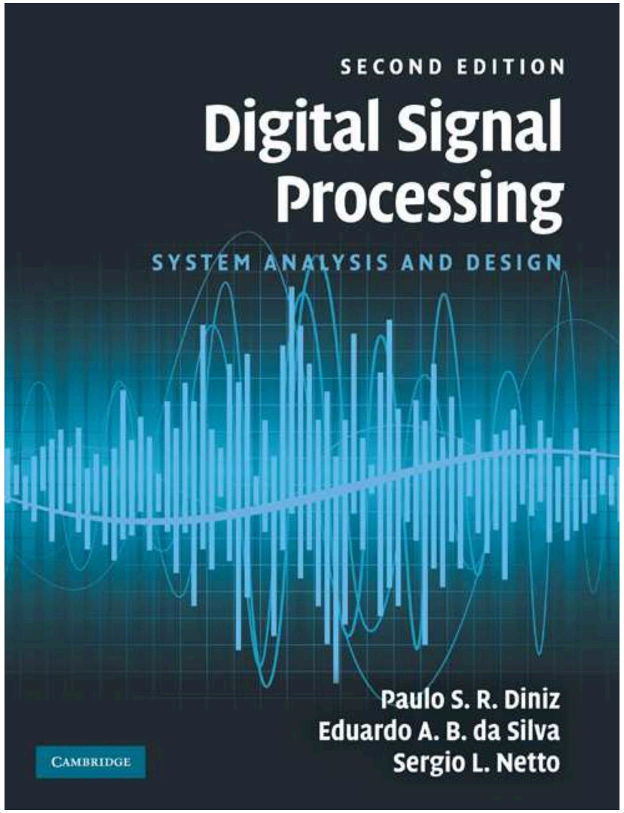
Discrete-time signals and systems



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Contents

Introduction

Discrete-time signal processing:

- Studies the rules governing the behavior of discrete-time signals
- Studies the systems that process them.
- Deals with the issues involved in processing continuous signals using digital techniques
- Pervades modern life; for example, is used in:
  - compact disc players
  - computer tomography
  - geological processing
  - mobile phones
  - electronic toys

  and many others.

Introduction

- Analog signal processing:
  - Processes a continuously varying quantity (analog signal)
  - Can be described by differential equations
- Digital signal processing:
  - Processes sequences of numbers (discrete-time signals) using some sort of digital hardware
  - Its power comes from the fact that, once a sequence of numbers is available to an appropriate digital hardware we can carry out any form of numerical processing on it.

Introduction

- For example, suppose we need to perform the following operation on a continuous-time signal:

$$y(t) = \frac{\cosh\left[\ln(|x(t)|) + x^3(t) + \cos^3\left(\sqrt{|x(t)|}\right)\right]}{5x^5(t) + e^{x(t)} + \tan(x(t))} \tag{1}$$

  - This would be clearly very difficult to implement using analog hardware.
  - However, if we sample the analog signal $x(t)$ and convert it into a sequence of numbers $x(n)$, it can be input to a digital computer, which can perform the above operation easily and reliably, generating a sequence of numbers $y(n)$.
  - If the continuous-time signal $y(t)$ can be recovered from $y(n)$, then the desired processing has been successfully performed.

Introduction

- This simple example highlights two important points.
  - One is how powerful digital signal processing is.
  - The other is that, if we want to process an analog signal using this sort of resource, we must have a way of converting a continuous-time signal into a discrete-time one, such that the continuous-time signal can be recovered from the discrete-time signal.
- However, it is important to note that very often discrete-time signals do not come from continuous-time signals, that is, they are originally discrete-time, and the results of their processing are only needed in digital form.

Discrete-time signals

- A discrete-time signal is one that can be represented by a sequence of numbers. For example, the sequence

$$\{x(n),\ \ n \in \mathbb{Z}\} \tag{2}$$

  where $\mathbb{Z}$ is the set of integer numbers, can represent a discrete-time signal where each number $x(n)$ corresponds to the amplitude of the signal at an instant nT.

- If $x_a(t)$ is an analog signal, we have that

$$x(n) = x_a(nT), \quad n \in \mathbb{Z} \tag{3}$$

  Since n is an integer, T represents the interval between two consecutive points at which the signal is defined.

  – It is important to note that T is not necessarily a time unit.

  – For example, if $x_a(t)$ is the temperature along a metal rod, then if T is a length unit, $x(n) = x_a(nT)$ may represent the temperature at sensors placed uniformly along this rod.

Discrete-time signals

- Here, we usually represent a discrete-time signal using the notation in equation (2). $x(n)$ is referred to as the nth sample of the signal (or the nth element of the sequence).

- An alternative notation, used in many texts, is to represent the signal as

$$\{x_a(nT), \ \ n \in \mathbb{Z}\} \tag{4}$$

  where the discrete-time signal is represented explicitly as samples of an analog signal $x_a(t)$.

  – In this case, the time interval between samples is explicitly shown, that is, $x_a(nT)$ is the sample at time nT.

Discrete-time signals

- Using the notation in equation (2), a discrete-time signal whose adjacent samples are 0.03 seconds apart would be represented as

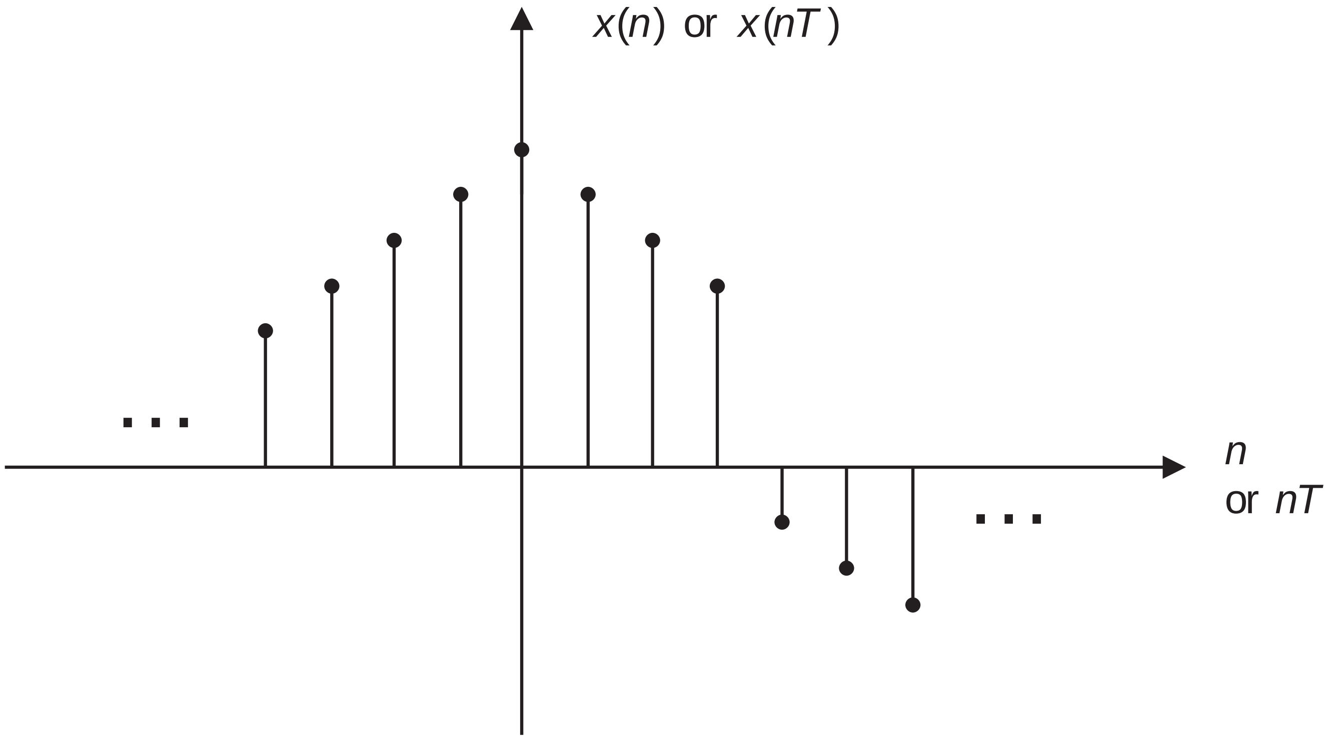
$$\ldots \quad x(0), x(1), x(2), x(3), x(4), \quad \ldots \tag{5}$$

whereas, using equation (4), it would be represented as

$$\ldots \quad x_a(0), x_a(0.03), x_a(0.06), x_a(0.09), x_a(0.12), \quad \ldots \tag{6}$$

Discrete-time signals

The graphical representation of a general discrete-time signal is shown below.

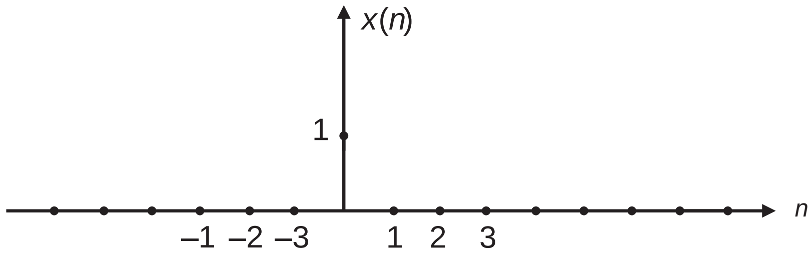


In what follows, we describe some of the most important discrete-time signals.

Discrete-time signals

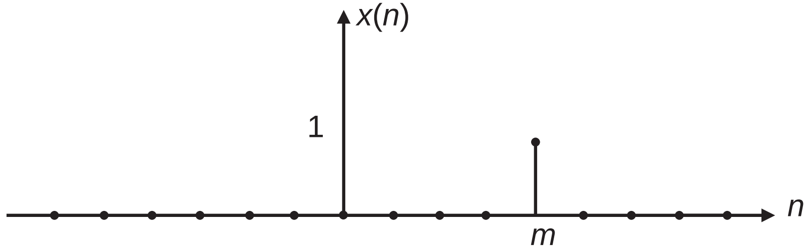
Unit impulse:

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



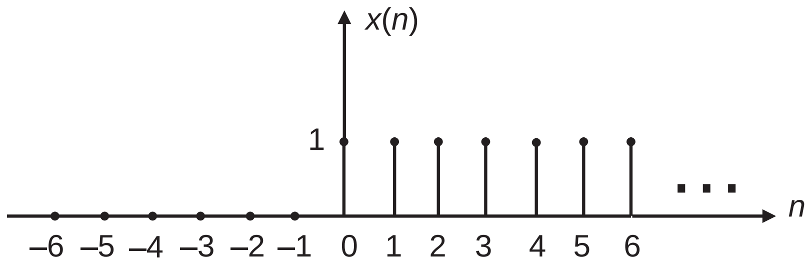
Delayed unit impulse:

$$\delta(n-m) = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

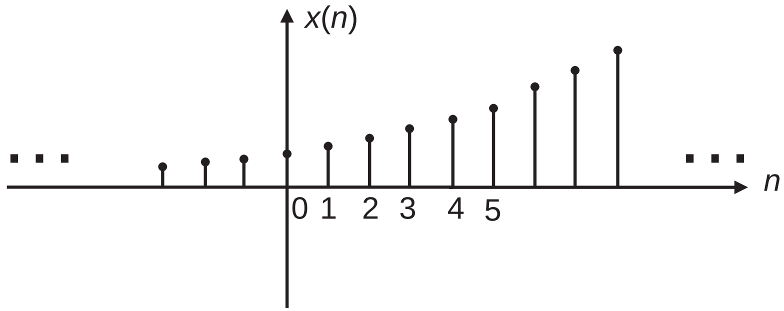


Unit step:

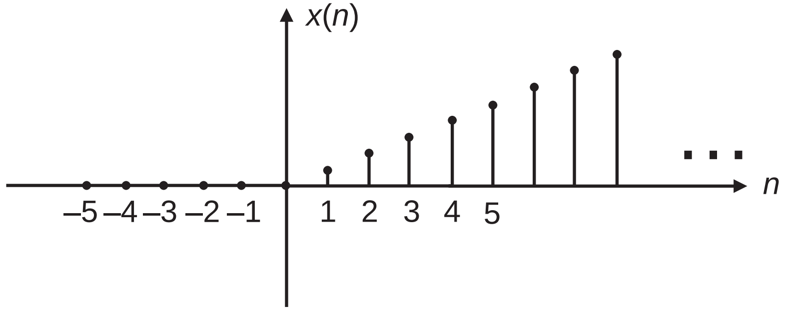
$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Discrete-time signals



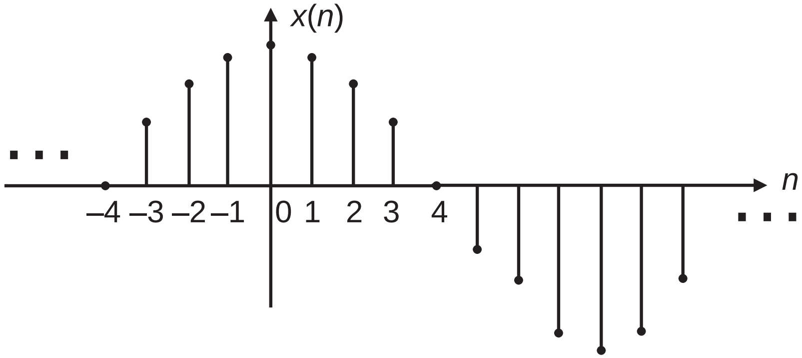
Real exponential function: $x(n) = e^{an}$



Unit ramp:
$$r(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Discrete-time signals

Cosine function: $x(n) = \cos(\omega n)$



- The angular frequency of this sinusoid is ω rad/sample and its frequency is $\frac{\omega}{2\pi}$ cycles/sample.

- For example, in the previous figure, the cosine function has angular frequency $\omega = \frac{2\pi}{16}$ rad/sample. This means that it completes one cycle, that equals 2π radians, in 16 samples.

Discrete-time signals

- If the sample separation represents time, ω can be given in rad/(time unit). It is important to note that, for $k \in \mathbb{Z}$,

$$\cos((\omega + 2k\pi)n) = \cos(\omega n + 2kn\pi) = \cos(\omega n) \tag{7}$$

$\Rightarrow$ This implies that, in the case of discrete signals, there is an ambiguity in defining the frequency of a sinusoid. In other words, when referring to discrete sinusoids, ω and $\omega + 2k\pi$, $k \in \mathbb{Z}$, are the same frequency.

Discrete-time signals

- By examining the above figures, we notice that any discrete-time signal is equivalent to a sum of shifted impulses multiplied by a constant, that is, the impulse shifted by k samples is multiplied by $x(k)$.
  - This can also be deduced from the definition of a shifted impulse in equation (11).
- For example, the unit step $u(n)$ in equation (11) can also be expressed as

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) \tag{8}$$

Likewise, any discrete-time signal $x(n)$ can be expressed as

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \tag{9}$$

Discrete-time signals

Periodic sequences:

- An important class of discrete-time signals or sequences is that of periodic sequences.
- A sequence $x(n)$ is periodic if and only if

There is $N \neq 0$ such that $x(n) = x(n+N)$ for all n

N is called the period of the sequence.

Discrete-time signals

- Using this definition, the period of the cosine function is an integer N such that

$$\cos(\omega n) = \cos(\omega(n + N)), \quad \text{for all } n \in \mathbb{Z} \tag{10}$$

  - This happens only if there is $k \in \mathbb{N}$ such that $\omega N = 2\pi k$. The smallest period is then

$$N = \min_{\substack{k \in \mathbb{N} \\ \frac{2\pi}{\omega} k \in \mathbb{N}}} \left\{ \frac{2\pi}{\omega} k \right\} \tag{11}$$

  - Therefore, we notice that not all discrete cosine sequences are periodic, as illustrated in Example 1.1 below.
  - An example of a periodic cosine sequence with period $N = 16$ samples is given in the illustration of the cosine function above.

Discrete-time signals

Example 1.1

Determine if the discrete signals above are periodic; if they are, determine their periods.

(a) $x(n) = \cos\left(\frac{12\pi}{5}n\right)$

(b) $x(n) = 10\sin^2\left(\frac{7\pi}{12}n + \sqrt{2}\right)$

(c) $x(n) = 2\cos\left(0.02n + 3\right)$

Discrete-time signals

Solution:

(a) $x(n) = \cos\left(\frac{12\pi}{5}n\right)$

In this case, we must have

$$\frac{12\pi}{5}(n+N) = \frac{12\pi}{5}n + 2k\pi \quad \Rightarrow \quad N = \frac{5k}{6} \tag{12}$$

This implies that the smallest N results for $k = 6$. Then the sequence is periodic with period $N = 5$. Note that in this case

$$\cos\left(\frac{12\pi}{5}n\right) = \cos\left(\frac{2\pi}{5}n + 2\pi n\right) = \cos\left(\frac{2\pi}{5}n\right) \tag{13}$$

and thus we have also that the frequency of this sinusoid, besides being $\omega = \frac{12\pi}{5}$, is also $\omega = \frac{2\pi}{5}$, as indicated by equation (7).

(b) $x(n) = 10\sin^2\left(\frac{7\pi}{12}n + \sqrt{2}\right)$

In this case, periodicity implies that

$$\sin^2\left(\frac{7\pi}{12}(n+N)\right) = \sin^2\left(\frac{7\pi}{12}n\right) \quad (14)$$

and then

$$\sin\left(\frac{7\pi}{12}(n+N)\right) = \pm\sin\left(\frac{7\pi}{12}n\right) \quad (15)$$

such that

$$\frac{7\pi}{12}(n+N) = \frac{7\pi}{12}n + k\pi \quad \Rightarrow \quad N = \frac{12k}{7} \quad (16)$$

The smallest N results for $k = 7$. Then this discrete-time signal is periodic with period $N = 12$.

(c) $x(n) = 2\cos(0.02n + 3)$

The periodicity condition requires that

$$\cos(0.02(n + N) + 3) = \cos(0.02n + 3) \tag{17}$$

such that

$$0.02(n + N) = 0.02n + 2k\pi \quad \Rightarrow \quad N = 100k\pi \tag{18}$$

Since no integer N satisfies the above equation, then the sequence is not periodic.

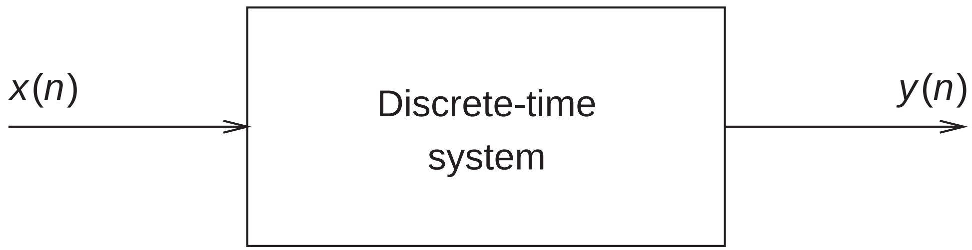
$\triangle$

Discrete-time systems

- A discrete-time system maps an input sequence $x(n)$ to an output sequence $y(n)$, such that

$$y(n) = \mathcal{H}\{x(n)\} \tag{19}$$

where the operator $\mathcal{H}\{\cdot\}$ represents a discrete-time system.



- A discrete-time system can be classified as:
  - linear or nonlinear
  - time invariant or time variant
  - causal or noncausal.

Discrete-time systems

Linearity:

- It is usually desirable to have a system that
  - When the amplitude of the input increases, the amplitude of the output increases without being distorted.
  - The output of a given combination of two signals is equivalent to the same combination applied on the outputs of the individual signals.
- A system with such properties is referred to as being **linear**.
- In more precise terms, a discrete-time system is linear if and only if

$$\mathcal{H}\{ax(n)\} = a\mathcal{H}\{x(n)\} \tag{20}$$

and

$$\mathcal{H}\{x_1(n) + x_2(n)\} = \mathcal{H}\{x_1(n)\} + \mathcal{H}\{x_2(n)\} \tag{21}$$

for any constant a, and any sequences $x(n)$, $x_1(n)$, and $x_2(n)$.

Discrete-time systems

Time invariance:

- It is sometimes desirable to have a system whose properties do not vary in time.
- Such a system is referred to as being **time invariant**.
- In more precise terms, a discrete-time system is time invariant if and only if, for any input sequence $x(n)$ and integer n_0, given that

$$\mathcal{H}\{x(n)\} = y(n) \tag{22}$$

then

$$\mathcal{H}\{x(n-n_0)\} = y(n-n_0) \tag{23}$$

- Some texts refer to the time-invariance property as the shift-invariance property, since a discrete system can process samples of a function not necessarily in time.

Discrete-time systems

Causality:

- Since one cannot "see into the future", in real-time applications one needs to have a system whose output depends only on past inputs.
- Such a system is usually referred to as being **causal**.
- In more precise terms, a discrete-time system is causal if and only if, when $x_1(n) = x_2(n)$ for $n < n_0$, then

$$\mathcal{H}\{x_1(n)\} = \mathcal{H}\{x_2(n)\}, \text{ for } n < n_0 \tag{24}$$

In other words, causality means that the output of a system at instant n does not depend on any input occurring after n.

Discrete-time systems

- Usually, in the case of a discrete-time signal, a noncausal system is not implementable in real time.
- This is because we would need input samples at instants of time greater than n in order to compute the output at time n.
- However, this would be allowed only if the time samples were pre-stored, as in off-line or batch implementations.
- If the signals to be processed do not consist of time samples acquired in real time, then there might be nothing equivalent to the concepts of past or future samples.

  $\Rightarrow$ In these cases, the role of causality is of a lesser importance.
- For example, for a discrete signal that corresponds to the temperature at sensors uniformly spaced along a metal rod, a processor can have access to all its samples simultaneously.
  - In this case, even a noncausal system can be easily implemented.

Discrete-time systems

Example 1.2

Characterize the following systems as being either linear or nonlinear, time invariant or time varying, causal or noncausal:

(a) $y(n) = (n + b)x(n - 4)$

(b) $y(n) = x^2(n + 1)$

Discrete-time systems

Solution

(a) $y(n) = (n+b)x(n-4)$

- Linearity:

$$\begin{aligned} \mathcal{H}\{ax(n)\} &= (n+b)ax(n-4) \\ &= a(n+b)x(n-4) \\ &= a\mathcal{H}\{x(n)\} \end{aligned} \tag{25}$$

and

$$\begin{aligned} \mathcal{H}\{x_1(n)+x_2(n)\} &= (n+b)[x_1(n-4)+x_2(n-4)] \\ &= (n+b)x_1(n-4)+(n+b)x_2(n-4) \\ &= \mathcal{H}\{x_1(n)\}+\mathcal{H}\{x_2(n)\} \end{aligned} \tag{26}$$

and therefore the system is linear.

- Time invariance:

$$y(n-n_0) = [(n-n_0)+b]x[(n-n_0)-4] \tag{27}$$

and then

$$\mathcal{H}\{x(n-n_0)\} = (n+b)x[(n-n_0)-4] \tag{28}$$

such that $y(n-n_0) \neq \mathcal{H}\{x(n-n_0)\}$, and the system is time varying.

