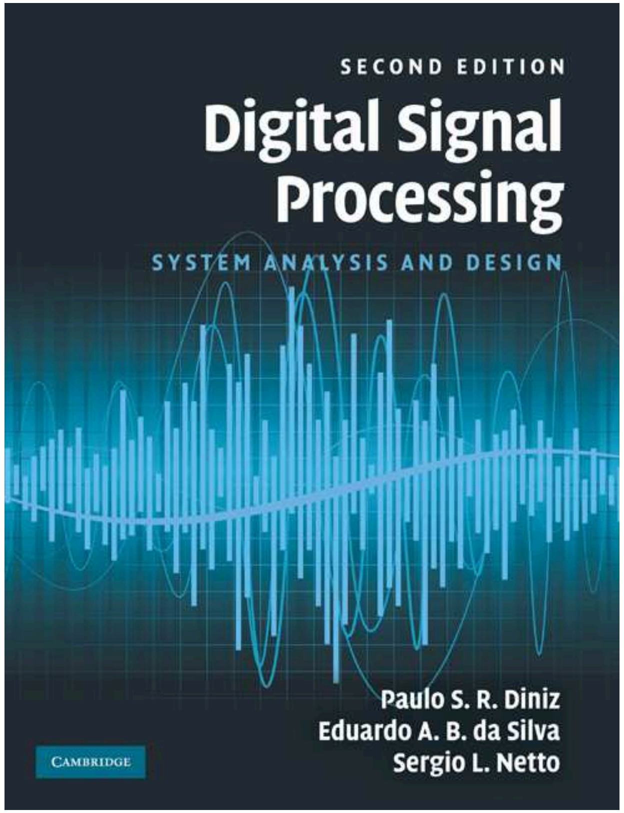
Efficient FIR Structures



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Introduction

- In this chapter, alternative realizations to those introduced in Chapter 5 for FIR filters are discussed.
- We first present the lattice realization, highlighting its application to the design of linear-phase perfect reconstruction filter banks. Then, the polyphase structure is revisited, discussing its application in parallel processing. We also present an FFT-based realization for implementing the FIR filtering operation in the frequency domain. Such a form can be very efficient in terms of computational complexity, and is particularly suitable for off-line processing, although widely used in real-time implementations. Next, the so-called recursive running sum is described as a special recursive structure for a very particular FIR filter, which has applications in the design of FIR filters with low arithmetic complexity.

Introduction

- In the case of FIR filters, the main concern is to examine methods which aim at reducing the number of arithmetic operations. These methods lead to more economical realizations with reduced quantization effects. In this chapter, we also present the prefilter, the interpolation, and the frequency response masking approaches for designing lowpass and highpass FIR filters with reduced arithmetic complexity. The frequency response masking method can be seen as a generalization of the other two schemes, allowing the design of passbands with general widths. For bandpass and bandstop filters, the quadrature approach is also introduced.

Lattice form

- Figure 1 depicts the block diagram of a nonrecursive lattice filter of order M, which is formed by concatenating basic blocks of the form shown in Figure 2.

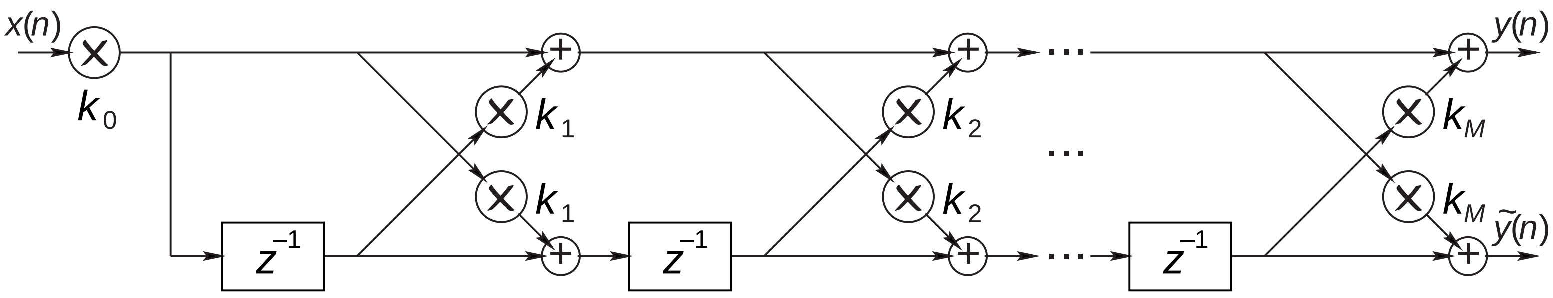

Figure 1: Lattice form realization of nonrecursive digital filters.

Lattice form

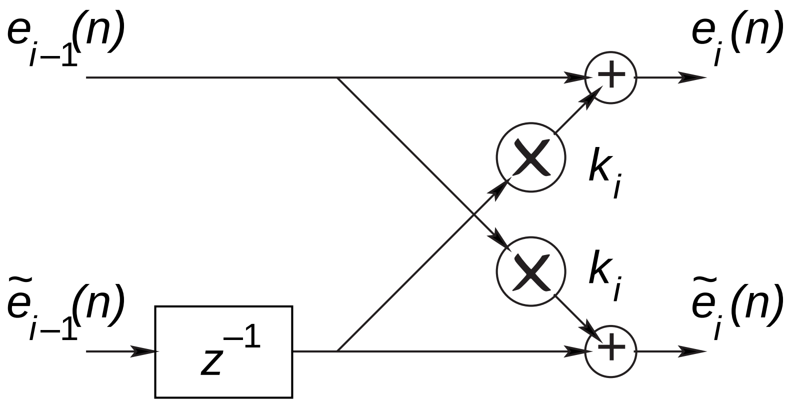

Figure 2: Basic block of the nonrecursive lattice form.

Lattice form

- To obtain a useful relation between the lattice parameters and the filter impulse response, we must analyze the recurrent relationships that appear in Figure 2. These equations are

$$e_i(n) = e_{i-1}(n) + k_i \tilde{e}_{i-1}(n-1) \tag{1}$$

$$\tilde{e}_i(n) = \tilde{e}_{i-1}(n-1) + k_i e_{i-1}(n) \tag{2}$$

for $i = 1, 2, \ldots, M$, with $e_0(n) = \tilde{e}_0(n) = k_0 x(n)$ and $e_M(n) = y(n)$.

- In the frequency domain, equations (1) and (2) become

$$\begin{bmatrix} E_i(z) \\ \tilde{E}_i(z) \end{bmatrix} = \begin{bmatrix} 1 & k_i z^{-1} \\ k_i & z^{-1} \end{bmatrix} \begin{bmatrix} E_{i-1}(z) \\ \tilde{E}_{i-1}(z) \end{bmatrix} \tag{3}$$

with $E_0(z) = \tilde{E}_0(z) = k_0 X(z)$ and $E_M(z) = Y(z)$.

