm{d}\tau\, \tau^3 I(\Omega, \mathrm{d}A, [A,A]) \\
& + \tfrac{3}{4}\int_0^1 \mathrm{d}\tau\, \tau^4 I(\Omega, [A,A], [A,A]) \qquad \text{(Jacobi identity } \Downarrow\text{)} \\
& + 6\int_0^1 \mathrm{d}\tau(\tau^3-\tau^2) I(\Omega, [A, \mathrm{d}A], A) + \tfrac{6}{2}\int_0^1 \mathrm{d}\tau(\tau^4-\tau^3) I(\Omega, \cancel{[A,[A,A]]}, A) \\
& + 6\int_0^1 \mathrm{d}\tau(\tau^3-\tau^2) I(\Omega, \mathrm{d}A, [A,A]) + \tfrac{6}{2}\int_0^1 \mathrm{d}\tau(\tau^4-\tau^3) I(\Omega, [A,A], [A,A]) \\
= {} & I(\Omega, \mathrm{d}A, \mathrm{d}A) + \tfrac{3}{4} I(\Omega, \mathrm{d}A, [A,A]) + \cancel{\tfrac{3}{20} I(\Omega, [A,A], [A,A])} \\
& - \tfrac{1}{2} I(\Omega, [A, \mathrm{d}A], A) - \tfrac{1}{2} I(\Omega, \mathrm{d}A, [A,A]) - \cancel{\tfrac{3}{20} I(\Omega, [A,A], [A,A])} \\
= {} & I(\Omega, \mathrm{d}A, \mathrm{d}A) + \tfrac{1}{4} I(\Omega, \mathrm{d}A, [A,A]) + \tfrac{1}{2} I(\Omega, [\mathrm{d}A, A], A),
\end{aligned}
$$

which is equation (12.32). Note that $-[A, \mathrm{d}A] = +[\mathrm{d}A, A]$ because under the wedge product a (real-valued) 1-form commutes with a 2-form.

Consider the particular trilinear symmetric invariant (12.27) that comes from a representation $\tilde{\rho}$ of $\mathfrak{g}$, for notational ease we write $\tilde{\rho}(A)$ instead of $\tilde{\rho}_A$:

$$I(A_1, A_2, A_3) := \frac{1}{3!}\sum_{\pi\in\mathscr{S}_3} \mathrm{tr}\{\tilde{\rho}(A_{\pi(1)})\tilde{\rho}(A_{\pi(2)})\tilde{\rho}(A_{\pi(3)})\}.$$

Then we have

$$I(\Omega, \mathrm{d}A, \mathrm{d}A) = \mathrm{tr}\{\tilde{\rho}(\Omega)\tilde{\rho}(\mathrm{d}A)\tilde{\rho}(\mathrm{d}A)\},$$

because all forms are of even degree and the trace is cyclic;

$$\begin{aligned} I(\Omega, \mathrm{d}A, [A,A]) &= \tfrac{1}{2}\,\mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho(\mathrm{d}A)\tilde\rho([A,A])\} + \tfrac{1}{2}\,\mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho([A,A])\tilde\rho(\mathrm{d}A)\} \\ &= \mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho(\mathrm{d}A)\tilde\rho(A)^2\} + \mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho(A)^2\tilde\rho(\mathrm{d}A)\}, \end{aligned}$$

because $\tilde\rho([A,A]) = 2\,\tilde\rho(A)^2$; and

$$\begin{aligned} I(\Omega, [\mathrm{d}A, A], A) &= \tfrac{1}{2}\,\mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho([\mathrm{d}A,A])\tilde\rho(A)\} - \tfrac{1}{2}\,\mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho(A)\tilde\rho([\mathrm{d}A,A])\} \\ &= \tfrac{1}{2}\,\mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho(\mathrm{d}A)\tilde\rho(A)^2\} - \underline{\tfrac{1}{2}\,\mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho(A)\tilde\rho(\mathrm{d}A)\tilde\rho(A)\}} \\ &\quad - \underline{\tfrac{1}{2}\,\mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho(A)\tilde\rho(\mathrm{d}A)\tilde\rho(A)\}} + \tfrac{1}{2}\,\mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho(A)^2\tilde\rho(\mathrm{d}A)\}, \end{aligned}$$