- Causality: If

$$x_1(n) = x_2(n), \quad \text{for } n < n_0 \tag{29}$$

then

$$x_1(n-4) = x_2(n-4), \quad \text{for } n-4 < n_0 \tag{30}$$

such that

$$x_1(n-4) = x_2(n-4), \quad \text{for } n < n_0 \tag{31}$$

and then

$$(n+b)x_1(n-4) = (n+b)x_2(n-4), \quad \text{for } n < n_0 \tag{32}$$

Hence, $\mathcal{H}\{x_1(n)\} = \mathcal{H}\{x_2(n)\}$, for all $n < n_0$ and, consequently, the system is causal.

(b) $y(n) = x^2(n+1)$

- Linearity:

$$\mathcal{H}\{ax(n)\} = a^2x^2(n+1) \neq a\mathcal{H}\{x(n)\} \tag{33}$$

and therefore the system is nonlinear.

- Time invariance:

$$\mathcal{H}\{x(n-n_0)\} = x^2[(n-n_0)+1] = y(n-n_0) \tag{34}$$

so the system is time invariant.

- Causality:

$$\mathcal{H}\{x_1(n)\} = x_1^2(n+1) \tag{35}$$

$$\mathcal{H}\{x_2(n)\} = x_2^2(n+1) \tag{36}$$

Therefore, if $x_1(n) = x_2(n)$, for $n < n_0$, and $x_1(n_0) \neq x_2(n_0)$, then, for $n = n_0 - 1 < n_0$,

$$\mathcal{H}\{x_1(n_0 - 1)\} = x_1^2(n_0) \tag{37}$$

$$\mathcal{H}\{x_2(n_0 - 1)\} = x_2^2(n_0) \tag{38}$$

and we have that $\mathcal{H}\{x_1(n)\} \neq \mathcal{H}\{x_2(n)\}$, and the system is noncausal.

$\triangle$

Impulse response and convolution sums

Suppose that $\mathcal{H}\{\cdot\}$ is a linear system, and we apply an excitation $x(n)$ to the system. Since, from equation (9), $x(n)$ can be expressed as a sum of shifted impulses

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \tag{39}$$

we can express its output as

$$y(n) = \mathcal{H}\left\{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right\} = \sum_{k=-\infty}^{\infty} \mathcal{H}\{x(k)\delta(n-k)\} \tag{40}$$

Since $x(k)$ in the above equation is just a constant, the linearity of $\mathcal{H}\{\cdot\}$ also implies that

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\mathcal{H}\{\delta(n-k)\} = \sum_{k=-\infty}^{\infty} x(k)h_k(n) \tag{41}$$

where $h_k(n) = \mathcal{H}\{\delta(n-k)\}$ is the response of the system to an impulse at $n = k$.

Impulse response and convolution sums

If the system is also time invariant, and we define

$$\mathcal{H}\{\delta(n)\} = h_0(n) = h(n) \tag{42}$$

then $\mathcal{H}\{\delta(n-k)\} = h(n-k)$, and the expression in equation (41) becomes

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{43}$$

indicating that a linear time-invariant system is completely characterized by its unit impulse response $h(n)$.

- Note that, when the system is linear but time varying, we would need, in order to compute $y(n)$, the values of $h_k(n)$, which depend on both n and k. This makes the computation of the summation in equation (41) quite complex.

Equation (43) is called a **convolution sum** or a **discrete-time convolution**.

Impulse response and convolution sums

If we make the change of variables $l = n - k$, equation (43) can be written as

$$y(n) = \sum_{l=-\infty}^{\infty} x(n-l)h(l) \tag{44}$$

that is, we can interpret $y(n)$ as the result of the convolution of the excitation $x(n)$ with the system impulse response $h(n)$.

- A shorthand notation for the convolution operation, as described in equations (43) and (44), is

$$y(n) = x(n) * h(n) = h(n) * x(n) \tag{45}$$

Impulse response and convolution sums

Suppose now that the output $y(n)$ of a system with impulse response $h(n)$ is the excitation for a system with impulse response $h'(n)$. In this case, we have

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{46}$$

$$y'(n) = \sum_{l=-\infty}^{\infty} y(l)h'(n-l) \tag{47}$$

Substituting equation (46) in equation (47), we have that

$$\begin{aligned} y'(n) &= \sum_{l=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x(k)h(l-k) \right) h'(n-l) \\ &= \sum_{k=-\infty}^{\infty} x(k) \left(\sum_{l=-\infty}^{\infty} h(l-k)h'(n-l) \right) \end{aligned} \tag{48}$$

Impulse response and convolution sums

By performing the change of variables $l = n - r$, the above equation becomes

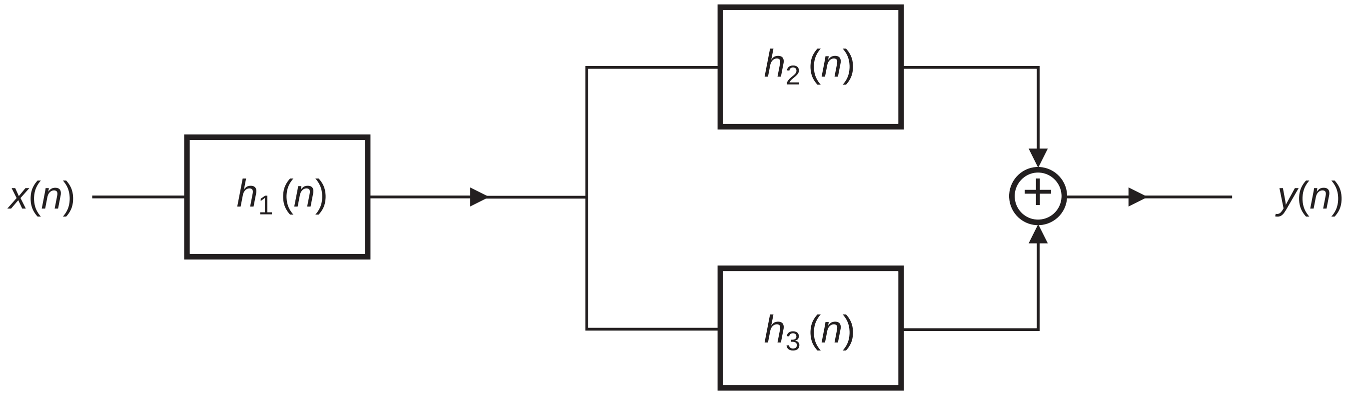
$$
\begin{aligned}
y'(n) &= \sum_{k=-\infty}^{\infty} x(k) \left(\sum_{r=-\infty}^{\infty} h(n-r-k)h'(r) \right) \\
&= \sum_{k=-\infty}^{\infty} x(k) \left(h(n-k) * h'(n-k) \right) \\
&= \sum_{k=-\infty}^{\infty} x(n-k) \left(h(k) * h'(k) \right)
\end{aligned}
\tag{49}
$$

- This shows that the impulse response of a linear time-invariant system formed by the series (cascade) connection of two linear time-invariant subsystems is the convolution of the impulse responses of the two subsystems.

Impulse response and convolution sums

Example 1.3

Compute $y(n)$ for the system depicted in the figure below as a function of the input signal and of the impulse responses of the subsystems.



Solution From the previous results, it is easy to conclude that

$$y(n) = (h_2(n) + h_3(n)) * h_1(n) * x(n) \tag{50}$$

$\triangle$

Stability

A system is referred to as bounded-input bounded-output (BIBO) stable if, for every input limited in amplitude, the output signal is also limited in amplitude. For a linear time-invariant system, equation (44) implies that

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |x(n-k)||h(k)| \tag{51}$$

The input being limited in amplitude is equivalent to

$$|x(n)| \leq x_{\text{max}} < \infty, \quad \text{for all } n \tag{52}$$

Therefore

$$|y(n)| \leq x_{\text{max}} \sum_{k=-\infty}^{\infty} |h(k)| \tag{53}$$

Stability

Hence, we can conclude that a sufficient condition for a system to be BIBO stable is

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \tag{54}$$

since this condition forces $y(n)$ to be limited.

To prove that this condition is also necessary, suppose that it does not hold, that is, the summation in equation (54) is infinite. If we choose an input such that

$$x(n_0 - k) = \begin{cases} +1, & \text{for } h(k) \geq 0 \\ -1, & \text{for } h(k) < 0 \end{cases} \tag{55}$$

we then have that

$$y(n_0) = |y(n_0)| = \sum_{k=-\infty}^{\infty} |h(k)| \tag{56}$$

that is, the output $y(n)$ is unbounded, showing that this condition is also necessary.

Difference equations and time-domain response

- In most applications, discrete-time systems can be described by difference equations
- They are the equivalent, for the discrete-time domain, to differential equations for the continuous-time domain.
- The input and output of a system described by a linear difference equation are generally related by

$$\sum_{i=0}^{N} a_i y(n-i) - \sum_{l=0}^{M} b_l x(n-l) = 0 \tag{57}$$

- This equation has an infinite number of solutions $y(n)$.
- Suppose that a particular $y_p(n)$ satisfies equation (57), that is

$$\sum_{i=0}^{N} a_i y_p(n-i) - \sum_{l=0}^{M} b_l x(n-l) = 0 \tag{58}$$

Difference equations and time-domain response

- Suppose also that $y_h(n)$ is a solution to the homogeneous equation, that is

$$\sum_{i=0}^{N} a_i y_h(n-i) = 0 \tag{59}$$

$\Rightarrow$ From equations (57)–(59), we can easily infer that $y(n) = y_p(n) + y_h(n)$ is also a solution to the same difference equation.

- The homogeneous solution $y_h(n)$ of a difference equation of order N, has N degrees of freedom (depends on N arbitrary constants).

$\Rightarrow$ One can only determine a solution for a difference equation if one supplies N auxiliary conditions.

  - One example of a se of auxiliary conditions is given by the values of $y(-1), y(-2), \ldots, y(-N)$.
  - Any N independent auxiliary conditions are enough to solve a difference equation. It is common to use N consecutive samples of $y(n)$.

Difference equations and time-domain response

Example 1.4

Find the solution for the following difference equation:

$$y(n) = ay(n-1) \tag{60}$$

as a function of the initial condition $y(0)$.

Difference equations and time-domain response

Solution

Running the difference equation $y(n) = ay(n-1)$ from $n = 1$ onwards, we have that

$$\left.\begin{aligned} y(1) &= ay(0) \\ y(2) &= ay(1) \\ y(3) &= ay(2) \\ &\vdots \\ y(n) &= ay(n-1) \end{aligned}\right\} \tag{61}$$

Multiplying the above equations, we have that

$$y(1)y(2)y(3)\ldots y(n) = a^n y(0)y(1)y(2)\ldots y(n-1) \tag{62}$$

and therefore, the solution of the difference equation is

$$y(n) = a^n y(0) \tag{63}$$

Difference equations and time-domain response

Example 1.5

Solve the following difference equation:

$$y(n) = e^{-\beta} y(n-1) + \delta(n) \tag{64}$$

Solution

From the previous example, any function of the form $y_h(n) = Ke^{-\beta n}$ satisfies

$$y_h(n) = e^{-\beta} y_h(n-1) \tag{65}$$

and is therefore a solution of the homogeneous difference equation.

One can verify by substitution that $y_p(n) = e^{-\beta n} u(n)$ is a particular solution.

Therefore, the general solution of the difference equation is given by

$$y(n) = y_p(n) + y_h(n) = e^{-\beta n} u(n) + Ke^{-\beta n} \tag{66}$$

where the value of K is determined by the auxiliary conditions.

Difference equations and time-domain response

First order difference equation: we need to specify only one condition.

If $y(-1) = \alpha$, the solution to equation (64) becomes

$$y(n) = \mathrm{e}^{-\beta n}u(n) + \alpha \mathrm{e}^{-\beta(n+1)} \qquad (67)$$

$\triangle$

Difference equations and time-domain response

- A linear system must satisfy $\mathcal{H}\{\alpha x\} = \alpha\mathcal{H}\{x\}$

$\Rightarrow$ for a linear system, $\mathcal{H}\{0\} = 0$

- If we restrict ourselves to inputs that are null prior to a certain sample, that is, $x(n) = 0$, for $n < n_0$, there is an interesting relation between linearity, causality, and the initial conditions of a system.
  - If the system is causal, the output at $n < n_0$ can not be influenced by any sample of the input $x(n)$ for $n \geq n_0$. Therefore, if $x(n) = 0$, for $n < n_0$, then $\mathcal{H}\{0\}$ and $\mathcal{H}\{x(n)\}$ must be identical for all $n < n_0$. Since, if the system is linear, $\mathcal{H}\{0\} = 0$, then necessarily $\mathcal{H}\{x(n)\} = 0$ for $n < n_0$.
  - This is equivalent to saying that the auxiliary conditions for $n < n_0$ must be null. Such a system is referred to as being **initially relaxed**.
  - Conversely, if the system is not initially relaxed, one can not guarantee that it is causal.

Difference equations and time-domain response

Example 1.6

For the linear system described by

$$y(n) = e^{-\beta}y(n-1) + u(n) \tag{68}$$

determine its output for the auxiliary conditions:

(a) $y(1) = 0$

(b) $y(-1) = 0$

and discuss the causality in both situations.

Difference equations and time-domain response

Solution The homogeneous solution of equation (68) is the same as in the previous example, that is

$$y_h(n) = Ke^{-\beta n} \tag{69}$$

By direct substitution in equation (68), it can be verified that the particular solution is of the form

$$y_p(n) = (a + be^{-\beta n})u(n) \tag{70}$$

where

$$a = \frac{1}{1 - e^{-\beta}}; \quad b = \frac{-e^{-\beta}}{1 - e^{-\beta}} \tag{71}$$

Thus, the general solution of the difference equation is given by

$$y(n) = \left(\frac{1 - e^{-\beta(n+1)}}{1 - e^{-\beta}}\right)u(n) + Ke^{-\beta n} \tag{72}$$

Difference equations and time-domain response

(a) For the auxiliary condition $y(1) = 0$, we have that

$$y(1) = \left(\frac{1 - e^{-2\beta}}{1 - e^{-\beta}} \right) + Ke^{-\beta} = 0 \tag{73}$$

yielding $K = -(1 + e^{\beta})$, and the general solution becomes

$$y(n) = \left(\frac{1 - e^{-\beta(n+1)}}{1 - e^{-\beta}} \right) u(n) - \left(e^{-\beta n} + e^{-\beta(n-1)} \right) \tag{74}$$

Since for $n < 0$, we have that $u(n) = 0$, then $y(n)$ simplifies to

$$y(n) = -\left(e^{-\beta n} + e^{-\beta(n-1)} \right) \tag{75}$$

Clearly, in this case, $y(n) \neq 0$, for $n < 0$, whereas the input $u(n) = 0$ for $n < 0$.

Thus, the system is not initially relaxed and therefore is noncausal.

Another way of verifying that the system is noncausal is by noting that, if the input is doubled, becoming $x(n) = 2u(n)$ instead of $u(n)$, then the particular solution is also doubled. Hence, the general solution of the difference equation becomes

$$y(n) = \left(\frac{2 - 2e^{-\beta(n+1)}}{1 - e^{-\beta}}\right) u(n) + Ke^{-\beta n} \tag{76}$$

If we require that $y(1) = 0$, then $K = 2 + 2e^{\beta}$, and, for $n < 0$, this yields

$$y(n) = -2\left(e^{-\beta n} + e^{-\beta(n-1)}\right) \tag{77}$$

Since this is different from the value of $y(n)$ for $u(n)$ as input, we see that the output for $n < 0$ depends on the input for $n > 0$, and therefore the system is noncausal.

(b) For the auxiliary condition $y(-1) = 0$, we have that $K = 0$, yielding the solution

$$y(n) = \left(\frac{1 - e^{-\beta(n+1)}}{1 - e^{-\beta}}\right) u(n) \tag{78}$$

In this case, $y(n) = 0$, for $n < 0$, that is, the system is initially relaxed and, therefore, causal, as discussed above. △

Difference equations and time-domain response

- An initially relaxed system described by a linear difference equation, besides being linear and causal, is also time invariant.
- Time invariance can be easily inferred if we consider that, for an initially relaxed system, the history of the system up to the application of the excitation is the same irrespective of the time sample position at which the excitation is applied.
  - This happens because the outputs are all zero up to, but not including, the time of the application of the excitation.

$\Rightarrow$ If time is measured having as a reference the time sample $n = n_0$, at which the input is applied, then the output will not depend on the reference n_0, because the history of the system prior to n_0 is the same irrespective of n_0.

  - This is equivalent to saying that if the input is shifted by k samples, then the output is just shifted by k samples, the rest remaining unchanged, thus characterizing a time-invariant system.

Recursive $\times$ non-recursive systems

- A general difference equation can be written as

$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{l=0}^{M} b_l x(n-l) \tag{79}$$

- Here, the output signal $y(n)$ depends on:
  - Samples of the input, $x(n), x(n-1), \ldots, x(n-M)$
  - Samples of the output, $y(n-1), y(n-2), \ldots, y(n-N)$
- In this general case, we say that the system is **recursive**, since, in order to compute the output, we need past samples of the output itself.
- When $a_1 = a_2 = \cdots = a_N = 0$, then the output at sample n depends only on values of the input signal.
  - In such case, the system is called **nonrecursive**.

– It is characterized by a difference equation of the form

$$y(n) = \sum_{l=0}^{M} b_l x(n-l) \tag{80}$$

- The above equation corresponds to a discrete system with impulse response $h(l) = b_l$.
  – Since there is only a finite number of coefficients b_l, such a system has a finite-duration impulse response.
  – Such discrete-time systems are often referred to as finite-duration impulse-response (**FIR**) filters.
  – In contrast, when $y(n)$ depends on its past values, in general the impulse response of the discrete system is not zero when $n \to \infty$.
  – Therefore, recursive digital systems are often referred to as infinite-duration impulse-response (**IIR**) filters.
- Note that there are cases in which a recursive system is FIR.

Recursive $\times$ non-recursive systems

Example 1.7

Find the impulse response of the system

$$y(n) - \frac{1}{\alpha} y(n-1) = x(n) \tag{81}$$

supposing that it is initially relaxed.

Recursive $\times$ non-recursive systems

Solution

$y(n) - \frac{1}{\alpha}y(n-1) = x(n)$

Since the system is initially relaxed, then $y(n) = 0$, for $n \leq -1$. Hence, for $n = 0$, we have that

$$y(0) = \frac{1}{\alpha}y(-1) + \delta(0) = \delta(0) = 1 \tag{82}$$

For $n > 0$, we have that

$$y(n) = \frac{1}{\alpha}y(n-1) \tag{83}$$

and, therefore, $y(n)$ can be expressed as

$$y(n) = \left(\frac{1}{\alpha}\right)^n u(n) \tag{84}$$

Note that $y(n) \neq 0$ for all $n \geq 0$, that is, the impulse response has infinite length.

$\triangle$

Solving difference equations

Consider the following homogeneous difference equation:

$$\sum_{i=0}^{N} a_i y(n-i) = 0 \tag{85}$$

Be $y_1(n)$ and $y_2(n)$ solutions to it. Then

$$\sum_{i=0}^{N} a_i y_1(n-i) = 0 \tag{86}$$

$$\sum_{i=0}^{N} a_i y_2(n-i) = 0 \tag{87}$$

Adding equation (86) multiplied by c_1 to equation (87) multiplied by c_2 we have that

Solving difference equations

$$c_1 \sum_{i=0}^{N} a_i y_1(n-i) + c_2 \sum_{i=0}^{N} a_i y_2(n-i) = 0$$

$$\Rightarrow \quad \sum_{i=0}^{N} a_i c_1 y_1(n-i) + \sum_{i=0}^{N} a_i c_2 y_2(n-i) = 0$$

$$\Rightarrow \quad \sum_{i=0}^{N} a_i (c_1 y_1(n-i) + c_2 y_2(n-i)) = 0 \tag{88}$$

The above equation means that $(c_1 y_1(n) + c_2 y_2(n))$ is also a solution to equation (85). This implies that, if $y_i(n)$, for $i = 0, 1, \ldots, (M-1)$, are solutions of an homogeneous difference equation, then

$$y_h(n) = \sum_{i=0}^{M-1} c_i y_i(n) \tag{89}$$

is also a solution.

Solving difference equations

As we have seen in a previous Example, a difference equation may have solutions of the form

$$y(n) = K\rho^{n} \tag{90}$$

Supposing that $y(n)$ from equation (90) is also a solution to the difference equation (85), we have that

$$\sum_{i=0}^{N} a_i K \rho^{(n-i)} = 0 \tag{91}$$

If we disregard the trivial solution $\rho = 0$ and divide the left-hand side of the above equation by $K\rho^{n}$, we get

$$\sum_{i=0}^{N} a_i \rho^{-i} = 0 \tag{92}$$

Solving difference equations

It has the same solutions of the following polynomial equation:

$$\sum_{i=0}^{N} a_i \rho^{(N-i)} = 0 \tag{93}$$

As a result, one can conclude that if $\rho_0, \rho_1, \ldots, \rho_{M-1}$, for $M \leq N$, are distinct zeros of the so-called characteristic polynomial in equation (93), then there are M solutions for the homogeneous difference equation given by

$$y(n) = c_k \rho_k^n, \quad k = 0, 1, \ldots, (M-1) \tag{94}$$

In fact, from equation (89), we have that any linear combination of these solutions is also a solution for the homogeneous difference equation. Then a general homogeneous solution can be written as

$$y_h(n) = \sum_{k=0}^{M-1} c_k \rho_k^n \tag{95}$$

where c_k, for $k = 0, 1, \ldots, (M-1)$, are arbitrary constants.

Solving difference equations

Example 1.8

Find the general solution for the Fibonacci equation

$$y(n) = y(n-1) + y(n-2) \tag{96}$$

with $y(0) = 0$ and $y(1) = 1$.

Solving difference equations

Solution

$y(n) = y(n-1) + y(n-2)$

The characteristic polynomial to the Fibonacci equation is

$$\rho^2 - \rho - 1 = 0 \tag{97}$$

whose roots are $\rho = \frac{1\pm\sqrt{5}}{2}$, leading to the general solution

$$y(n) = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \tag{98}$$

Applying the auxiliary conditions $y(0) = 0$ and $y(1) = 1$ to the above equation, we have that

$$\begin{cases} y(0) = c_1 + c_2 = 0 \\ y(1) = \left(\frac{1+\sqrt{5}}{2}\right) c_1 + \left(\frac{1-\sqrt{5}}{2}\right) c_2 = 1 \end{cases} \tag{99}$$

Solving difference equations

Thus, $c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$, and the solution to the Fibonacci equation becomes

$$y(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \tag{100}$$

$\triangle$

Solving difference equations

If the characteristic polynomial in equation (93) has a pair of complex conjugate roots ρ and ρ^* of the form $a \pm jb = re^{\pm j\phi}$, the associated homogeneous solution is given by

$$
\begin{aligned}
y_h(n) &= \hat{c}_1(re^{j\phi})^n + \hat{c}_2(re^{-j\phi})^n \\
&= r^n(\hat{c}_1 e^{j\phi n} + \hat{c}_2 e^{-j\phi n}) \\
&= r^n[(\hat{c}_1 + \hat{c}_2)\cos(\phi n) + j(\hat{c}_1 - \hat{c}_2)\sin(\phi n)] \\
&= c_1 r^n \cos(\phi n) + c_2 r^n \sin(\phi n)
\end{aligned}
\tag{101}
$$

If the characteristic polynomial in equation (93) has multiple roots, solutions distinct from equation (94) are required. For example, if ρ is a double root, there also exists a solution of the form

$$
y_h(n) = cn\rho^n \tag{102}
$$

where c is an arbitrary constant.