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Lattice form

- Defining the auxiliary polynomials

$$N_i(z) = k_0 \frac{E_i(z)}{E_0(z)} \tag{4}$$

$$\tilde{N}_i(z) = k_0 \frac{\tilde{E}_i(z)}{\tilde{E}_0(z)} \tag{5}$$

one can demonstrate by induction, using equation (3), that these polynomials obey the following recurrence formulas:

$$N_i(z) = N_{i-1}(z) + k_i z^{-i} N_{i-1}(z^{-1}) \tag{6}$$

$$\tilde{N}_i(z) = z^{-i} N_i(z^{-1}) \tag{7}$$

for $i = 1, 2, \ldots, M$, with $N_0(z) = \tilde{N}_0(z) = k_0$ and $N_M(z) = H(z)$.

Lattice form

- Therefore, from equations (3) and (7), we have that

$$N_{i-1}(z) = \frac{1}{1-k_i^2}\left(N_i(z) - k_i z^{-i} N_i(z^{-1})\right) \tag{8}$$

for $i = 1, 2, \ldots, M$.

- Note that the recursion in equation (3) implies that $N_i(z)$ has degree i. Therefore, $N_i(z)$ can be expressed as

$$N_i(z) = \sum_{m=0}^{i} h'_{m,i} z^{-m} \tag{9}$$

Lattice form

- Since $N_{i-1}(z)$ has degree $(i-1)$ and $N_i(z)$ has degree i, in equation (8) the highest degree coefficient of $N_i(z)$ must be canceled by the highest degree coefficient of $k_i z^{-i} N_i(z^{-1})$. This implies that

$$h'_{i,i} - k_i h'_{0,i} = 0 \tag{10}$$

- Since, from equation (9), $h'_{0,i} = N_i(\infty)$, and, from equation (6), $N_i(\infty) = N_{i-1}(\infty) = \cdots = N_0(\infty) = 1$, we have that equation (10) is equivalent to $h'_{i,i} = k_i$. Therefore, given the filter impulse response, the k_i coefficients are determined by successively computing the polynomials $N_{i-1}(z)$ from $N_i(z)$ using equation (8) and making

$$k_i = h'_{i,i} \tag{11}$$

for $i = M, \ldots, 1, 0$.

Lattice form

- To determine the filter impulse response $N_M(z) = H(z) = \sum_{m=0}^{M} h_m z^{-m}$ from the set of lattice coefficients k_i, we must first use equation (6) to compute the auxiliary polynomials $N_i(z)$, starting with $N_0(z) = k_0$. The desired coefficients are then given by

$$h_m = h'_{m,M} \tag{12}$$

for $m = 0, 1, \ldots, M$.

Filter banks using the lattice form

- Structures similar to the lattice form are useful for the realization of critically decimated filter banks. They are referred to as lattice realizations of filter banks.
- As discussed in Chapter 9, using orthogonal filter banks one can only design trivial 2-band filter banks with linear-phase analysis/synthesis filters. Hence, for a more general case, the solution is to employ biorthogonal filter banks. We now show one example of such structures, which implement linear-phase 2-band perfect reconstruction filter banks.

Filter banks using the lattice form

- In order for a 2-band filter bank to have linear phase, all its analysis and synthesis filters, $H_0(z)$, $H_1(z)$, $G_0(z)$, and $G_1(z)$, must have linear phase. From Subsection 4.2.3, if we suppose that all the filters in the filter bank have the same odd order $2M+1$, then they have linear phase if

$$\left.\begin{aligned} H_i(z) &= \pm z^{-2M-1} H_i(z^{-1}) \\ G_i(z) &= \pm z^{-2M-1} G_i(z^{-1}) \end{aligned}\right\} \qquad (13)$$

for $i = 0, 1$.

Filter banks using the lattice form

- The perfect reconstruction property holds if the analysis and synthesis polyphase matrices are related by

$$\mathbf{R}(z) = z^{-\Delta}\mathbf{E}^{-1}(z) \tag{14}$$

where the analysis and synthesis polyphase matrices are defined by

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \mathbf{E}(z^2)\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \mathbf{R}^{\mathrm{T}}(z^2)\begin{bmatrix} z^{-1} \\ 1 \end{bmatrix} \tag{16}$$

Filter banks using the lattice form

- For perfect reconstruction the synthesis filters with $c = -1$ should satisfy

$$G_0(z) = z^{2(l-\Delta)} H_1(-z) \quad (17)$$

$$G_1(z) = -z^{2(l-\Delta)} H_0(-z) \quad (18)$$

- Recall from the discussions in Section 9.5 that

$$P(z) - P(-z) = H_0(z)H_1(-z) - H_0(-z)H_1(z) = 2z^{-2l-1} \quad (19)$$

- Hence, $P(z)$ has linear phase, and has to be symmetric with all odd-index coefficients equal to zero, except the central coefficient of index $2l + 1$ which must be equal to 1.

- In addition, the end terms of $P(z)$ have to be of an even index, in order to cancel in $P(z) - P(-z)$.

Filter banks using the lattice form

- All these restrictions on $P(z)$ lead to one of the following constraints on the order of the analysis filters $H_0(z)$ and $H_1(z)$ (see details in Section 9.5):
  - If both filter orders are even, they must differ by an odd multiple of 2 and the filter impulse responses must be symmetric.
  - If both orders are odd, they must differ by a multiple of 4 (which includes the case where both filters are of the same order) and one filter must be symmetric and the other antisymmetric.
  - If one filter order is even and the other is odd, one filter must be symmetric and the other antisymmetric, and the filter $P(z)$ degenerates into only 2 non-zero coefficients with all zeros on the unit circle.

Filter banks using the lattice form

- Suppose, now, that we define the polyphase matrix of the analysis filters, $\mathbf{E}(z)$, as having the following general form:

$$\mathbf{E}(z) = \mathcal{K}_M \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \left(\prod_{i=M}^{1} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & k_i \\ k_i & 1 \end{bmatrix} \right) \tag{20}$$

with

$$\mathcal{K}_M = \frac{1}{2} \prod_{i=1}^{M} \left(\frac{1}{1 - k_i^2} \right) \tag{21}$$

Filter banks using the lattice form

- If we define the polyphase matrix of the synthesis filters, $\mathbf{R}(z)$, as

$$\mathbf{R}(z) = \left(\prod_{i=1}^{M} \begin{bmatrix} 1 & -k_i \\ -k_i & 1 \end{bmatrix} \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{22}$$

we then have that

$$\mathbf{R}(z) = z^{-M}\mathbf{E}^{-1}(z) \tag{23}$$

and perfect reconstruction is guaranteed irrespective of the values of k_i.

Filter banks using the lattice form

- Also, from equations (15) and (16), as well as equations (20) and (22), we have that

$$\left.\begin{aligned} H_0(z) &= z^{-2M-1}H_0(z^{-1}) \\ H_1(z) &= -z^{-2M-1}H_1(z^{-1}) \\ G_0(z) &= z^{-2M-1}G_0(z^{-1}) \\ G_1(z) &= -z^{-2M-1}G_1(z^{-1}) \end{aligned}\right\} \tag{24}$$

and linear phase is also guaranteed irrespective of the values of k_i.

Example 12.1

- Prove by induction that the formulation of equation (20) leads to linear-phase analysis filters.