because $\tilde\rho([\mathrm{d}A, A]) = \tilde\rho(\mathrm{d}A)\tilde\rho(A) - \tilde\rho(A)\tilde\rho(\mathrm{d}A)$. Finally we can compute the Adler-Bardeen anomaly:

$$\begin{aligned} \mathfrak{a}(\Omega) &= I(\Omega, \mathrm{d}A, \mathrm{d}A) + \tfrac{1}{4}\,I(\Omega, \mathrm{d}A, [A,A]) + \tfrac{1}{2}\,I(\Omega, [\mathrm{d}A, A], A) \\ &= \mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho(\mathrm{d}A)\tilde\rho(\mathrm{d}A)\} + \tfrac{1}{4}\,[\mathrm{tr}\{\tilde\rho(\Omega)\underline{\tilde\rho(\mathrm{d}A)\tilde\rho(A)^2\}} + \mathrm{tr}\{\tilde\rho(\Omega)\underline{\tilde\rho(A)^2\tilde\rho(\mathrm{d}A)\}}] \\ &\quad + \tfrac{1}{2}\,[\tfrac{1}{2}\,\mathrm{tr}\{\tilde\rho(\Omega)\underline{\tilde\rho(\mathrm{d}A)\tilde\rho(A)^2\}} - \mathrm{tr}\{\tilde\rho(\Omega)\tilde\rho(A)\tilde\rho(\mathrm{d}A)\tilde\rho(A)\} \\ &\qquad + \tfrac{1}{2}\,\mathrm{tr}\{\tilde\rho(\Omega)\underline{\tilde\rho(A)^2\tilde\rho(\mathrm{d}A)\}}] \\ &= \mathrm{tr}\{\tilde\rho(\Omega)[\tilde\rho(\mathrm{d}A)\tilde\rho(\mathrm{d}A) + \tfrac{1}{2}\,\tilde\rho(\mathrm{d}A)\tilde\rho(A)^2 - \tfrac{1}{2}\,\tilde\rho(A)\tilde\rho(\mathrm{d}A)\tilde\rho(A) + \tfrac{1}{2}\,\tilde\rho(A)^2\tilde\rho(\mathrm{d}A)]\} \\ &= \mathrm{tr}\{\tilde\rho(\Omega)\,\mathrm{d}[\tilde\rho(A)\wedge\mathrm{d}\tilde\rho(A) + \tfrac{1}{2}\,\tilde\rho(A)^3]\}. \end{aligned}$$

Problem 12.3

Verify that (12.46) defines a representation of the Lie algebra $\mathfrak{E}$.

Solution 12.3

$$W(\Omega, v)A = \mathrm{d}\Omega + [A, \Omega] - L_v A + \mathrm{d}i_v\mathring{A} + [A, i_v\mathring{A}].$$

It is sufficient to treat three sub-problems: $[(\Omega', 0), (\Omega, 0)]$, $[(0, v'), (0, v)]$ and $[(0, v), (\Omega, 0)]$.

- $[\Omega', \Omega]$

$$\begin{aligned} W(\Omega', 0)W(\Omega, 0)A &= W(\Omega', 0)(\mathrm{d}\Omega + [A, \Omega]) = [\mathrm{d}\Omega' + [A, \Omega'], \Omega] \\ &= [\mathrm{d}\Omega', \Omega] + [A, [\Omega', \Omega]] + [[A, \Omega], \Omega'], \end{aligned}$$

$$
\begin{aligned}
[W(\Omega',0),W(\Omega,0)]A = {} & [\mathrm{d}\Omega',\Omega]+[A,[\Omega',\Omega]]+[[A,\Omega],\Omega'] \\
& -[\mathrm{d}\Omega,\Omega']\underline{-[A,[\Omega,\Omega']]}-[[A,\Omega'],\Omega] \\
= {} & [\mathrm{d}\Omega',\Omega]+[A,[\Omega',\Omega]]+\cancel{[[A,\Omega],\Omega']}-[\mathrm{d}\Omega,\Omega'] \\
& \underline{-\cancel{[[A,\Omega],\Omega']}-[\cancel{\Omega},\cancel{[A,\Omega']}]}-[\cancel{[A,\Omega']},\cancel{\Omega}] \\
= {} & \mathrm{d}[\Omega',\Omega]+[A,[\Omega',\Omega]]=W([\Omega',\Omega],0)A.
\end{aligned}
$$