Solving difference equations

In general, if ρ is a root of multiplicity m, then the associated solution is of the form

$$y_h(n) = \sum_{l=0}^{m-1} d_l n^l \rho^n \tag{103}$$

where d_l, for $l = 0, 1, \ldots, (m-1)$, are arbitrary constants.

From the above, we can conclude that the homogeneous solutions of difference equations for each root type of the characteristic polynomial follow the rules summarized in Table 1.

Solving difference equations

Table 1: Typical homogeneous solutions.

Root type [Multiplicity]	**Homogeneous solution** $y_h(n)$
Real ρ_k [1]	$c_k \rho_k^n$
Real ρ_k [m_k]	$\sum_{l=0}^{m_k-1} d_l n^l \rho_k^n$
Complex conjugates $\rho_k, \rho_k^* = re^{\pm j\phi}$ [1]	$r^n[c_1 \cos(\phi n) + c_2 \sin(\phi n)]$
Complex conjugates $\rho_k, \rho_k^* = re^{\pm j\phi}$ [m_k]	$\sum_{l=0}^{m_k-1} \left[d_{1,l} n^l r^n \cos(\phi n) + d_{2,l} n^l r^n \sin(\phi n)\right]$

Solving difference equations

Finding a particular solution

A widely used method to find a particular solution for a difference equation of the form

$$\sum_{i=0}^{N} a_i y_p(n-i) = \sum_{l=0}^{M} b_l x(n-l) \tag{104}$$

is the so-called method of undetermined coefficients. This method can be used when the input sequence is the solution of a difference equation with constant coefficients. In order to do so, we define a delay operator $D\{\cdot\}$ as follows

$$D^{-i}\{y(n)\} = y(n-i) \tag{105}$$

Such a delay operator is linear, since

$$\begin{aligned} D^{-i}\{c_1 y_1(n) + c_2 y_2(n)\} &= c_1 y_1(n-i) + c_2 y_2(n-i) \\ &= c_1 D^{-i}\{y_1(n)\} + c_2 D^{-i}\{y_2(n)\} \end{aligned} \tag{106}$$

Solving difference equations

Also, the cascade of delay operators satisfies the following

$$D^{-i}\{D^{-j}\{y(n)\}\} = D^{-i}\{y(n-j)\} = y(n-i-j) = D^{-(i+j)}\{y(n)\} \quad (107)$$

Using delay operators, equation (104) can be rewritten as

$$\left(\sum_{i=0}^{N} a_i D^{-i}\right)\{y_p(n)\} = \left(\sum_{l=0}^{M} b_l D^{-l}\right)\{x(n)\} \quad (108)$$

The key idea is to find a difference operator $Q(D)$ of the form

$$Q(D) = \sum_{k=0}^{R} d_k D^{-k} = \prod_{r=0}^{R}(1 - \alpha_r D^{-1}) \quad (109)$$

such that it annihilates the excitation, that is

$$Q(D)\{x(n)\} = 0 \quad (110)$$

Solving difference equations

Applying $Q(D)$ to equation (108) we get

$$\begin{aligned} Q(D)\left\{\left(\sum_{i=0}^{N} a_i D^{-i}\right)\{y_p(n)\}\right\} &= Q(D)\left\{\left(\sum_{l=0}^{M} b_l D^{-l}\right)\{x(n)\}\right\} \\ &= \left(\sum_{l=0}^{M} b_l D^{-l}\right)\{Q(D)\{x(n)\}\} \\ &= 0 \end{aligned} \quad (111)$$

This allows the non-homogeneous difference equation to be solved using the same procedures used to find the homogeneous solutions.

For example, for a sequence $x(n) = s^n$, we have that $x(n-1) = s^{n-1}$; then,

$$x(n) = sx(n-1) \quad \Rightarrow \quad [1 - sD^{-1}]\{x(n)\} = 0$$

and therefore the annihilator polynomial for $x(n) = s^n$ is $Q(D) = (1 - sD^{-1})$. The annihilator polynomials for some typical inputs are summarized in Table 2.

Solving difference equations

Table 2: Annihilator polynomials for different input signals.

Input $x(n)$	**Polynomial** $Q(D)$
s^n	$1 - sD^{-1}$
n^i	$\left(1 - D^{-1}\right)^{i+1}$
$n^i s^n$	$\left(1 - sD^{-1}\right)^{i+1}$
$\cos(\omega n)$ or $\sin(\omega n)$	$\left(1 - e^{j\omega}D^{-1}\right)\left(1 - e^{-j\omega}D^{-1}\right)$
$s^n\cos(\omega n)$ or $s^n\sin(\omega n)$	$\left(1 - se^{j\omega}D^{-1}\right)\left(1 - se^{-j\omega}D^{-1}\right)$
$n\cos(\omega n)$ or $n\sin(\omega n)$	$\left[\left(1 - e^{j\omega}D^{-1}\right)\left(1 - e^{-j\omega}D^{-1}\right)\right]^2$

Using the concept of annihilator polynomials, we can determine the form of the particular solution for certain types of input signals, which may include some undetermined coefficients. Some useful cases are shown in Table 3.

Solving difference equations

Table 3: Typical particular solutions for different input signals.

Input $x(n)$	**Particular solution** $y_p(n)$
s^n, $s \neq \rho_k$	αs^n
s^n, $s = \rho_k$ with multiplicity m_k	$\alpha n^{m_k} s^n$
$\cos(\omega n + \phi)$	$\alpha \cos(\omega n + \phi)$
$\left[\sum_{i=0}^{I} \beta_i n^i\right] s^n$	$\left[\sum_{i=0}^{I} \alpha_i n^i\right] s^n$

It is important to notice that there are no annihilator polynomials for inputs containing $u(n - n_0)$ or $\delta(n - n_0)$. Therefore, if a difference equation has inputs such as these, the above techniques can only be used for either $n \geq n_0$ or $n < n_0$, as discussed in the following Example.

Solving difference equations

Example 1.9

Solve the difference equation

$$y(n) + a^2 y(n-2) = b^n \sin\left(\frac{\pi}{2} n\right) u(n) \tag{112}$$

assuming that $a \neq b$ and $y(n) = 0$, for $n < 0$.

Solving difference equations

Solution

$y(n) + a^2 y(n-2) = b^n \sin\left(\frac{\pi}{2}n\right) u(n)$

Using operator notation the above equation becomes

$$\left(1 + a^2 D^{-2}\right)\{y(n)\} = b^n \sin\left(\frac{\pi}{2}n\right) u(n) \tag{113}$$

The homogeneous equation is

$$y_h(n) + a^2 y_h(n-2) = 0 \tag{114}$$

Then, the characteristic polynomial equation from which we derive the homogeneous solution is

$$\rho^2 + a^2 = 0 \tag{115}$$

Since its roots are $\rho = a e^{\pm j\frac{\pi}{2}}$, then the two solutions for the homogeneous equation are $a^n \sin\left(\frac{\pi}{2}n\right)$ and $a^n \cos\left(\frac{\pi}{2}n\right)$, as given in Table 1.

Solving difference equations

Then, the general homogeneous solution becomes

$$y_h(n) = a^n \left[c_1 \sin\left(\frac{\pi}{2}n\right) + c_2 \cos\left(\frac{\pi}{2}n\right)\right] \quad (116)$$

- If one applies the correct annihilation to the excitation signals, the original difference equation is transformed into a higher order homogeneous equation.
- The solutions of this higher order homogeneous equation include the homogeneous and particular solutions of the original difference equation.
- However, **there is no annihilator polynomial** for $b^n \sin\left(\frac{\pi}{2}n\right) u(n)$.

$\Rightarrow$ One can only compute the solution to the difference equation for $n \geq 0$, when the term to be annihilated becomes just $b^n \sin\left(\frac{\pi}{2}n\right)$.

Therefore, for $n \geq 0$, according to Table 2, for the given input signal the annihilator polynomial is given by

$$Q(D) = \left(1 - be^{j\frac{\pi}{2}}D^{-1}\right)\left(1 - be^{-j\frac{\pi}{2}}D^{-1}\right) = \left(1 + b^2D^{-2}\right) \quad (117)$$

Solving difference equations

Applying the annihilator polynomial on the difference equation, we obtain

$$\left(1 + b^2 D^{-2}\right)\left(1 + a^2 D^{-2}\right)\{y(n)\} = 0 \tag{118}$$

The corresponding polynomial equation is

$$(\rho^2 + b^2)(\rho^2 + a^2) = 0 \tag{119}$$

It has four roots, two of the form $\rho = ae^{\pm j\frac{\pi}{2}}$ and two of the form $\rho = be^{\pm j\frac{\pi}{2}}$. Since $a \neq b$, for $n \geq 0$ the complete solution is then given by

$$y(n) = b^n\left[d_1 \sin\left(\frac{\pi}{2}n\right) + d_2 \cos\left(\frac{\pi}{2}n\right)\right] + a^n\left[d_3 \sin\left(\frac{\pi}{2}n\right) + d_4 \cos\left(\frac{\pi}{2}n\right)\right] \tag{120}$$

the constants d_i, for $i = 1, 2, 3, 4$, are computed such that $y(n)$ is a particular solution to the non-homogeneous equation.

- Notice that the term involving a^n corresponds to the solution of the homogeneous equation.

Solving difference equations

$\Rightarrow$ We do not need to substitute it on the equation since it will be annihilated for every d_3 and d_4.

One can then compute d_1 and d_2 by substituting only the term involving b^n in the non-homogeneous equation (112), leading to the following algebraic development:

$$
\begin{aligned}
& b^n\left[d_1 \sin\left(\frac{\pi}{2}n\right) + d_2 \cos\left(\frac{\pi}{2}n\right)\right] \\
& + a^2 b^{n-2}\left[d_1 \sin\left(\frac{\pi}{2}(n-2)\right) + d_2 \cos\left(\frac{\pi}{2}(n-2)\right)\right] = b^n \sin\left(\frac{\pi}{2}n\right) \\
\Rightarrow \quad & \left[d_1 \sin\left(\frac{\pi}{2}n\right) + d_2 \cos\left(\frac{\pi}{2}n\right)\right] \\
& + a^2 b^{-2}\left[d_1 \sin\left(\frac{\pi}{2}n - \pi\right) + d_2 \cos\left(\frac{\pi}{2}n - \pi\right)\right] = \sin\left(\frac{\pi}{2}n\right) \\
\Rightarrow \quad & \left[d_1 \sin\left(\frac{\pi}{2}n\right) + d_2 \cos\left(\frac{\pi}{2}n\right)\right] \\
& + a^2 b^{-2}\left[-d_1 \sin\left(\frac{\pi}{2}n\right) - d_2 \cos\left(\frac{\pi}{2}n\right)\right] = \sin\left(\frac{\pi}{2}n\right) \\
\Rightarrow \quad & d_1(1 - a^2 b^{-2}) \sin\left(\frac{\pi}{2}n\right) + d_2(1 - a^2 b^{-2}) \cos\left(\frac{\pi}{2}n\right) = \sin\left(\frac{\pi}{2}n\right)
\end{aligned}
$$

Solving difference equations

Therefore, we conclude that

$$d_1 = \frac{1}{1 - a^2 b^{-2}}; \quad d_2 = 0 \tag{121}$$

and the overall solution for $n \geq 0$ is

$$y(n) = \frac{b^n}{1 - a^2 b^{-2}} \sin\left(\frac{\pi}{2} n\right) + a^n \left[d_3 \sin\left(\frac{\pi}{2} n\right) + d_4 \cos\left(\frac{\pi}{2} n\right)\right] \tag{122}$$

- We now compute the constants d_3 and d_4 using the auxiliary conditions generated by the condition $y(n) = 0$, for $n < 0$.
- This implies that we should use $y(-1) = 0$ and $y(-2) = 0$.
- However, one can not use equation (122) since it is valid only for $n \geq 0$.

Solving difference equations

⇒ We need to run the difference equation starting from the auxiliary conditions $y(-2) = y(-1) = 0$ to compute $y(0)$ and $y(1)$:

$$\left\{\begin{array}{ll} n = 0: & y(0) + a^2 y(-2) = b^0 \sin\left(\frac{\pi}{2} \times 0\right) u(0) = 0 \\ n = 1: & y(1) + a^2 y(-1) = b^1 \sin\left(\frac{\pi}{2}\right) u(1) = b \end{array}\right. \Rightarrow \left\{\begin{array}{l} y(0) = 0 \\ y(1) = b \end{array}\right. \tag{123}$$

Using these auxiliary conditions in equation (122), we get

$$\left\{\begin{array}{l} y(0) = \frac{1}{1 - a^2 b^{-2}} \sin\left(\frac{\pi}{2} \times 0\right) + \left[d_3 \sin\left(\frac{\pi}{2} \times 0\right) + d_4 \cos\left(\frac{\pi}{2} \times 0\right)\right] = 0 \\ y(1) = \frac{b}{1 - a^2 b^{-2}} \sin\left(\frac{\pi}{2}\right) + a\left[d_3 \sin\left(\frac{\pi}{2}\right) + d_4 \cos\left(\frac{\pi}{2}\right)\right] = b \end{array}\right. \tag{124}$$

Solving difference equations

and then

$$\begin{cases} d_4 = 0 \\ \dfrac{b}{1 - a^2 b^{-2}} + a d_3 = b \end{cases} \Rightarrow \begin{cases} d_3 = -\dfrac{a b^{-1}}{1 - a^2 b^{-2}} \\ d_4 = 0 \end{cases} \tag{125}$$

Substituting these values in equation (122), the general solution becomes

$$\begin{cases} y(n) = 0, & n < 0 \\ y(n) = \left(\dfrac{b^n - a^{n+1} b^{-1}}{1 - a^2 b^{-2}} \right) \sin\left(\dfrac{\pi}{2} n \right), & n \geq 0 \end{cases} \tag{126}$$

which can be expressed in compact form as

$$y(n) = \left(\frac{b^n - a^{n+1} b^{-1}}{1 - a^2 b^{-2}} \right) \sin\left(\frac{\pi}{2} n \right) u(n) \tag{127}$$

Solving difference equations

An interesting case arises if the excitation is a pure sinusoid, that is, if $a = 1$, when the above solution can de written as

$$y(n) = \frac{b^n}{1 - b^{-2}} \sin\left(\frac{\pi}{2}n\right) u(n) - \frac{b^{-1}}{1 - b^{-2}} \sin\left(\frac{\pi}{2}n\right) u(n) \tag{128}$$

- If $b > 1$, for large values of n, the first term of the right-hand side grows without bound (since b^n tends to infinity), and therefore the system is unstable.
- If $b < 1$, then b^n tends to zero as n grows, and therefore the solution becomes the pure sinusoid

$$y(n) = -\frac{b^{-1}}{1 - b^{-2}} \sin\left(\frac{\pi}{2}n\right) \tag{129}$$

We refer to this as a steady-state solution of the difference equation. Such solutions are very important in practice, and in Chapter 2 other techniques to compute them will be studied. $\triangle$

Solving difference equations

Example 1.10

Determine the solution of the difference equation in the previous example supposing that $a = b$ (observe that the annihilator polynomial has common zeros with the homogeneous equation).

Solving difference equations

Solution

$a = b \quad \Rightarrow \quad y(n) + a^2 y(n-2) = a^n \sin\left(\frac{\pi}{2}n\right) u(n)$

In this case there are repeated roots in the difference equation, and as a result the complete solution has the following form for $n \geq 0$:

$$y(n) = na^n \left[d_1 \sin\left(\frac{\pi}{2}n\right) + d_2 \cos\left(\frac{\pi}{2}n\right)\right] + a^n \left[d_3 \sin\left(\frac{\pi}{2}n\right) + d_4 \cos\left(\frac{\pi}{2}n\right)\right] \tag{130}$$

As in the case for $a \neq b$, we notice that the right-hand side of the summation is the homogeneous solution, and thus it will be annihilated for all d_3 and d_4.

For finding d_1 and d_2 one should substitute the left-hand side of the summation in the original equation (112), for $n \geq 0$. This yields

$$
\begin{aligned}
& n a^n \left[d_1 \sin\left(\frac{\pi}{2} n\right) + d_2 \cos\left(\frac{\pi}{2} n\right) \right] \\
& + a^2 (n-2) a^{n-2} \left[d_1 \sin\left(\frac{\pi}{2}(n-2)\right) + d_2 \cos\left(\frac{\pi}{2}(n-2)\right) \right] = a^n \sin\left(\frac{\pi}{2} n\right) \\
\Rightarrow \quad & n \left[d_1 \sin\left(\frac{\pi}{2} n\right) + d_2 \cos\left(\frac{\pi}{2} n\right) \right] \\
& + (n-2) \left[d_1 \sin\left(\frac{\pi}{2} n - \pi\right) + d_2 \cos\left(\frac{\pi}{2} n - \pi\right) \right] = \sin\left(\frac{\pi}{2} n\right) \\
\Rightarrow \quad & n \left[d_1 \sin\left(\frac{\pi}{2} n\right) + d_2 \cos\left(\frac{\pi}{2} n\right) \right] \\
& + (n-2) \left[-d_1 \sin\left(\frac{\pi}{2} n\right) - d_2 \cos\left(\frac{\pi}{2} n\right) \right] = \sin\left(\frac{\pi}{2} n\right) \\
\Rightarrow \quad & [n d_1 - (n-2) d_1] \sin\left(\frac{\pi}{2} n\right) + [n d_2 - (n-2) d_2] \cos\left(\frac{\pi}{2} n\right) = \sin\left(\frac{\pi}{2} n\right) \\
\Rightarrow \quad & 2 d_1 \sin\left(\frac{\pi}{2} n\right) + 2 d_2 \cos\left(\frac{\pi}{2} n\right) = \sin\left(\frac{\pi}{2} n\right) \qquad (131)
\end{aligned}
$$

Therefore, we conclude that

$$
d_1 = \frac{1}{2}; \quad d_2 = 0 \qquad (132)
$$

Solving difference equations

The overall solution for $n \geq 0$ is then

$$y(n) = \frac{na^n}{2} \sin\left(\frac{\pi}{2}n\right) + a^n \left[d_3 \sin\left(\frac{\pi}{2}n\right) + d_4 \cos\left(\frac{\pi}{2}n\right)\right] \tag{133}$$

As in the case for $a \neq b$, in order to compute the constants d_3 and d_4 one must use auxiliary conditions for $n \geq 0$, since equation (130) is only valid for $n \geq 0$. Since $y(n) = 0, \;$ for $n < 0$, we need to run the difference equation starting from the auxiliary conditions $y(-2) = y(-1) = 0$ to compute $y(0)$ and $y(1)$.

$$\left\{ \begin{array}{ll} n = 0: & y(0) + a^2 y(-2) = a^0 \sin\left(\frac{\pi}{2} \times 0\right) u(0) = 0 \\ n = 1: & y(1) + a^2 y(-1) = a^1 \sin\left(\frac{\pi}{2}\right) u(1) = a \end{array} \right. \Rightarrow \left\{ \begin{array}{l} y(0) = 0 \\ y(1) = a \end{array} \right. \tag{134}$$

Solving difference equations

Using these auxiliary conditions in equation (133), we get

$$\begin{cases} y(0) = d_4 = 0 \\ y(1) = a\left[\frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right] + a\left[d_3 \sin\left(\frac{\pi}{2}\right) + d_4 \cos\left(\frac{\pi}{2}\right)\right] = a \end{cases} \tag{135}$$

and then

$$\frac{a}{2} + ad_3 = a \quad \Rightarrow \quad d_3 = \frac{1}{2}; \quad d_4 = 0 \tag{136}$$

and since $y(n) = 0$, for $n < 0$, the solution is

$$y(n) = \left(\frac{n+1}{2}\right) a^n \sin\left(\frac{\pi}{2}n\right) u(n) \tag{137}$$

$\triangle$

Computing impulse responses

If we want to find the impulse response of a system, we need to solve the following difference equation:

$$\sum_{i=0}^{N} a_i y(n-i) = \delta(n) \tag{138}$$

For a linear system to be causal it must be initially relaxed, that is, the auxiliary conditions prior to the input must be zero. For causal systems, since the input $\delta(n)$ is applied at $n = 0$, we must have

$$y(-1) = y(-2) = \cdots = y(-N) = 0 \tag{139}$$

For $n > 0$, equation (138) becomes homogeneous, that is

$$\sum_{i=0}^{N} a_i y(n-i) = 0 \tag{140}$$

and it can be solved using the techniques presented earlier in this section.