Example 12.1 - Solution

- For $M = 1$, the expression for the analysis filter bank, from equations (15) and (20), becomes

$$\begin{aligned} \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}_{M=1} &= \mathcal{K}_1 \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-2} \end{bmatrix} \begin{bmatrix} 1 & \kappa_1 \\ \kappa_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \\ &= \mathcal{K}_1 \begin{bmatrix} 1 + \kappa_1 z^{-1} + \kappa_1 z^{-2} + z^{-3} \\ -1 - \kappa_1 z^{-1} + \kappa_1 z^{-2} + z^{-3} \end{bmatrix} \end{aligned} \tag{25}$$

- Since $H_0(z)$ and $H_1(z)$ are symmetric and antisymmetric, respectively, the filter bank is linear-phase for $M = 1$. Now, in order to finish the proof, we need to show that if the filter bank is linear-phase for a given M, then it is also linear-phase for $M + 1$.

Example 12.1 - Solution

- For a lattice structure with $(M+1)$ stages, we have that

$$
\begin{aligned}
S &= \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}_{M+1} \\
&= \mathcal{K}_{M+1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \left(\prod_{i=M+1}^{1} \begin{bmatrix} 1 & k_i \\ k_i z^{-2} & z^{-2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \\
&= \mathcal{K}_{M+1} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & k_{M+1} \\ k_{M+1} z^{-2} & z^{-2} \end{bmatrix} \left(\prod_{i=M}^{1} \begin{bmatrix} 1 & k_i \\ k_i z^{-2} & z^{-2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \\
&= \frac{1}{1-k_{M+1}^2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & k_{M+1} \\ k_{M+1} z^{-2} & z^{-2} \end{bmatrix} \mathcal{K}_M \left(\prod_{i=M}^{1} \begin{bmatrix} 1 & k_i \\ k_i z^{-2} & z^{-2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}
\end{aligned} \tag{26}
$$

Example 12.1 - Solution

- Hence

$$
\begin{aligned}
S &= \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}_{M+1} \\
&= \frac{1}{1-k_{M+1}^2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & k_{M+1} \\ k_{M+1}z^{-2} & z^{-2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}_M \\
&= \frac{1}{2(1-k_{M+1}^2)} \begin{bmatrix} (1+k_{M+1})(1+z^{-2}) & (1-k_{M+1})(-1+z^{-2}) \\ (1+k_{M+1})(-1+z^{-2}) & (1-k_{M+1})(1+z^{-2}) \end{bmatrix} \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}_M \\
&= \frac{1}{2} \begin{bmatrix} \dfrac{1+z^{-2}}{1-k_{M+1}} & \dfrac{-1+z^{-2}}{1+k_{M+1}} \\ \dfrac{-1+z^{-2}}{1-k_{M+1}} & \dfrac{1+z^{-2}}{1+k_{M+1}} \end{bmatrix} \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}_M
\end{aligned} \tag{27}
$$

Example 12.1 - Solution

- Assuming that $[H_0(z)]_M$ is symmetric and $[H_1(z)]_M$ is antisymmetric of equal orders $(2M+1)$ and similar coefficients, as given in equation (25) for the case $M = 1$, we may write that

$$
\begin{aligned}
[H_0(z)]_M &= z^{-2M-1}\left[H_0(z^{-1})\right]_M && (28)\\
[H_1(z)]_M &= -z^{-2M-1}\left[H_1(z^{-1})\right]_M && (29)
\end{aligned}
$$

- Therefore, for finishing the proof, we must show that $[H_0(z)]_{M+1}$ is symmetric and $[H_1(z)]_{M+1}$ is antisymmetric. Since their orders are equal to $(2(M+1)+1) = (2M+3)$, we must show, according to equations (28) and (29), that

$$
\begin{aligned}
[H_0(z)]_{M+1} &= z^{-2M-3}\left[H_0(z^{-1})\right]_{M+1} && (30)\\
[H_1(z)]_{M+1} &= -z^{-2M-3}\left[H_1(z^{-1})\right]_{M+1} && (31)
\end{aligned}
$$

Example 12.1 - Solution

- From equation (27) we have that

$$z^{-2M-3}\begin{bmatrix} H_0(z^{-1}) \\ H_1(z^{-1}) \end{bmatrix}_{M+1} = \frac{z^{-2M-3}}{2}\begin{bmatrix} \frac{1+z^2}{1-k_{M+1}} & \frac{-1+z^2}{1+k_{M+1}} \\ \frac{-1+z^2}{1-k_{M+1}} & \frac{1+z^2}{1+k_{M+1}} \end{bmatrix}\begin{bmatrix} H_0(z^{-1}) \\ H_1(z^{-1}) \end{bmatrix}_{M} \tag{32}$$

Example 12.1 - Solution

- Substituting $[H_0(z^{-1})]_M$ and $[H_1(z^{-1})]_M$ from equations (28) and (29) in the above equation we get

$$
\begin{aligned}
P &= z^{-2M-3}\begin{bmatrix} H_0(z^{-1}) \\ H_1(z^{-1}) \end{bmatrix}_{M+1} \\
&= \frac{z^{-2M-3}}{2}\begin{bmatrix} \dfrac{1+z^2}{1-k_{M+1}} & \dfrac{-1+z^2}{1+k_{M+1}} \\ \dfrac{-1+z^2}{1-k_{M+1}} & \dfrac{1+z^2}{1+k_{M+1}} \end{bmatrix} z^{2M+1}\begin{bmatrix} H_0(z) \\ -H_1(z) \end{bmatrix}_M \\
&= \frac{1}{2}\begin{bmatrix} \dfrac{1+z^{-2}}{1-k_{M+1}} & \dfrac{1-z^{-2}}{1+k_{M+1}} \\ \dfrac{1-z^{-2}}{1-k_{M+1}} & \dfrac{1+z^{-2}}{1+k_{M+1}} \end{bmatrix}\begin{bmatrix} H_0(z) \\ -H_1(z) \end{bmatrix}_M \qquad (33)
\end{aligned}
$$

Example 12.1 - Solution

- And then

$$
\begin{aligned}
P &= z^{-2M-3} \begin{bmatrix} H_0(z^{-1}) \\ H_1(z^{-1}) \end{bmatrix}_{M+1} \\
&= \frac{1}{2} \begin{bmatrix} \dfrac{1+z^{-2}}{1-k_{M+1}} & \dfrac{-1+z^{-2}}{1+k_{M+1}} \\ -\dfrac{-1+z^{-2}}{1-k_{M+1}} & -\dfrac{1+z^{-2}}{1+k_{M+1}} \end{bmatrix} \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix}_{M} \\
&= \begin{bmatrix} H_0(z) \\ -H_1(z) \end{bmatrix}_{M+1}
\end{aligned}
\tag{34}
$$

where the last step derives from equation (27).

Example 12.1 - Solution

- Comparing the above equation with equations (30) and (31) we see that the lowpass and highpass filters for $(M+1)$ stages are also symmetric and antisymmetric, which completes the proof by induction.
- The realizations of the analysis and synthesis filters are shown in Figures 3a and 3b, respectively, for the same-order case, as developed in Example 12.1 above.

