Discrete Models of Financial Markets
Solutions to Exercises

Chapter 2

Exercise 2.1.
Show that the option price increases if U increases. Show that it also increases if D goes down.
Solution. Recall: $C(0) = x_C S(0) + y_C A(0)$ with

$$\begin{cases} x_C = \dfrac{S(0)(1+U) - K}{S(0)(U-D)}, \\ \\ y_C = -\dfrac{(1+D)(S(0)(1+U)-K)}{A(0)(U-D)(1+R)}. \end{cases} \tag{1}$$

so inserting these values gives

$$C(U) = \frac{S(0)(1+U) - K}{U-D} - \frac{(1+D)(S(0)(1+U)-K)}{(U-D)(1+R)} \tag{2}$$

and

$$\begin{aligned} C'(U) &= \frac{S(0)(U-D) - [S(0)(1+U)-K]}{(U-D)^2} \\ &\quad - \frac{(1+D)S(0)(U-D)(1+R) - (1+R)(1+D)[S(0)(1+U)-K]}{(U-D)^2(1+R)^2} \\ &= \frac{K - S(0)(1+D)}{(U-D)^2} - \frac{(1+D)[K-S(0)(1+D)]}{(U-D)^2(1+R)} \\ &= \frac{[K-S(0)(1+D)](R-D)}{(U-D)^2(1+R)} \geq 0 \end{aligned}$$

if $K \geq S(0)(1+D)$ as assumed to avoid trivial cases. A similar calculation proves the second part.
Exercise 2.2
Show that the option price does not increase if the 'spread' $|U-D|$ increases by analysing the following examples: for $R = 0$, $S(0) = 1$, $X = 1$, $T = 1$ consider two cases: $U = 0.05$, $D = -0,05$ or $U = 0.01$, $D = -0.19$.
Solution. From Theorem 2.8 we find $C(0) = 0.025$ in the first case, and $|U-D| = 0.1$, while in the second case we have $|U-D| = 0.2$, but $C(0) = 0.0095$. Note that in the first case the spread is symmetrical, while in the second it is heavily weighted towards the downside risk.
Exercise 2.3
Prove that for $\Omega = \{\mathrm{u}, \mathrm{d}\}$ we have $\mathrm{Var}(X) = p(1-p)(X(\mathrm{u}) - X(\mathrm{d}))^2$.

Solution. $\mathbb{E}[X] = pX(\mathrm{u}) + (1 - pX(\mathrm{d})$, and $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ so

$$\begin{aligned}\mathrm{Var}(X) &= pX(\mathrm{u})^2 + (1-p)X(\mathrm{d})^2 - (pX(\mathrm{u}) + (1-p)X(\mathrm{d}))^2 \\ &= p(1-p)(X(\mathrm{u})^2 - 2X(\mathrm{u})X(\mathrm{d}) + X(\mathrm{d})^2) \\ &= p(1-p)(X(\mathrm{u}) - X(\mathrm{d}))^2\end{aligned}$$

Exercise 2.4
Show that $\mathrm{Var}(aX) = a^2\mathrm{Var}(X)$.
Solution. $\mathrm{Var}(aX) = \mathbb{E}[aX^2] - (\mathbb{E}[aX])^2 = a^2(\mathbb{E}[X^2] - (\mathbb{E}[X])^2) = a^2\mathrm{Var}(X)$.
Exercise 2.5
Show that the excess mean returns for a derivative $H = h(S(1))$ and an underlying stock S in the single-step binomial model are related by

$$\mu_{K_H} - R = \beta_H(\mu_{K_S} - R).$$

By considering the risk neutral probabilities, deduce that for a European call option C we always have $\beta_C \geq 1$.

Solution. We compare the excess mean returns. The replicating portfolio (x_H, y_H) for H is as given in (1) and we know that for each scenario $\omega = \mathrm{u}, \mathrm{d}$ we have

$$x_H S(1,\omega) + y_H(1+R) = H(\omega),$$

while the fair price for H is

$$H(0) = y_H + x_H S(0).$$

So we have

$$x_H S^{\mathrm{u}} - H^{\mathrm{u}} = -y_H(1+R) = x_H S^{\mathrm{d}} - H^{\mathrm{d}}$$

and each of these quantities equals

$$(1+R)[x_H S(0) - H(0)].$$

For any $p \in (0,1)$ we can therefore write

$$\begin{aligned} & p[x_H S^{\mathrm{u}} - H^{\mathrm{u}}] + (1-p)[x_H S^{\mathrm{d}} - H^{\mathrm{d}}] \\ = \;& (1+R)[x_H S(0) - H(0)].\end{aligned}$$

Grouping terms on the left, we obtain

$$\begin{aligned} & x_H[pS^{\mathrm{u}} + (1-p)S^{\mathrm{d}}] - [pH^{\mathrm{u}} + (1-p)H^{d}] \\ = \;& (1+R)[x_H S(0) - H(0)].\end{aligned}$$

Multiply and divide terms on the left by $S(0)$ (resp. $H(0)$) and subtract 1 from each term inside the brackets, so that

$$\begin{aligned} & x_H S(0)[\frac{pS^{\mathrm{u}} + (1-p)S^{\mathrm{d}}}{S(0)} - 1] - H(0)[\frac{pH^{\mathrm{u}} + (1-p)H^{\mathrm{d}}}{H(0)} - 1] \\ = \;& R[x_H S(0) - D(0)].\end{aligned}$$

The terms in the brackets on the left are simply the mean returns μ_S and μ_H on S and H respectively, so we have

$$x_H S(0)\mu_S - H(0)\mu_H = R[x_H S(0) - H(0)]$$

and this can be re-ordered as:

$$x_H S(0)[\mu_S - R] = H(0)[\mu_H - R].$$

But we saw that $\beta_H = \frac{x_H S(0)}{H(0)}$, so finally

$$\mu_H - R = \beta_H[\mu_S - R].$$

For the second part we write the risk-neutral probability in terms of the stock prices: $q = \frac{R-D}{U-D} = \frac{(1+R)S(0)-S^{\mathrm{d}}}{S^{\mathrm{u}}-S^{\mathrm{d}}}$ and $1-q = \frac{S^{\mathrm{u}}-(1+R)S(0)}{S^{\mathrm{u}}-S^{\mathrm{d}}}$. Now consider the risk-neutrality condition

$$\begin{aligned} H(0) &= \frac{1}{1+R}[qH^{\mathrm{u}} + (1-q)H^{\mathrm{d}}] \\ &= \frac{1}{1+R}\frac{(1+R)S(0)-S^{\mathrm{d}}}{S^{\mathrm{u}}-S^{\mathrm{d}}}H^{\mathrm{u}} + \frac{1}{1+R}\frac{S^{\mathrm{u}}-(1+R)S(0)}{S^{\mathrm{u}}-S^{\mathrm{d}}}H^{d} \end{aligned}$$

which means that

$$\begin{aligned} & (1+R)(S^{\mathrm{u}} - S^{\mathrm{d}}]H(0) \\ = \;& (1+R)S(0)[H^{\mathrm{u}} - H^{\mathrm{d}}] + [S^{\mathrm{u}}H^{\mathrm{d}} - S^{\mathrm{d}}H^{\mathrm{u}}]. \end{aligned}$$

But if $H = C$ is a European call with strike $K \in (S^{\mathrm{d}}, S^{\mathrm{u}})$ then $C^{\mathrm{d}} = 0$ and $C^{\mathrm{u}} = S^{\mathrm{u}} - K > 0$, so that the final bracket in the last equation is negative, and the equation simplifies to the inequality

$$[S^{\mathrm{u}} - S^{\mathrm{d}}]C(0) \leq S(0)[C^{\mathrm{u}} - C^{\mathrm{d}}]. \tag{3}$$

But $x_C = \frac{C^{\mathrm{u}}-C^{\mathrm{d}}}{S^{\mathrm{u}}-S^{\mathrm{d}}}$. so $\beta_C = \frac{S(0)}{C(0)}x_C \geq 1$.

If $K \leq S^{\mathrm{d}}$ the second bracket above reads

$$\begin{aligned} & S^{\mathrm{u}}[S^{\mathrm{d}} - K] - S^{\mathrm{d}}[S^{\mathrm{u}} - K] \\ = \;& K[S^{\mathrm{d}} - S^{\mathrm{u}}] < 0, \end{aligned}$$

while if $K > S^{\mathrm{u}}$, both values of $C(1)$ are 0, so $x_C = 0$, i.e. $C(0) = y_C > 0$, so there is arbitrage. Hence in all non-trivial cases (3) holds.

Exercise 2.6

Verify that for any given p (with $P(\mathrm{u}) = p$)

$$\mathbb{E}_P(K_H) - \mathbb{E}_Q(K_H) = (p-q)\frac{H^{\mathrm{u}} - H^{\mathrm{d}}}{H(0)}.$$

Solution. By definition

$$\begin{aligned}
\mathbb{E}_P[K_H] - \mathbb{E}_Q[K_H] &= \frac{pH^{\mathrm{u}} + (1-p)H^{\mathrm{d}}}{H(0)} - 1 - \frac{qH^{\mathrm{u}} + (1-q)H^{\mathrm{d}}}{H(0)} - 1 \\
&= \frac{1}{H(0)}[(p-q)H^{\mathrm{u}} + (1-p-(1-q))H^{\mathrm{d}}] \\
&= (p-q)\frac{H^{\mathrm{u}} - H^{\mathrm{d}}}{H(0)}.
\end{aligned}$$

Exercise 2.7 (Corrected: in the book a typesetting error produced σK_H, it should read σ_{K_H}, as below)

Assuming $\mathbb{E}_P(K_H) \geq R$ show that

$$\mathbb{E}_P(K_H) - R = (p-q)\frac{\sigma_{K_H}}{\sqrt{p(1-p)}},$$

where q is the risk-neutral probability, implying in particular that $p \geq q$.

Solution. Since $\sigma_{K_H} = \sqrt{p(1-p)}\frac{H^{\mathrm{u}}-H^{\mathrm{d}}}{H(0)}$ and $\mathbb{E}_Q[K_H] = R$, the result is immediate from Exercise 2.6.

Exercise 2.8

Find the relationship between the risk neutral probability and the market price of risk, defined as

$$m_S = \frac{\mu_{K_S} - R}{\sigma_{K_S}}.$$

Solution. Since $q = \frac{R-D}{U-D}$ we have

$$m_S = \frac{D - R + (U-D)}{\sqrt{p(1-p)}(U-D)} = \frac{q-p}{\sqrt{p(1-p)}}.$$

Exercise 2.9

Show that for any derivative security H on the stock S its market price of risk $m_H = \frac{\mu_{K_H} - R}{\sigma_{K_H}}$ is the same as m_S. Also give a heuristic explanation of this result.

Solution. Intuitively this is obvious, as the price of H is determined by the price of S, and involves not additional source of randomness. To prove that $m_H = m_S$ we use the formula for m_S in terms of the risk-neutral probability q given above. Recall that $\mu_{K_H} = \frac{pH^u + (1-p)H - H(0)^d}{H(0)}$ and $\sigma_{K_H} =$

$\sqrt{p(1-p)}|\frac{H^u - H^d}{H(0)}|$, so that

$$\begin{aligned} m_H &= \frac{1}{\sqrt{p(1-p)}}(\frac{pH^u + (1-p)H^d - (1+R)H(0)}{H^u - H^d}) \\ &= \frac{1}{\sqrt{p(1-p)}}(\frac{pH^u + (1-p)H^d - \mathbb{E}_Q(H(1))}{H^u - H^d}) \text{ (Q is risk-neutral)} \\ &= \frac{1}{\sqrt{p(1-p)}}(\frac{pH^u + (1-p)H^d - (qH^u + (1-q)H^d)}{H^u - H^d}) \\ &= \frac{q-p}{\sqrt{p(1-p)}} = m_S. \end{aligned}$$

Exercise 2.10

Find a random variable G playing the role of a 'density' of Q with respect to P, considered on $\Omega = \{\mathrm{u}, \mathrm{d}\}$, i.e., satisfying $q = G(\mathrm{u})p$, $1 - q = G(\mathrm{d})(1-p)$. Prove that $\mathbb{E}_P(G) = 1$, so that a sequence $G(0) = 1$, $G(1) = G$ is a martingale with respect to P.