Computing impulse responses

- In order to do so, we need N auxiliary conditions.
- However, since equation (140) is valid only for $n > 0$, we can not use the auxiliary conditions from equation (139), but need N auxiliary conditions for $n > 0$ instead.
  - For example, these conditions can be $y(1), y(2), \ldots, y(N)$, which can found, starting from the auxiliary conditions in equation (139), by running the difference equation (138) from $n = 0$ to $n = N$, leading to

Computing impulse responses

$$\begin{cases} n=0: \; y(0) = \dfrac{\delta(0)}{a_0} - \dfrac{1}{a_0}\displaystyle\sum_{i=1}^{N} a_i y(-i) = \dfrac{1}{a_0} \\ n=1: \; y(1) = \dfrac{\delta(1)}{a_0} - \dfrac{1}{a_0}\displaystyle\sum_{i=1}^{N} a_i y(1-i) = -\dfrac{a_1}{a_0^2} \\ \quad \vdots \\ n=N: \; y(N) = \dfrac{\delta(N)}{a_0} - \dfrac{1}{a_0}\displaystyle\sum_{i=1}^{N} a_i y(N-i) = -\dfrac{1}{a_0}\displaystyle\sum_{i=1}^{N} a_i y(N-i) \end{cases} \tag{141}$$

Computing impulse responses

Example 1.11

Compute the impulse response of the system governed by the following difference equation:

$$y(n) - \frac{1}{2}y(n-1) + \frac{1}{4}y(n-2) = x(n) \tag{142}$$

Computing impulse responses

Solution

$y(n) - \frac{1}{2}y(n-1) + \frac{1}{4}y(n-2) = x(n)$

For $n > 0$ the impulse response satisfies the homogeneous equation. The corresponding polynomial equation is

$$\rho^2 - \frac{1}{2}\rho + \frac{1}{4} = 0 \qquad (143)$$

whose roots are $\rho = \frac{1}{2}e^{\pm j\frac{\pi}{3}}$. Therefore, for $n > 0$, the solution is

$$y(n) = c_1 2^{-n} \cos\left(\frac{\pi}{3}n\right) + c_2 2^{-n} \sin\left(\frac{\pi}{3}n\right) \qquad (144)$$

Computing impulse responses

Considering the system to be causal, we have that $y(n) = 0$, for $n < 0$. Therefore, we need to compute the auxiliary conditions for $n > 0$ as follows:

$$\begin{cases} n = 0: \; y(0) = \delta(0) + \frac{1}{2}y(-1) - \frac{1}{4}y(-2) = 1 \\ n = 1: \; y(1) = \delta(1) + \frac{1}{2}y(0) - \frac{1}{4}y(-1) = \frac{1}{2} \\ n = 2: \; y(2) = \delta(2) + \frac{1}{2}y(1) - \frac{1}{4}y(0) = 0 \end{cases} \tag{145}$$

Applying the above conditions to the solution in equation (144), we have

$$\begin{cases} y(1) = c_1 2^{-1} \cos\left(\frac{\pi}{3}\right) + c_2 2^{-1} \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} \\ y(2) = c_1 2^{-2} \cos\left(\frac{2\pi}{3}\right) + c_2 2^{-2} \sin\left(\frac{2\pi}{3}\right) = 0 \end{cases} \tag{146}$$

Hence

$$\begin{cases} \frac{1}{4}c_1 + \frac{\sqrt{3}}{4}c_2 = \frac{1}{2} \\ -\frac{1}{8}c_1 + \frac{\sqrt{3}}{8}c_2 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 1 \\ c_2 = \frac{\sqrt{3}}{3} \end{cases} \tag{147}$$

and the impulse response becomes

$$y(n) = \begin{cases} 0, & n < 0 \\ 1, & n = 0 \\ \frac{1}{2}, & n = 1 \\ 0, & n = 2 \\ 2^{-n}\left[\cos\left(\frac{\pi}{3}n\right) + \frac{\sqrt{3}}{3}\sin\left(\frac{\pi}{3}n\right)\right], & n \geq 2 \end{cases} \tag{148}$$

which, by inspection, can be expressed in a compact form as

$$y(n) = 2^{-n}\left[\cos\left(\frac{\pi}{3}n\right) + \frac{\sqrt{3}}{3}\sin\left(\frac{\pi}{3}n\right)\right]u(n) \tag{149}$$

Sampling of continuous-time signals

- In many cases, a discrete-time signal $x(n)$ consists of samples of a continuous-time signal $x_a(t)$, that is

$$x(n) = x_a(nT) \tag{150}$$

- If we want to process the continuous-time signal $x_a(t)$ using a discrete-time system, we need to:
  - Convert it using equation (150),
  - Process the discrete-time input digitally,
  - Convert the discrete-time output back to the continuous-time domain.
- In order for this operation to be effective, it is essential that we have capability of restoring a continuous-time signal from its samples.
- In this section, we derive conditions under which a continuous-time signal can be recovered from its samples, and devise ways of performing this recovery.

Sampling of continuous-time signals

The Fourier transform of a continuous-time signal $f(t)$ is given by

$$F(j\Omega) = \int_{-\infty}^{\infty} f(t)e^{-j\Omega t}dt \tag{151}$$

where Ω is referred to as the frequency and is measured in radians per second (rad/s). The corresponding inverse relationship is expressed as

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\Omega)e^{j\Omega t}d\Omega \tag{152}$$

If $x(t) = a(t)b(t)$, then its Fourier transform can be expressed as

$$X(j\Omega) = \frac{1}{2\pi}A(j\Omega) * B(j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(j\Omega - j\Omega')B(j\Omega')d\Omega' \tag{153}$$

where $X(j\Omega)$, $A(j\Omega)$, and $B(j\Omega)$ are the Fourier transforms of $x(t)$, $a(t)$, and $b(t)$, respectively.

Sampling of continuous-time signals

In addition, if a signal $x(t)$ is periodic with period T, then we can express it by its Fourier series defined by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \tag{154}$$

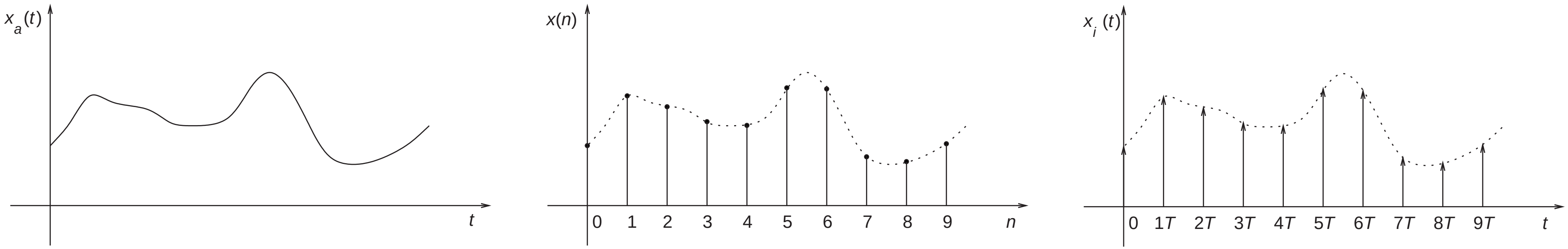
where the a_ks are called the series coefficients which are determined as

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\frac{2\pi}{T}t}dt \tag{155}$$

Sampling of continuous-time signals

Given $x(n) = x_a(nT)$, we start by defining a continuous-time signal $x_i(t)$ consisting of a train of impulses at $t = nT$, each of area equal to $x(n)$, as follows:

$$x_i(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT) \tag{156}$$



Since, from equation (150), $x(n) = x_a(nT)$, then equation (156) becomes

$$x_i(t) = \sum_{n=-\infty}^{\infty} x_a(nT)\delta(t - nT) = x_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = x_a(t)p(t) \tag{157}$$

Sampling of continuous-time signals

This indicates that $x_i(t)$ can also be obtained by multiplying the continuous-time signal $x_a(t)$ by a train of impulses $p(t)$ defined as

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{158}$$

In the equations above, we have defined a continuous-time signal $x_i(t)$ that can be obtained from the discrete-time signal $x(n)$ in a straightforward manner. In what follows, we relate the Fourier transforms of $x_a(t)$ and $x_i(t)$, and study the conditions under which $x_a(t)$ can be obtained from $x_i(t)$.

The Fourier transform of $x_i(t)$ is such that

$$X_i(j\Omega) = \frac{1}{2\pi} X_a(j\Omega) * P(j\Omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega - j\Omega')P(j\Omega')d\Omega' \tag{159}$$

Therefore, in order to arrive at an expression for the Fourier transform of $x_i(t)$, we must first determine the Fourier transform of $p(t)$, $P(j\Omega)$.

Sampling of continuous-time signals

$p(t)$ is a periodic function with period T and that we can decompose it in a Fourier series, as described in equations (154) and (155). Since, from equation (158), $p(t)$ in the interval $[-\frac{T}{2}, \frac{T}{2}]$ is equal to just an impulse $\delta(t)$, the coefficients a_k in the Fourier series of $p(t)$ are given by

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jk\frac{2\pi}{T}t}dt = \frac{1}{T} \tag{160}$$

and the Fourier series for $p(t)$ becomes

$$p(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j\frac{2\pi}{T}kt} \tag{161}$$

Sampling of continuous-time signals

As the Fourier transform of $f(t) = e^{j\Omega_0 t}$ is equal to $F(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$, then, from equation (161), the Fourier transform of $p(t)$ becomes

$$P(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi}{T}k\right) \tag{162}$$

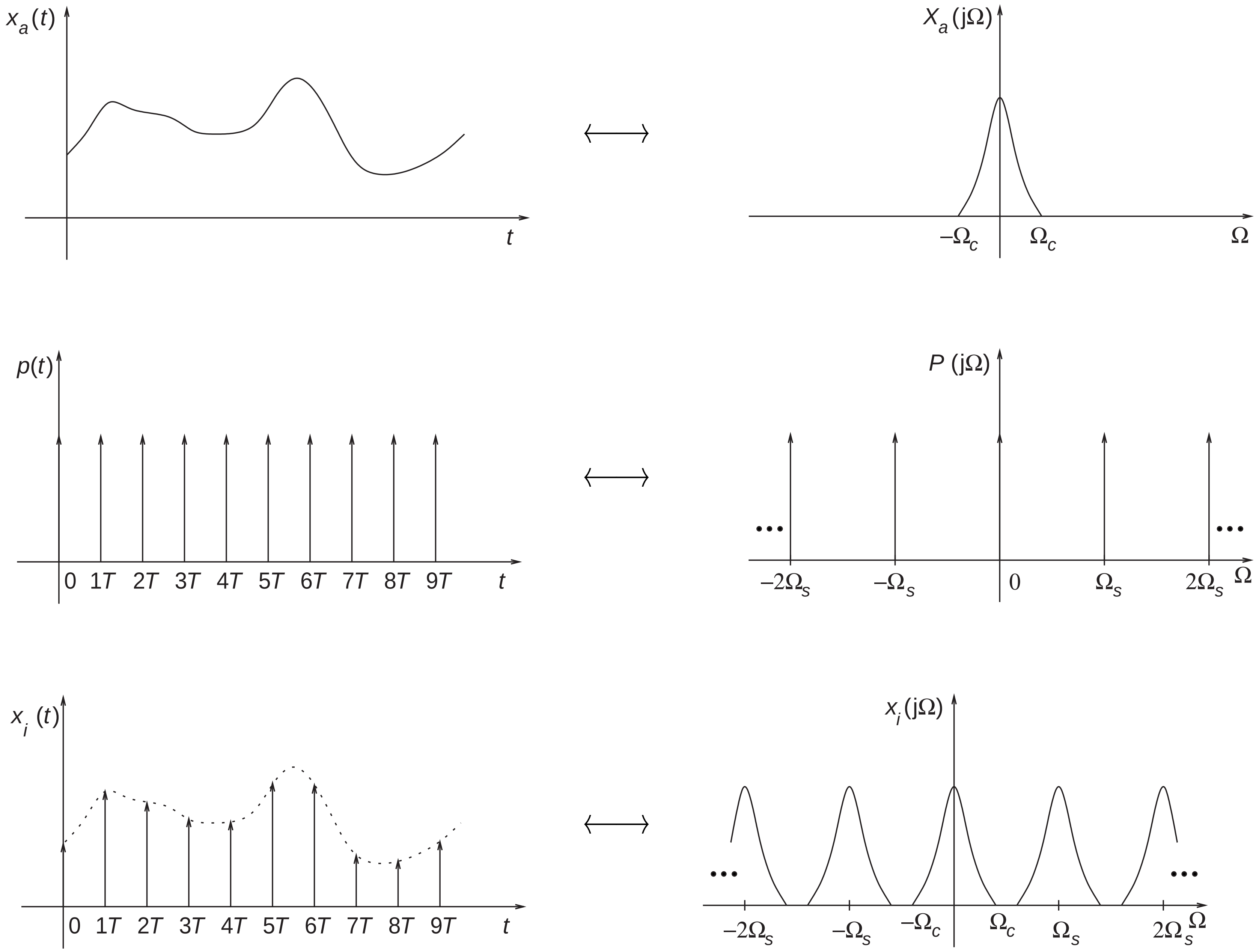
Substituting this expression for $P(j\Omega)$ in equation (159), we have that

$$\begin{aligned} X_i(j\Omega) &= \frac{1}{2\pi} X_a(j\Omega) * P(j\Omega) \\ &= \frac{1}{T} X_a(j\Omega) * \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi}{T}k\right) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(j\Omega - j\frac{2\pi}{T}k\right) \end{aligned} \tag{163}$$

where, in the last step, we used the fact that the convolution of a function $F(j\Omega)$ with a shifted impulse $\delta(\Omega - \Omega_0)$ is the shifted function $F(j\Omega - j\Omega_0)$.

Sampling of continuous-time signals

Equation (163) shows that the spectrum of $x_i(t)$ is composed of infinite shifted copies of the spectrum of $x_a(t)$, with the shifts in frequency being multiples of the sampling frequency, $\Omega_s = \frac{2\pi}{T}$. The next figure shows examples of signals $x_a(t)$, $p(t)$, and $x_i(t)$, and their respective Fourier transforms.

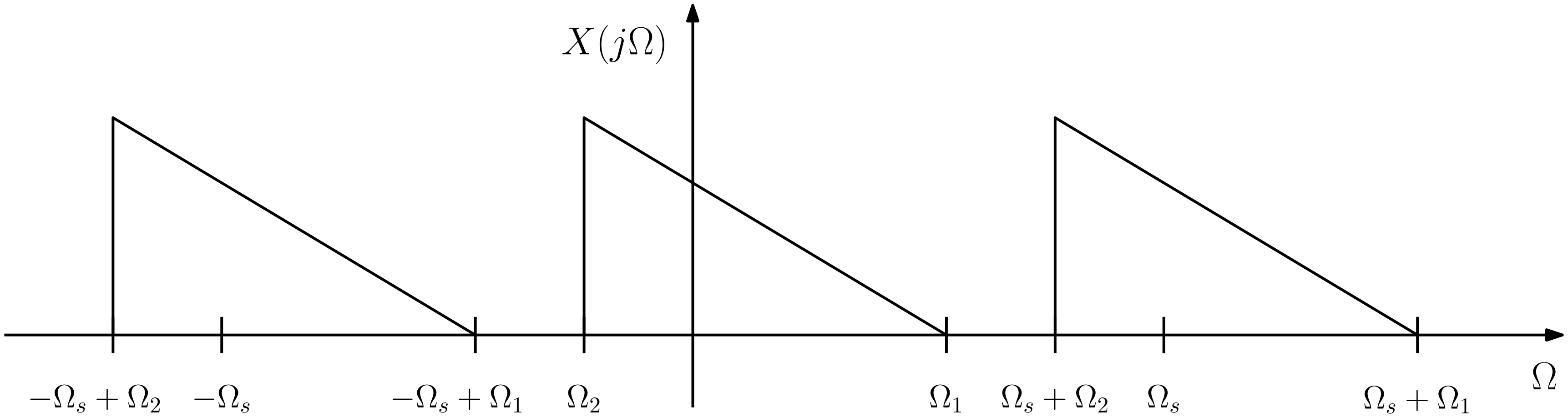

$x_a(t)$
t
$X_a(j\Omega)$
$-\Omega_c$
Ω_c
Ω
$p(t)$
0 1T 2T 3T 4T 5T 6T 7T 8T 9T t
$P(j\Omega)$
$-2\Omega_s$ $-\Omega_s$ 0 Ω_s $2\Omega_s$ Ω
$x_i(t)$
0 1T 2T 3T 4T 5T 6T 7T 8T 9T t
$x_i(j\Omega)$
$-2\Omega_s$ $-\Omega_s$ $-\Omega_c$ Ω_c Ω_s $2\Omega_s$ Ω

Sampling of continuous-time signals

We then see that, in order to avoid the repeated copies of the spectrum of $x_a(t)$ interfering with one another:

- The signal should be band-limited.
- Its bandwidth Ω_c should be such that the upper edge of the spectrum centered at zero is smaller than the lower edge of the spectrum centered at Ω_s.

Referring to general complex case depicted below, we must have $\Omega_s + \Omega_2 > \Omega_1$, or equivalently, $\Omega_s > \Omega_1 - \Omega_2$.



Sampling of continuous-time signals

In the case of real signals, since the spectrum is symmetric around zero, the one-sided bandwidth of the continuous-time signal Ω_c is such that $\Omega_c = \Omega_1 = -\Omega_2$, and then we must have $\Omega_s > \Omega_c - (-\Omega_c)$, implying that

$$\Omega_s > 2\Omega_c \tag{164}$$

that is, the sampling frequency must be larger than double the one-sided bandwidth of the continuous-time signal. The frequency $\Omega = 2\Omega_c$ is called the **Nyquist frequency** of the real continuous-time signal $x_a(t)$.

In addition, if the condition in equation (164) is satisfied, the original continuous signal $x_a(t)$ can be recovered by isolating the part of the spectrum of $x_i(t)$ that corresponds to the spectrum of $x_a(t)$. This can be achieved by filtering the signal $x_i(t)$ with an ideal lowpass filter having bandwidth $\frac{\Omega_s}{2}$.

Sampling of continuous-time signals

- On the other hand, if the condition in equation (164) is not satisfied, the repetitions of the spectrum interfere with one another, and the continuous-time signal can not be recovered from its samples.
- This superposition of the repetitions of the spectrum of $x_a(t)$ in $x_i(t)$, when the sampling frequency is smaller than $2\Omega_c$, is commonly referred to as **aliasing**.

Figure 1b–d shows the spectra of $x_i(t)$ for Ω_s equal to, smaller than, and larger than $2\Omega_c$, respectively. The aliasing phenomenon is clearly identified in the next figure.

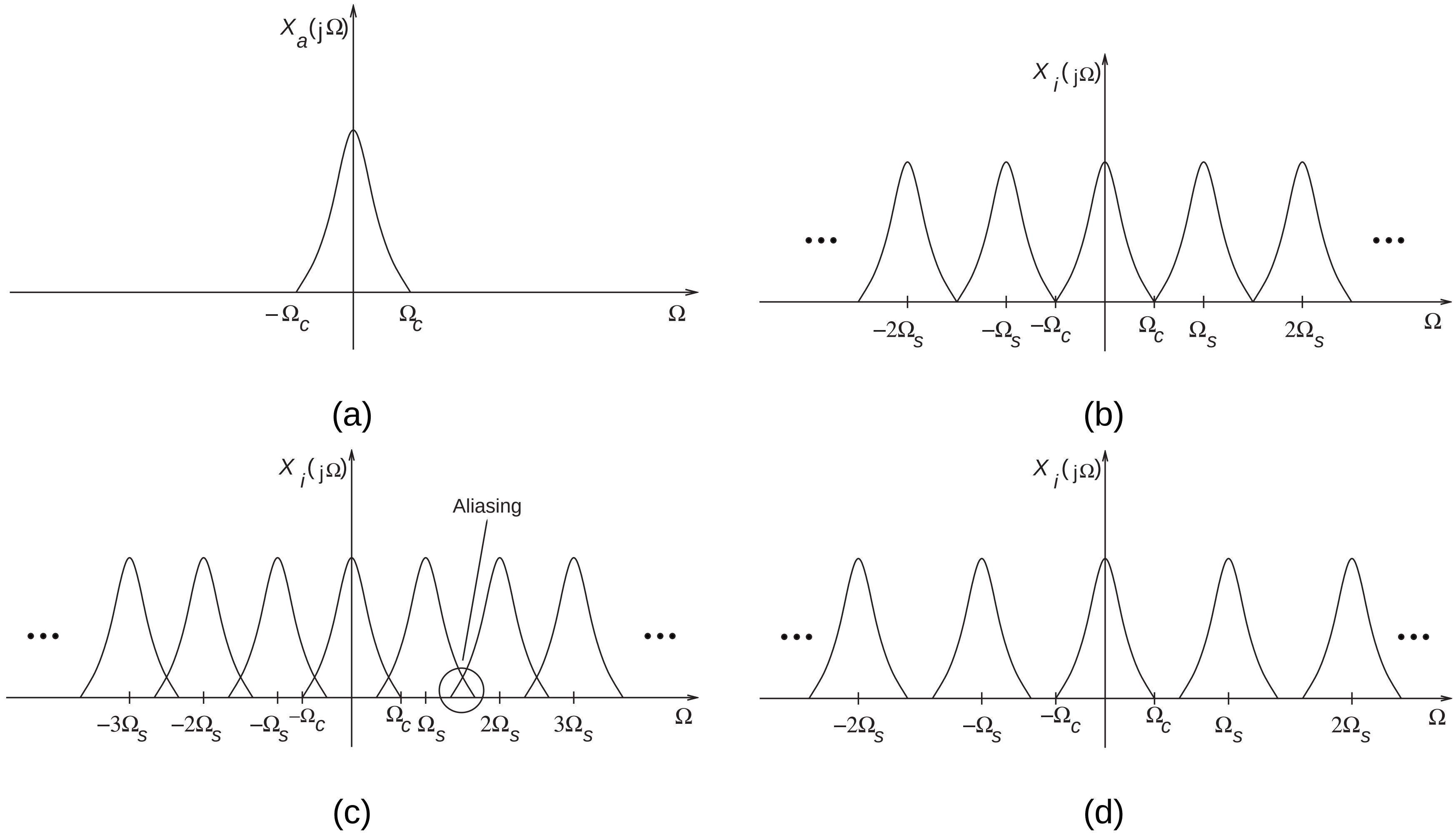


Figure 1: (a) Spectrum of the continuous-time signal. Spectra of $x_i(t)$ for: (b) $\Omega_s = 2\Omega_c$; (c) $\Omega_s < 2\Omega_c$; (d) $\Omega_s > 2\Omega_c$.

Sampling of continuous-time signals

In fact, any of the spectrum repetitions have the full information about $x_a(t)$. However, if we isolate a repetition of the spectrum not centered at $\Omega = 0$, we get a modulated version of $x_a(t)$, which should be demodulated. Since its demodulation is equivalent to shifting the spectrum back to the origin, it is usually better to take the repetition of the spectrum centered at the origin in the first place.

We are now ready to enunciate a very important result:

SAMPLING THEOREM

If a continuous-time signal $x_a(t)$ is band-limited, that is, its Fourier transform is such that $X_a(j\Omega) = 0$, for $|\Omega| > |\Omega_c|$, then $x_a(t)$ can be completely recovered from the discrete-time signal $x(n) = x_a(nT)$ if the sampling frequency Ω_s satisfies $\Omega_s > 2\Omega_c$.

$\diamond$

Sampling of continuous-time signals

Example 1.12

Consider the discrete-time sequence

$$x(n) = \sin\left(\frac{6\pi}{4}n\right) \tag{165}$$

Assuming that the sampling frequency is $f_s = 40$ kHz, find two continuous-time signals that could have generated this sequence.

Sampling of continuous-time signals

Solution

Supposing that the continuous-time signal is of the form

$$x_a(t) = \sin(\Omega_c t) = \sin(2\pi f_c t) \tag{166}$$

We have that when sampled with sampling frequency $f_s = \frac{1}{T_s}$ it generates the following discrete signal:

$$\begin{aligned} x(n) &= x_a(nT_s) \\ &= \sin(2\pi f_c n T_s) \\ &= \sin\left(2\pi \frac{f_c}{f_s} n\right) \\ &= \sin\left(2\pi \frac{f_c}{f_s} n + 2k\pi n\right) \\ &= \sin\left[2\pi\left(\frac{f_c}{f_s} + k\right) n\right] \end{aligned} \tag{167}$$

for any integer k.