Example 12.1 - Solution

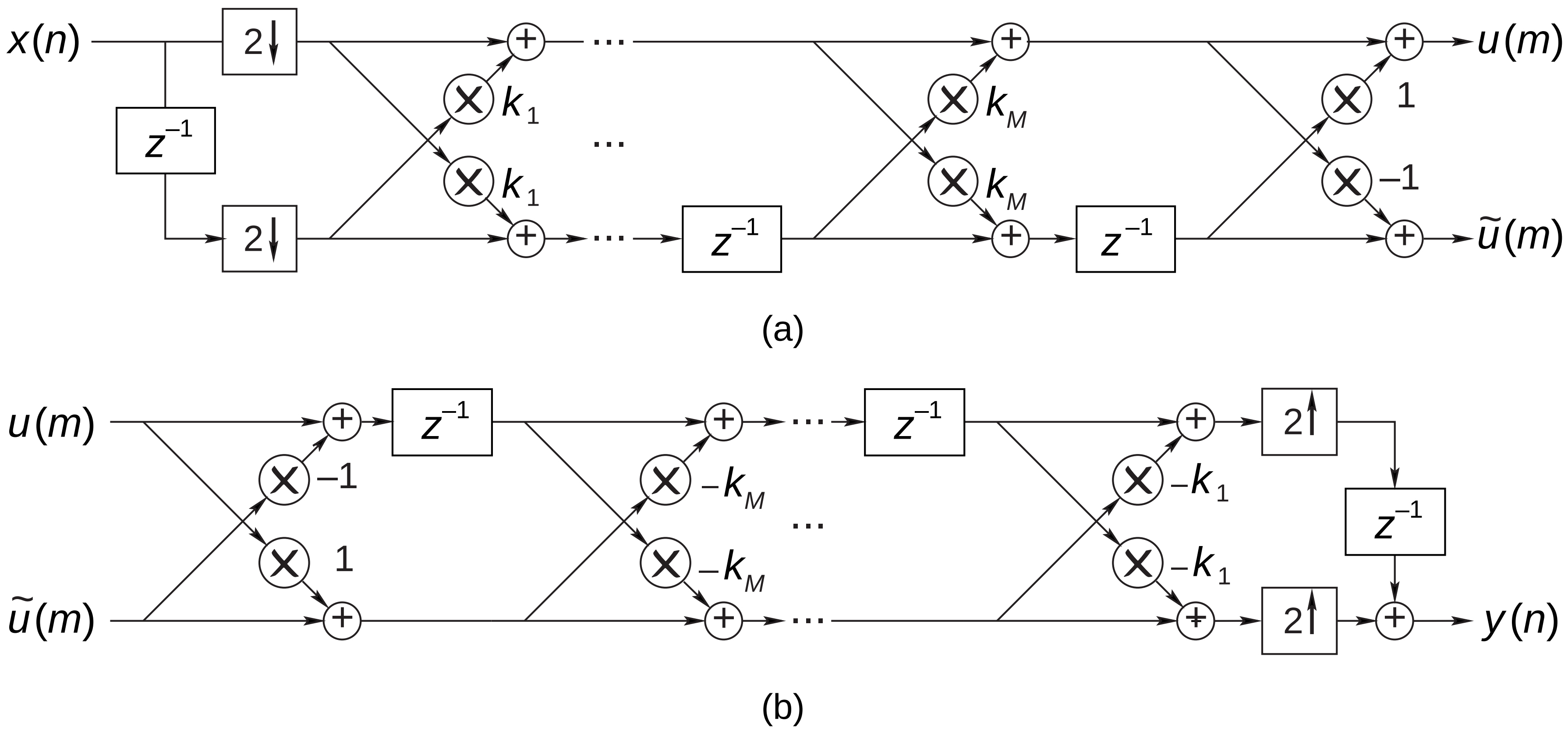


Figure 3: Lattice form realization of a linear-phase filter bank: (a) analysis filters; (b) synthesis filters.

Example 12.1 - Solution

- Obs: There are rare cases of perfect reconstruction FIR filter banks where the corresponding lattices can not be synthesized as some coefficient k_i may be equal to 1.

- As seen above, one important property of such realizations is that both perfect reconstruction and linear phase are guaranteed irrespective of the values of k_i.

- They are often referred to as having structurally induced perfect reconstruction and linear phase. Therefore, one possible design strategy for such filter banks is to carry out a multi-variable optimization on k_i, for $i = 1, 2, \ldots, M$, using a chosen objective function.

Example 12.1 - Solution

- For example, one could minimize the L_2 norm of the deviation of both the lowpass filter $H_0(z)$ and the highpass filter $H_1(z)$ in both their passbands and stopbands using the following objective function:

$$\begin{aligned} \xi(k_1, k_2, \ldots, k_M) = & \int_0^{\omega_p} \left(1 - \left|H_0(e^{j\omega})\right|\right)^2 d\omega + \int_{\omega_r}^{\pi} \left|H_0(e^{j\omega})\right|^2 d\omega \\ & + \int_{\omega_r}^{\pi} \left(1 - \left|H_1(e^{j\omega})\right|\right)^2 d\omega + \int_0^{\omega_p} \left|H_1(e^{j\omega})\right|^2 d\omega \end{aligned} \tag{35}$$

which corresponds essentially to a least-squares error function.

Example 12.2

- Implement the transfer function below using a lattice structure:

$$H_1(z) = z^{-5} + 2z^{-4} + 0.5z^{-3} - 0.5z^{-2} - 2z^{-1} - 1 \tag{36}$$

- Determine $H_0(z)$ and the synthesis filters obtained.

Example 12.2 - Solution

- Using $M = 2$ into equation (20), the polyphase matrix of the analysis filter bank is then given by

$$\begin{aligned}\mathbf{E}(z) &= \mathcal{K}_2 \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & \kappa_2 \\ \kappa_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & \kappa_1 \\ \kappa_1 & 1 \end{bmatrix} \\ &= \mathcal{K}_2 \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \kappa_2 \\ \kappa_2 z^{-1} & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & \kappa_1 \\ \kappa_1 z^{-1} & 1 \end{bmatrix} \end{aligned} \tag{37}$$

- Hence, from equation (15), the highpass filter becomes

$$H_1(z) = \mathcal{K}_2 \left[-1 - \kappa_1 z^{-1} + \kappa_2 (1 - \kappa_1) z^{-2} - \kappa_2 (1 - \kappa_1) z^{-3} + \kappa_1 z^{-4} + z^{-5} \right] \tag{38}$$

Example 12.2 - Solution

- Equating this expression to the given transfer function, one gets

$$\left.\begin{aligned} \kappa_1 &= 2 \\ \kappa_2 &= 0.5 \end{aligned}\right\} \tag{39}$$

- Note that with this lattice structure the resulting filter banks are equal to the desired $H_0(z)$ and $H_1(z)$ up to a constant, since the value of $\mathcal{K}_2$ must be given by equation (21), which, in this case, is

$$\mathcal{K}_2 = \left(\frac{1}{2}\right)\left(\frac{1}{1-2^2}\right)\left(\frac{1}{1-(0.5)^2}\right) = -\frac{2}{9} \tag{40}$$

Example 12.2 - Solution

- Therefore, the filter $H_1(z)$ becomes

$$H_1(z) = -\frac{2}{9}(-1-2z^{-1}-0.5z^{-2}-0.5z^{-3}+2z^{-4}+z^{-5}) \tag{41}$$