- $[v',v]$

$$
\begin{aligned}
W(0,v')W(0,v)A = {} & W(0,v')(-L_vA+\mathrm{d}i_v\mathring{A}+[A,i_v\mathring{A}]) \\
= {} & -L_v(-L_{v'}A+\mathrm{d}i_{v'}\mathring{A}+[A,i_{v'}\mathring{A}])+[-L_vA+\mathrm{d}i_v\mathring{A}+[A,i_v\mathring{A}],i_v\mathring{A}] \\
= {} & L_vL_{v'}A-\mathrm{d}i_v\mathrm{d}i_{v'}\mathring{A}-[L_vA,i_{v'}\mathring{A}]-[A,i_v\mathrm{d}i_{v'}\mathring{A}]-[L_{v'}A,i_v\mathring{A}] \\
& +[\mathrm{d}i_{v'}\mathring{A},i_v\mathring{A}]+[[A,i_{v'}\mathring{A}],i_v\mathring{A}],
\end{aligned}
$$

$$
\begin{aligned}
[W(0,v'),W(0,v)]A = {} & (L_vL_{v'}-L_{v'}L_v)A-(\mathrm{d}i_v\mathrm{d}i_{v'}-\mathrm{d}i_{v'}\mathrm{d}i_v)\mathring{A}-[A,(i_v\mathrm{d}i_{v'}-i_{v'}\mathrm{d}i_v)\mathring{A}] \\
& +[\mathrm{d}i_{v'}\mathring{A},i_v\mathring{A}]-[\mathrm{d}i_v\mathring{A},i_{v'}\mathring{A}]+\underline{[[A,i_{v'}\mathring{A}],i_v\mathring{A}]}-[[A,i_v\mathring{A}],i_{v'}\mathring{A}] \\
= {} & \underline{\underline{-L_{[v',v]}A}}+\underline{(\mathrm{d}i_{v'}\mathrm{d}i_v-\mathrm{d}i_v\mathrm{d}i_{v'})}\mathring{A}+\underline{[A,(i_{v'}\mathrm{d}i_v-i_v\mathrm{d}i_{v'})\mathring{A}]} \\
& +\mathrm{d}\underline{[i_{v'}\mathring{A},i_v\mathring{A}]}+\underline{[A,[i_{v'}\mathring{A},i_v\mathring{A}]]+\cancel{[[A,i_v\mathring{A}],i_{v'}\mathring{A}]}}-\cancel{[[A,i_v\mathring{A}],i_{v'}\mathring{A}]}.
\end{aligned}
$$

Equation (12.40) tells us that we expect on the right-hand side $W(\Omega,[v',v])$ with $\Omega=-i_{v'}i_v\mathring{F}$:

$$
W(-i_{v'}i_v\mathring{F},[v',v])A=-\mathrm{d}i_{v'}i_v\mathring{F}-[A,i_{v'}i_v\mathring{F}]-L_{[v',v]}A+\mathrm{d}i_{[v',v]}\mathring{A}+[A,i_{[v',v]}\mathring{A}].
$$

With

$$
\tfrac{1}{2}\,i_{v'}i_v[\mathring{A},\mathring{A}]=i_{v'}[i_v\mathring{A},\mathring{A}]=[i_v\mathring{A},i_{v'}\mathring{A}]=-[i_{v'}\mathring{A},i_v\mathring{A}],
$$

and

$$
i_{[v',v]}\mathring{A}=(L_{v'}i_v-i_vL_{v'})\mathring{A}=(i_{v'}\mathrm{d}i_v+\cancel{\mathrm{d}i_{v'}i_v}-i_v\mathrm{d}i_{v'}-i_vi_{v'}\mathrm{d})\mathring{A},
$$

we compute the right-hand side:

$$
\begin{aligned}
W(-i_{v'} i_v \mathring{F}, [v', v])A = & - \mathrm{d} i_{v'} i_v \mathrm{d}\mathring{A} - \mathrm{d}\, \tfrac{1}{2}\, i_{v'} i_v [\mathring{A}, \mathring{A}] - [A, i_{v'} i_v \mathrm{d}\mathring{A}] - [A, \tfrac{1}{2}\, i_{v'} i_v [\mathring{A}, \mathring{A}]] \\
& - L_{[v',v]} A + \mathrm{d} i_{[v',v]} \mathring{A} + [A, i_{[v',v]} \mathring{A}] \\
= & - \cancel{\mathrm{d} i_{v'} i_v \mathrm{d}} \mathring{A} + \mathrm{d}[i_{v'} \mathring{A}, i_v \mathring{A}] - [A, \cancel{i_{v'}} \cancel{i_v} \mathrm{d}\mathring{A}] + \underline{[A, [i_{v'} \mathring{A}, i_v \mathring{A}]]} \\
& \underline{\underline{- L_{[v',v]} A}} + \mathrm{d}(i_{v'} \mathrm{d} i_v - i_v \mathrm{d} i_{v'} - \cancel{i_v i_{v'} \mathrm{d}}) \mathring{A} \\
& + [A, (i_{v'} \mathrm{d} i_v - i_v \mathrm{d} i_{v'} - \cancel{i_v} \cancel{i_{v'}} \mathrm{d}) \mathring{A}] = \qquad [W(0, v'), W(0, v)]A.
\end{aligned}
$$

- $[v, \Omega]$

$$
\begin{aligned}
W(0, v) W(\Omega, 0) A = & W(0, v)(\mathrm{d}\Omega + [A, \Omega]) = [-L_v A + \mathrm{d} i_v \mathring{A} + [A, i_v \mathring{A}], \Omega] \\
= & - [L_v A, \Omega] + [\mathrm{d} i_v \mathring{A}, \Omega] + [A, [i_v \mathring{A}, \Omega]] + [[A, \Omega], i_v \mathring{A}], \\
W(\Omega, 0) W(0, v) A = & W(\Omega, 0)(-L_v A + \mathrm{d} i_v \mathring{A} + [A, i_v \mathring{A}]) \\
= & - L_v(\mathrm{d}\Omega + [A, \Omega]) + [\mathrm{d}\Omega + [A, \Omega], i_v \mathring{A}] \\
= & - L_v \mathrm{d}\Omega - [L_v A, \Omega] - [A, L_v \Omega] + [\mathrm{d}\Omega, i_v \mathring{A}] + [[A, \Omega], i_v \mathring{A}], \\
[W(0, v), W(\Omega, 0)]A = & - \cancel{[L_v A, \Omega]} + \underline{[\mathrm{d} i_v \mathring{A}, \Omega]} + [A, [i_v \mathring{A}, \Omega]] + \cancel{[[A, \Omega], i_v \mathring{A}]} \\
& + L_v \mathrm{d}\Omega + \cancel{[L_v A, \Omega]} + [A, L_v \Omega] - \underline{[\mathrm{d}\Omega, i_v \mathring{A}]} - \cancel{[[A, \Omega], i_v \mathring{A}]} \\
= & \mathrm{d} L_v \Omega + [A, L_v \Omega] + \mathrm{d}[i_v \mathring{A}, \Omega] + [A, [i_v \mathring{A}, \Omega]] = \star.
\end{aligned}
$$

On the other hand, by equation (12.41) we have:

$$
W([(0, v), (\Omega, 0)])A = W(L_v \Omega + [i_v \mathring{A}, \Omega], 0)A = \star.
$$

<u>Problem 12.4</u>
Calculate the square of the operator s defined in (12.56).

<u>Solution 12.4</u>
Let $Q \in \Lambda^l(\mathfrak{E}, P)$ and $E_i \in \mathfrak{E}$ for $i = -1, 0, 1, \ldots, l$. Then,

$$
\begin{aligned}
(sQ)(E_0, E_1, \ldots, E_l) = & \frac{1}{l!} \sum_{\pi \in \mathscr{S}_{l+1}} \mathrm{sig}\pi\, W(E_{\pi(0)}) Q(E_{\pi(1)}, \ldots, E_{\pi(l)}) \\
& - \frac{1}{2(l-1)!} \sum_{\pi \in \mathscr{S}_{l+1}} \mathrm{sig}\pi\, Q([E_{\pi(0)}, E_{\pi(1)}], E_{\pi(2)}, \ldots, E_{\pi(l)}),
\end{aligned}
$$

