1

A Student's Guide to the Navier-Stokes Equations: A Supplement on the Reynolds Transport Theorem

A very common, and somewhat quick, way of deriving the integral form of the governing equations involves what is known as the Reynolds transport theorem. The Reynolds transport theorem involves taking the derivative of an integral whose limits are not constants but are instead functions of the variable of the derivative. The Reynolds transport theorem is essentially an extension of the Leibniz integration rule, also known as differentiation under the integral sign rule. You may see a number of forms for the Reynolds transport theorem but the most common is:

$$\boxed{\frac{d}{dt}\iiint_{\mathcal{V}(t)} f\, d\mathcal{V} = \iiint_{\mathcal{V}(t)} \frac{\partial f}{\partial t} d\mathcal{V} + \oiint_{A(t)} f\vec{V}\cdot\vec{n}\, dA} \tag{1.1}$$

where f is some function (which does not have to be a scalar but also could be a higher order tensor). The velocity in this case is the velocity of an small differential area segment on a moving fluid element. Notice on the left-hand side of the equation that we are attempting to take the time derivative of an integral whose limits are also functions of time (i.e., the $\mathcal{V}(t)$ is a function of time). When we came across this type of situation before, we cheated a little and attempted to "sneak" the derivative into integral with a trick where the derivative of the differential volume was related to the divergence of velocity (see Section 2.4). This trick can be avoided by simply using the Reynolds transport theorem.

Instead of deriving the Reynolds transport theorem, we are going to look at some examples of its use. The simplest example of the use of the Reynolds transport theorem is if we set $f = 1$, giving us:

$$\underbrace{\frac{d}{dt}\iiint_{\mathcal{V}(t)} d\mathcal{V}}_{\mathcal{V}(t)} = \iiint_{\mathcal{V}(t)} \overbrace{\cancel{\frac{\partial 1}{\partial t}}}^{0} d\mathcal{V} + \underbrace{\oiint_{A(t)} \vec{V}\cdot\vec{n}dA}_{\iiint_{\mathcal{V}(t)}\vec{\nabla}\cdot\vec{V}d\mathcal{V}}$$

$$\rightarrow \frac{D\mathcal{V}(t)}{Dt} = \iiint_{\mathcal{V}(t)} \vec{\nabla}\cdot\vec{V}d\mathcal{V}$$

$$\xrightarrow{\text{shrink}} \frac{D\left(d\mathcal{V}(t)\right)}{Dt} = \vec{\nabla}\cdot\vec{V}d\mathcal{V}$$

Notice that the result of setting $f = 1$ leads us right to the idea that the material derivative of volume of a moving fluid element is proportional to the divergence of velocity, which is something we found out in Chapter 2.

The next example we can look at is to set $f = \rho$, which is just mass per volume (or density). This leads us to:

$$\frac{d}{dt}\iiint_{\mathcal{V}(t)} \rho d\mathcal{V} = \iiint_{\mathcal{V}(t)} \frac{\partial\rho}{\partial t} d\mathcal{V} + \oiint_{A(t)} \rho\vec{V}\cdot\vec{n}dA \tag{1.2}$$

By setting the right-hand side of Equation 1.2 to zero, we obtain the integral form of the continuity equation:

$$\iiint_{\mathcal{V}(t)} \frac{\partial\rho}{\partial t} d\mathcal{V} + \oiint_{A(t)} \rho\vec{V}\cdot\vec{n}dA = 0$$

We can do the same thing with momentum as we did with mass. By setting $f = \rho\vec{V}$ (i.e., momentum per volume as opposed to mass per volume), the Reynolds transport theorem now states:

$$\frac{d}{dt}\iiint_{\mathcal{V}(t)} \rho\vec{V} d\mathcal{V} = \iiint_{\mathcal{V}(t)} \frac{\partial\left(\rho\vec{V}\right)}{\partial t} d\mathcal{V} + \oiint_{A(t)} \rho\vec{V}\vec{V}\cdot\vec{n}dA \tag{1.3}$$

The key thing here is recognizing that the $\vec{V}\vec{V}$ term on the right-hand side of Equation 1.3 is actually the outer product of the velocity vector with itself, i.e. $\vec{V}\otimes\vec{V}$. Therefore:

$$\frac{d}{dt}\iiint_{\mathcal{V}(t)} \rho\vec{V} d\mathcal{V} = \iiint_{\mathcal{V}(t)} \frac{\partial\left(\rho\vec{V}\right)}{\partial t} d\mathcal{V} + \oiint_{A(t)} \rho\vec{V}\otimes\vec{V}\cdot\vec{n}dA \tag{1.4}$$

The right-hand side of Equation 1.4, when set equal to the summation of forces, will lead to the integral form of the Navier-Stokes equations.

If we set $f = \rho e$ (or energy per volume), we get:

$$\frac{d}{dt}\iiint_{\mathcal{V}(t)} \rho e d\mathcal{V} = \iiint_{\mathcal{V}(t)} \frac{\partial(\rho e)}{\partial t} d\mathcal{V} + \oiint_{A(t)} \rho e \vec{V} \cdot \vec{n} dA$$

Setting the right-hand side of the above equation equal to $\dot{Q} - \dot{W} + Source$ gives the starting point for our energy equation in integral form.

As you can probably tell, we could have started with the Reynolds transport theorem and derived all of the equations from this theorem. However, the approach taken in the book, in the author's opinion, ends up offering a more intuitive approach to the governing equations.