- The polyphase components of the highpass filter of the analysis filter bank are such that

$$\left.\begin{array}{l} E_{10}(z) = \mathcal{K}_2\left[-1-\kappa_2(\kappa_1-1)z^{-1}+\kappa_1 z^{-2}\right] = -\frac{2}{9}\left(-1-0.5z^{-1}+2z^{-2}\right) \\ E_{11}(z) = \mathcal{K}_2\left[-\kappa_1-\kappa_2(1-\kappa_1)z^{-1}+z^{-2}\right] = -\frac{2}{9}\left(-2+0.5z^{-1}+z^{-2}\right) \end{array}\right\} \tag{42}$$

Example 12.2 - Solution

- And for the lowpass filter, we have that

$$\left.\begin{array}{l} E_{00}(z) = \mathcal{K}_2\left[1+\kappa_2(1+\kappa_1)z^{-1}+\kappa_1 z^{-2}\right] = -\frac{2}{9}\left(1+1.5z^{-1}+2z^{-2}\right) \\ E_{01}(z) = \mathcal{K}_2\left[\kappa_1+\kappa_2(1+\kappa_1)z^{-1}+z^{-2}\right] = -\frac{2}{9}\left(2+1.5z^{-1}+z^{-2}\right) \end{array}\right\} \tag{43}$$

- The lowpass filter of the analysis filter bank has the following expression

$$\begin{aligned} H_0(z) &= E_{00}(z^2) + z^{-1}E_{01}(z^2) \\ &= -\frac{2}{9}\left(1+2z^{-1}+1.5z^{-2}+1.5z^{-3}+2z^{-4}+z^{-5}\right) \end{aligned} \tag{44}$$

Example 12.2 - Solution

- The determinant of the matrix $\mathbf{E}(z)$ has the following expression:

$$\begin{aligned}\det[\mathbf{E}(z)] &= E_{00}(z)E_{11}(z) - E_{10}(z)E_{01}(z) \\ &= \left(\frac{2}{9}\right)^2 \left[\left(1 + 1.5z^{-1} + 2z^{-2}\right)\left(-2 + 0.5z^{-1} + z^{-2}\right)\right. \\ &\quad \left. - \left(-1 - 0.5z^{-1} + 2z^{-2}\right)\left(2 + 1.5z^{-1} + z^{-2}\right)\right] \\ &= -\frac{2}{9}z^{-2} \end{aligned} \tag{45}$$

Example 12.2 - Solution

- As a result, from equation (23), the polyphase matrix of the synthesis filter is given by

$$\mathbf{R}(z) = \frac{z^{-2}}{\det\left[\mathbf{E}(z)\right]}\begin{bmatrix} E_{11}(z) & -E_{01}(z) \\ -E_{10}(z) & E_{00}(z) \end{bmatrix}$$
$$= \begin{bmatrix} \left(-2+0.5z^{-1}+z^{-2}\right) & \left(-2-1.5z^{-1}-z^{-2}\right) \\ \left(1+0.5z^{-1}-2z^{-2}\right) & \left(1+1.5z^{-1}+2z^{-2}\right) \end{bmatrix} \tag{46}$$

- And the synthesis filters are, from equation (16),

$$\left.\begin{aligned} G_0(z) &= 1-2z^{-1}+0.5z^{-2}+0.5z^{-3}-2z^{-4}+z^{-5} \\ G_1(z) &= 1-2z^{-1}+1.5z^{-2}-1.5z^{-3}+2z^{-4}-z^{-5} \end{aligned}\right\} \tag{47}$$

Polyphase form

- A nonrecursive transfer function of the form $H(z) = \sum_{n=0}^{N-1} h(n)z^{-n}$ when $N = K\overline{N}$ (as seen in Section 8.8), can also be expressed as

$$
\begin{aligned}
H(z) &= \sum_{n=0}^{\overline{N}-1} h(Kn)z^{-Kn} + \sum_{n=0}^{\overline{N}-1} h(Kn+1)z^{-Kn-1} + \cdots \\
&\quad + \sum_{n=0}^{\overline{N}-1} h(Kn+K-1)z^{-Kn-K+1} \\
&= \sum_{n=0}^{\overline{N}-1} h(Kn)z^{-Kn} + z^{-1}\sum_{n=0}^{\overline{N}-1} h(Kn+1)z^{-Kn} + \cdots \\
&\quad + z^{-K+1}\sum_{n=0}^{\overline{N}-1} h(Kn+K-1)z^{-Kn}
\end{aligned}
\tag{48}
$$

Polyphase form

- This last form of writing $H(z)$ can be directly mapped onto the realization shown in Figure 4, where each $H_i(z)$ is given by

$$H_i(z^K) = \sum_{n=0}^{\overline{N}-1} h(Kn+i)z^{-Kn} \tag{49}$$

for $i = 0, 1, \ldots, (K-1)$.

- Such a realization is referred to as the polyphase realization and it finds a large range of applications in the study of multirate systems and filter banks, as seen in Chapter 9.

Polyphase form

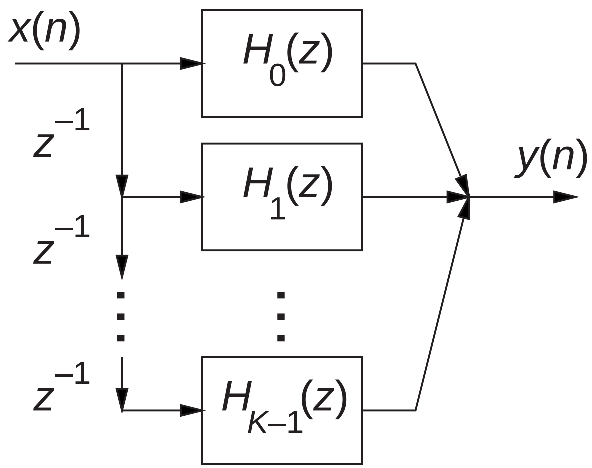


Figure 4: Block diagram of realization of equation (48).

Frequency-domain form

- The output of an FIR filter corresponds to the linear convolution of the input signal with the finite-length impulse response of the filter $h(n)$.
- Therefore, from Section 3.4 we have that, if the input signal $x(n)$ of a nonrecursive digital filter is known for all n, and null for $n < 0$ and $n > L$, an alternative approach to computing the output $y(n)$ can be derived using the fast Fourier transform (FFT).
- If the filter length is N, by completing these sequences with the necessary number of zeros (zero-padding procedure) and determining the resulting $(N + L)$-element FFTs of $h(n)$, $x(n)$, and $y(n)$, we have that

$$\text{FFT}\{y(n)\} = \text{FFT}\{h(n)\}\text{FFT}\{x(n)\} \tag{50}$$

and then

$$y(n) = \text{FFT}^{-1}\{\text{FFT}\{h(n)\}\text{FFT}\{x(n)\}\} \tag{51}$$

Frequency-domain form

- Using this approach, we are able to compute the entire sequence $y(n)$ with a number of arithmetic operations per output sample proportional to $\log_2(L+N)$.
- In the case of direct evaluation, the number of operations per output sample is of the order of L. Clearly, for large values of L and not too large values of N, the FFT method is the most efficient one.
- In the above approach, the entire input sequence must be available to allow one to compute the output signal. In this case, if the input is extremely long, the complete computation of $y(n)$ can result in a long computational delay, which is undesirable in several applications. For such cases, the input signal can be sectioned and each data block processed separately using the so-called overlap-and-save and overlap-and-add methods, as described in Chapter 3.