Solution The stated requirements imply that $G(\mathrm{u}) = \frac{q}{p}$ and $G(\mathrm{d}) = \frac{1-q}{1-p}$, so that $\mathbb{E}_P(G) = p(\frac{q}{p}) + (1-p)(\frac{1-q}{1-p}) = 1$. So G is a martingale for P.

Exercise 2.11

Show that there exist real numbers $a \neq 0$, and b such that $K_1 = aK_2 + b$.

Solution The two equations we need to solve for a and b are

$$\begin{aligned} U_1 &= aU_2 + b \\ D_1 &= aD_2 + b \end{aligned}$$

which gives

$$\begin{aligned} a &= \frac{U_1 - D_1}{U_2 - D_2} \\ b &= \frac{D_1U_2 - D_dU_1}{U_2 - D_2}. \end{aligned}$$

Exercise 2.12

Show that the correlation coefficient for the returns is $\rho = 1$ if $a > 0$ and $\rho = -1$ otherwise.

Solution. $K_1 = (U_1, D_1)$ and $K_2 = (U_2, D_2)$ are vectors in $\mathbb{R}^2$ and their correlation coefficient ρ is the cosine of the angle between them. Since $K_1 = aK_2 + b$, the angle is 0 when $a > 0$ so $\rho = 1$, while it is π when $a < 0$, hence $\rho = -1$.

Exercise 2.13

Find h using the relation between the returns.

Solution. We need h such that $U_2 = aU_1 + b$ and $D_2 = aD_1 + b$. Now by Exercise 2.11, $U_2 = \frac{1}{a}(U_1 - b)$ and $D_2 = \frac{1}{a}(D_1 - b)$, so $h(z) = \frac{1}{a}(z - b)$ for z in $\mathbb{R}$.

Exercise 2.14

Find the replicating portfolio (x, y) such that $S_2(1) = xS_1(1) + yA(1)$.

Solution The previous Exercise shows that $h(z) = \frac{z}{a} - \frac{b}{a}$, so $S_2(1) = h(S_1(1) = \frac{1}{a}S(1) - \frac{b}{a}$, and the replicating portfolio is $(x, y) = (\frac{1}{a}, -\frac{b}{aA(0)(1+R)})$..

Exercise 2.15

Find an arbitrage if $S_1(0) = 50$, $U_1 = 20\%$, $D_1 = -10\%$, $S_2(0) = 80$, $U_2 = 15\%$, $D_2 = -5\%$, $A(0) = 1$, $R = 10\%$.

Solution First observe that here $q_1 = \frac{0.1+-0.1}{0.2+0.1} = \frac{2}{3}$ and $q_2 = \frac{0.1+0.05}{0.15+0.05} = \frac{3}{4}$ so $Q_1 \neq Q_2$, hence an arbitrage is possible. To construct one we need to solve for x, y, z in the following

$$\begin{aligned} xS_1(0) + yA(0) + zS_2(0) &= 0 \\ xS_1(1) + yA(0)(1+R) + zS_2(1) &\geq 0 \end{aligned}$$

and the inequality consists of two parts. For simplicity let $A(0) = 10$, then the first equation is $50x + 10y + 80z = 0$, which reduces to $5x + y + 8z = 0$. Look for x, y, z making the inequality in the down state at time 1 an equality, i.e., $45x + 11y + 76z = 0$. From these two equations we eliminate y and find that $z = -\frac{5}{6}x$. Taking $x = 6$, $y = 10$, $z = -5$ we see that both equations are satisfied, while the inequality in the up state reads $60(6) + 11(10) + 92(-5) = 10 > 0$. So this choice of x, y, z provides an arbitrage.

Exercise 2.16

Find the form of S_2^{d} as a function of the remaining parameters so that there is no arbitrage.

Solution To avoid arbitrage we need $Q_1 = Q_2$, that is,

$$\frac{R - D_1}{U_1 - D_1} = \frac{R - D_2}{U_2 - D_2},$$

which gives

$$D_2 = \frac{(U_1 - D_1)R - (R - D_1)U_2}{U_1 - R}$$

and so

$$S_2^{\mathrm{d}} = S_2(0)(1 + D_2) = S_2(0)(1 + \frac{(U_1 - D_1)R - (R - D_1)U_2}{U_1 - R}).$$

Exercise 2.17

Given a trinomial single-stock model with $R = 0$, $S(0) = 10$ and $S^{\mathrm{u}} = 20$, $S^{\mathrm{m}} = 15$, $S^{\mathrm{d}} = 7.5$ show that the derivative security H can be replicated if and only if $3H^{\mathrm{u}} - 5H^{\mathrm{m}} + 2H^{\mathrm{d}} = 0$.

Solution We again take $A(0) = 10$ for ease of calculation. To replicate H we must have a solution for the system of values at time 1, which is (since $R = 0$)

$$\begin{aligned} 20x + 10y &= H^{\mathrm{u}} \\ 15x + 10y &= H^{\mathrm{m}} \\ 7.5x + 10y &= H^{\mathrm{d}} \end{aligned}$$

This system is consistent iff (subtracting the second equation from the first) $5x = H^{\mathrm{u}} - H^{\mathrm{m}}$ and $10y = 4H^{\mathrm{m}} - 3H^{\mathrm{u}}$ fit the third equation, that is,

$$\frac{3}{2}(H^{\mathrm{u}} - H^{\mathrm{m}}) + (4H^{\mathrm{m}} - 3H^{\mathrm{u}}) = H^{\mathrm{d}}$$

which simplifies to $3H^{\mathrm{u}} - 5H^{\mathrm{m}} + 2H^{\mathrm{d}} = 0$.

Exercise 2.18

Let $S(0) = 100$, $A(0) = 1$, $U = 20\%$, $M = 10\%$, $D = -15\%$, $R = 5\%$, $H^{\mathrm{u}} = 25$, $H^{\mathrm{m}} = 5$. Find H^{d} such that there is a unique replicating portfolio.

Solution The system to be solved uniquely is

$$\begin{aligned} 120x + 10.5y &= 25 \\ 110x + 1.05y &= 5 \\ 85x + 1.05y &= H^{\mathrm{d}}. \end{aligned}$$

We obtain $x = 2$ from the first two equations, so $y = \frac{215}{1.05}$. The third equation then forces $85(2) + 215 = H^{\mathrm{d}}$, so $H^{\mathrm{d}} = -45$.

Exercise 2.19

For general H^{u}, H^{m}, H^{d} find a relation between these three numbers so that there is a unique replicating portfolio with arbitrary $S(0)$, U, M, D, R.

Solution We solve the equations

$$\begin{aligned} xS^{\mathrm{u}} + yA(0)(1+R) &= H^{\mathrm{u}} \\ xS^{\mathrm{d}} + yA(0)(1+R) &= H^{\mathrm{d}} \end{aligned}$$

for x, y to obtain

$$\begin{aligned} x &= \frac{H^{\mathrm{u}} - H^{\mathrm{d}}}{S^{\mathrm{u}} - S^{\mathrm{d}}} \\ y &= \frac{1}{A(0)(1+R)}\left(\frac{H^{\mathrm{d}}S^{\mathrm{u}} - H^{\mathrm{u}}S^{\mathrm{d}}}{S^{\mathrm{u}} - S^{\mathrm{d}}}\right). \end{aligned}$$

Inserting this solution into the third equation,

$$xS^{\mathrm{m}} + yA(0)(1+R) = H^{\mathrm{m}}$$

shows that

$$H^{\mathrm{m}}(U - D) = H^{\mathrm{u}}(M - D) + H^{\mathrm{d}}(U - M).$$

Exercise 2.20

Let $S(0) = 30$, $A(0) = 1$, $U = 20\%$, $M = 10\%$, $D = -10\%$, $R = 5\%$, $H(1) = (S(1) - 32)^{+}$. Find the super-replication price.

Solution We minimise $V(0) = 30x + y$ subject to the linear constraints

$$\begin{aligned} V^{\mathrm{u}}(1) &= 36x + 1.05y \geq 4 \\ V^{\mathrm{m}}(1) &= 33x + 1.05y \geq 1 \\ V^{\mathrm{d}}(1) &= 27x + 1.05y \geq 0 \end{aligned}$$

finding (Solver, Excel) $x = 0.1135388, y = -0.0832358$ with $V(0) = 3.32292886$.

Exercise 2.21

Using the data of Exercise 2.20 find a sub-replicating portfolio price.

Solution We maximise $V(0) = 30x + y$ subject to the linear constraints

$$\begin{aligned} V^{\mathrm{u}}(1) &= 36x + 1.05y \leq 4 \\ V^{\mathrm{m}}(1) &= 33x + 1.05y \leq 1 \\ V^{\mathrm{d}}(1) &= 27x + 1.05y \leq 0 \end{aligned}$$

finding (Solver, Excel) $x = 0.52287589, y = -14.117649$ with $V(0) = 1.56862767$.

Exercise 2.22

Given a trinomial single-stock model with $R = 10\%$, $S(0) = 10$ and $S^{\mathrm{u}} = 20, S^{\mathrm{m}} = 15, S^{\mathrm{d}} = 7.5$ find all risk neutral probabilities.

Solution We need $\mathbb{E}_Q(\frac{1}{1.1}S(1)) = S(0)$, so with $Q = (q_1, q_2, q_3)$ we need $q_i \in (0,1)$ for $i = 1, 2, 3$ and

$$\frac{1}{1.1}(20q_1 + 15q_2 + 7.5(1 - q_1 - q_2)) = 10,$$

which gives

$$25q_1 + 15q_2 = 7.$$

Set $q_1 = \lambda$, then $q_2 = \frac{7}{25} - \frac{5}{3}\lambda$, which means that we need $\lambda \in (0,1) \cap (-\frac{8}{25}, \frac{7}{25}) \cap (-\frac{4}{5}, \frac{7}{10}) = (0, \frac{7}{25})$ in order to ensure that all $q_i \in (0,1)$.

Exercise 2.23

Show that if $H^{\mathrm{sub}} = H^{\mathrm{super}}$ then there exists a replicating portfolio.

Solution In the definition of H^{sub}, H^{super} we can take min , max since the sets are closed. So there are portfolios $x^{\mathrm{sub}}, y^{\mathrm{sub}}$ subreplicating $H(1)$ and $x^{\mathrm{super}}, y^{\mathrm{super}}$ super-replicating $H(1)$ such that

$$x^{\mathrm{sub}}S(0) + y^{\mathrm{sub}} = x^{\mathrm{super}}S(0) + y^{\mathrm{super}}$$

and hence

$$(x^{\mathrm{zub}} - x^{\mathrm{super}})S(0) = y^{\mathrm{super}} - y^{\mathrm{sub}}$$

Moreover

$$\begin{aligned} x^{\mathrm{sub}}S(0)\frac{1+U}{1+R} + y^{\mathrm{sub}} &\leq x^{\mathrm{super}}S(0)\frac{1+U}{1+R} + y^{\mathrm{super}} \\ x^{\mathrm{sub}}S(0)\frac{1+M}{1+R} + y^{\mathrm{sub}} &\leq x^{\mathrm{super}}S(0)\frac{1+M}{1+R} + y^{\mathrm{super}} \\ x^{\mathrm{sub}}S(0)\frac{1+D}{1+R} + y^{\mathrm{sub}} &\leq x^{\mathrm{super}}S(0)\frac{1+D}{1+R} + y^{\mathrm{super}} \end{aligned}$$

and inserting $y^{\mathrm{super}} - y^{\mathrm{sub}}$ into the inequalities we get

$$\begin{aligned} (x^{\mathrm{sub}} - x^{\mathrm{super}})S(0)\frac{1+U}{1+R} &\leq +(x^{\mathrm{sub}} - x^{\mathrm{super}})S(0) \\ ((x^{\mathrm{sub}} - x^{\mathrm{super}})S(0)\frac{1+M}{1+R} &\leq +(x^{\mathrm{sub}} - x^{\mathrm{super}})S(0) \\ (x^{\mathrm{sub}} - x^{\mathrm{super}})S(0)\frac{1+D}{1+R} &\leq +(x^{\mathrm{sub}} - x^{\mathrm{super}})S(0) \end{aligned}$$

Hence if $x^{\mathrm{sub}} - x^{\mathrm{super}} \neq 0$

$$\begin{aligned}\frac{1+U}{1+R} &\leq 1\\ \frac{1+M}{1+R} &\leq 1\\ \frac{1+D}{1+R} &\leq 1\end{aligned}$$

that is U, M, D are less or equal to R, contradicting the NAP. It follows that the two portfolios are the same, and thus form a replicating portfolio,

Exercise 2.24

Show that the replicating strategy is unique.