Sampling of continuous-time signals

Therefore, in order to a sinusoid following equation (166), when sampled, yield the discrete signal in equation (165), we must have that

$$2\pi\left(\frac{f_c}{f_s}+k\right)=\frac{6\pi}{4} \quad \Rightarrow \quad f_c=\left(\frac{3}{4}-k\right)f_s \tag{168}$$

For example,

$$k=0 \quad \Rightarrow \quad f_c=\frac{3}{4}f_s=30 \text{ kHz} \quad \Rightarrow \quad x_1(t)=\sin(60000\pi t) \tag{169}$$

$$k=-1 \quad \Rightarrow \quad f_c=\frac{7}{4}f_s=70 \text{ kHz} \quad \Rightarrow \quad x_2(t)=\sin(140000\pi t) \tag{170}$$

We can verify that by computing $x_i(t)$ for the two above signals according to equation (157):

$$\begin{aligned}
x_{1_i}(t) &= \sum_{n=-\infty}^{\infty} x_1(t)\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} \sin(60000\pi t)\delta\left(t-\frac{n}{40000}\right) \\
&= \sum_{n=-\infty}^{\infty} \sin\left(60000\pi\frac{n}{40000}\right)\delta\left(t-\frac{n}{40000}\right) \\
&= \sum_{n=-\infty}^{\infty} \sin\left(\frac{3\pi}{2}n\right)\delta\left(t-\frac{n}{40000}\right) \qquad (171) \\
x_{2_i}(t) &= \sum_{n=-\infty}^{\infty} x_2(t)\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} \sin(140000\pi t)\delta\left(t-\frac{n}{40000}\right) \\
&= \sum_{n=-\infty}^{\infty} \sin\left(140000\pi\frac{n}{40000}\right)\delta\left(t-\frac{n}{40000}\right) \\
&= \sum_{n=-\infty}^{\infty} \sin\left(\frac{7\pi}{2}n\right)\delta\left(t-\frac{n}{40000}\right) \qquad (172)
\end{aligned}$$

Sampling of continuous-time signals

Since

$$\sin\left(\frac{7\pi}{2}n\right) = \sin\left[\left(\frac{3\pi}{2} + 2\pi\right)n\right] = \sin\left(\frac{3\pi}{2}n\right) \tag{173}$$

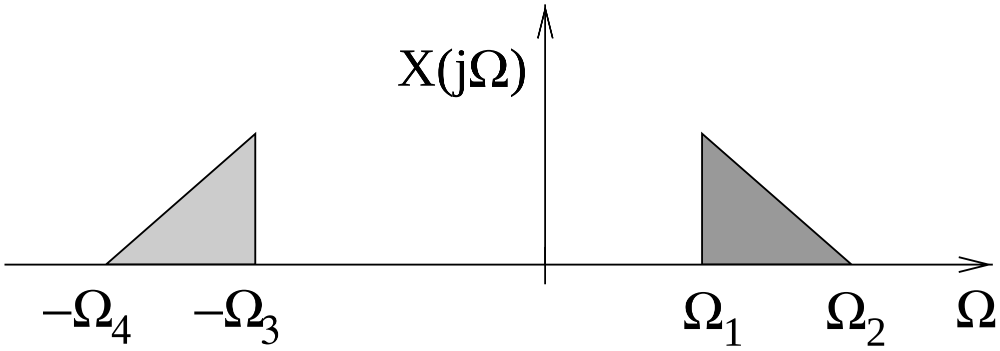
then we have that the signals $x_{1_i}(t)$ and $x_{2_i}(t)$ are identical.

$\triangle$

Sampling of continuous-time signals

Example 1.13

In the figure below, assuming that $\Omega_2 - \Omega_1 < \Omega_1$ and $\Omega_4 - \Omega_3 < \Omega_3$:



1. Using a single sampler, what would be the minimum sampling frequency such that no information is lost?

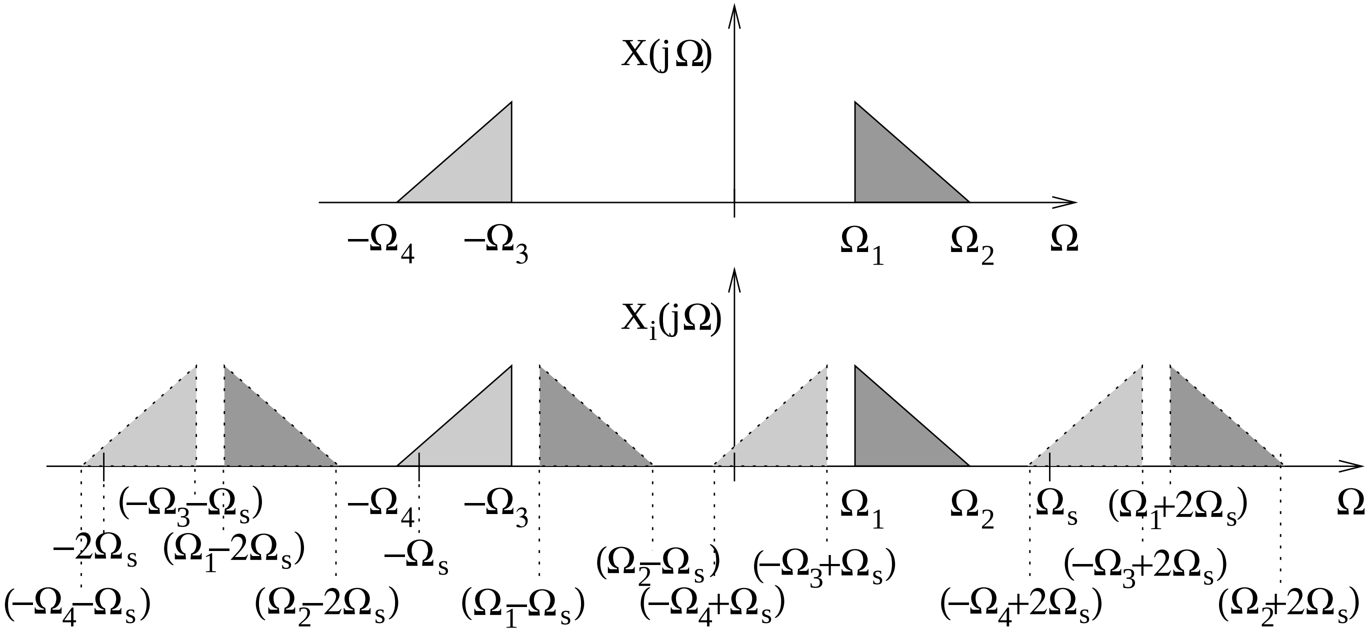
2. Using an ideal filter and two samplers, what would be the minimum sampling frequencies such that no information is lost? Depict the configuration used in this case.

Sampling of continuous-time signals

Solution

(a) By examining the above figure we see that:

- A sampling rate $\Omega_s > \Omega_2 + \Omega_4$ would avoid aliasing.
- However, since it is given that $\Omega_2 - \Omega_1 < \Omega_1$ and $\Omega_4 - \Omega_3 < \Omega_3$, then in the empty spectrum between $-\Omega_3$ and Ω_1 we can accommodate one copy of the spectrum in the interval $[\Omega_1, \Omega_2]$ and one copy of the spectrum in the interval $[-\Omega_4, -\Omega_3]$.
- According to equation (163), when a signal is sampled its spectrum is repeated at multiples of Ω_s.
- Therefore, we can choose Ω_s so that the spectrum of the sampled signal would be as in the lower part of the next figure.



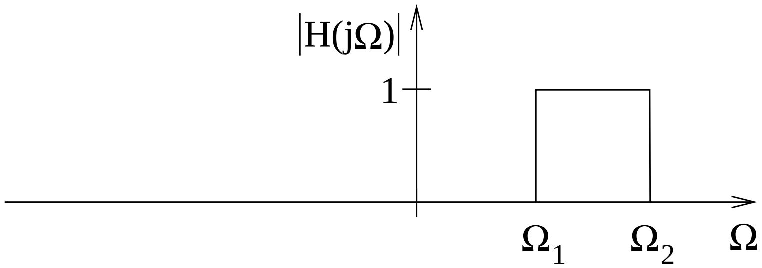
To avoid spectrum superposition, we must have

$$\begin{cases} \Omega_1 - \Omega_s > -\Omega_3 \\ -\Omega_4 + \Omega_s > \Omega_2 - \Omega_s \end{cases} \Rightarrow \frac{\Omega_2 + \Omega_4}{2} < \Omega_s < \Omega_1 + \Omega_3 \qquad (174)$$

Therefore, the minimum sampling frequency would be $\Omega_s = \dfrac{\Omega_2 + \Omega_4}{2}$, provided that $\Omega_s < \Omega_1 + \Omega_3$.

Sampling of continuous-time signals

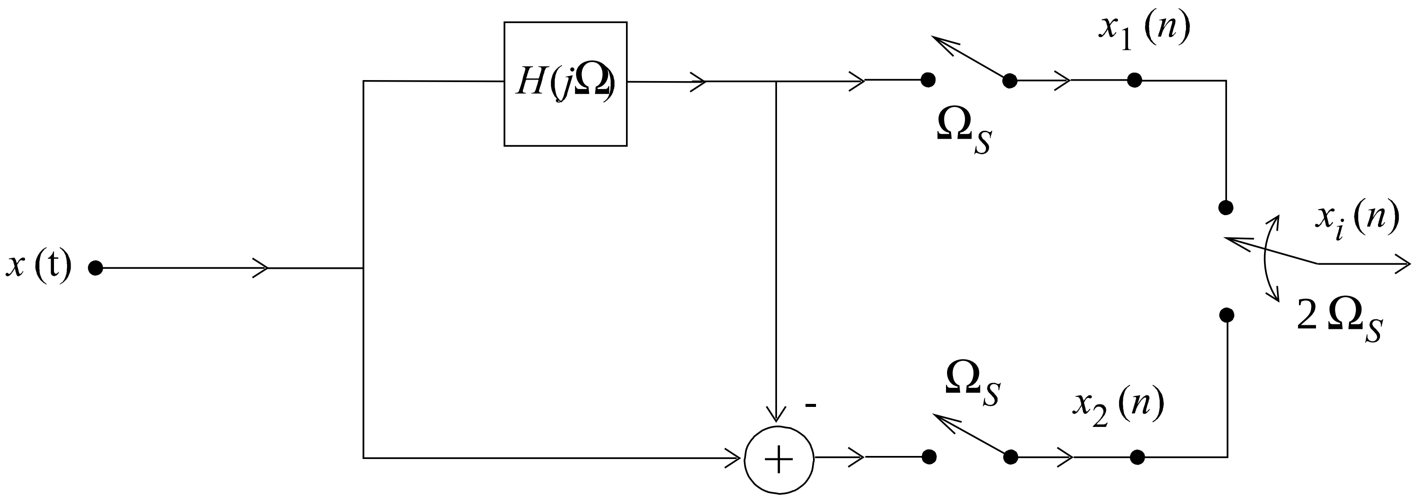
(b) If we have an ideal filter as depicted below we can isolate both parts of the spectrum, and then sample them at a much lower rate.



For example, we can sample the output of this filter with a frequency $\Omega_{s_1} > \Omega_2 - \Omega_1$.

Sampling of continuous-time signals

If we use the scheme in the next figure, we take the filter output and subtract it from the input signal. The result will have just the left side of the spectrum of the original signal, which can be sampled with a frequency $\Omega_{s_2} > \Omega_4 - \Omega_3$.



If we use a single sampling frequency, its value should satisfy

$$\Omega_s > \max\{\Omega_2 - \Omega_1, \Omega_4 - \Omega_3\} \quad (175)$$

Note that the output is composed of one sample of each signal $x_1(n)$ and $x_2(n)$, and therefore the effective sampling frequency is $2\Omega_s$. $\triangle$

Sampling of continuous-time signals

- As illustrated in the example above, for some bandpass signals $x(t)$, sampling may be performed below the limit $2\Omega_{\text{max}}$, where Ω_{max} represents the maximum absolute value of the frequency present in $x(t)$.

- Although one can not obtain a general expression for the minimum Ω_s in these cases, it must always satisfy $\Omega_s > \Delta\Omega$, where $\Delta\Omega$ represents the net bandwidth of $x(t)$.

Sampling of continuous-time signals

- The original continuous-time signal $x_a(t)$ can be recovered from the signal $x_i(t)$ by filtering $x_i(t)$ with an ideal lowpass filter having cutoff frequency $\frac{\Omega_s}{2}$.
  - More specifically, if the signal has bandwidth Ω_c, then it suffices that the cutoff frequency of the ideal lowpass filter is Ω_{LP}, such that $\Omega_c \leq \Omega_{LP} \leq \frac{\Omega_s}{2}$.
- Therefore, the Fourier transform of the impulse response of such a filter should be

$$H(j\Omega) = \begin{cases} T, & \text{for } |\Omega| < \Omega_{LP} \\ 0, & \text{for } |\Omega| \geq \Omega_{LP} \end{cases} \tag{176}$$

where the passband gain T compensates for the factor $\frac{1}{T}$ in equation (163). This ideal frequency response is illustrated in Figure 2a.

Sampling of continuous-time signals

Computing the inverse Fourier transform of $H(j\Omega)$ using equation (152), we see that the impulse response $h(t)$ of the filter is

$$h(t) = \frac{T\sin(\Omega_{LP}t)}{\pi t} \tag{177}$$

which is depicted in Figure 2b.

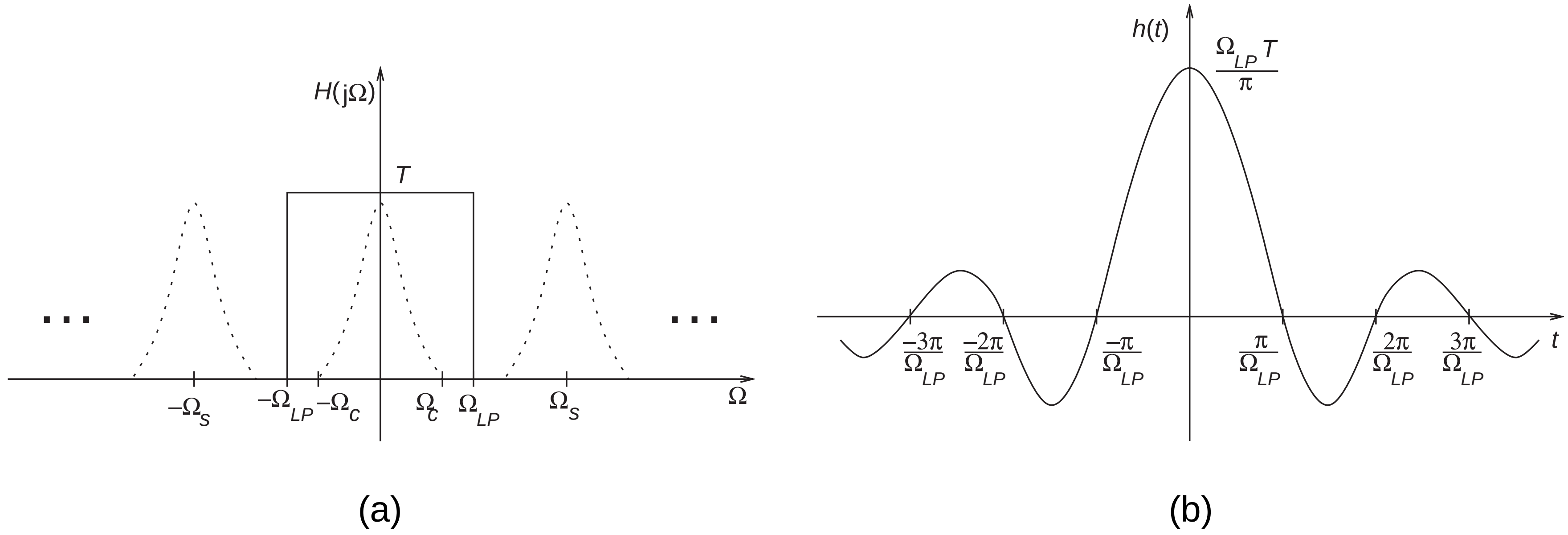


Figure 2: Ideal lowpass filter: (a) frequency response; (b) impulse response.

Sampling of continuous-time signals

Then, given $h(t)$, the signal $x_a(t)$ can be recovered from $x_i(t)$ by the following convolution integral

$$x_a(t) = \int_{-\infty}^{\infty} x_i(\tau)h(t-\tau)d\tau \tag{178}$$

Replacing $x_i(t)$ by its definition in equation (156), we have that

$$\begin{aligned} x_a(t) &= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n)\delta(\tau - nT)h(t-\tau)d\tau \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(n)\delta(\tau - nT)h(t-\tau)d\tau \\ &= \sum_{n=-\infty}^{\infty} x(n)h(t-nT) \end{aligned} \tag{179}$$

Sampling of continuous-time signals

and using $h(t)$ from equation (177),

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{T \sin(\Omega_{LP}(t-nT))}{\pi(t-nT)} \tag{180}$$

If we make Ω_{LP} equal to half the sampling frequency, that is, $\Omega_{LP} = \frac{\Omega_s}{2} = \frac{\pi}{T}$, then equation (180) becomes

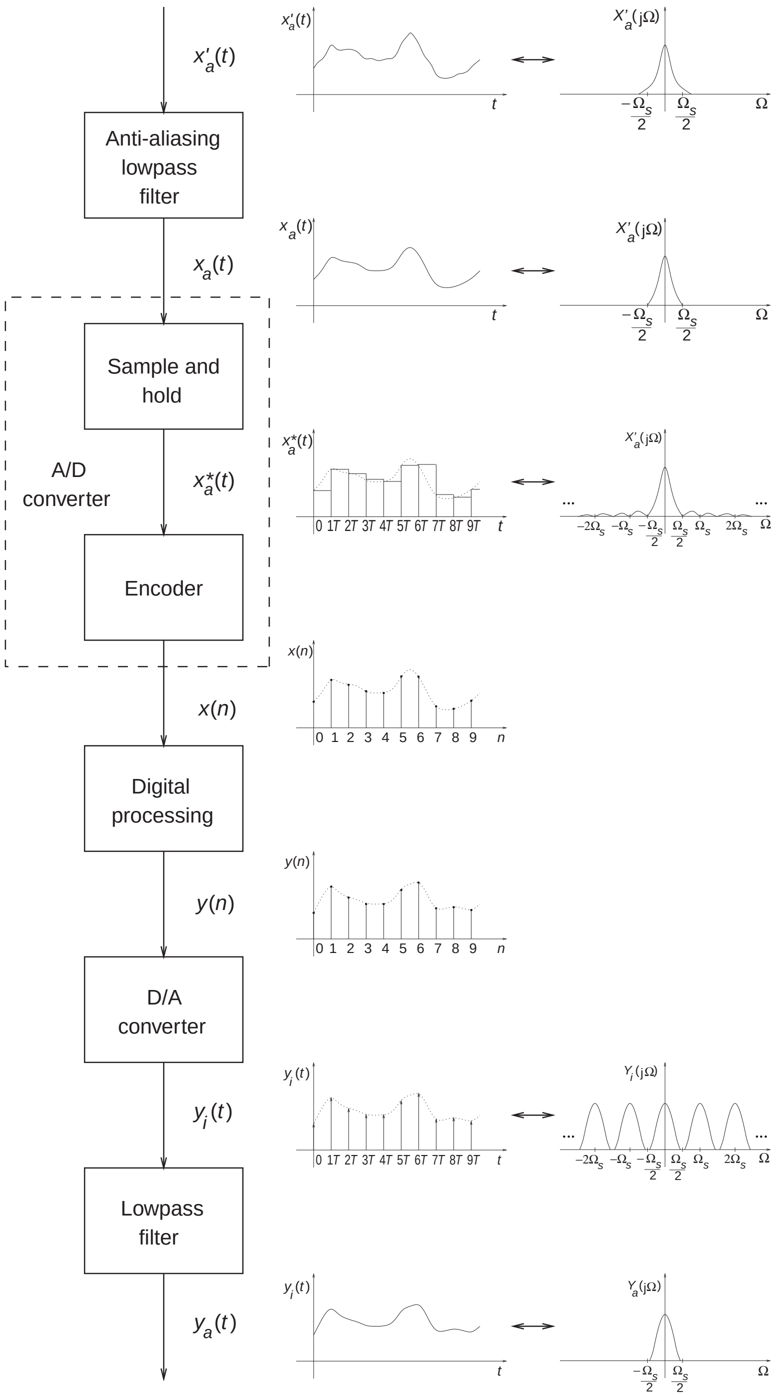
$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left(\frac{\Omega_s}{2}t - n\pi\right)}{\frac{\Omega_s}{2}t - n\pi} = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left[\pi\left(\frac{t}{T} - n\right)\right]}{\pi\left(\frac{t}{T} - n\right)} \tag{181}$$

- Equations (180) and (181) represent interpolation formulas to recover the continuous-time signal $x_a(t)$ from its samples $x(n) = x_a(nT)$.
- However, since, in order to compute $x_a(t)$ at any time t_0, all the samples of $x(n)$ have to be known, these interpolation formulas are not practical.

Sampling of continuous-time signals

- This is the same as saying that the lowpass filter with impulse response $h(t)$ is not realizable.
- This is because it is noncausal and can not be implemented via any differential equation of finite order.
- Clearly, the closer the lowpass filter is to the ideal, the smaller is the error in the computation of $x(t)$ using equation (180).
- In Chapters 4 and 5 methods to approximate such ideal filters will be extensively studied.

From what we have studied in this section, we can develop a block diagram of the several phases constituting the processing of an analog signal using a digital system. The next figure depicts each step of the procedure.


$x'_a(t)$
Anti-aliasing lowpass filter
$x_a(t)$
Sample and hold
A/D converter
$x^*_a(t)$
Encoder
$x(n)$
Digital processing
$y(n)$
D/A converter
$y_i(t)$
Lowpass filter
$y_a(t)$

Sampling of continuous-time signals

- The first block in the diagram of Figure 122 is a lowpass filter, that guarantees the analog signal is band-limited, with bandwidth $\Omega_c \leq \frac{\Omega_s}{2}$.
- The second block is a sample-and-hold system, which samples the signal $x_a(t)$ at times $t = nT$ and holds the obtained value for T seconds, that is, until the value for the next time interval is sampled.
  More precisely, $x_a^*(t) = x_a(nT)$, for $nT < t < (n+1)T$.
- The third block, the encoder, converts each sample value output by the sample-and-hold, $x_a^*(t)$, for $nT < t < (n+1)T$, to a number $x(n)$. Since this number is input to digital hardware, it must be represented with a finite number of bits. This operation introduces an error in the signal, which gets smaller as the number of bits used in the representation increases. The second and third blocks constitute what we usually call the analog-to-digital (A/D) conversion.