Recursive running sum form

- The direct realization of the transfer function

$$H(z) = \sum_{i=0}^{M} z^{-i} \tag{52}$$

  where all the multiplier coefficients are equal to one, requires a large number of additions for large M.

- An alternative way to implement such a transfer function results from interpreting it as the sum of the terms of a geometric series. This yields

$$H(z) = \frac{1 - z^{-M-1}}{1 - z^{-1}} \tag{53}$$

  This equation leads to the realization in Figure 5, widely known as the recursive running sum (RRS).

Recursive running sum form

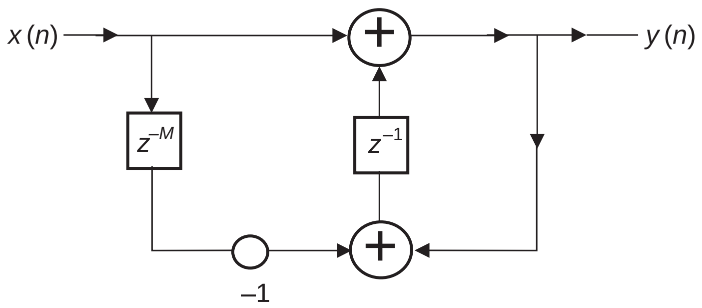


Figure 5: Block diagram of the recursive running sum (RRS) realization.

Recursive running sum form

- The RRS corresponds to a very simple lowpass filter, comprising $(M + 1)$ delays and only 2 additions per output sample. The RRS bandwidth can be controlled by appropriately choosing the value of M, as illustrated in Example 12.3 below.
- The RRS DC gain is equal to $(M + 1)$. To compensate for that, a scaling factor of $\frac{1}{(M+1)}$ should be employed at the filter input, which generates an output roundoff noise with variance of about $[(M + 1)q]^2/12$.
- To reduce this effect, one may eliminate the input scaling and perform the RRS internal computations with higher dynamic range, by increasing the internal binary wordlength accordingly.
- In this case, the output-noise variance is reduced to $q^2/12$, since the signal is quantized only at the output. To guarantee this, the number of extra bits must be the smallest integer larger than or equal to $\log_2(M + 1)$.

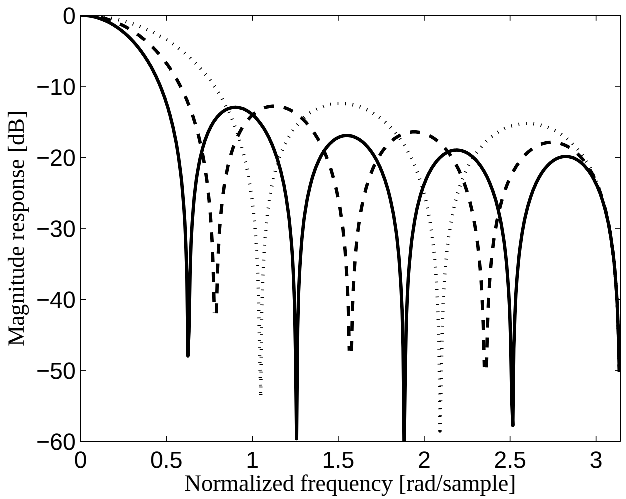
Example 12.3

- Determine the magnitude responses and pole-zero constellations for the RRS blocks with $M = 5, 7, 9$.

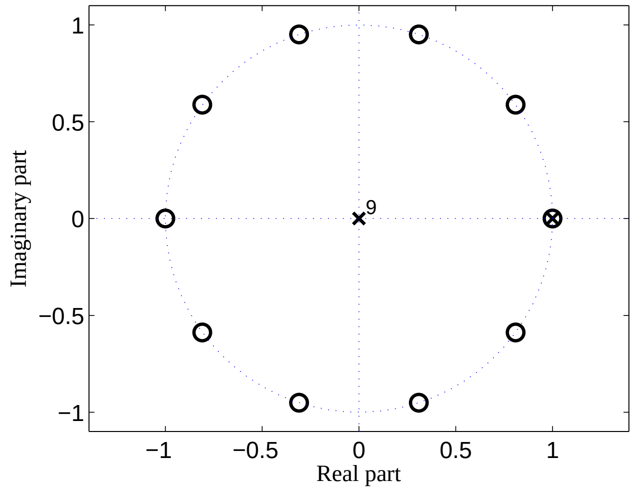
Example 12.3 - Solution

- The RRS has a pole at $z = 1$ that is canceled by a zero at the same position, leading to an FIR filter with a recursive realization.
- By examining the numerator polynomial of the RRS filter its zeros are equally spaced on the unit circle, placed at $z = e^{\frac{2\pi}{M+1}}$.
- The magnitude responses of the RRS for $M = 5, 7, 9$ are depicted in Figure 6a, whereas the pole-zero constellation when $M = 9$ is seen in Figure 6b.

Example 12.3 - Solution



(a)



(b)

Figure 6: RRS characteristics: (a) magnitude response for $M = 5$ (dotted line), $M = 7$ (dashed line), and $M = 9$ (solid line); (b) pole-zero constellation for $M = 9$.

Modified-sinc filter

- The RRS filter is certainly one of the most widely used lowpass filter with very efficient implementation. The main drawback of the RRS filter is its low stopband attenuation, that can not be increased in a straightforward manner.

- A simple and yet efficient extension of the RRS is the modified-sinc filter, which has the following general transfer function:

$$
\begin{aligned}
H(z) &= \frac{1}{(M+1)^3}\left(\frac{1 - bz^{-(M+1)} + bz^{-2(M+1)} - z^{-3(M+1)}}{1 - az^{-1} + az^{-2} - z^{-3}}\right) \\
&= \frac{1}{(M+1)^3}\left(\frac{1-2\cos(M+1)\omega_0 z^{-(M+1)}+z^{-2(M+1)}}{1-2\cos\omega_0 z^{-1}+z^{-2}}\right)\left(\frac{1-z^{-(M+1)}}{1-z^{-1}}\right)
\end{aligned}
\tag{54}
$$

where $a = (1 + 2\cos\omega_0)$ and $b = [1 + 2\cos(M + 1)\omega_0]$.

Modified-sinc filter

- Figure 7 shows a canonic structure for the modified-sinc filter, that is, utilizing the minimum number of multipliers, delays, and adders.

- The modified-sinc filter places equally spaced triplets of zeros on the unit circle. The first triplet is located around DC (one zero at DC and two at $\pm\omega_0$), and is cancelled by the filter poles, generating an improved lowpass filter, as compared to the RRS filter, as long as $\omega_0 < \frac{\pi}{M+1}$.