Solution Suppose x_1, y_1 and x_2, y_2 replicate the claim H :

$$x_1 S(1) + y_1 A(1) = x_2 S(1) + y_2 A(1)$$

Their inital values must be the same

$$x_1 S(0) + y_1 = x_2 S(0) + y_2$$

so

$$\begin{aligned}(x_1 - x_2)S(1) &= (y_2 - y_1)A(1)\\ &= (x_1 - x_2)S(0)A(1)\end{aligned}$$

and as a result either $x_1 = x_2$ or $S(1)$ is deterministic.

Exercise 2.25

Show that the existence of replicating strategy implies uniqueness of the risk neutral probability.

Solution.

$$\begin{aligned}\mathbb{E}_Q(K) &= R\\ \mathbb{E}_Q(H(1)) &= H(0)(1+R)\end{aligned}$$

Two equations in 2 variables (in the trinomial model, the value of q_3 is given by the first two) have unique solution assuming non-degeneracy of $H(1)$. (Uniqueness holds if this is assumed for all derivative securities.)

Pedestrian use of the definitions gives

$$Uq^{\mathrm{u}} + Mq^{\mathrm{m}} + (1 - q^{\mathrm{u}} - q^{\mathrm{m}})D = R$$

$$q^{\mathrm{u}} = \frac{R - D + (D - M)q^{\mathrm{m}}}{U - D}$$

$$H(1,\mathrm{u})q^{\mathrm{u}} + H(1,\mathrm{m})q^{\mathrm{m}} + H(1,\mathrm{d})(1 - q^{\mathrm{u}} - q^{\mathrm{m}}) = H(0)(1+R)$$

$$
\begin{aligned}
& H(1,\mathrm{u})\frac{R-D+(D-M)q^{\mathrm{m}}}{U-D}+H(1,\mathrm{m})q^{\mathrm{m}} \\
& +H(1,\mathrm{d})(1-\frac{R-D+(D-M)q^{\mathrm{m}}}{U-D}-q^{\mathrm{m}}) \\
= \; & H(0)(1+R)
\end{aligned}
$$

$$
\begin{aligned}
& H(1,\mathrm{u})(R-D+(D-M)q^{\mathrm{m}})+H(1,\mathrm{m})q^{\mathrm{m}}(U-D) \\
& +H(1,\mathrm{d})(U-D-(R-D+(D-M)q^{\mathrm{m}})-q^{\mathrm{m}}(U-D)) \\
= \; & H(0)(1+R)(U-D)
\end{aligned}
$$

So

$$
\begin{aligned}
& q^{\mathrm{m}}[H(1,\mathrm{u})(D-M)+H(1,\mathrm{m})(U-D)+H(1,\mathrm{d})(D-M)-(U-D)] \\
= \; & H(0)(1+R)(U-D)-H(1,\mathrm{u})(R-D)-H(1,\mathrm{d})(U-R)
\end{aligned}
$$

A problem appears to arise if

$$
\begin{aligned}
H(1,\mathrm{u})(D-M)+H(1,\mathrm{m})(U-D)+H(1,\mathrm{d})(D-M)-(U-D) &= 0 \\
H(0)(1+R)(U-D)-H(1,\mathrm{u})(R-D)-H(1,\mathrm{d})(U-R) &= 0
\end{aligned}
$$

If the RHS is zero, the probability will be degenerate, but then the meaning of the second line is that $H(0)$ is the price in the binomial model - with the middle branch ignored - and the first line specifies the middle branch so that we have degeneracy in the derivative, that is, the value of $H(1)$ on this branch is determined by the other two branches.

Exercise 2.26 Correction: In the book there is a misprint: $C(0)=\frac{255}{12}$ **should have read** $C(0)=\frac{255}{22}$.

Consider a trinomial model for stock prices with $S(0)=120$ and $S(1)=135,125,115$, respectively. Assume that $R=10\%$. Consider a call with strike 120 as the second security. Show that $C(0)=\frac{120}{11}$ allows arbitrage and that there is a unique degenerate probability which makes discounted stock and call prices a martingale. Carry out the same analysis for $C(0)=\frac{255}{12}$ and draw a conclusion about admissible call prices.

Solution For $\mathbb{E}_Q(\frac{1}{1.1}S(1))=S(0)$ and $\mathbb{E}_Q(\frac{1}{1.1}C(1))=C(0)$ we need $Q=(q_1,q_2,1-q_1-q_2)$ to solve

$$
\begin{aligned}
135q_1+125q_2+115(1-q_1-q_2) &= 132 \\
15q_1+5q_2+0(1-q_1-q_2) &= 12
\end{aligned}
$$

which reduces to $q_1=0.7$, $q_2=0.3$ and so $q_3=1-q_1-q_2=0$. Thus Q is the unique degenerate solution. Take sell one option, $z=-1$, buy one share, $x=-1$, and invest 109.09 risk free. At maturity we have the value 0 in U and M scenarios and 5 in the D case.

For $C(0)=255/11$ we have $q_1=-5.1$ so no risk neutral probability. If $C(0)=255/22$, $q_1=0$, $q_2=0.85$, $q_3=0.15$ and $z=-1$, $x=0.75$, with 78.41

borrowed risk-free gives 2.5 in the middle scenario, 0 in the other two. We leave the other cases to the reader.

Exercise 2.27

If the returns on stocks are $5\%, 8\%, -20\%$ on stock one and $-5\%, 10\%, a\%$ on stock two, find a so that there exists a risk-neutral probability, where $R = 5\%$.

Solution Write $Q = (q_1, q_2, q_3)$ for the desired risk-neutral probability. We a solution, with all q_i in $(0, 1)$, of the system

$$\begin{aligned} 5q_1 + 8q_2 - 20q_3 &= 5 \\ -5q_1 + 10q_2 + aq_3 &= 5 \\ q_1 + q_2 + q_3 &= 1. \end{aligned}$$

The coefficient matrix A is invertible iff $\det(A) \neq 0$, i.e. $5(10 - a) - 8(-5 - a) + (-20)(-5 - 10) \neq 0$. So we need $a \neq -130$ to obtain a solution, and we must choose a to ensure that the q_i lie in $(0, 1)$. Taking $a = 0$ we find $q_3 = \frac{1}{13}$, $q_2 = \frac{25}{39}$ and $q_1 = \frac{11}{39}$ as a possible solution.

Exercise 2.28

Find the price of a basket option (where the strike is compared with the sum of two stock prices) with payoff $H(1) = \max\{(S_1(1) + S_2(1) - X), 0\}$ where $S_1(0) = 100$, $S_2(0) = 50$, $A(0) = 1$, $X = 150$, the returns are $20\%, 5\%, -20\%$ for stock one and $-10\%, 8\%, 2\%$ for stock two, and $R = 5\%$. Use replication as well as a risk-neutral probability and compare the prices obtained.

Solution For replication we must solve the matrix equation

$$\begin{bmatrix} 120 & 45 & 1.05 \\ 105 & 54 & 1.05 \\ 80 & 51 & 1.05 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \\ 0 \end{bmatrix}$$

which has the solution $(x_1, x_2, y) = (0.3666, -0.0555, -25, 2380)$. So the option price is 8.6508, approximately.

For the risk-neutral probability $Q = (q_1, q_2, q_3)$ we need to solve

$$\begin{bmatrix} 0.2 & 0.05 & -0.2 \\ -0.1 & 0.08 & 0.02 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.05 \\ 1 \end{bmatrix}$$

which yields $Q = (0.1388, 0.7777, 0.0833)$ and so the option price is $\mathbb{E}_Q((1 + R)^{-1}H(1)) = 8.6508$ as above

Exercise 2.29

Show that $S' < S$ implies $C(S) - C(S') \leq S - S'$ and $P(S') - P(S) \leq S - S'$.

Solution Suppose the strike is K.By call-put parity $C(S) - P(S) = S - K(1 + R)^{-N}$, and $C(S') - P(S') = S' - K(1 + R)^{-N}$ so

$$[C(S) - C(S')] + [P(S') - P(S)] = S - S'$$

and the result follows since if the sum of two non-negative numbers is less than $S - S'$, each of them must be less than $S - S'$.

Exercise 2.30

Prove that the call price $C(S)$ is a convex function of the stock price S:

$$C(\frac{S' + S''}{2}) \leq \frac{C(S') + C(S'')}{2}.$$

Employing call-put parity, show that the same is true for the put

Solution Suppose the strike is K. Write $S' = x'S$, $S'' = x''S$. The function $f(x) = (x - K)^+$ is convex, so

$$(\frac{x' + x''}{2} S(N) - K)^+ \leq \frac{1}{2}(x'S(N) - K)^+ + \frac{1}{2}(x''S(N) - K)^+.$$

This compares the payoffs of two derivatives, with H' on the left (a call on $\frac{x'+x''}{2}$ shares in S, strike K) and H on the right (the average of two calls on x' and x'' shares with strike K), so by Proposition 2.43, $H(0) \geq H'(0)$, which yields the desired inequality of their initial prices.

For the put we have $C(S) - P(S) = S - K(1 + R)^{-N}$ by parity for all three calls. Adding the right-hand sides for S' and S'' provides $2C(\frac{S'+S''}{2}) - 2P(\frac{S'+S''}{2})$, so the result for the put follows.

Exercise 2.31

Show that if $K' < K$ then

$$\begin{aligned} C(K') - C(K) &\leq (1 + R)^{-1}(K - K'), \\ P(K) - P(K') &\leq (1 + R)^{-1}(K - K'). \end{aligned}$$

Solution Call-put parity gives $C(K) - P(K) = S - K(1 + R)^{-N}$, $C(K') - P(K') = S - K'(1 + R)^{-N}$ and this gives

$$[C(K') - C(K)] + [P(K) - P(K')] = (K - K')(1 + R)^{-N}.$$

Both terms on the left are non-negative so each must be smaller than the right-hand side.

Exercise 2.32

Show that the call price is a convex function of the strike price.

Solution Inequalities of payoffs imply inequalities for prices so the result follows from

$$(S(N) - \frac{K' + K''}{2})^+ \leq \frac{1}{2}(S(N) - K')^+ + \frac{1}{2}(S(N) - K'')^+$$

since $f(x) = (S(N) - x)^+$ is convex.

Chapter 3

Exercise 3.1
Find the tree of the values of the derivatives with the folllowing payoffs:

$$
\begin{aligned}
H_1 &= (\max\{S(n) : n = 0, 1, 2\} - 100)^+ , \\
H_2 &= \left(\frac{S(0) + S(1) + S(3)}{3} - 100 \right)^+ .
\end{aligned}
$$

(Note that these are path-dependent derivatives.)