Sampling of continuous-time signals

- The fourth block carries out the digital signal processing, transforming the discrete-time signal $x(n)$ into a discrete-time signal $y(n)$.
- The fifth block transforms the numbers representing the samples $y(n)$ back into the analog domain, in the form of a train of impulses $y_i(t)$, constituting the process known as digital-to-analog (D/A) conversion.
- The sixth block is a lowpass filter necessary to eliminate the repetitions of the spectrum contained in $y_i(t)$, in order to recover the analog signal $y_a(t)$ corresponding to $y(n)$.
  - In practice, sometimes the fifth and sixth blocks are implemented in one operation. For example, we can transform the samples $y(n)$ into an analog signal $y_a(t)$ using a D/A converter plus a sample-and-hold operation, similar to the one of the second block.

The sample-and-hold operation is equivalent to filtering the train of impulses

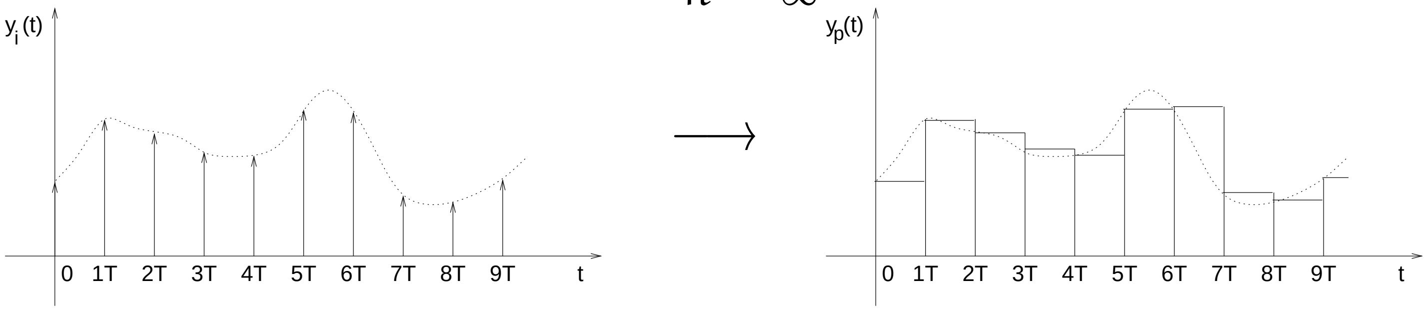
$$y_i(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - nT) \tag{182}$$

with a filter having impulse response

$$h(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{183}$$

yielding the analog signal

$$y_p(t) = \sum_{n=-\infty}^{\infty} y(n)h(t - nT) \tag{184}$$



In this case, the recovery of the analog signal is not perfect, but is a good enough approximation in some practical cases.

Sampling of continuous-time signals

Example 1.14

For the sample-and-hold operation described by equations (182) to (184), determine:

(a) An expression for the Fourier transform of $y_p(t)$ in equation (184) as a function of the Fourier transform of $x_a(t)$ (suppose that $y(n) = x_a(nT)$).

(b) The frequency response of an ideal lowpass filter that outputs $x_a(t)$ when $y_p(t)$ is applied to its input. Such filter would compensate for the artifacts introduced by the sample-and-hold operation in a D/A conversion.

Sampling of continuous-time signals

Solution

(a) The train of pulses given by

$$y_p(t) = \sum_{n=-\infty}^{\infty} y(n)\, h(t - nT) \tag{185}$$

is the result of the convolution of an impulse train with the given pulse as follows:

$$y_p(t) = y_i(t) * h(t) \tag{186}$$

Using equation (163), in the frequency domain the above equation becomes

$$Y_p(j\Omega) = Y_i(j\Omega)\, H(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_a\left(j\Omega - j\frac{2\pi}{T}k\right) H(j\Omega) \tag{187}$$

Since, from equation (183),

Sampling of continuous-time signals

$$
\begin{aligned}
H(j\Omega) &= \int_{-\infty}^{\infty} h(t)e^{-j\Omega t}dt \\
&= \int_{0}^{T} e^{-j\Omega t}dt \\
&= \frac{1}{j\Omega}\left(1 - e^{-j\Omega T}\right) \\
&= \frac{e^{-\frac{\Omega T}{2}}}{j\Omega}\left(e^{\frac{j\Omega T}{2}} - e^{-\frac{j\Omega T}{2}}\right) \\
&= \frac{e^{-\frac{\Omega T}{2}}}{j\Omega}\left[2j\sin\left(\frac{\Omega T}{2}\right)\right] \\
&= Te^{-\frac{\Omega T}{2}}\left[\frac{\sin\left(\frac{\Omega T}{2}\right)}{\frac{\Omega T}{2}}\right]
\end{aligned} \tag{188}
$$

Sampling of continuous-time signals

hence

$$Y_p(j\Omega) = e^{-\frac{\Omega T}{2}} \left[\frac{\sin\left(\frac{\Omega T}{2}\right)}{\frac{\Omega T}{2}} \right] \sum_{k=-\infty}^{\infty} X_a\left(j\Omega - j\frac{2\pi}{T}k\right) \tag{189}$$

(b) In order to recuperate signal $x_a(t)$, we have to compensate the distortion introduced by the frequency spectrum of the pulse $h(t)$. This can be done by processing $y_p(t)$ with a lowpass filter with the following desirable frequency response:

$$G(j\Omega) = \begin{cases} 0, & |\Omega| \geq \frac{\pi}{T} \\ \frac{T}{H(j\Omega)} = e^{\frac{\Omega T}{2}} \left[\frac{\frac{\Omega T}{2}}{\sin\left(\frac{\Omega T}{2}\right)} \right], & |\Omega| < \frac{\pi}{T} \end{cases} \tag{190}$$

$\triangle$

Sampling of continuous-time signals

Example 1.15

- Cinema is a way of storing and displaying moving pictures.
- Before it was invented, all that was known was a way to store and display single images, by taking photographs.
- Cinema was invented when one decided to capture a moving image as a series of pictures equally spaced in time.
  - For example, today, in a commercial movie one captures 24 pictures/second.
  - This scheme works because the human visual system enjoys a property called persistence of vision: When a light is flashed, one still sees the light for some time after it is gone.
  - Because of this, when displaying the pictures in sequence, a human viewer has the illusion of continuous movement.

Sampling of continuous-time signals

In more precise terms:

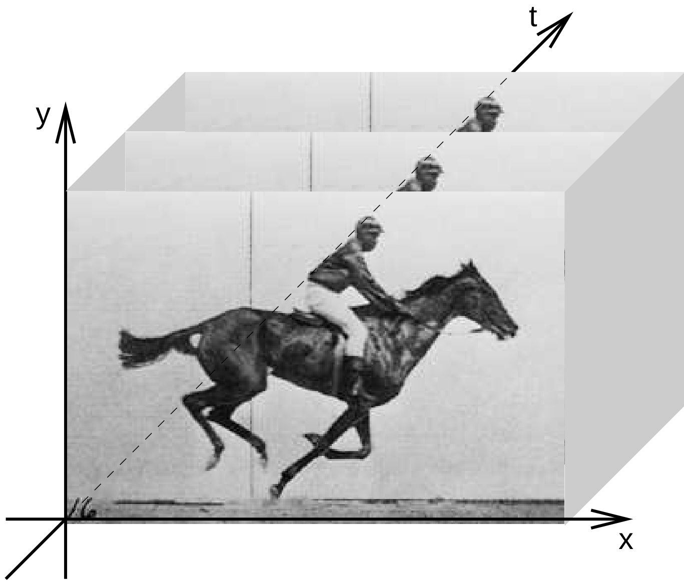
- A moving picture is a three-dimensional signal, with two continuous spatial dimensions (representing, for instance, horizontal and vertical directions) and one continuous time dimension.
- A photograph is a time sample of this three-dimensional signal.
- When one displays this sequence of time samples (photographs), the human visual system sees it as a continuously moving picture, that is, as a continuous-time signal.

From what has been said, we can note that cinema can be regarded a discrete-time signal processing system.

Identify in the cinema context which signal processing operation corresponds to each step of the processes of recording and displaying moving pictures, highlighting the associated design constraints.

Sampling of continuous-time signals

Solution The light field of a gray-scale moving picture can be described as a three-dimensional function $f_a(x, y, t)$, where x and y are spatial coordinates and t is a time coordinate, as represented in the figure below.

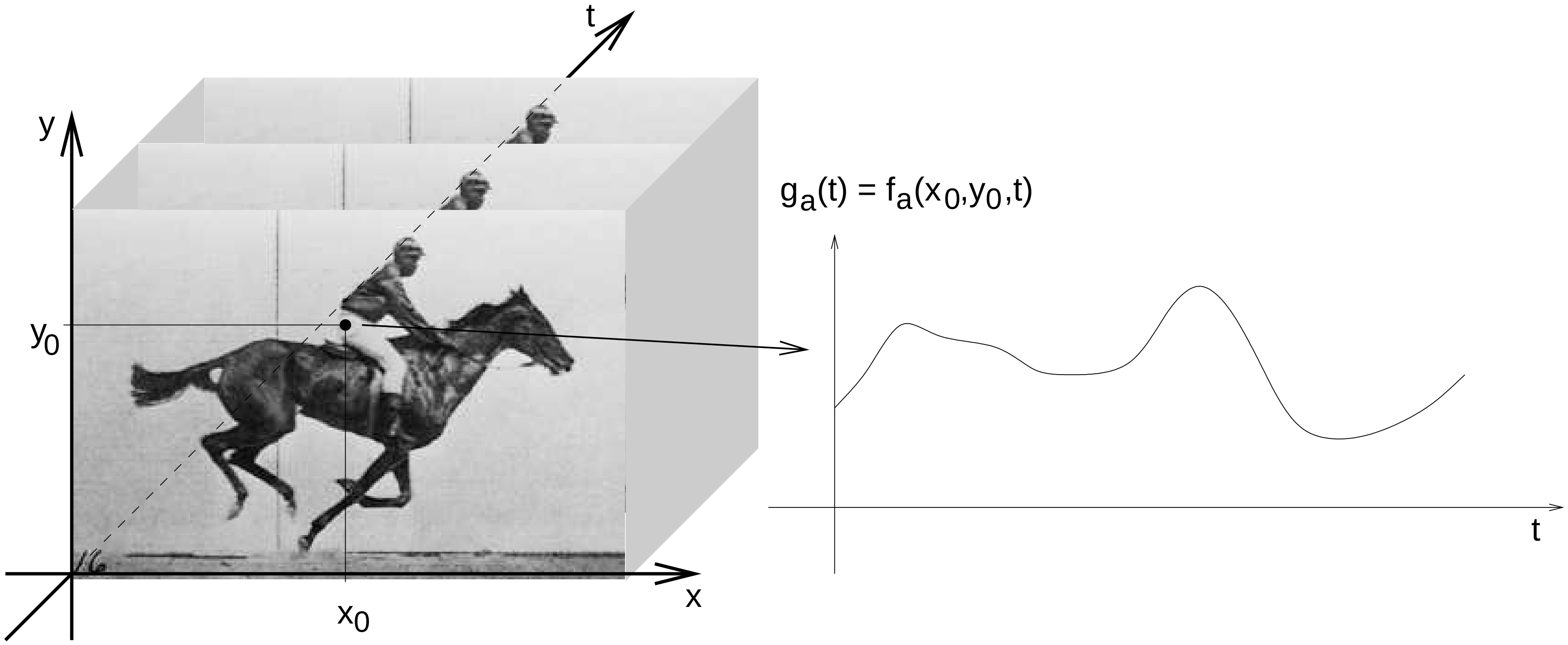


Sampling of continuous-time signals

The light intensity (luminance) of a point of coordinates (x_0, y_0) is thus the one-dimensional time signal

$$g_a(t) = f_a(x_0, y_0, t) \tag{191}$$

as depicted below.



Sampling of continuous-time signals

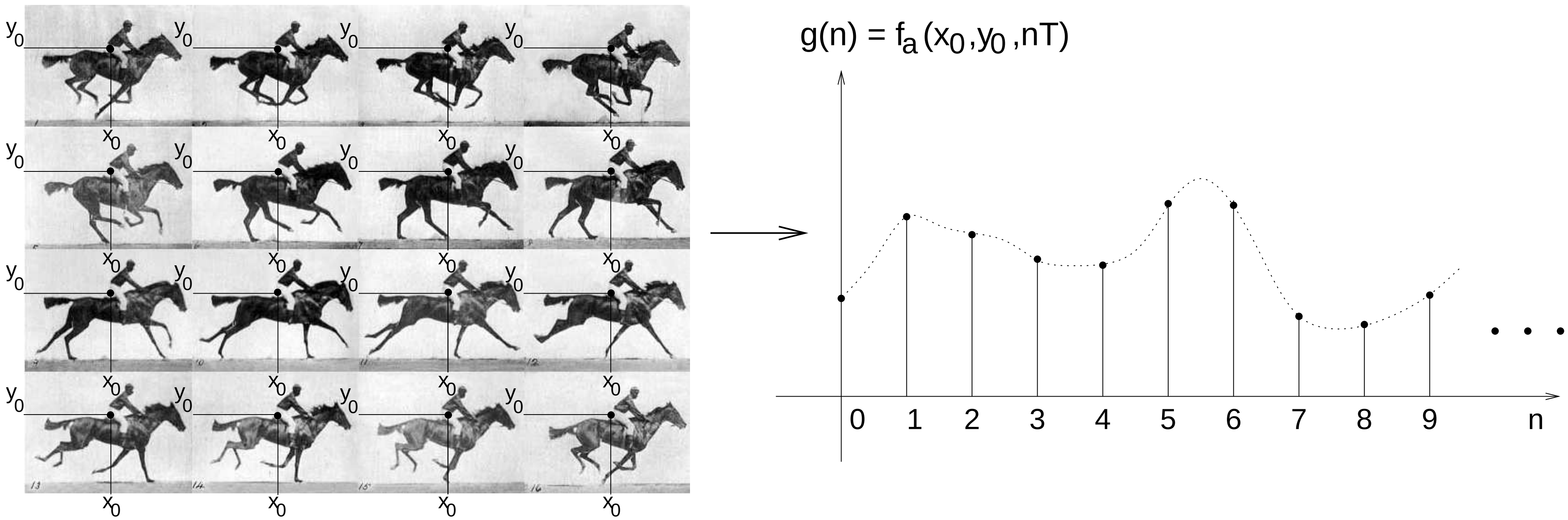
When one takes photographs of a moving picture every T time units, the three-dimensional signal is sampled in time, generating the discrete-time signal

$$f(x, y, n) = f_a(x, y, nT) \tag{192}$$

The discrete-time one-dimensional signal corresponding to the intensity of the point (x_0, y_0) is then

$$g(n) = g_a(nT) = f(x_0, y_0, n) = f_a(x_0, y_0, nT) \tag{193}$$

as represented in the next figure.



- In order to avoid aliasing, one should sample the signal $g_a(t)$ with a sampling frequency larger than twice its bandwidth.

- Supposing that the bandwidth of $g_a(t) = f_a(x_0, y_0, t)$ is W_{x_0,y_0}, then in order to avoid aliasing the time interval between photographs should be determined by the largest time bandwidth among all coordinates of the picture, that is,

$$T < \frac{2\pi}{\max_{x_0,y_0}\{2W_{x_0,y_0}\}} \tag{194}$$

Sampling of continuous-time signals

- It is reasonable to think that the faster the objects in a scene move, the larger is the time bandwidth of the light intensity of its points.
- Therefore, the above equation says that, in order to avoid aliasing, one should film the scene by taking enough pictures per second such that the time interval between them satisfies a maximum constraint.
  - For example, when filming a hummingbird, if there is too long an interval between photographs, it might flap its wings many times between photographs.
  - Depending on the speed one takes the photographs, its wings may even be approximately in the same position in every photograph.
    - This would have the effect that its wings would appear static when playing the film.
  - This is a good example of the aliasing that occurs when a three dimensional space-time signal is inadequately sampled in time.

Sampling of continuos-time signals

- We can see that the process of filming, that is, taking pictures of a moving scene equally spaced in time, is equivalent to the “Sample and hold” and “Encoder” blocks.
- It is interesting to note that in this photograph-shooting context there can not be any anti-aliasing filter.
  - This is so because one can not change the way a scene varies in time before filming it.
  - Therefore, depending on how fast a scene is moving, aliasing can not be avoided.

Sampling continuous-time signals

- Through the sampling process, one is able to represent a two-dimensional moving picture (a three-dimensional signal) as a sequence of photographs.
  - These photographs can be stored and processed.
  - After that, one has to display a moving picture from these photographs.
- In the cinema, each photograph has to be in the form of a transparency.
- With the help of a system of lenses, one transparency can be projected on a screen by turning on a lamp behind it.
- In order to display a sequence of transparencies on the screen, there must be a way of replacing the transparency currently in front of the lamp by the next one.
  - It is easier to do this if the lamp is turned off during the transparency replacement process.
  - This is the same as having a flashing light source projecting the transparencies on the screen such that a different transparency is in front of the light source each time it flashes.

Sampling continuous-time signals

- By modeling the intensity of a flashing light as an impulse, a light flashing periodically can be represented by a train of impulses, allowing one to depict the moving-picture displaying process as given in Figure 3.
  - There, the light intensity of a given point on the screen has been modeled as a train of impulses, each impulse having as amplitude the intensity of the point at a given time.
- If the light field projected on the screen is $f_i(x, y, t)$, we have that

$$f_i(x,y,t) = \sum_{n=-\infty}^{\infty} f_a(x,y,nT)\delta(t-nT) \tag{195}$$

- Thus, using equation (193), the intensity at point (x_0, y_0) is

$$g_i(t) = f_i(x_0,y_0,t) = \sum_{n=-\infty}^{\infty} f_a(x_0,y_0,nT)\delta(t-nT) = \sum_{n=-\infty}^{\infty} g(n)\delta(t-nT) \tag{196}$$

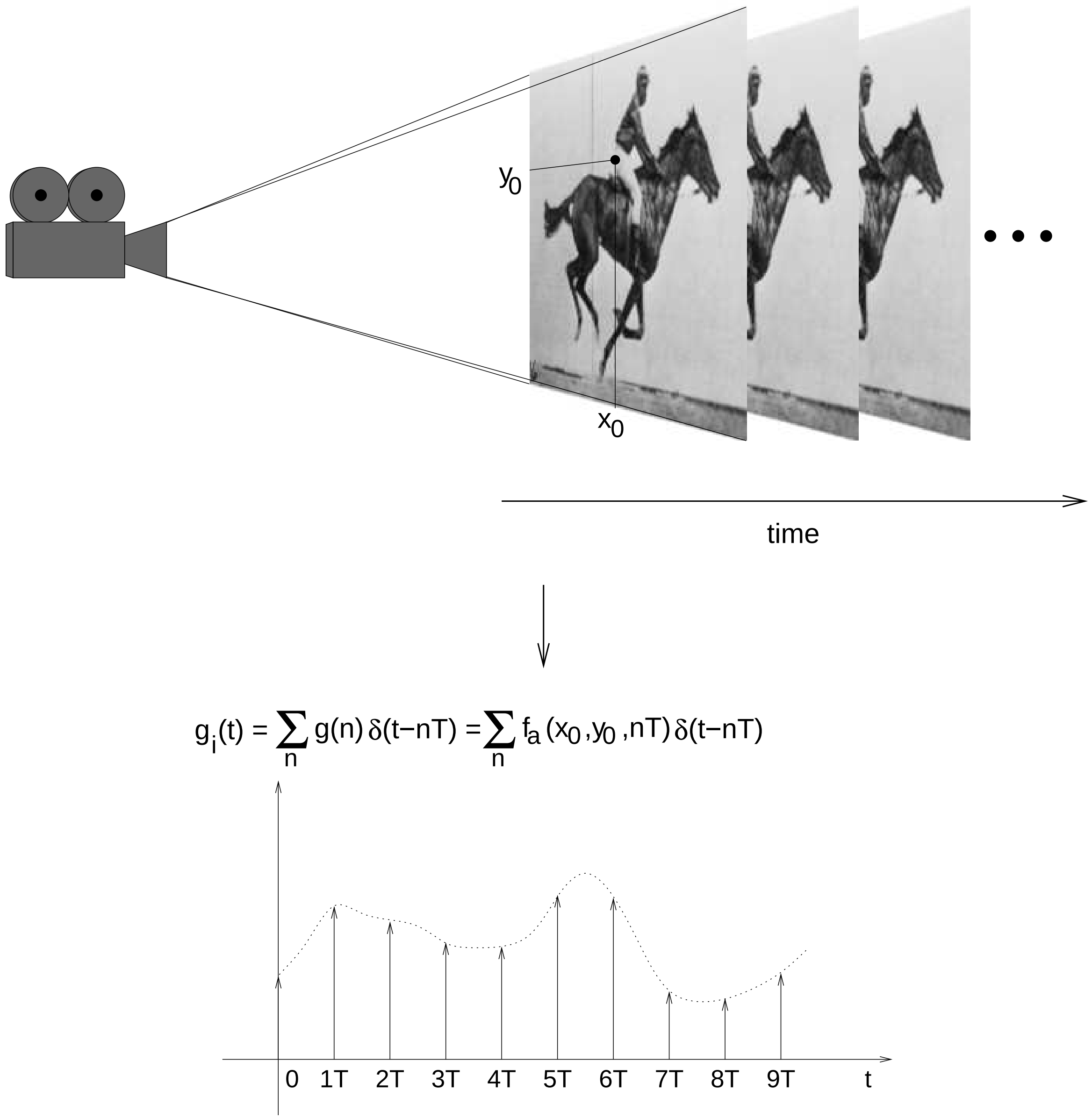
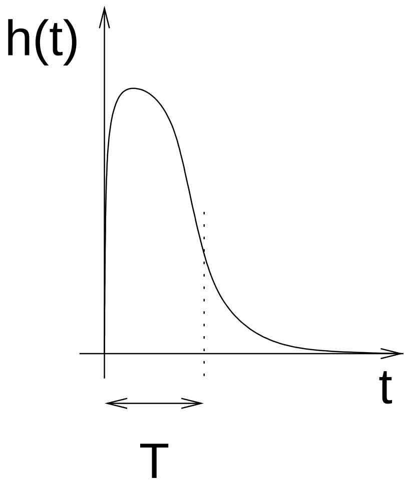


Figure 3: The film projector is equivalent to replacing the intensity of each sample in a picture by an impulse.

Sampling of continuous-time signals

- The human visual system enjoys a property called persistence of vision.
- In a nutshell, if a light is flashed before your eyes, persistence of vision is the reason why you keep seeing the light for some time after it goes off.
  - This is depicted below, where $h(t)$ represents the human visual system response to a flash of light at $t = 0$.



Sampling of continuous-time signals

- Due to the persistence of vision, when you look at a sequence of pictures flashing in the screen, you do not actually see the pictures flashing provided that the flashes are frequent enough.
  - Instead, you have the impression that the picture is moving continuously.
- In mathematical terms, the function $h(t)$ in Figure 142 is the response of the human visual system to an impulse of light $\delta(t)$.