Modified-sinc filter

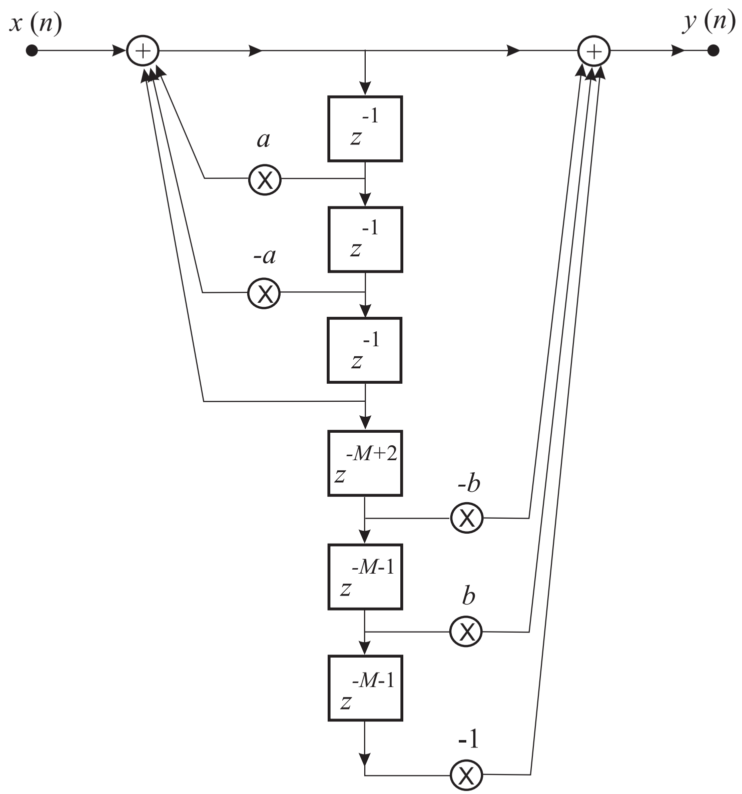
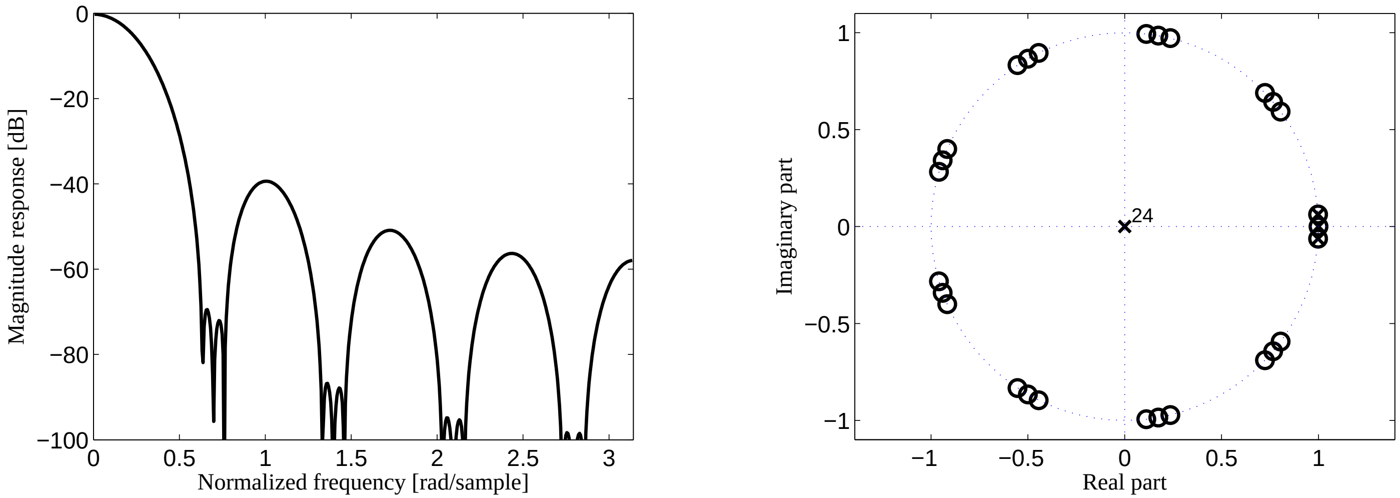


Figure 7: Canonic realization of the modified-sinc filter.

Modified-sinc filter

- Figure 8 depicts the frequency response of the modified-sinc filter for $M = 8$ and $\omega_0 = \frac{\pi}{50}$.



(a) (b)

Figure 8: Characterization of modified-sic filter for $M = 8$ and $\omega_0 = \frac{\pi}{50}$: (a) magnitude response; (b) pole-zero constellation.

Realizations with reduced number of arithmetic operations

- The main drawback of FIR filters is the large number of arithmetic operations required to satisfy practical specifications. However, especially in the case of filters with narrow passbands or transition bands, there is a correlation among the values of filter multiplier coefficients.
- This fact can be exploited in order to reduce the number of arithmetic operations required to satisfy such specifications.
- This section presents some of the most widely used methods for designing FIR filters with reduced computational complexity.

Prefilter approach

- The main idea of the prefilter method consists of generating a simple FIR filter, with reduced multiplication and addition counts, whose frequency response approximates the desired response as far as possible.
- Then this simple filter is cascaded with an amplitude equalizer, designed so that the overall filter satisfies the prescribed specifications.
- The reduction in the computational complexity results from the fact that the prefilter greatly relieves the specifications for the equalizer.
- This happens because the equalizer has wider transition bands to approximate, thus requiring a lower order.

Prefilter approach

- Several prefilter structures are given in the literature, and choosing the best one for a given specification is not an easy task.
- A very simple lowpass prefilter is the RRS filter, seen in Section 12.5.
- From equation (53), the frequency response of the Mth-order RRS filter is given by

$$H(e^{j\omega}) = \frac{\sin\left[\frac{\omega(M+1)}{2}\right]}{\sin\left(\frac{\omega}{2}\right)} e^{-\frac{j\omega M}{2}} \tag{55}$$

Prefilter approach

- This indicates that the RRS frequency response has several ripples at the stopband with decreasing magnitudes as ω approaches π. The first zero in the RRS frequency response occurs at

$$\omega_{z1} = \frac{2\pi}{M+1} \tag{56}$$

- Using the RRS as a prefilter, this first zero must be placed above and as close as possible to the stopband edge ω_r. In order to achieve this, the RRS order M must be such that

$$M = \left\lfloor \frac{2\pi}{\omega_r} - 1 \right\rfloor \tag{57}$$

where $\lfloor x \rfloor$ represents the largest integer less than or equal to x.

Prefilter approach

- More efficient prefilters can be generated by cascading several RRS sections. For example, if we cascade two prefilters, so that the first one satisfies equation (57), and the second is designed to cancel the secondary ripples of the first, we could expect a higher stopband attenuation for the resulting prefilter.
- This would relieve the specifications for the equalizer even further. In particular, the first stopband ripple belonging to the first RRS must be attenuated by the second RRS, without introducing zeros in the passband.
- Although the design of prefilters by cascading several RRS sections is always possible, practice has shown that there is little to gain in cascading more than three sections. The modified-sinc filter is also a very efficient prefilter.