Solution We have to specify the missing data (using the values from Example 3.4) $S(0) = 100$, $U = 20\%$, $D = -10\%$ $r = 5\%$.

The option with payoff H_1 :

0	1	2	path
		44	UU
	30.48		
		20	UD
16.33			
		8	DU
	3.81		
		0	DD

The option with payoff H_2 :

0	1	2	3	path
			30.93	UUU
		22.6		
			16.53	UUD
	15.81			
			16.53	UDU
		10.6		
			5.73	UDD
8.23				
			6.53	DUU
		3.11		
			0	DUD
	1.48			
			0	DDU
		0		
			0	DDD

Exercise 3.2
Prove that K_1, K_2, K_3 are independent, which by definition means that for each pair we have condition $P(\{K_i = x_i\} \cap \{K_j = x_j\}) = P(\{K_i = x_i\})P(\{K_j = x_j\})$ for $i \neq j$,

and also

$$P(\bigcap_{k=1}^{3}\{\omega : K_i(\omega) = x_i\}) = \prod_{k=1}^{3} P(\{\omega : K_i(\omega) = x_i\}).$$

Solution The verification for each pair is the same as for the two-step case. To verify the above identity, note that by definition both sides are 0 unless each x_i one of the two values U, D. There are 8 possible cases, grouped accoring to the number of time U occurs: for UUU we have p^3 on the left by definition of P, and this equals the RHS. Similarly of UUD, UDU or DUU we obtain $p^2(1-p)$ on each side, for UDD, DUD, DDU it is $p(1-p)^2$, and for DDD we obtain $(1-p)^3$.

Exercise 3.3

Extend the pricing scheme of the previous section to find the option price and observe that it is not equal to the expectation computed above.

Solution With the data $S(0) = 100$, $U = 20\%$, $D = -10\%$ $r = 5\%$. $p = 0.6$, we compute $q = 0.5$ and then $\mathbb{E}_Q(\frac{1}{(1+r)^3}(S(3) - K)^+ = 17.45$.

Exercise 3.4

Illustrate the martingale property of stock prices under the risk-neutral probability by numerical computations.

Solution With the data $S(0) = 100$, $U = 20\%$, $D = -10\%$ $r = 5\%$. we find $q = 0.5$ and the tree of expected discounted stock values

0	1	2
		144
	120	
100		108
	90	
		81

Exercise 3.5

Take $N = 3$ and let $S(0) = 100$, $U = 20\%$, $D = -10\%$, $R = 5\%$. Consider a call with exercise price $K = 100$ at time 3 and find the process $C(n)$.

Solution The tree of option prices:

0	1	2	3
			72.8
		48.76	
	29.93		29.6
17.45		14.1	
	6.71		0
		0	
			0

Exercise 3.6

Consider the filtration generated by the sequence $C(n)$, $n = 0, 1, 2$. Find the parameters so that it is identical to the filtration generated by the stock prices. Can we have a constant filtration $\mathcal{P}_0 = \mathcal{P}_1 = \mathcal{P}_2$?

Solution For the data $S(0) = 100$, $U = 20\%$, $D = -10\%$, $R = 5\%$ the filtration of option prices coincides with the filtration generated by stock prices

$$\begin{aligned}
\mathcal{P}_0 &= \{\Omega\} \\
\mathcal{P}_1 &= \{B_U, B_D\} \\
\mathcal{P}_2 &= \{B_{UU}, B_{UD}, B_{DU}, B_{DD}\}
\end{aligned}$$

To have the same partitions the random variables $C(n)$ must be constant so that they are $\mathcal{P}_0$-measurable. For this it is sufficient to take K larger that all stock pricess at time 3, i.e. $K \geq 100 \times 1.2^2 = 144$. If for instance $K = 130$ we have $C(n)$ of the form

0	1	2
		14
	6.67	
3.18		0
	0	
		0

and

$$\begin{aligned}
\mathcal{P}_0 &= \{\Omega\} \\
\mathcal{P}_1 &= \{B_U, B_D\} \\
\mathcal{P}_2 &= \{B_{UU}, B_{UD}, B_{DU} \cup B_{DD}\}
\end{aligned}$$

Exercise 3.7
Find the process of prices of the Asian option with payoff

$$H(5) = \max\{\frac{1}{6}\sum_{k=0}^{5} S(k) - K, 0\},$$

where $S(0) = 60$, $U = 12\%$, $U = -6\%$, $R = 4\%$, $K = 62$.

Solution

0	1	2	3	4	5	*sum*
					19.15	486.91
				17.2	16.32	469.92
			14.42		13.79	454.74
				12.24	11.41	440.48
		11.08			11.53	441.2
				10.07	9.16	426.93
			7.9		7.03	414.2
				5.91	5.04	402.23
	7.51				9.52	429.1
				8.14	7.14	414.84
			6.04		5.02	402.1
				3.97	3.02	390.13
		3.72			3.12	390.73
				2.15	1.13	378.76
			1.15		0	368.07
				0	0	358.03
4.58					7.72	418.3
				6.41	5.34	404.04
			4.38		3.22	391.3
				2.24	1.22	379.33
		2.5			1.32	379.93
				0.71	0	367.96
			0.38		0	357.27
				0	0	347.23
	1.34				0	369.78
				0	0	357.81
			0		0	347.12
				0	0	337.08
		0			0	337.58
				0	0	327.53
			0		0	318.56
				0	0	310.13

Exercise 3.8

A popular combination of a call and a put is a *bottom straddle,* which involves buying both options with the same strike price K and expiry N. Verify that the payoff function is given by $H(N) = |S(N) - K|$. With the data from the previous exercise, compute the prices and values of the hedging strategy for a straddle with expiry $N = 5$, that is, consider $H(5) = |S(5) - K|$. Find the prices of the security with payoff $H(5) = 1$ if $S(5) \geq K$ and $H(5) = 0$ otherwise.

Solution The straddle prices

0	1	2	3	4	5
					43.74
				34.8	
			26.97		26.75
		20.14		19.62	
	14.83		13.42		12.48
11.24		9.53		6.88	
	7.77		5.53		0.51
		6.28		4.35	
			7.78		9.53
				12.77	
					17.97

Hedging:

0	1	2	3	4
$x(n)$				1
			1	
		1		1
	0.88		1	
0.65		0.69		1
	0.32		0.24	
		-0.24		-0.9
			-0.94	
				-1

0	1	2	3	4
$y(n)$				-59.61
			-57.33	
		-55.12		-59.62
	-44.31		-57.33	
-27.76		-34.06		-59.62
	-10.28		-8.72	
		19		54.58
			54.62	
				59.61

For payoff $H(5) = 1$ if $S(5) \geq K$ and $H(5) = 0$ otherwise, we have the values

0	1	2	3	4	5
					1
				0.96	
			0.92		1
		0.88		0.96	
	0.82		0.92		1
0.72		0.81		0.96	
	0.65		0.74		1
		0.51		0.53	
			0.28		0
				0	
					0

Hedging

0	1	2	3	4
$x(n)$				0
			0	
		0		0
	0.01		0	
0.02		0.02		0
	0.03		0.04	
		0.05		0.1
			0.06	
				0

0	1	2	3	4
$y(n)$				0.96
			0.92	
		0.88		0.96
	0.15		0.92	
-0.48		-0.45		0.96
	-1.04		-1.64	
		-2.14		-5.05
			-2.71	
				0

Exercise 3.9

A financial advisor selects a stock and receives a bonus of 100 for each up move of stock prices payable at time 5. Find the process of values of this bonus regarded as a derivative (use the same parameters as in the previous exercises).

Solution

0	1	2	3	4	5
					500
				438.03	
			380.09		400
		325.96		341.88	
	275.43		287.64		300
228.31		237.06		245.73	
	189.95		195.18		200
		148.16		149.57	
			102.73		100
				53.42	
					0

Exercise 3.10

Build a modification of the above computations where the shares are not divisible, that is, the stock position must be an integer, but you have sold and will hedge a package of 10 options. How will the results influence the initial price?

Solution We have $S(0) = 100$, $U = 20\%$, $D = -10\%$, $R = 10\%$, $K = 105$, $N = 3$, the stock positions have to be rounded up, giving $x^{uu}(3) = 10$, $x^{ud}(3) = 8$, $x^{dd}(3)$, $x^{u}(2) = 10$, $x^{d}(2) = 6$, $x(1) = 9$, with the risk free position chosen for superhedging the final payoff, with initial value 252.89for 10 options.

Chapter 4

Exercise 4.1

Show that if $M(n)$ is $\mathcal{P}_n$-measurable, and $\mathbb{E}(M(n+1)-M(n)|\mathcal{P}_n)=0$ then M is a martingale.

Solution Let $0 \leq n < m \leq N$, and consider $M(m)-M(n)$ as the telescoping sum $\sum_{k=n}^{m-1}(M(k+1)-M(k))$, so

$$\begin{aligned}
\mathbb{E}[M(m)|\mathcal{P}_n)-M(n] &= \mathbb{E}[(M(m)-M(n)|\mathcal{P}_n] \\
&\quad \text{(as } M(n) \text{ is } \mathcal{P}_n\text{-measurable)} \\
&= \mathbb{E}[\sum_{k=n}^{m-1}(M(k+1)-M(k))|\mathcal{P}_n] \\
&= \sum_{k=n}^{m-1}\mathbb{E}[(M(k+1)-M(k))|\mathcal{P}_n] \\
&= 0.
\end{aligned}$$

Exercixe 4.2

Give an example showing that the reverse implication does not hold, i.e. $\mathbb{E}(M(0))=\mathbb{E}(M(1))=\mathbb{E}(M(2))$ does not necessarily mean that $M(0), M(1), M(2)$ is a martingale.

Solution For example in a two-step binomial model, take $M(0)=0$, so that we need $p(M(\mathrm{u})+(1-p)M(\mathrm{d})=0$, which suggests $M(1)=(1-p,-p)$, and to obtain $\mathbb{E}(M(2))=0$ we then need $M(2)=(x,y,z)$ with $p^2x+2p(1-p)y+(1-p)^2z=0$. Then $px+(1-p)y=1-p$ would be required for M to be a martingale. So we choose $(x,y,z.)$ to satisfy the first identity and not the second. Usin $p=\frac{1}{2}$, for simplicity, $M(0)=0$, $M(1)=(\frac{1}{2},-\frac{1}{2})$, $M(2)=(\frac{4}{3},\frac{1}{3},-2)$ provides one possible solution.

Exercise 4.3

Prove a version of the previous proposition, where $V(0)$ and predictable $y(n)$ are given and $x(n)$ has to be constructed so that the strategy is self-financing.

Solution For (x,y) to be self-financing, we require $V_{(x,y)}(n)=x(n+1)S(n)+y(n+1)A(n)$. Given $V(0)$ and y we construct $x(1)=\frac{1}{S(0)}(V(0)-y(1)A(0))$ and by induction

$$x(n+1)=\frac{1}{S(n)}(V_{(x,y)}(n)-y(n+1)A(n))$$

for $1 \leq n < N$. This defines a predictable sequence $x(n)$ since all terms on the right above $\mathcal{P}_n$-measurable. Mutliplying both sides by $S(n)$ and rearranging we see that the self-financing condition holds.

Exercise 4.4

Give an example where the values of a strategy become negative. Find conditions on the sequence $x(n)$ so that $V_{(x,y)}(n) \geq 0$ for all n.

Solution Suppose $V(0)=0$, and consider a single-step binomial model with $x(1)=1$, $y(1)=-S(0)$ (with $A(0)=1$). If the stock price goes down,

$V(1) = S(0)(1+D) - S(0)(1+R) < 0$. If $V(0) > 0$, for $x(1) > 0$ we consider the unfavourable case, the down movement, and solve

$$V(1) = x(1)S(0)(1+D) + (V(0) - x(1)S(0))(1+R) \geq 0$$

to get

$$x(1) \leq \frac{V(0)(1+R)}{S(0)(R-D)}$$

with routine extension for further steps.