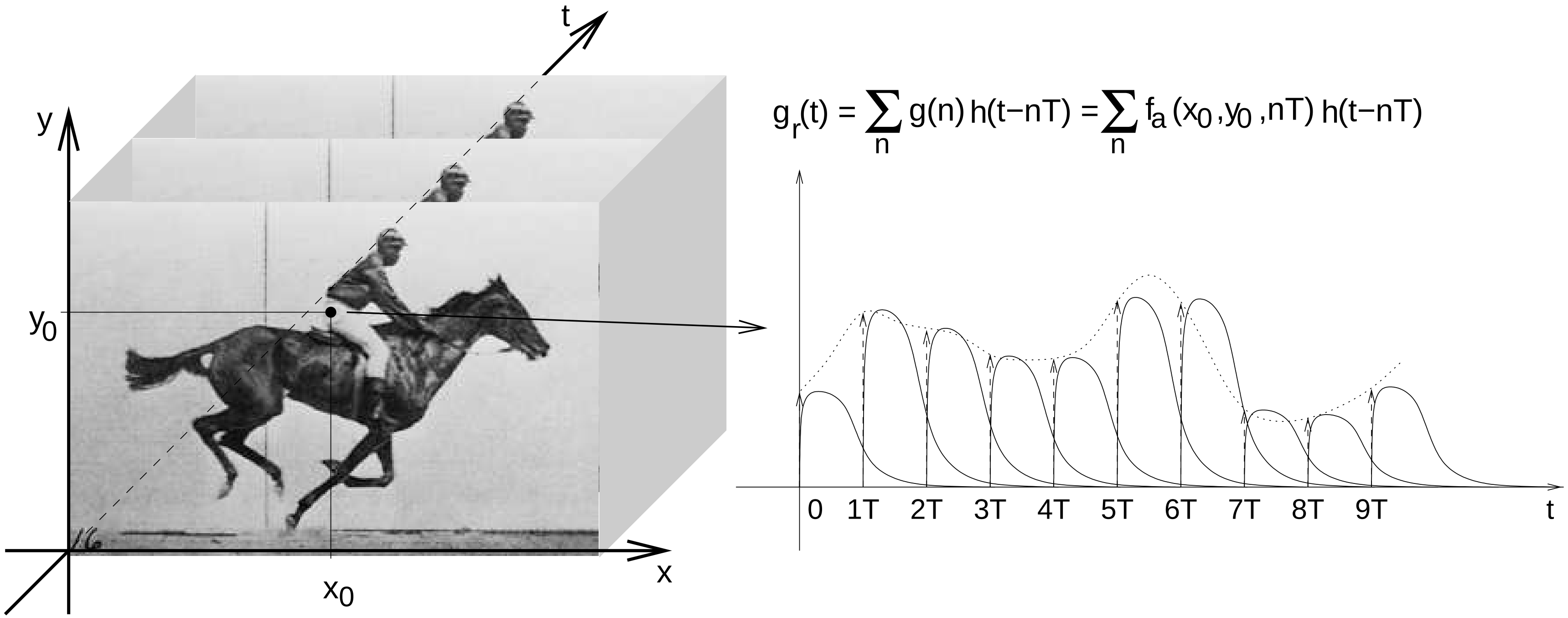
$\Rightarrow$ From equation (196), the response of the human visual system to a point (x_0, y_0) flashing on the cinema screen is given by

$$g_r(t) = g_i(t)*h(t) = \sum_{n=-\infty}^{\infty} g(n)h(t-nT) = \sum_{n=-\infty}^{\infty} f_a(x,y,nT)h(t-nT) \tag{197}$$

which is equivalent to equations (178) and (179).

Sampling of continuous-time signals

- By referring to Figure 144, the human visual system replaces each impulse by a function $h(t)$, thus perceiving the sequence of light impulses as the right-hand side signal in the figure below.



Sampling of continuous-time signals

With this interpretation, we can make interesting design considerations about the cinema system.

- The time interval between photographs is limited by the aliasing effect.
- In a regular signal processing system, one would choose the impulse response of the reconstructing filter according to equations (176) and (177),

$$h(t) = \frac{T\sin(\Omega_{LP}t)}{\pi t} \qquad H(j\Omega) = \begin{cases} T, & \text{for } |\Omega| < \Omega_{LP} \\ 0, & \text{for } |\Omega| \geq \Omega_{LP} \end{cases}$$

  so that the spectrum repetitions are filtered out from the train of impulses and the original analog signal is recovered.

Sampling of continuous-time signals

- However, in cinema, the impulse response of the reconstructing filter is given by the persistence of vision, that is a physiological characteristic, and thus can not be changed.
  - Specifically in modern cinema, in order to avoid aliasing in most scenes of interest, it is enough that we take pictures at a rate of 24 pictures per second, that is, $T \leq 1/24$ s.
  - Therefore, in order to filter out the spectrum repetitions, the bandwidth of the time impulse response of the human visual system $h(t)$ should be $\Omega_{LP} < 24$ Hz.
  - However, psychovisual tests have determined that, for the human visual system, $h(t)$ is a lowpass function with $\Omega_{LP} \approx 48$ Hz.

  $\Rightarrow$ With this natural cutoff frequency, the spectrum repetitions can not be filtered out, and thus one loses the impression of a continuous movement.
  - In this case, one perceives that the pictures are flashing, a phenomenon referred to as flickering.

Sampling of continuous-time signals

- In order to avoid this, there are two options:

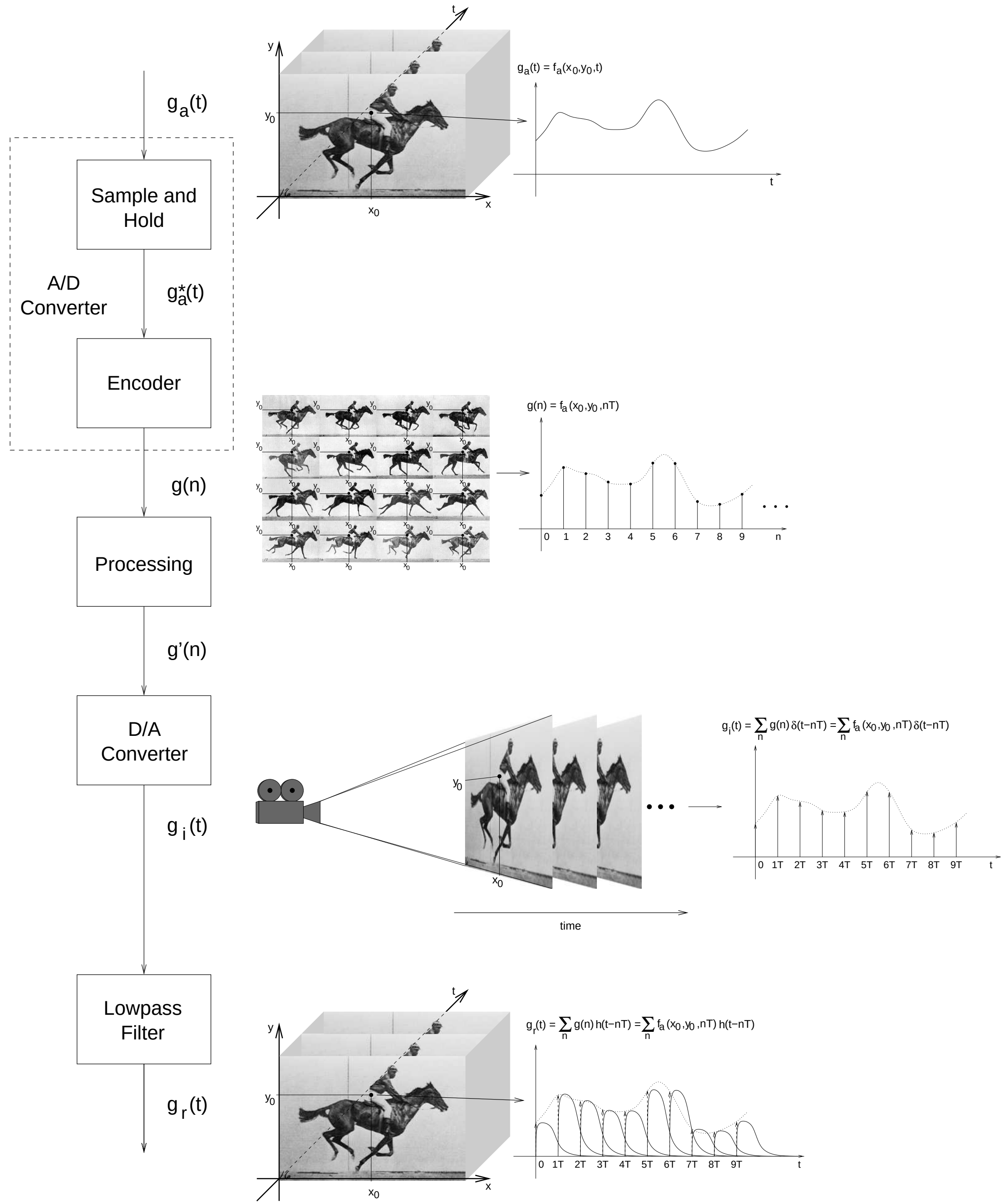
  1. To halve the sampling period, matching T to $\Omega_{\rm LP} = 48$ Hz.

     – This solution has the disadvantage of requiring twice as much pictures as the ones required to avoid aliasing. This is not cost effective, since only 24 pictures per second are enough to avoid aliasing.

  2. The solution employed by modern cinema, that is to repeat each picture twice so that the interval between pictures is $1/48$ s.

     – This procedure avoids aliasing allowing one to filter out the spectrum repetitions

The whole movie generating process is summarized in the next figure.



$g_a(t)$
Sample and Hold
A/D Converter
$g_a^*(t)$
Encoder
$g(n)$
Processing
$g'(n)$
D/A Converter
$g_i(t)$
Lowpass Filter
$g_r(t)$
$g_a(t) = f_a(x_0, y_0, t)$
$g(n) = f_a(x_0, y_0, nT)$
$g_i(t) = \sum_n g(n)\,\delta(t-nT) = \sum_n f_a(x_0, y_0, nT)\,\delta(t-nT)$
time
$g_r(t) = \sum_n g(n)\,h(t-nT) = \sum_n f_a(x_0, y_0, nT)\,h(t-nT)$

Random signals

- In nature, we are commonly forced to work with signals whose waveforms are not exactly known at each time instant.
- Some examples may include the outcome of a die, the suit of a card randomly drawn, a resistor value, or a stock price at a particular time.
- In such cases, even without knowing the exact signal value, one can still extract valuable information about the process of interest using the mathematical tools presented in this section.

Random variable

- A random variable is a mapping of the result of an experiment onto the set of real numbers.
- By doing so, we can extract numerical information about the problem at hand, as indicated below.
- The cumulative distribution function (CDF) is determined by the probability of a given random variable X to be less than or equal to a particular value x, that is

$$F_X(x) = P\{X \leq x\} \tag{198}$$

  where $P\{\mathcal{E}\}$ denotes the probability of the event $\mathcal{E}$.

- The corresponding probability density function (PDF) is given by

$$f_X(x) = \frac{dF_X(x)}{dx} \tag{199}$$

Random variable

- From their definitions, it is simple to infer that the CDF and PDF present the following properties:
  - $\lim_{x \to -\infty} F_X(x) = 0$
  - $\lim_{x \to +\infty} F_X(x) = 1$
  - $F_X(x)$ is a nondecreasing function with x, such that $0 \leq F_X(x) \leq 1$
  - $f_X(x) \geq 0$, for all x
  - $\int_{-\infty}^{\infty} f_X(x) dx = 1$
- These two functions unify the statistical treatment of both discrete- and continuous-valued random variables, despite the fact that each variable type has probability functions with different characteristics (discrete random variables, for instance, present impulsive PDFs).

Random variable

- There is a plethora of PDFs in the associated literature to characterize all kinds of random phenomena.
- For our purposes, the most interesting PDFs are the uniform (continuous $u_{X,c}(x)$ and discrete $u_{X,d}(x)$) and Gaussian $\phi_X(x)$ distributions, defined as

$$u_{X,c}(x) = \begin{cases} \dfrac{1}{a-b}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases} \tag{200}$$

$$u_{X,d}(x) = \begin{cases} \dfrac{1}{M}\delta(x - x_i), & x_i = x_1, x_2, \ldots, x_M \\ 0, & \text{otherwise} \end{cases} \tag{201}$$

$$\phi_X(x) = \frac{1}{\sqrt{2\pi\overline{\sigma}^2}} \mathrm{e}^{-\frac{(x-\overline{\mu})^2}{2\overline{\sigma}^2}} \tag{202}$$

respectively, where a, b, M, $\overline{\mu}$, and $\overline{\sigma}^2$ are the corresponding parameters for these distributions.

Random variable

- Using the PDF, we can extract meaningful measures that characterize the statistical behavior of a given random variable.
- In particular, we define the ith-order moment as

$$E\{X^i\} = \int_{-\infty}^{\infty} x^i f_X(x) dx \tag{203}$$

where the two most important special cases are:

  - The first-order moment (statistical mean or statistical expectation) $\mu_X = E\{X\}$
  - The second-order moment (energy or mean-squared value) $E\{X^2\}$.

- If Y is a function of the random variable X, that is, $Y = g(X)$, for the first-order moment of Y, we can write that

$$\mu_Y = E\{Y\} = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx \tag{204}$$

Random variable

- We are often interested in measures that are not influenced by a random variable's mean value.
- For these cases, we may also define the so-called ith-order central moments as

$$E\{(X-\mu_X)^i\} = \int_{-\infty}^{\infty} (x-\mu_X)^i f_X(x)dx \tag{205}$$

- By far, the most important central moment is the second-order one, also known as the variance $\sigma_X^2 = E\{(X-\mu_X)^2\}$, the squared-root of which is referred to as the standard deviation.
  - Larger variances indicate that the value of the random variable is more spread around the mean, whereas smaller σ_X^2 occurs when the values of X are more concentrated around its mean.
  - A simple algebraic development indicates that

$$\sigma_X^2 = E\{(X-\mu_X)^2\} = E\{X^2\} - 2E\{X\mu_X\} + \mu_X^2 = E\{X^2\} - \mu_X^2 \tag{206}$$

Random Variable

Example 1.16

Determine the values for the statistical mean, energy, and variance of a random variable X characterized by the discrete uniform PDF given in equation (201) with $x_i = 1, 2, \ldots, M$.

Solution

Using the corresponding definitions, where the integral operations on the delta functions degenerate in simple sums, we get

$$\mu_X = \sum_{i=1}^{M} i \frac{1}{M} = \frac{1}{M}\left[\frac{1}{2}M(M+1)\right] = \frac{M+1}{2} \tag{207}$$

$$E\{X^2\} = \sum_{i=1}^{6} i^2 \frac{1}{M} = \frac{1}{M}\left[\frac{M(M+1)(2M+1)}{6}\right] = \frac{2M^2+3M+1}{6} \tag{208}$$

The variance can be determined by the relationship expressed in equation (206), yielding

$$\sigma_X^2 = \frac{M^2 - 1}{12} \tag{209}$$

- All statistics presented above are based on the PDF of the random variable. In that sense, we are exchanging the requirement of knowing the value of a variable by the requirement of knowing its PDF.

- This may seem like trading one problem for another. However, in several cases, even though we are not able to determine the value of X, we can still determine its PDF.
  - One simple example is the outcome of casting a fair die, which can not be predicted but whose associated PDF is easily determined.
  - This allows one to know important statistics associated to the event, as illustrated in this example by considering $M = 6$.

$\triangle$

Random variable

When dealing with two random variables X and Y simultaneously, we may define the joint CDF by

$$F_{X,Y}(x, y) = P\{X \leq x, Y \leq y\} \tag{210}$$

and the corresponding joint PDF as

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} \tag{211}$$

The two variables are said to be statistically independent if we may write

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \tag{212}$$

where independence indicates that the outcome of a given variable does not affect the value of the other.

Random variable

The concept of moments is easily extended to two random variables X and Y by defining the joint moment of order (i, j) as

$$E\{X^i Y^j\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^i y^j f_{X,Y}(x, y) dx dy \tag{213}$$

and the joint central moment of order (i, j) as

$$E\{(X - \mu_X)^i (Y - \mu_Y)^j\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)^i (y - \mu_Y)^j f_{X,Y}(x, y) dx dy \tag{214}$$

Random variable

When the orders are $(1, 1)$, we get the cross-correlation $r_{X,Y}$ and the cross-covariance $c_{X,Y}$ between X and Y, that is

$$\begin{aligned} r_{X,Y} &= E\{XY\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy \qquad (215) \\ c_{X,Y} &= E\{(X-\mu_X)(Y-\mu_Y)\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-E\mu_Y)f_{X,Y}(x,y)dxdy \qquad (216) \end{aligned}$$

- In the case of complex random variables, the cross-correlation is given by $r_{X,Y} = E\{XY^*\}$, where the superscript asterisk denotes the complex conjugation operation.

- For the sake of simplicity, the remaining of this chapter is restricted to real signals.

Random processes

- A random process is an ordered collection of random variables.
  - The most common form of ordering the set of random variables is associating each of them with different time-instants, giving rise to the common interpretation of a random process as a set of random variables ordered in time.
- Consider the complete set of utterances of the vowel $/A/$ by a particular person:
  - We refer to the mth discrete-time utterance as $a_m(n)$, for $m = 1, 2, \ldots$
  - For a given m it can be regarded as a sample or a realization of the random process $\{A\}$.
  - The entire set of realizations is referred to as the ensemble.
  - In the context of signal processing, a realization may also be referred to as a random signal.

– If we consider a particular time instant n_1, the value of each random signal at n_1 defines the random variable $A(n_1)$, that is,

$$A(n_1) = \{a_1(n_1), a_2(n_1), \ldots\} \tag{217}$$

– Naturally, for the whole random process $\{A\}$ we can define an infinite number of random variables $A(n)$, each one associated to a particular time instant n and all of them obeying the intrinsic order established by the time variable.

- In a random process $\{X\}$, each random variable $X(n)$ has its own PDF $f_{X(n)}(x)$, which can be used to determine the ith-order moments of the associated random variable.

- Similarly, two random variables $X(n_1)$ and $X(n_2)$ of the same random process have a joint PDF defined through equation (211) from its joint CDF

$$F_{X(n_1),X(n_2)}(x_1, x_2) = P\{X(n_1) \leq x_1, X(n_2) \leq x_2\} \tag{218}$$

Random processes

- Based on this function, we can define the so-called autocorrelation function of the random process,

$$
\begin{aligned}
R_X(n_1, n_2) &= r_{X(n_1),X(n_2)} \\
&= E\{X(n_1)X(n_2)\} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(n_1),X(n_2)}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2
\end{aligned}
\tag{219}
$$

- As its name indicates, the autocorrelation function represents the statistical relation between two random variables (associated to the time-instants n_1 and n_2) of a given random process.

Random processes

- A random process is called wide-sense stationary (WSS) if its mean value and autocorrelation function present the following properties:

$$E\{X(n)\} = c, \quad \text{for all } n \tag{220}$$

$$R_X(n, n+\nu) = R_X(\nu), \quad \text{for all } n, \nu \tag{221}$$

- The first relation indicates that the mean value of all random variables $X(n)$ is constant throughout the entire process.

- The second property means that the autocorrelation function of a WSS process depends only on the time interval between two random variables and not on their absolute time instants.

- Overall, the two relationships indicate that the first- and second-order statistics of the random process do not change over time, indicating the (wide-sense) stationary nature of the process from a statistical point of view.

Random processes

- If the statistics of all orders are invariant in time, the process is referred to as strict-sense stationary (SSS).
  - Hence, SSS processes are also WSS, whereas the opposite is not necessarily true.
- Since it is very hard to verify the invariance property for all orders, in practice we often work with the WSS characterization, that only requires first- and second-order invariance.
- Later, we verify that WSS random processes, for instance, can be well characterized also in the frequency domain.
- The stationarity concept can be extended to two distinct processes $\{X\}$ and $\{Y\}$
  - They are said to be jointly WSS if their first-order moments are constant for all n and if their cross-correlation function $R_{XY}(n, n+\nu) = E\{X(n)Y(n+\nu)\}$ is only a function of ν.

Random processes

Example 1.17

Consider the random process $\{X\}$ described by its mth realization as

$$x_m(n) = \cos(\omega_0 n + \Theta_m) \tag{222}$$

where Θ is a continuous random variable with uniform PDF within the interval $[0, 2\pi]$. Determine the statistical mean and the autocorrelation function for this process, verifying whether or not it is WSS.

Random processes

Solution By writing that $X(n) = g(\Theta)$, from equations (204) and (219), respectively, we get

$$\begin{aligned} E\{X(n)\} &= \int_0^{2\pi} g(\theta) f_\Theta(\theta)\, d\theta \\ &= \int_0^{2\pi} \cos(\omega_0 n + \theta) \frac{1}{2\pi}\, d\theta \\ &= \frac{1}{2\pi} \sin(\omega_0 n + \theta)\Big|_{\theta=0}^{\theta=2\pi} \\ &= \frac{1}{2\pi}\left[\sin(\omega_0 n + 2\pi) - \sin(\omega_0 n)\right] \\ &= 0, \quad \text{for all } n \end{aligned} \tag{223}$$

$$
\begin{aligned}
R_X(n_1, n_2) &= E\{\cos(\omega_0 n_1 + \Theta)\cos(\omega_0 n_2 + \Theta)\} \\
&= \frac{1}{2}E\{\cos(\omega_0 n_1 - \omega_0 n_2)\} + \frac{1}{2}E\{\cos(\omega_0 n_1 + \omega_0 n_2 + 2\Theta)\}
\end{aligned}
\tag{224}
$$

In this last development for $R_X(n_1, n_2)$, the first term is not random, and the corresponding expected value operator can be dropped, whereas for the second term, one has that

$$
\begin{aligned}
\frac{1}{2}E\{\cos(\omega_0 n_1 + \omega_0 n_2 + 2\Theta)\} &= \frac{1}{2}\int_0^{2\pi} \cos(\omega_0 n_1 + \omega_0 n_2 + 2\theta)\frac{1}{2\pi}\,d\theta \\
&= \frac{1}{8\pi}\sin(\omega_0 n_1 + \omega_0 n_2 + 2\theta)\Big|_{\theta=0}^{\theta=2\pi} \\
&= \frac{1}{8\pi}\left[\sin(\omega_0 n_1 + \omega_0 n_2 + 4\pi) - \sin(\omega_0 n_1 + \omega_0 n_2)\right] \\
&= 0
\end{aligned}
\tag{225}
$$

and thus

$$R_X(n_1, n_2) = \frac{1}{2}\cos(\omega_0 n_1 - \omega_0 n_2) = \frac{1}{2}\cos[\omega_0(n_1 - n_2)], \quad \text{for all } n_1, n_2 \tag{226}$$

Therefore, from equations (223) and (226), we conclude that the random process $\{X\}$ is WSS.