and

$$
\begin{aligned}
&(s^2Q)(E_{-1}, E_0, \ldots, E_l) \\
&= \frac{1}{(l+1)!} \sum_{\pi \in \mathscr{S}_{l+2}} \operatorname{sig}\pi \, W(E_{\pi(-1)})(sQ)(E_{\pi(0)}, \ldots, E_{\pi(l)}) \\
&\quad - \frac{1}{2l!} \sum_{\pi \in \mathscr{S}_{l+2}} \operatorname{sig}\pi \, (sQ)([E_{\pi(-1)}, E_{\pi(0)}], E_{\pi(1)}, \ldots, E_{\pi(l)}) \\
&= \frac{l+1}{(l+1)!} \sum_{\pi \in \mathscr{S}_{l+2}} \operatorname{sig}\pi \, \cancel{W(E_{\pi(-1)})W(E_{\pi(0)})}Q(E_{\pi(1)}, \ldots, E_{\pi(l)}) \\
&\quad - \frac{(l+1)l}{2(l+1)!} \sum_{\pi \in \mathscr{S}_{l+2}} \operatorname{sig}\pi \, W(E_{\pi(-1)})Q([E_{\pi(0)}, E_{\pi(1)}], E_{\pi(2)}, \ldots, E_{\pi(l)}) \\
&\quad - \frac{1}{2l!} \sum_{\pi \in \mathscr{S}_{l+2}} \operatorname{sig}\pi \, \cancel{W([E_{\pi(-1)}, E_{\pi(0)}])}Q(E_{\pi(1)}, \ldots, E_{\pi(l)}) \\
&\quad + \frac{l}{2l!} \sum_{\pi \in \mathscr{S}_{l+2}} \operatorname{sig}\pi \, W(E_{\pi(1)})Q([E_{\pi(-1)}, E_{\pi(0)}], E_{\pi(2)}, \ldots, E_{\pi(l)}) \\
&\quad - \frac{*}{2l!} \sum_{\pi \in \mathscr{S}_{l+2}} \operatorname{sig}\pi \, W(E_{\pi(1)})Q([[\cancel{E_{\pi(-1)}, E_{\pi(0)}}], E_{\pi(2)}], E_{\pi(3)}, \ldots, E_{\pi(l)}) \\
&\quad - \frac{*}{2l!} \sum_{\pi \in \mathscr{S}_{l+2}} \operatorname{sig}\pi \, W(E_{\pi(1)})Q([\cancel{E_{\pi(-1)}, E_{\pi(0)}}], [\cancel{E_{\pi(2)}}, E_{\pi(3)}], E_{\pi(4)}, \ldots, E_{\pi(l)})
\end{aligned}
$$

The last term vanishes because

$$
\begin{aligned}
\sum_{\pi \in \mathscr{S}_4} \operatorname{sig}\pi \, Q([E_{\pi(-1)}, E_{\pi(0)}], [E_{\pi(2)}, E_{\pi(3)}], .) &= - \sum_{\pi \in \mathscr{S}_4} \operatorname{sig}\pi \, Q([E_{\pi(2)}, E_{\pi(3)}], [E_{\pi(-1)}, E_{\pi(0)}], .) \\
&= - \sum_{\pi \in \mathscr{S}_4} \operatorname{sig}\pi \, Q([E_{\pi(-1)}, E_{\pi(0)}], [E_{\pi(2)}, E_{\pi(3)}], .)
\end{aligned}
$$

where the first equality holds since Q is alternating and the second holds since the permutation

$$\pi : (2, 3, -1, 0) \mapsto (-1, 0, 2, 3)$$

is even.

The next to last term vanishes due to the Jacobi identity.

The two remaining terms cancel because the permutation

$$\pi : (-1, 0, 1) \mapsto (1, -1, 0)$$

is even.

13 Anomalies from graphs

Problem 13.1

Show that in four-dimensional spacetime the invariant (13.16) vanishes identically.

Solution 13.1

In $n = 4$ dimensions, $n = 2j - 2$ implies $j = 3$ and we have to show that

$$\left[\prod_{i=1}^{2} \frac{\frac{1}{2}a_i}{\sinh \frac{1}{2}a_i} \right]_3 = 0.$$

Now

$$\frac{\sinh \frac{1}{2}a_i}{\frac{1}{2}a_i} = 1 + \frac{(\frac{1}{2}a_i)^2}{3!} + \frac{(\frac{1}{2}a_i)^4}{5!} + \cdots$$

contains only even powers of a_i. Therefore its inverse contains only even powers as well. Homogeneous polynomials of degree three are consequently absent in the above product.