Exercise 4.5

Find a condition on $x(n)$ so that $y(n) \geq 0$ (no risk-free borrowing) and the strategy is self-financing.

Solution We need

$$V_{(x,y)}(n) = x(n)S(n) + y(n)A(n) = x(n+1)S(n) + y(n+1)A(n),$$

so for all $y(n)$ to be non-negative we need, for $1 \leq n < N$,

$$\max(x(n), x(n+1)) \leq \frac{V(n)}{S(n)}.$$

Exercise 4.6

Given an $\mathcal{F}_n$-adapted sequence of d-vectors with $Z_j(n) > 0$ for all $n = 0, 1, \ldots, N$, $j = 0, 1, \ldots, d$ with $(\mathbf{x}, y)$ and $(\mathbf{S}, A)$ as above, show that the strategy $(\mathbf{x},y)$ is self-financing for the price process $(\mathbf{S}, A)$ if and only if it is self-financing for the price process $(Z_1S_1, \ldots, Z_dS_d, Z_0A)$.

Solution Write $V_{(x,y)}(n) = \mathbf{x}(n){\cdot}\mathbf{S}(n)$, where, by abuse of notation, we write $\mathbf{S} = (S_1, ..., S_d, A)$ and $\mathbf{x} = (x_1, ..., x_d, y)$ as vectors in $\mathbb{R}^{d+1}$. Dropping the subscript $(\mathbf{x}, y)$ for simplicity, the self-financing condition becomes

$$\Delta V(n) = \mathbf{x}(n){\cdot}(\Delta\mathbf{S}(n))$$

which is the same as saying

$$(\Delta\mathbf{x}(n)) \cdot \mathbf{S}(n-1) = 0,$$

since

$$\begin{aligned}
\Delta V(n) &= \mathbf{x}(n) \cdot \mathbf{S}(n) - \mathbf{x}(n-1) \cdot \mathbf{S}(n-1) \\
&= \mathbf{x}(n) \cdot (\mathbf{S}(n) - \mathbf{S}(n-1)) + (\mathbf{x}(n) - \mathbf{x}(n-1)) \cdot \mathbf{S}(n-1) \\
&= \mathbf{x}(n){\cdot}(\Delta\mathbf{S}(n)) + (\Delta\mathbf{x}(n)) \cdot \mathbf{S}(n-1).
\end{aligned}$$

Changing $\mathbf{S}$ to $\mathbf{ZS} = (Z_1S_1, \ldots, Z_dS_d, Z_0A)$ we obtain

$$(\Delta\mathbf{x}(n)) \cdot (\mathbf{ZS}(n-1)) = 0,$$

and conversely.

Exercise 4.7

Explain why the result of the previous exercise shows that the converse of Theorem 4.29 also holds.

Solution Exercise 4.6 shows that if the strategy is self-financing for the discounted prices then it is also self-financing for the original prices, since the two price processes differ by multiplication by a strictly positive adapted process (in fact, a deterministic one).

Exercise 4.8

Build a binomial model with $S(0) = 100$, $R = 10\%$ if calls with strikes 100 and 105 have prices 10 and 8, respectively.

Solution Consider a single step model. Then, assuming $S(0)(1+D) \le K \le S(0)(1+U)$

$$C(0) = \frac{q(S(0)(1+U) - K)}{1+R}$$

which gives a system

$$\begin{aligned} 11 &= \frac{0.1 - D}{U - D}(100(1+U) - 100) \\ 8.8 &= \frac{0.1 - D}{U - D}(100(1+U) - 105) \end{aligned}$$

with a solution: $U = 0.25, D = -0.017857$.

Exercise 4.9

Suppose that on you have entered a long forward contract with forward price $F(0,2) = 124.75$. At time 1 you would like to close this position. How much money would you have to pay (or receive) if $S(1) = 120$ and $R = 9\%$?

Solution The payoff at time 2 for the holder of the contract is $S(2)-F(0,2)$ which can be replicated at time 1 by buying one share and borrowing risk free the amount $F(0,2)$. The value of this replicating strategy is $120-114.45 = 5.55$, an amount to be received. In other words, assuming that the risk-free rate was the same at time 0, we can find $S(0) = 124.75 \times (1 + 9\%)^{-2} = 105$, and the value of the forward position at time 1 is $S(1) - S(0)(1 + R) = 5.55$. The value is positive if the stock goes up more than by the risk-free rate.

Exercise 4.10

Suppose that the stock prices on consecutive days are 100, 100.33, 100.33, 100.15, 99, 101. Find the cash flow of the long futures position if the annual risk-free rate is 9%. .

Solution With constant rate $f(n,N) = F(n,N) = S(0)(1+R_{\text{day}})^{N-n}$, and assuming $N = 5$ days, with $R_{\text{day}} = 0.0002361$, we have

$$\begin{aligned} F(1,5) - F(0,5) &= 0.3 \\ F(2,5) - F(1,5) &= -0.02 \\ F(3,5) - F(2,5) &= -0.2 \\ F(4,5) - F(3,5) &= -1.18 \\ F(5,5) - F(4,5) &= 1.98 \end{aligned}$$

Exercise 4.11

Conduct a similar analysis of the protective put, which consists of long positions (of one unit each) in the underlying stock and a put option on the stock.

Solution The initial outlay is $P(0)+S(0)$ which equals $C(0)+K(1+R)^{-N}$ by call-put parity. Our final position depends on which of $S(N)$ and K is larger. If $S(N) \geq K$ our put is worthless, and our net position is $S(N)-(S(0)+P(0))$, while is $S(N) < K$ it is $K-(S(0)+P(0))$. While there is a potential loss of the initial outlay, we are protected by the put without having to forgo the profit when $S(N)$ is much larger than K. The worst final postion is $K-(S(0)+P(0))$, so the choice of strike is significant.

Chapter 5

Exercise 5.1

Find the price of an American straddle, i.e., the derivative with payoff $I(n) = |S(n) - K|$ for $N = 4$, where $S(n)$ follows a binomial tree with $S(0) = 50$, $U = 12\%$, $D = -6\%$, $R = 5\%$, $K = 50$.

Solution Tree of payoffs:

$$
\begin{array}{ccccc}
 & & & & 28.68 \\
 & & & 20.25 & \\
 & & 12.72 & & 16.03 \\
 & 6 & & 8.96 & \\
0 & & 2.64 & & 5.42 \\
 & 3 & & 0.52 & \\
 & & 5.82 & & 3.49 \\
 & & & 8.47 & \\
 & & & & 10.96
\end{array}
$$

Tree of prices:

$$
\begin{array}{ccccc}
 & & & & 28.68 \\
 & & & 22.63 & \\
 & & 17.37 & & 16.03 \\
 & 13.17 & & 11.34 & \\
10.24 & & 8.25 & & 5.42 \\
 & 6.96 & & 4.45 & \\
 & & 5.82 & & 3.49 \\
 & & & 8.47 & \\
 & & & & 10.96
\end{array}
$$

Exercise 5.2

Investigate the dependence on the strike K of the difference between the prices of European and American puts on the same stock.

Solution Consider the first step in the backward computations.

$$
\begin{aligned}
P^{\mathrm{E}}(N-1) &= \frac{1}{1+R}(q(K-S(N))^{+} + (1-q)(K-S(N))^{+}), \\
P^{\mathrm{A}}(N-1) &= \max(K-(S(N-1))^{+}, \frac{1}{1+R}(K-q(S(N))^{+} \\
&\quad +(1-q)(K-S(N))^{+}), \\
P^{\mathrm{A}}(N-1) - P^{\mathrm{E}}(N-1) &= (K-S(N-1))^{+} - \frac{1}{1+R}(q(K-S(N))^{+} \\
&\quad +(1-q)(K-S(N))^{+})
\end{aligned}
$$

So the difference between the American and European put price increases with K since if we replace K by $K+\varepsilon$, the first term grows by ε (suppose it is positive) while the second term (which is subtracted) increases by no more that by $\frac{\varepsilon}{1+R}$. By induction the property is preserved with equality at earlier steps.

Exercise 5.3

Prove that $\tau : \Omega \to \{1, 2, \ldots, N\}$ is a stopping time if and only if $\{\omega : \tau(\omega) = n\} \in \mathcal{F}_n$ for all n.

Solution Suppose the condition holds for τ and fix $n \leq N$. As $\mathcal{F}_k \subset \mathcal{F}_n$ for all $k \leq n$, $\{\omega : \tau(\omega) \leq n\} = \cup_{k=1}^{n} \{\omega : \tau(\omega) = k\}$ is in $\mathcal{F}_n$, so τ is a stopping time. Conversely, given a stopping time τ and $n \leq N$, the set $\{\omega\{\tau(\omega) = n\} = \{\omega : \tau(\omega) \leq n\} \cap (\cap_{k<n} \{\omega : \tau(\omega) > k\}$ is in $\mathcal{F}_n$ since $\mathcal{F}_k \subset \mathcal{F}_n$ for $k \leq n$.

Exercise 5.4 In a binomial tree with three steps consider all possible stopping times and find numerically the one that maximises the expected discounted payoff (expectation with respect to the risk-neutral probability).

Solution There are $4^8 = 65536$ possible random variables with values in $\{0, 1, 2, 3]$ but the measurability condition restricts the number to 22, as shown below. Note that for instance if $\tau(\omega) = 1$ for some path, then this value has to be taken on all other paths with the same initial movement since $\{\tau = 1\} \in \mathcal{P}_1 = \{B^U, B^D\}$. (In the case at hand is easier to list the stopping times rather than write a computer code.) Given all stopping time, computing expected payoffs and taking the maximal one is straightforward and this returns the theoretical no-arbitrage option price.

UUU	0	1	2	3	1	1	2	3	3	2	2	2	2	2	2	3	3	3	2	3	3	3
UUD	0	1	2	3	1	1	2	3	3	2	2	2	2	2	2	3	3	3	2	3	3	3
UDU	0	1	2	3	1	1	2	3	2	3	2	2	2	3	3	3	2	2	3	2	3	3
UDD	0	1	2	3	1	1	2	3	2	3	2	2	2	3	3	3	2	2	3	2	3	3
DUU	0	1	2	3	2	3	1	1	2	2	3	2	3	2	3	2	3	2	3	3	2	3
DUD	0	1	2	3	2	3	1	1	2	2	3	2	3	2	3	2	3	2	3	3	2	3
DDU	0	1	2	3	2	3	1	1	2	2	2	3	3	3	2	2	2	3	3	3	3	2
DDD	0	1	2	3	2	3	1	1	2	2	2	3	3	3	2	2	2	3	3	3	3	2

Exercise 5.5

Find the optimal exercise time for the straddle described in Exercise 5.1

Solution For $\omega \in B^{DD}$, $\tau_{\text{opt}}(\omega) = 2$, otherwise $\tau_{\text{opt}}(\omega) = 5$ with $\mathbb{E}_Q(H(\tau)) = 10.24$.

Exercise 5.6

Find the exact form of the compensator and check that A is predictable for the American put in our 5-step model with $S(0) = 100$, $U = 15\%$, $D = -10\%$, $K = 100$, $R = 5\%$.

Solution

0	1	2	3	4	5
					0
				0	
					0
			0		
					0
				0	
					0
		0			
					0
				0	
					0
			0		
					0
				0	
					4.76
0	0				
					1.08
				1.08	
					1.08
			1.08		
					1.08
				1.08	
					5.84
		1.08			
					5.79
				5.79	
					10.55
			5.79		
					15.32
				10.55	
					15.32

Exercise 5.7

Find the compensator for the straddle with $I(n) = |S(n) - 50|$ for $N = 4$, where $S(n)$ follows a binomial tree with $S(0) = 50$, $U = 12\%$, $D = -6\%$, $R = 5\%$.