$\triangle$

Random processes

The Nth-order autocorrelation matrix $\mathbf{R}_X$ of a WSS random process $\{X\}$ is defined as

$$\mathbf{R}_X = \begin{bmatrix} R_X(0) & R_X(1) & R_X(2) & \cdots & R_X(N-1) \\ R_X(-1) & R_X(0) & R_X(1) & \cdots & R_X(N-2) \\ R_X(-2) & R_X(-1) & R_X(0) & \cdots & R_X(N-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_X(1-N) & R_X(2-N) & R_X(3-N) & \cdots & R_X(0) \end{bmatrix} \quad (227)$$

with $R_X(\nu) = E\{X(n)X(n+\nu)\}$ as before.

Random processes

- We classify a random process as ergodic if all statistics of the ensemble can be determined by averaging on a single realization across the different samples.
  - For example, when the discrete variable corresponds to time, time averaging is performed in order to determine the statistics.
- Despite being a quite strong assumption, ergodicity is commonly resorted to in situations where only a few or possibly one realization(s) of the random process is available.
  - In such cases, we may drop the realization index m, denoting the available random signal by just $x(n)$.
  - The mean value, variance, autocorrelation function, and so on, of the entire process $\{X\}$, are estimated based solely on $x(n)$.

Filtering a random signal

Consider the input-output relationship of a linear time-invariant system, described by its impulse response $h(n)$, given by the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{228}$$

If $x(n)$ is a random signal, the nature of the output signal shall also be random. Let us proceed to characterize the output signal $y(n)$ when $x(n)$ is a WSS random signal.

Filtering a random signal

Determining the mean value of $y(n)$, one gets

$$\begin{aligned} E\{y(n)\} &= E\left\{\sum_{k=-\infty}^{\infty} x(n-k)h(k)\right\} \\ &= \sum_{k=-\infty}^{\infty} E\{x(n-k)\}h(k) \\ &= E\{x(n)\}\sum_{k=-\infty}^{\infty} h(k) \end{aligned} \tag{229}$$

where we have used the facts that $h(n)$ is a deterministic signal and that $E\{x(n)\}$ is constant for all n, since $\{X\}$ is assumed to be WSS.

Filtering a random signal

- An interesting interpretation of the above equation comes from the fact that the output $w(n)$ of a system to a constant input $v(n) = c$ is

$$w(n) = \sum_{k=-\infty}^{\infty} h(k)v(n-k) = c \sum_{k=-\infty}^{\infty} h(k) \tag{230}$$

  - This means that the output of a linear system to a constant input is also constant, and equal to the input multiplied by the constant

$$H_0 = \sum_{k=-\infty}^{\infty} h(k) \tag{231}$$

    that can be regarded as the system DC gain.

  - Hence, equation (229) indicates that the statistical mean of the output signal is the mean value of the WSS input signal multiplied by the system DC gain, which is quite an intuitive result.

Filtering a random signal

The autocorrelation function of the output signal is given by

$$
\begin{aligned}
R_Y(n_1,n_2) &= E\{y(n_1)y(n_2)\} \\
&= E\left\{\left(\sum_{k_1=-\infty}^{\infty} x(n_1-k_1)h(k_1)\right)\left(\sum_{k_2=-\infty}^{\infty} x(n_2-k_2)h(k_2)\right)\right\} \\
&= E\left\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(n_1-k_1)x(n_2-k_2)h(k_1)h(k_2)\right\} \\
&= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} E\{x(n_1-k_1)x(n_2-k_2)\}h(k_1)h(k_2) \\
&= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} R_X(n_1-k_1,n_2-k_2)h(k_1)h(k_2) \qquad (232)
\end{aligned}
$$

Filtering a random signal

Since $\{X\}$ is WSS, then one may write that

$$R_Y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} R_X((n_1 - n_2) - k_1 + k_2) h(k_1) h(k_2) \tag{233}$$

which is also a function of $\nu = (n_1 - n_2)$ for all time instants n_1, n_2

$\Rightarrow$ One can conclude that if the input $\{X\}$ is WSS, then the output process $\{Y\}$ to a linear time-invariant system is also WSS.

Random signals

- The scope of random variables and processes is quite vast, and this section has only presented the tip of the iceberg on these matters.
- The main aspect that should be kept in mind is that even though it may not be feasible to determine the exact value of a given signal, one can still determine its PDF or autocorrelation function, and extract many pieces of information (statistical mean, variance, stationarity behavior, and so on) about it using the methodology introduced here.
- In several cases, the results from these analyzes are all that one needs to describe a given signal or process, as will be verified in several parts of this book.

Do-it-yourself: discrete-time signals and systems

Experiment 1.1:

In Example 1.7, we were able to determine a closed-form expression for the impulse response associated to the difference equation

$$y(n) - \frac{1}{\alpha} y(n-1) = x(n)$$

A numerical solution for this problem can be determined, for $\alpha = 1.15$ and $0 \leq n \leq 30$, using the MATLAB commands:

```
alpha = 1.15; N = 30;
x = [1 zeros(1,N)];
y = filter(1,[1 -1/alpha],x);
stem(y);
```

which yield the plot seen in Figure 4.

Do-it-yourself: discrete-time signals and systems

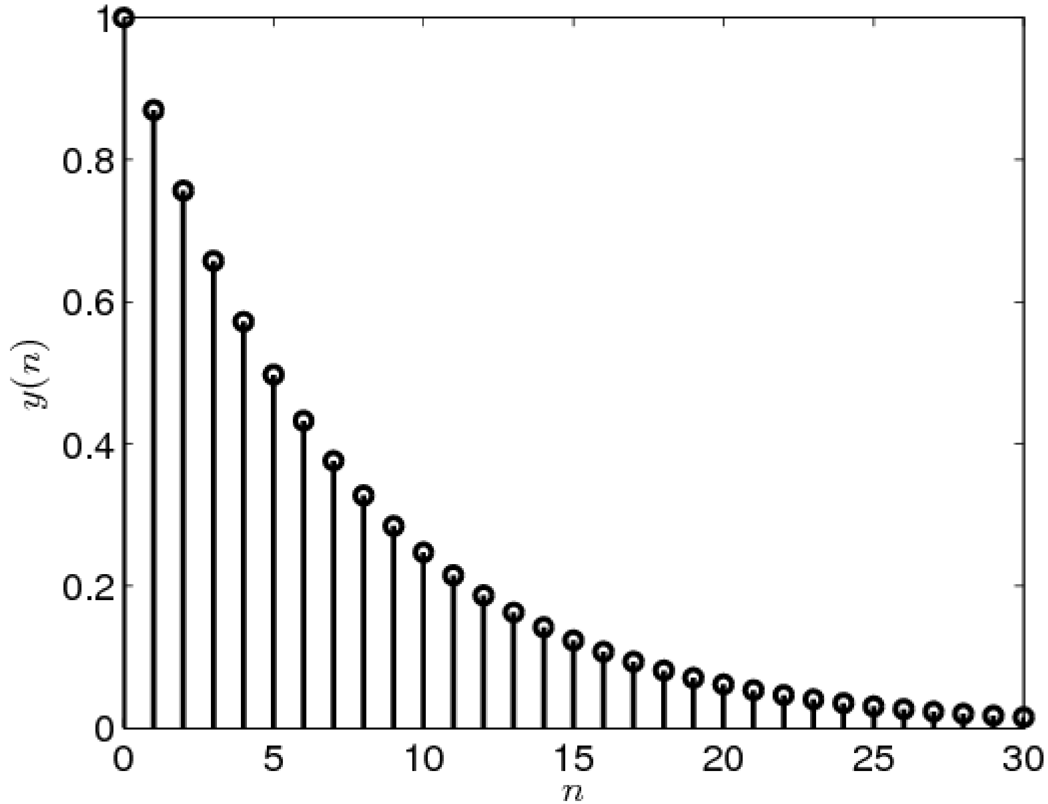


Figure 4: Solution of difference equation in Experiment 1.1.

Do-it-yourself: discrete-time signals and systems

In general, the MATLAB command

```
y = filter([b_0 b_1 ... b_M],[1 a_1 ... a_N],x,Z_i);
```

determines the solution of the general difference equation

$$y(n)+a_1y(n-1)+\ldots+a_Ny(n-N) = b_0x(n)+b_1x(n-1)+\ldots+b_Mx(n-M) \tag{234}$$

when the input signal is provided in the vector `x` and the vector `Z_i` contains its initial conditions.

Do-it-yourself: discrete-time signals and systems

Experiment 1.2:

The process of sampling can be seen as a mapping of a continuous-time function into a set of discrete-time samples. In general, however, there are infinite functions that can generate the same set of samples.

To illustrate such a notion, consider a general function $f_1(t)$. Using a sampling frequency of f_s samples per second, the sampling process yields the discrete-time function $f_1(nT_s)$, with $T_s = 1/f_s$ and integer n.

Sampling any function of the form $f_2(t) = f_1(\alpha t)$, with any positive α, using a sampling frequency $f_s' = \alpha f_s$, we get $T_s' = 1/f_s' = T_s/\alpha$. Hence, $f_2(nT_s') = f_1(\alpha(nT_s')) = f_1(nT_s)$, which corresponds to the same set of samples as before.

Do-it-yourself: discrete-time signals and systems

Therefore, in general, a given set of samples does not specify the original continuous-time function in a unique way. To reduce this uncertainty, we must specify the sampling frequency employed to generate the given samples. By doing so, the algebraic reasoning above breaks down and we eliminate (almost) all continuous-time candidate functions for a given sample set. There is, however, one last candidate that must be eliminated to avoid duplicity. Let us illustrate such case by emulating a sampling procedure using MATLAB.

Consider the $3-$Hz cosine function $f_1(t) = \cos(2\pi 3t)$ sampled at $F_s = 10$ samples per second, for a $1-$s time interval, using the MATLAB command:

```
time = 0:0.1:0.9;
f_1 = cos(2*pi*3.*time)
```

An identical sample list can be obtained, with the same sampling rate and time interval, from a $7-$Hz cosine function $f_2(t) = \cos(2\pi 7t)$ as given by:

```
f_2 = cos(2*pi*7.*time)
```

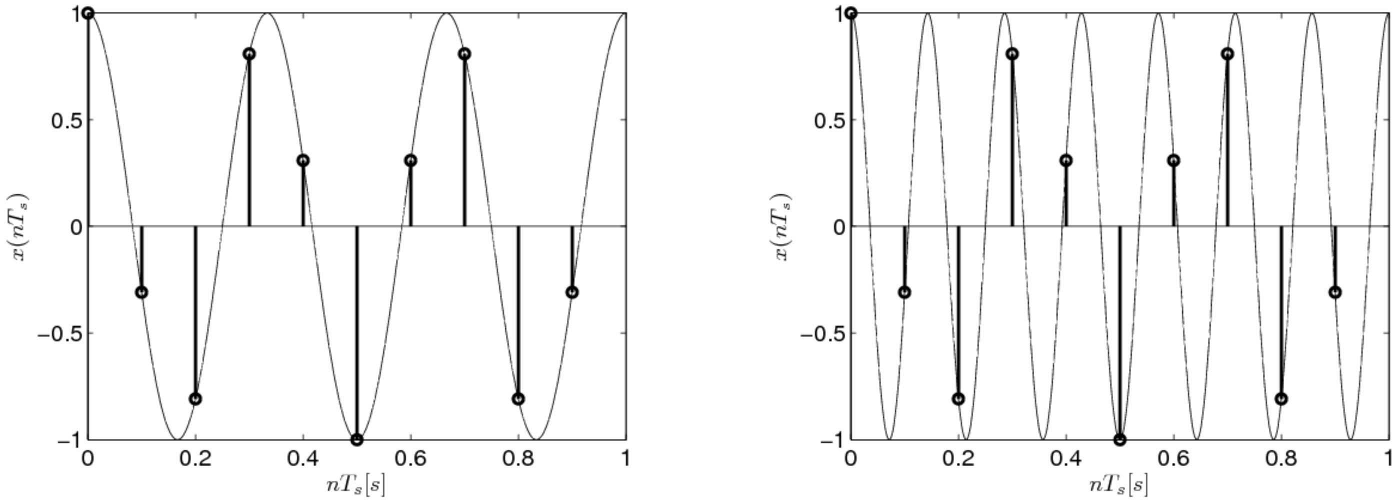
with the variable `time` specified as above.

Do-it-yourself: discrete-time signals and systems

The resulting samples are visualized in Figure 5, generated by the commands:

```
time_aux = 0:0.001:(1-0.001);
figure(1);
stem(time,f_1);
hold on;
plot(time_aux, cos(2*pi*3.*time_aux));
hold off;
figure(2);
stem(time,f_2);
hold on;
plot(time_aux, cos(2*pi*7.*time_aux));
hold off;
```

In this sequence, the `hold on` commands allow plotting more than one function in the same figure and the `time_aux` variable is used to emulate a continuous time counter in the `plot` commands to draw the background functions.



(a) (b)

Figure 5: Sampling of cosine functions of frequency f using $F_s = 10$ samples per second: (a) $f = 3$ Hz; (b) $f = 7$ Hz.

To eliminate the duplicity illustrated in Figure 5, we must refer to the Sampling Theorem. That result indicates that a $7-$Hz cosine function shall not be sampled with $F_s = 10$ Hz, since the minimum sampling frequency in this case should be above $F_s = 14$ Hz. Therefore, if Nyquist's sampling criterion is satisfied, there is only one continuous-time function associated to a given set of discrete-time samples and a particular sampling frequency.

Do-it-yourself: Discrete-time signals and systems

Experiment 1.3:

Suppose the signal $x(t) = 5\cos(2\pi 5t) + 2\cos(2\pi 50t)$, sampled with $F_s = 1000$ samples per second, as shown in Figure 6a, is corrupted by a small amount of noise, forming the signal shown in Figure 6b generated by the commands:

```
amplitude_1 = 5; freq_1 = 5;
amplitude_2 = 2; freq_2 = 50;
F_s = 1000; time = 0:1/F_s:(1-1/F_s);
sine_1 = amplitude_1*sin(2*pi*freq_1.*time);
sine_2 = amplitude_2*sin(2*pi*freq_2.*time);
noise = randn(1,length(time));
x_clean = sine_1 + sine_2;
x_noisy = x_clean + noise;
figure(1);
plot(time,x_clean);
figure(2);
plot(time,x_noisy);
```

Do-it-yourself: Discrete-time signals and systems

In particular, the `randn` command generates the specified number of samples of a pseudo-random signal with Gaussian distribution with zero mean and unit variance.

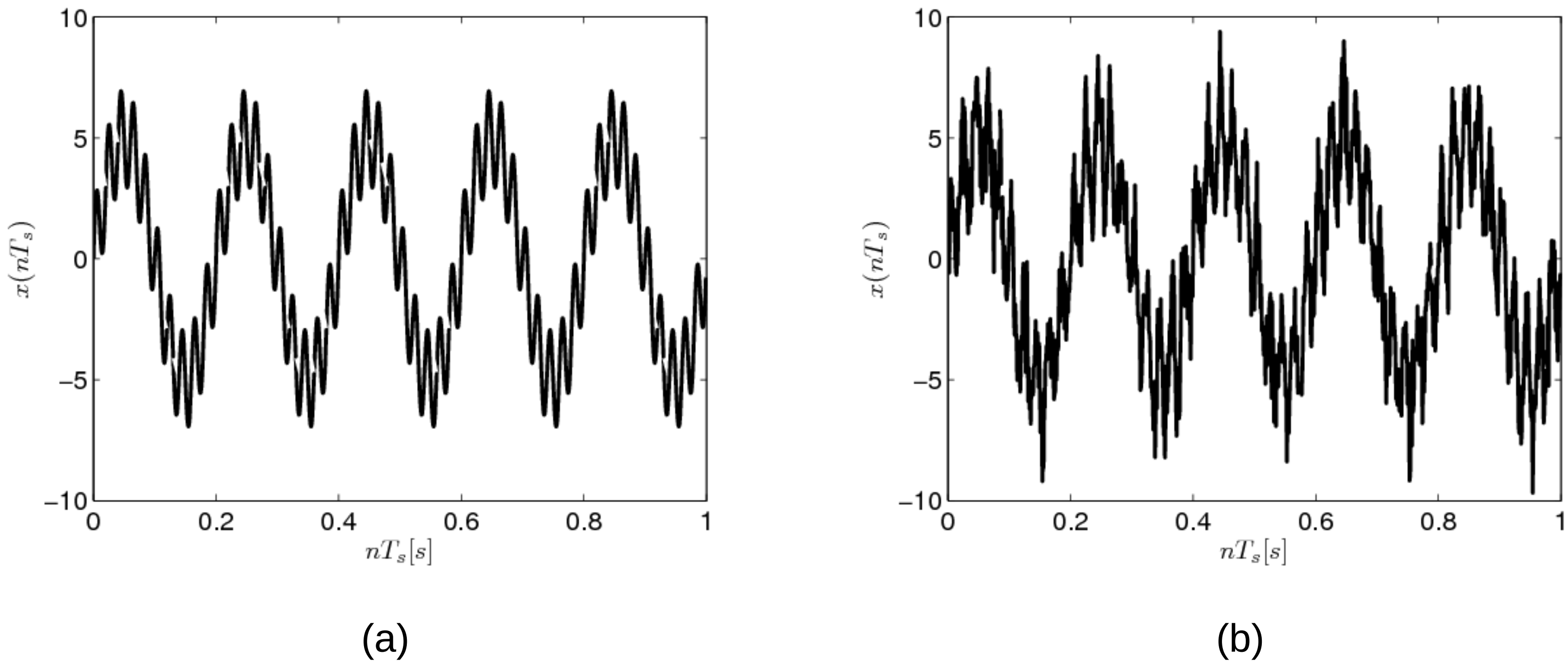


Figure 6: Sum of two sinusoidal components: (a) clean signal; (b) noisy signal.

Do-it-yourself: Discrete-time signals and systems

We can minimize the noisy effect by averaging N successive samples of $x(n) =$`x_noisy`, implementing the following difference equation

$$y(n) = \frac{x(n) + x(n-1) + \ldots + x(n-N+1)}{N} \tag{235}$$

As mentioned in Experiment .1, we can perform such processing by specifying the value of N and using the MATLAB commands

```
b = ones(1,N);
y = filter(b,1,x_noisy);
```

which yield the plots shown in Figure 7 for $N = 3$, $N = 6$, $N = 10$, and $N = 20$.

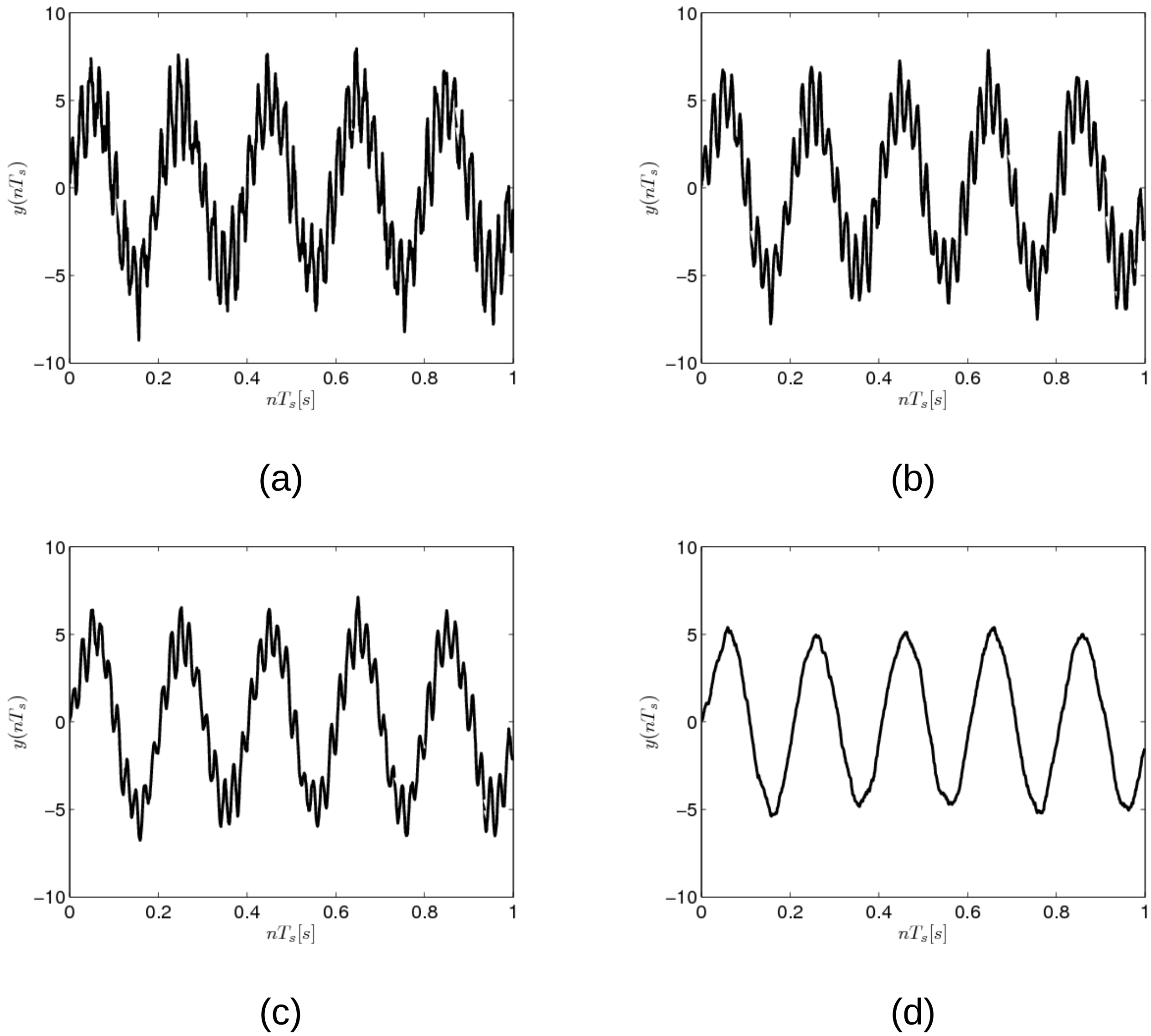


Figure 7: Averaged-in-time signals using N consecutive samples: (a) $\mathrm{N} = 3$; (b) $\mathrm{N} = 6$; (c) $\mathrm{N} = 10$; (d) $\mathrm{N} = 20$.

Do-it-yourself: Discrete-time signals and systems

- Figure 7 indicates that the averaging technique is quite effective on reducing the amount of noise from the corrupted signal.
- In this case, the larger the value of N, the higher the ability to remove the noise component.
- If, however, N is too large, as observed in Figure 7d, the averaging procedure almost eliminates the high-frequency sinusoidal component.

Do-it-yourself: Discrete-time signals and systems

- One then may argue if it is possible to reduce the noise component without affecting significantly the original signal.
  - Perhaps a more elaborate processing may preserve better the sinusoidal components.
- The theory and design of tools for answering these types of questions is the main subject of this signal processing course.
- In the following chapters several techniques for processing a wide range of signals are investigated.
- Although intuition is important in some practical situations, as illustrated in this experiment, our presentation follows a formal and technically justified path.
- In the end, the reader may be able not only to employ the methods presented throughout the book, but also to understand them, selecting the proper tool for each particular application.