Solution

0	1	2	3	4
				0
			0	
				0
		0		
				0
			0	
				0
0	0			
				0
			0	
				0
		0		0.1
			0.1	
				2.48

Exercise 5.8

Show that for a call option the compensator is zero (use the data from the previous Exercise or work with a general case) which suggests that it is never optimal to exercise a call before maturity.

Solution The construction of Snell envelope compares the payoff with the discounted expected value at the next step, which can be regarded as European derivative. The first step, defining the option prices at $N-1$, returns the discounted expected payoff at N since in Proposition 5.14 it is shown that the European option price dominates the payoff. By induction this shows that at each node the immediate payoff will be dominated and the tree of Snell envelope will concide with European call option prices.

Exercise 5.9

Show that in a 4-step binomial model with $U = 10\%$, $D = -10\%$, $R = 0$, a put with $K = S(0)$ can be optimally exercised at time 4. Find τ_{opt}, and explain why this example does not contradict Remark 5.9, despite the fact that $\tau_{\text{opt}}(\omega) < 4$ for some ω

Solution Here the optimal time is $\tau_{\text{opt}}(\omega) = 2$ for $\omega \in B_{UU}$, $\tau_{\text{opt}}(\omega) = 3$ for $\omega \in B_{DUD} \cup B_{UDD}$ and it is 4otherwise. However this is decided on the basis of payoff being equal to the expected future value, not strictly greater. hence postponing the exercise does not create a loss. In fact, taking $\tau = 4$ for all ω gives the same expected payoff, so this constant τ is optimal al well. The compensator here is identically 0 at all nodes which is a consequence of the irrelevance of the choice of the exercise time.

Exercise 5.10

Investigate the dependence of American call price on a dividend expressed as a percentage of the stock price

Solution Within the data of the example (Example 5.1) it can be seen that if d is the rate at which the dividend is paid, for $d \geq 7.7\%$ the call price stays at 14.54, and then increases as d decreases with maximal value 22.91 for $d = 0$.

Exercise 5.11

Show that European and American derivative securities not paying dividends have the same prices if the payoff is of the form $h(S(n))$ for convex $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$. Find a counterexample if the last condition is violated.

Solution To show that the American option price is equal to that of its European counterpart it is enough to verify that the discounted American option price process is a martingale under the risk neutral probability. We shall prove by backward induction that for each $n = 0, \ldots, T$

$$Z(n) \geq h(S(n)). \tag{4}$$

Indeed, for $n = N$ we have $Z(N) = h(S(N))$. Now suppose that (4) holds for some n. Then, using the fact that h is a convex function and applying Jensen's inequality, we have

$$\begin{aligned}
\frac{1}{1+R}\mathbb{E}_Q(Z(n)|\mathcal{F}_{n-1}) &\geq \frac{1}{1+R}\mathbb{E}_Q(h(S(n))|\mathcal{F}_{n-1}) \\
&= \mathbb{E}_Q(\frac{1}{1+R}h(S(n)) + \frac{R}{1+R}h(0)|\mathcal{F}_{n-1}) \\
&\quad (\text{since } 0 = \frac{R}{1+R}h(0)) \\
&\geq \mathbb{E}_Q(h(\frac{1}{1+R}S(n))|\mathcal{F}_{n-1}) \quad (\text{convexity of } h) \\
&\geq h(\mathbb{E}_Q(\frac{1}{1+R}S(n)|\mathcal{F}_{n-1})) \quad (\text{Jensen inequality}) \\
&= h(S(n-1)),
\end{aligned}$$

so that

$$Z(n-1) = \max\{h(S(n-1)), \frac{1}{1+R}\mathbb{E}_Q(Z(n)|\mathcal{F}_{n-1})\} \geq h(S(n-1)),$$

which completes the proof by induction. It follows that

$$Z(n) = \max\{h(S(n)), \frac{1}{1+R}\mathbb{E}_Q(Z(n+1)|\mathcal{F}_n)\} = \frac{1}{1+R}\mathbb{E}_Q(Z(n+1)|\mathcal{F}_n),$$

so that $(1+R)^{-n}Z(n)$ is indeed a martingale under the risk neutral probability. As a result,

$$Z(0) = \frac{1}{(1+R)^N}\mathbb{E}_Q(h(S_N)),$$

which means that the price $Z(0)$ of the American option with expiry time N and payoff $h(S(n))$ is equal to that of its European counterpart with expiry time N and payoff $h(S(N))$.

An example with $h(0) > 0$, and where this fails, is the put option.

Exercise 5.12

Suppose the underlying stock S pays a dividend D at time $n < N$. Show that

$$S(0) - D(1+R)^{-n} - K \leq C^{\mathrm{A}} - P^{\mathrm{A}} \leq S(0) - K(1+R)^{-N}.$$

Solution If $S(0) - K(1+R)^{-N} < C_{\mathrm{A}} - P_{\mathrm{A}}$, then write and sell a call, buy a put, and buy a share, investing (or borrowing, if negative) the balance risk-free. Then we can write and sell a call, and buy a put and a share, financing the transactions in the money market. If the holder of the American call chooses to exercise it at time $n \leq N$, then we shall receive K for the share and settle the money market position, ending up with the put and a positive amount

$$\begin{aligned} & K + (C_{\mathrm{A}} - P_{\mathrm{A}} - S(0))(1+R)^n \\ &= (K(1+R)^{-n} + C_{\mathrm{A}} - P_{\mathrm{A}} - S(0))(1+R)^n \\ &\geq (K(1+R)^{-N} + C_{\mathrm{A}} - P_{\mathrm{A}} - S(0))(1+R)^n > 0. \end{aligned}$$

If the call option is not exercised at all, we can sell the share for K by exercising the put at time N and close the money market position, also ending up with a positive amount

$$K + (C_{\mathrm{A}} - P_{\mathrm{A}} - S(0))(1+R)^N > 0.$$

The above argument is for zero dividend, now the arbitrage profit may also include the dividend if the call is exercised after the dividend becomes due. (Nevertheless, the dividend cannot be included in this inequality because the option may be exercised and the share sold before the dividend is due.)

If

$$C_{\mathrm{A}} - P_{\mathrm{A}} < S(0) - D(1+R)^{-n} - K,$$

then at time 0 sell short a share, write and sell a put, and buy a call option, investing the balance risk-free. If the put is exercised at time $n < N$, you will have to buy a share for K, borrowing the amount at the rate r. As the dividend becomes due, borrow the amount risk-free and pay it to the owner of the share. At time N return the share to the owner, closing the short sale. You will be left with the call option and a positive amount

$$\begin{aligned} & (S(0) + P_{\mathrm{A}} - C_{\mathrm{A}} - D(1+R)^{-n})(1+R)^N - K(1+R)^{N-n} \\ &> K(1+R)^N - K(1+R)^{(N-n)} \\ &\geq 0. \end{aligned}$$

(If the put is exercised before the dividend becomes due, you can increase your arbitrage profit by closing out the short position in stock immediately, in which case you would not have to pay the dividend.) If the put is not exercised before expiry N, then at time 0 sell short a share, write and sell a put, and buy a call option, investing the balance on the money market. When a dividend becomes due on the shorted share, borrow the amount and pay it to the owner of the stock. At time N close the money market position, buy a share for K by exercising the call if $S(N) > K$ or settling the put if $S(N) \leq K$, and close the short position in stock. Your arbitrage profit will be

$$(-C_{\mathrm{E}} + P_{\mathrm{E}} + S(0) - D(1+R)^{-n})(1+R)^{N} - K > 0.$$

Exercise 5.13

Show that the following hold for a stock that pays a dividend D at time $n \leq N$:

$$\begin{array}{rcl} \max\{0, S(0) - D(1+R)^{-n} - K(1+R)^{-N}, S(0) - K\} & \leq & C^{\mathrm{A}}, \\ \max\{0, K(1+R)^{-N} + D(1+R)^{-n} - S(0), K - S(0)\} & \leq & P^{\mathrm{A}}. \end{array}$$

Solution The lower bounds for European options imply

$$S(0) - D(1+R)^{-n} - K(1+R)^{-N} \leq C_{\mathrm{E}} \leq C_{\mathrm{A}},$$

$$-S(0) + D(1+R)^{-n} + K(1+R)^{-N} \leq P_{\mathrm{E}} \leq P_{\mathrm{A}}.$$

But because the price of an American option cannot be less than its payoff at any time, we also have $S(0) - K \leq C_{\mathrm{A}}$ and $K - S(0) \leq P_{\mathrm{A}}$. Moreover, the upper bounds $C_{\mathrm{A}} < S(0)$ and $P_{\mathrm{A}} < K$ can be established in a similar manner as for non-dividend-paying stock. All these inequalities can be summarised as follows: for dividend-paying stock

$$\begin{array}{rcl} \max\{0, S(0) - D(1+R)^{-n} - K(1+R)^{-N}, S(0) - K\} & \leq & C_{\mathrm{A}} < S(0), \\ \max\{0, -S(0) + D(1+R)^{-n} + K(1+R)^{-N}, -S(0) + K\} & \leq & P_{\mathrm{A}} < K. \end{array}$$

Exercise 5.14

Show that the American put option price $P^{\mathrm{A}}(S)$ is a nonincreasing function of the underlying S.

Solution Suppose that $P_{\mathrm{A}}(S') < P_{\mathrm{A}}(S'')$ for some $S' < S''$, where $S' = x'S(0)$ and $S'' = x''S(0)$. We can write and sell a put on a portfolio with x'' shares and buy a put on a portfolio with x' shares, both options having the same strike price K and expiry time N. The balance $P_{\mathrm{A}}(S'') - P_{\mathrm{A}}(S')$ of these transactions can be invested in the money market. If the written option is exercised at time $n \leq N$, then we can meet the liability by exercising the other option immediately. Indeed, since $x' < x''$, the payoffs satisfy $(K - x'S(n))^{+} \geq (K - x''S(n))^{+}$. The investment in the money market provides an arbitrage profit.

Exercise 5.15

Show that American call and put prices are both convex functions of the underlying.

Solution We claim

$$\begin{array}{rcl} C_{\mathrm{A}}(\alpha S' + (1-\alpha)S'') & \leq & \alpha C_{\mathrm{A}}(S') + (1-\alpha)C_{\mathrm{A}}(S''), \\ P_{\mathrm{A}}(\alpha S' + (1-\alpha)S'') & \leq & \alpha P_{\mathrm{A}}(S') + (1-\alpha)P_{\mathrm{A}}(S''). \end{array}$$

The inequality for calls follows from the fact that American and European call prices are the same and the convexity of European prices. Let $S = \alpha S' + (1-\alpha)S''$ and let $S' = x'S(0)$, $S'' = x''S(0)$ and $S = xS(0)$. Suppose that

$$P_{\mathrm{A}}(S) > \alpha P_{\mathrm{A}}(S') + (1-\alpha)P_{\mathrm{A}}(S'').$$

We can write and sell a put on a portfolio with x shares, and purchase α puts on a portfolio with x' shares and $1-\alpha$ puts on a portfolio with x'' shares, all three options sharing the same strike price K and expiry time N. The positive balance $P_{\mathrm{A}}(S) - \alpha P_{\mathrm{A}}(S') - (1-\alpha)P_{\mathrm{A}}(S'')$ of these transactions can be invested in the money market. If the written option is exercised at time $n \leq N$, then we shall have to pay $(K - xS(n))^+$, where $x = \alpha x' + (1-\alpha)\, x''$. This is an arbitrage strategy because the other two options cover the liability:

$$(K - xS(n))^+ \leq \alpha\,(K - x'S(n))^+ + (1-\alpha)\,(K - x''S(n))^+ .$$

Exercise 5.16

Provide arbitrage arguments to verify the following inequalities when $K' < K''$:

$$\begin{aligned} C^{\mathrm{A}}(K') - C^{\mathrm{A}}(K'') &< K'' - K', \\ P^{\mathrm{A}}(K'') - P^{\mathrm{A}}(K') &< K'' - K'. \end{aligned}$$

Solution The inequality for calls follows from the same inequality for European options. Suppose that $K' < K''$, but $P_{\mathrm{A}}(K'') - P_{\mathrm{A}}(K') > K'' - K'$. Let us write and sell a put with strike K'', buy a put with strike K' and invest the balance $P_{\mathrm{A}}(K'') - P_{\mathrm{A}}(K')$ in the money market. If the written option is exercised at time $n \leq T$, we will have to pay $K'' - S(n)$. To do so, we can exercise the option with strike K', receiving the payoff $K' - S(n)$. Added to the amount invested in the money market with interest accumulated up to time n, it gives

$$\begin{aligned} K' - S(n) + (P_{\mathrm{A}}(K'') - P_{\mathrm{A}}(K'))\,(1+R)^n &> K' - S(n) + (K'' - K')\,(1+R)^n \\ &\geq K'' - S(n). \end{aligned}$$

The positive surplus above $K'' - S(n)$ would be our arbitrage profit.

Chapter 6

Exercise 6.1.

Show that $r_m < r_k$ if $m > k \geq 1$ (taking $r_1 = r$).

Solution $(1 + R_{\frac{1}{m}})^m = 1 + r$ with $r_m = mR_{\frac{1}{m}}$ implies $(1 + \frac{1}{m}r_m)^m = 1 + r$ so $r_m = m[(1+r)^{\frac{1}{m}} - 1]$ and with $f(x) = x((1+r)^{\frac{1}{x}} - 1)$, we have $f'(x) = ((1+r)^{\frac{1}{x}} - 1) + x(1+r)^{\frac{1}{x}} \ln(1+r) \ln x > 0$ for $x > 1$.

For each particular step we have returns of the form $R_{\frac{1}{m}} = \frac{1}{m}r_m$, or $R_h = hr_m$, representing the so called *simple interest rule*: the annual interest rate is linearly scaled to the period of length h. This reflects that fact that no transactions take place between consecutive steps. The total growth is best described as a product of single step growth factors $1 + R_{\frac{1}{m}}$ since after each step the interest is added to the account, so that $A(n+1) = A(n)(1+R_{\frac{1}{m}})$, the interest being $A(n)R_{\frac{1}{m}}$. The increased sum is the basis for the next step.

When we are only concerned with the value of a risk-free investment (or a loan) at a future instant n, with no cash flows in the meantime then the simple interest convention is used to annualise the return and the rate which is quoted for such transactions is given by

$$r(0,n) = \frac{1}{nh}\frac{A(n) - A(0)}{A(0)}$$

For $h = \frac{1}{m}$. $n = m$ we have $r(0,m) = r$. Note that above for each step we in fact employ the simple interest convention as the expression $1 + \frac{1}{m}r_m$ for the single step growth factor shows ($h = \frac{1}{m}$, $n = 1$).

Exercise 6.2.

Find the formula expressing $r(0,n)$ by means of the interest rate r. Show that the sequence $(r(0,n))_{n \leq N}$ is increasing.

Solution Inserting the form of $A(n)$ we get $r(0,n) = \frac{1}{nh}\left[(1+r)^n - 1\right]$, and with $f(x) = \frac{1}{xh}\left[(1+r)^x - 1\right]$, $f'(x) = \frac{1}{h}\ln x \left[(1+r)^x - 1\right] + \frac{1}{xh}(1+r)^x \ln(1+r) > 0$.

Exercise 6.3

Find the rates $L(0,n)$ implied by the bond prices of Example 6.1. Compute the bond prices $B(1,n)$ assuming that the rates stay constant: $L(1,n) = L(0,n)$, $n > 1$.

Solution

n	$B(0,n)$	$L(0,n)$
1	0.9991	1.08%
2	0.9974	1.56%
3	0.9956	1.77%
4	0.9939	1.84%
5	0.9921	1.91%
6	0.9903	1.96%
7	0.9884	2.01%
8	0.9866	2.04%
9	0.9847	2.07%
10	0.9829	2.09%
11	0.9810	2.11%
12	0.9791	2.13%

Clearly $B(1,n) = B(0,n-1)$, $n \geq 2$.

Exercise 6.4.

Using the results of Exercise 6.3 find the rate $L(1,12)$ required for $B(1,12)$ to fall below $B(0,12)$.

Solution

$$\begin{aligned} B(1,12) &= \frac{1}{1+L(1,12)\frac{11}{12}} \leq B(0,12), \\ L(1,12) &\geq \frac{12}{11}\left(\frac{1}{B(0,12)} - 1\right) = 2.33\% \end{aligned}$$

Exercise 6.5. (Corrected formulation)

An investor gambles on a decrease in interest rates and wishes to earn a return $K(0,k)$ higher by 0.1% than the return implied by the current rates. Sketch the graph of the function $k \mapsto L(k,n)$ which would allow one to achieve this at any $0 < k < n$. First try the data from Example 6.1.

Solution Transforming the formula for $K(0,k)$ we get

$$L(k,12) = \frac{1}{(12-k)/12}\left(\frac{1+L(0,12)}{K(0,k)+1} - 1\right).$$

For the increase of return by 0.1%, the future rates would have to be

k	$L(k,12)$
1	2.1173%
2	2.1211%
3	2.1121%
4	2.1162%
5	2.1040%
6	2.0877%
7	2.0404%
8	2.0001%
9	1.8921%
10	1.7372%
11	1.1503%

Anyway, a uniform increase is consistent with the fact that the return depends on the length of the period, so this increase is in fact highly non-uniform.

Exercise 6.6

Find the forward prices $B(0,m,12)$ for the data from Examplke 6.1

Solution We have

$$B(0,m,12) = \frac{B(0,12)}{B(0,m)}$$

so

m	$B(0,m,12)$
1	0.9800
2	0.9817
3	0.9834
4	0.9851
5	0.9869
6	0.9887
7	0.9906
8	0.9924
9	0.9943
10	0.9961
11	0.9981

Exercise 6.7.

Illustrate this theorem using the data from Example 6.1. Are the rates $F(k,m,n)$ non-negative?

Solution

$$F(0,m,n) = -\frac{B(0,n) - B(0,m)}{(n-m)hB(0,n)},$$

m	$F(0,m,12)$
0	2.1346%
1	2.2284%
2	2.2429%
3	2.2470%
4	2.2674%
5	2.2761%
6	2.2878%
7	2.2796%
8	2.2980%
9	2.2878%
10	2.3287%
11	2.3287%

Exercise 6.8.

Is it possible to have $F(k,m,n) < 0$? Give an example. Formulate a property which would guarantee $F(k,m,n) \geq 0$.

Solution

Condition for positive forward rates

$$F(0,m,n) > 0 \quad \text{iff} \quad B(0,n) < B(0,m).$$

This may happen in practice if n is close to m, and this indicates the prediction of decrease of the yields for some future times, a prediction reflected in the bond prices.

Exercise 6.9

Find the forward rates $F(0,n)$ for the data from Example 6.1.

Solution We use the following formula for $n = 0, 1, \ldots, 11$:

$$F(0,n) = -\frac{B(0,n+1) - B(0,n)}{hB(0,n+1)}$$

obtaining

n	$F(0,n)$
0	1.0810%
1	2.0453%
2	2.1695%
3	2.0525%
4	2.1772%
5	2.1812%
6	2.3068%
7	2.1893%
8	2.3154%
9	2.1976%
10	2.3242%
11	2.3287%

Exercise 6.10.

Assume that for all $k \le n$, $B(k,n) = (1+r)^{-(n-k)}$ (with $h=1$), Compute all simple forward and spot rates at time k.

Solution

$$F(k,m,n) = -\frac{B(k,n)-B(k,m)}{(n-m)hB(k,n)} = -\frac{(1+r)^{-n}-(1+r)^{-m}}{(n-m)h(1+r)^{-n}} = -\frac{1-(1+r)^{n-m}}{(n-m)h},$$

$$F(k,n) = F(k,n,n+1) = \frac{r}{h},$$

with the spot rates $r(k) = F(k,k) = \frac{r}{h}$ as well.

Exercise 6.11.

Assume that the unit is one month and find the zero-coupon bonds implied by a 4-month zero-coupon bond trading at 98, and two coupon bonds with semi-anual coupons of 10, one maturing after 14 months trading at 101, and one maturing after 12 months trading at 103.

Solution

$$\begin{aligned} B(0,4) &= 0.98, \\ 10B(0,2) + 10B(0,8) + 110B(0,14) &= 101, \\ 10B(0,6) + 110B(0,12) &= 103. \end{aligned}$$

We have two equations and 5 variables so some additional assumptions are needed. First, assume that the rates r_2, r_4 (monthly compounding, anual rates) implied by the bonds $B(0,2)$ and $B(0,4)$ are the same

$$\begin{aligned} B(0,4) &= \frac{1}{(1+\frac{1}{12}r_4)^4} = 0.98, \\ r_2 &= r_4 = 6.08\% \\ B(0,2) &= \frac{1}{(1+\frac{1}{12}r_2)^2} = 0.9899. \end{aligned}$$

Then assume that (an alternative, to impose linear interpolation for the rates is better, but more complicated numerically):

$$\begin{aligned} B(0,6) &= \frac{1}{2}B(0,8) + \frac{1}{2}B(0,4) \\ B(0,8) &= \frac{2}{3}B(0,6) + \frac{1}{3}B(0,12) \end{aligned}$$

to get a system with two variables yielding

$$\begin{aligned} B(0,2) &= 0.9899 \\ B(0,4) &= 0.98 \\ B(0,6) &= 0.9476 \\ B(0,8) &= 0.9151 \\ B(0,12) &= 0.8502 \\ B(0,14) &= 0.745 \end{aligned}$$

Exercise 6.12.

Let $h = 1$ and prove that the coupon rate equals the implied interest rate if and only if $P(0) = F$.

Solution $C = rF$,

$$F = \sum_{j=1}^{J} rFB(0,j) + FB(0,J)$$

with

$$B(0,k) = \frac{1}{(1+r)^k}$$

gives a true identity

$$1 = \sum_{j=1}^{J} r\frac{1}{(1+r)^j} + \frac{1}{(1+r)^J}.$$

Exercise 6.13.

Having sold a variable coupon bond with face value F design a hedging (replicating) strategy.

Solution

(This gives an alternative proof of Theorem 6.11.) We replicate the coupon C_i by a self-financing strategy:

At time t buy F bonds maturing at T_{i-1} and sell F bonds maturing at T_i. This will cost $F(B(t,T_{i-1}) - B(t,T_i))$.

At time T_{i-1} the T_{i-1}-bonds will mature and you will receive the amount F. Invest this amount in T_i-bonds. This will give $\frac{F}{B(T_{i-1},T_i)}$ such bonds.

At time T_i the T_i-bonds will mature. Since you are holding $\frac{F}{B(T_{i-1},T_i)} - F$ of such bonds, you will receive the amount $\frac{F}{B(T_{i-1},T_i)} - F$, that is, the coupon value C_i.

We have replicated the cash flow provided by the ith coupon using a self-financing strategy with initial cost $F\,(B(t,T_{i-1}) - B(t,T_i))$. This, then, must be the value of the coupon at time t. The time t value of the floating coupon bond must therefore be

$$P(t) = \sum_{i=1}^{N} F\,(B(t,T_{i-1}) - B(t,T_i)) + FB(t,T_N) = FB(t,T_0).$$

In particular, on its emission date T_0 the price of a floating coupon bond is equal to the face value,

$$P(T_0) = F.$$

Exercise 6.14

Find the swap rates $r_{\text{swap}}(0,n)$ for the data from Example 6.1. Perturb one of the rates obtained by adding or subtracting x and reconstruct the bond prices. Analyse the impact of x on the prices obtained.

Solution We have $h_{\text{coupon}} = \frac{1}{12}$, the formula

$$r_{\text{swap}} = \frac{1 - B(0,J)}{h_{\text{coupon}} \sum_{j=1}^{J} B(0,j)}.$$

gives the rate $r_{\text{swap}}(0,J)$ and so

J	$r_{\text{swap}}(0,J)$
1	1.0810%
2	1.5627%
3	1.7646%
4	1.8364%
5	1.9043%
6	1.9503%
7	2.0009%
8	2.0243%
9	2.0564%
10	2.0704%
11	2.0933%
12	2.1127%

Given the swap rates we can reconstruct the bond prices inductively

$$h_{\text{coupon}} r_{\text{swap}}(0,J) = \frac{1 - B(0,J)}{\sum_{j=1}^{J-1} B(0,j) + B(0,J)}.$$

so

$$B(0,J) = \frac{1 - \sum_{j=1}^{J-1} B(0,j) h_{\text{coupon}} r_{\text{swap}}(0,J)}{1 + h_{\text{coupon}} r_{\text{swap}}(0,J)}$$

Any perturbation of one swap rate increases the corresponding bond price.

Exercise 6.15.

Find an arbitrage strategy for the above example.

Solution

Consider $B(n,3)$ as a derivative security with $B(n,2)$ as the underlying, $n = 0,1$, exercise time $N = 1$. So computing dicounted expected payoff

$$\mathbb{E}_{Q_2}(B(1,2)) = B(0,1)(q_2 B^{\mathrm{u}}(1,3) + (1-q_2) B^{\mathrm{d}}(1,3)) = 0.98970997 > 0.9897 = B(0,3)$$

so arbitrage requires buying $B(0,3)$ and selling a portfolio of $B(0,1)$ and $B(0,2)$ replicating the payoff $B(1,3)$.

Exercise 6.16.

Within the scheme of Example 6.13 show that if $B(0,2)(1+R) = \mathbb{E}(B(1,2))$, $B(0,3)(1+R) = \mathbb{E}(B(1,3))$ then $q_3 = q_2$ for any p. Formulate a condition linking the expectations and variances of $B(1,2)$ and $B(1,3)$ so that $q_2 = q_3$ for all p. Show that if this condition does not hold, $q_2 \neq q_3$ for any $p \in (0,1)$.

Solution The conditions

$$\begin{aligned} pB^{\mathrm{u}}(1,2)+(1-p)B^{\mathrm{d}}(1,2) &= B(0,2)(1+R), \\ pB^{\mathrm{u}}(1,3)+(1-p)B^{\mathrm{d}}(1,3) &= B(0,3)(1+R), \end{aligned}$$

imply that p is the risk-neutral probability for each bond, so $p = q_2 = q_3$.

The condition guaranteeing equality of the risk neutral probabilities is that the market prices of risk coincide

$$\frac{m_2 - R}{\sigma_2} = \frac{m_3 - R}{\sigma_3}.$$

If this is violated, the risk-neutral probabilities are different (Exercise 2.8).

Exercise 6.17.

Change the value of $B^{\mathrm{u}}(1,2)$ in Example 6.13 so that $q_2 = q_3$ and analyse the expectations and variances after the change.

Solution

With $B^{\mathrm{u}}(1,2) = 0.996603016$ we ahve $q_2 = q_3 = 0.92014277$ and $\mathbb{E}(B(1,2)) = 0.99651$, $\sqrt{\mathrm{Var}(B(1,2))} = 0.00009$.

Exercise 6.18

Build the initial tree for a bond maturing at time 4 with initial price $B(0,4) = 0.9859$, such that the price of 100,000 puts with strike $K = 0.9898$ is 80 and find the perturbation $\varepsilon_{\mathrm{4to3}}$. Build the complete tree of prices $B(k,4)$ using the perturbations found in the previous example.

Solution With $\varepsilon_{\mathrm{4to3}} = 0.1084\%$ we get

						$B^{\mathrm{uuu}}(3,4)$	0.9972
				$B^{\mathrm{uu}}(2,4)$	0.9942		
						$B^{\mathrm{uud}}(3,4)$	0.9966
		$B^{\mathrm{u}}(1,4)$	0.9903				
						$B^{\mathrm{udu}}(3,4)$	0.9961
				$B^{\mathrm{ud}}(2,4)$	0.9925		
						$B^{\mathrm{u}}dd(3,4)$	0.9955
$B(0,4)$	0.9859						
						$B^{\mathrm{duu}}(3,4)$	0.9967
				$B^{\mathrm{du}}(2,4)$	0.9927		
						$B^{\mathrm{dud}}(3,4)$	0.9961
		$B^{\mathrm{d}}(1,4)$	0.9882				
						$B^{\mathrm{ddu}}(3,4)$	0.9957
				$B^{\mathrm{dd}}(2,4)$	0.991		
						$B^{\mathrm{ddd}}(3,4)$	0.995

Exercise 6.19

Find the short rates $r(3)$ following Exercise 6.18 and recover the bond prices $B(k,4)$ from the short rates.

Solution Using

$$r(k) = \frac{1}{h}\left(\frac{1}{B(k,k+1)} - 1\right)$$

we get

ω	$r(3)$
UUU	3.32%
UUD	4.08%
UDU	4.62%
UDD	5.38%
DUU	3.87%
DUD	4.63%
DDU	5.17%
DDD	5.92%

Given the rates we can recover $B(3,4)$ from $r(3)$ but for earlier bond prices we need risk neutral probabilities and rates $r(0), r(1), r(2)$. With rates as in the example and $q = 0.5$ we get the prices as in Exercise 3.18, but the values will be different if some other q is used.

Exercise 6.20.

Find the tree of bond prices in the above example taking $q = 0.5$ and then taking the above values of $q(0), q(1)$. Extend this to the case considered in Exercise 6.18.

Solution With $q = 0.5$ we find $B(k,3)$ for $k = 0, 1$,

0	1
	0.99392
0.98964	
	0.99256

For $q(0) = 0.330928$, $q(1) = 0.607029$ we have

0	1
	0.99399
0.9897	
	0.99263

Taking $q = 0.5$ and the rates from Exercise 6.19 we get $B(k,4)$ as below

0	1	2	3
			0.9972
		0.9943	0.9966
	0.9903		0.9962
		0.9926	0.9955
0.9861			0.9968
		0.9929	0.9962
	0.9885		0.9957
		0.9912	0.9951

To fit the inital bond price, taking $q(0)$, $q(1)$ as above, we find $q(2) = 0.4378$ and get

0	1	2	3
			0.9972
		0.9942	0.9966
	0.9905		0.9962
		0.9925	0.9955
0.9859			0.9968
		0.9929	0.9962
	0.9887		0.9957
		0.9912	0.9951

Exercise 6.21.
Consider the above models with $h = \frac{1}{12}$, $k = 0, 1, 2$ and try to calibrate them to the initial bond prices $B(0, 2) = 0.9932$, $B(0, 3) = 0.9897$ taking the rate implied by $B(0, 1) = 0.9966$ as the initial short rate.
Solution
First we find $r(0) = 4.0939\%$. Merton's model with $q = 0.5$,

$$\begin{aligned} r^{\mathrm{u}}(1) &= r(0) + a\frac{1}{12} - \sigma\sqrt{\frac{1}{12}}, \\ r^{\mathrm{u}}(1) &= r(0) + a\frac{1}{12} + \sigma\sqrt{\frac{1}{12}} \end{aligned}$$

$$\begin{aligned} B^u(1,2) &= \frac{1}{1 + r^u(1)/12} \\ B^d(1,2) &= \frac{1}{1 + r^d(1)/12} \\ B(0,2) &= \frac{1}{1 + r(0)/12}\left(\frac{1}{2}B^u(1,2) + \frac{1}{2}B^d(1,2)\right) = 0.9932 \end{aligned}$$

gives $a = 0.02$, $\sigma = 0.07$ (with some accuracy: the price obtained is 0.9932006). These values give the initial price for $N = 3$ equal to 0.98979365.
Vasicek model in the first step has the form

$$\begin{aligned} r^{\mathrm{u}}(1) &= r(0) + [a - br(0)]\,\frac{1}{12} - \sigma\sqrt{\frac{1}{12}}, \\ r^{\mathrm{u}}(1) &= r(0) + [a - br(0)]\frac{1}{12} + \sigma\sqrt{\frac{1}{12}} \end{aligned}$$

and with the same form for bond prices, for $a = 0.004$, $b = 0.0435$, $\sigma = 0.8$ we get better accuracy. For $N = 3$ the values $a = 006556$, $b = 0.0001$, $\sigma = 0.0004$ give $B(0, 3) = 0.989700005$.

Exercise 6.22.

Let $h = \frac{1}{12}$, let $n = 3$ and let the short rates and the prices for the bond with maturity 3 be as in the following tree:

$$
\begin{array}{ccccc}
 & & & \nearrow & B^{\mathrm{uu}}(2,3) = 0.9991 \\
 & & \begin{array}{l} B^{\mathrm{u}}(1,3) = 0.9962 \\ r^{\mathrm{u}}(1) = 2.40\% \end{array} & & \\
 & \nearrow & & \searrow & B^{\mathrm{ud}}(2,3) = 0.9960 \\
\begin{array}{l} B(0,3) = 0.9927 \\ r(0) = 4.11\% \end{array} & & & & \\
 & \searrow & & \nearrow & B^{\mathrm{du}}(2,3) = 0.9986 \\
 & & \begin{array}{l} B^{\mathrm{d}}(1,3) = 0.9959 \\ r^{\mathrm{d}}(1) = 3.00\% \end{array} & & \\
 & & & \searrow & B^{\mathrm{dd}}(2,3) = 0.9969
\end{array}
$$

First, compute the risk neutral probabilities $q(0)$, $q^{\mathrm{u}}(1)$, $q^{\mathrm{d}}(1)$ (of upward movement) using (6.7) and then compute the prices of the remaining bonds.

Solution $q(0) = \frac{2}{3}$, $q^{\mathrm{u}}(1) = 0.707225806$, $q^{\mathrm{d}}(1) = 0.876323529$, next the bond prices $B^{\mathrm{u}}(1,2)$, $B^{\mathrm{d}}(1,2)$ and $B(0,1)$ are given directly by the short rates, and

$$B(0,2) = B(0,1)(q(0)B^{\mathrm{u}}(1,2) + (1 - q(0))B^{\mathrm{d}}(1,2)) = 0.994432.$$

Exercise 6.23.

Build a concrete model for $N = 3$ with initial term structure determined by the familiar bond prices: $B(0,1) = 0.9966$, $B(0,2) = 0.9932$, $B(0,3) = 0.9897$, taking $q = \frac{1}{2}$, $\delta = 0.99923$.

Solution

		$B^{UU}(2,3)$	0.99724
$B^U(1,3)$	0.99384		
		$B^{UD}(2,3)$	0.99648
$B^D(1,3)$	0.99231		
		$B^{DD}(2,3)$	0.99571

$B^U(1,2)$	0.99697
$B^D(1,2)$	0.9962

Exercise 6.24.

Adjust the δ in the previous exercise so that the pack of 100,000 calls written on 3-bond with exercise time $n = 2$ and unit exercise price $K = 0.9964$ is worth 24.47 today.

Solution $\delta = 0.99923996008143$.

Exercise 6.25.

Find the short rates, and compute the expectations and variances following Exercise 6.23.

Solution Short rates

$$\begin{array}{ccc} & & 3.32\% \\ & 3.64\% & \\ 4.09\% & & 4.24\% \\ & 4.56\% & \\ & & 5.15\% \end{array}$$

Expected rates $\mathbb{E}(r(1)) = 4.101\%$, $\mathbb{E}(r(2)) = 4.237\%$, variances: $\mathrm{Var}(r(1)) = 0.000021$.