Prefilter approach

- We show now that the Chebyshev (minimax) approach for designing optimal FIR filters presented in Chapter 5 can be adapted to design the equalizer, by modifying the error function definition, in the following way.
- The response obtained by cascading the prefilter with the equalizer is given by

$$H(z) = H_p(z)H_e(z) \tag{58}$$

where $H_p(z)$ is the prefilter transfer function, and only the coefficients of $H_e(z)$ are to be optimized.

Prefilter approach

- The error function can then be rewritten as

$$
\begin{aligned}
|E(\omega)| &= \left|W(\omega)\left(D(\omega) - H_p(e^{j\omega})H_e(e^{j\omega})\right)\right| \\
&= \left|W(\omega)H_p(e^{j\omega})\left(\frac{D(\omega)}{H_p(e^{j\omega})} - H_e(e^{j\omega})\right)\right| \\
&= \left|W'(\omega)\left(D'(\omega) - H_e(e^{j\omega})\right)\right|
\end{aligned}
\tag{59}
$$

Prefilter approach

- where

$$D'(\omega) = \begin{cases} \dfrac{1}{H_p(e^{j\omega})}, & \omega \in \text{passbands} \\ 0, & \omega \in \text{stopbands} \end{cases} \tag{60}$$

$$W'(\omega) = \begin{cases} \left|H_p(e^{j\omega})\right|, & \omega \in \text{passbands} \\ \dfrac{\delta_p}{\delta_r}\left|H_p(e^{j\omega})\right|, & \omega \in \text{stopbands} \end{cases} \tag{61}$$

- It is worth mentioning that $H_p(e^{j\omega})$ often has zeros at some frequencies, which causes problems to the optimization algorithm. One way to circumvent this is to replace $|H_p(e^{j\omega})|$ in the neighborhoods of its zeros by a small number such as 10^{-6}.

Example 12.4

- Design a highpass filter using the standard minimax and the prefilter methods satisfying the following specifications:

$$\left.\begin{aligned} A_r &= 40 \text{ dB} \\ \Omega_r &= 6600 \text{ Hz} \\ \Omega_p &= 7200 \text{ Hz} \\ \Omega_s &= 16\,000 \text{ Hz} \end{aligned}\right\} \tag{62}$$

Example 12.4 - Solution

- The prefilter approach described above applies only to narrowband lowpass filters. However, it requires only a slight modification in order to be applied to narrowband highpass filter design.
- The modification consists of designing the lowpass filter approximating $D(\pi - \omega)$ and then replacing z^{-1} by $-z^{-1}$ in the realization.
- Therefore, the lowpass specifications are

$$\Omega'_p = \frac{\Omega_s}{2} - \Omega_p = 8000 - 7200 = 800 \text{ Hz} \tag{63}$$

$$\Omega'_r = \frac{\Omega_s}{2} - \Omega_r = 8000 - 6600 = 1400 \text{ Hz} \tag{64}$$

- Using the minimax approach, the resulting direct-form filter has order 42, thus requiring 22 multiplications per output sample.

Example 12.4 - Solution

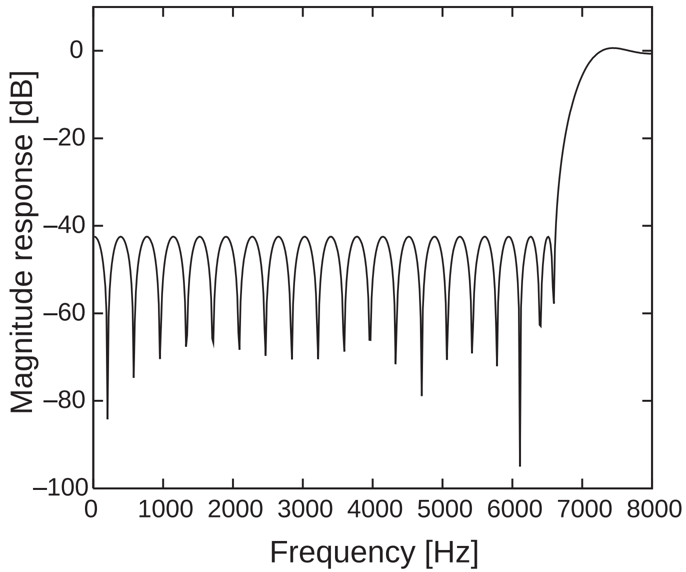
- Using the prefilter approach, with an RRS of order 10, the resulting equalizer has order 34, requiring only 18 multiplications per output sample. Only half of the equalizer coefficients are shown in Table 1, as the other coefficients can be obtained as $h(34 - n) = h(n)$.

Table 1: Equalizer coefficients.

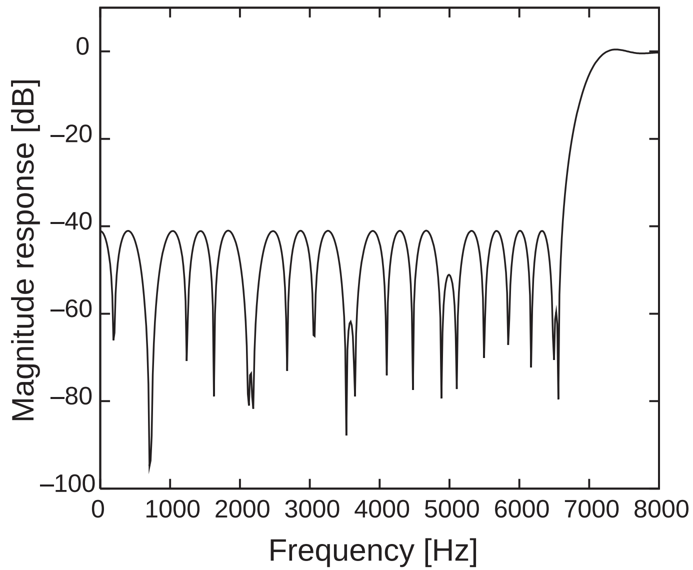
$h(0)$ to $h(17)$		
$h(0) = -5.7525\text{E}{-03}$	$h(6) = 3.9039\text{E}{-03}$	$h(12) = -2.9552\text{E}{-03}$
$h(1) = 1.4791\text{E}{-04}$	$h(7) = -5.3685\text{E}{-03}$	$h(13) = 7.1024\text{E}{-03}$
$h(2) = -1.4058\text{E}{-03}$	$h(8) = 6.1928\text{E}{-03}$	$h(14) = -1.1463\text{E}{-02}$
$h(3) = 6.0819\text{E}{-04}$	$h(9) = -5.9842\text{E}{-03}$	$h(15) = 1.5271\text{E}{-02}$
$h(4) = 6.3692\text{E}{-04}$	$h(10) = 4.4243\text{E}{-03}$	$h(16) = -1.7853\text{E}{-02}$
$h(5) = -2.2099\text{E}{-03}$	$h(11) = 9.1634\text{E}{-04}$	$h(17) = 1.8756\text{E}{-02}$

Example 12.4 - Solution

- The magnitude responses of the direct-form and the prefilter-equalizer filters are depicted in Figure 9.



(a)



(b)

Figure 9: Magnitude responses: (a) direct-form minimax approach; (b) prefilter approach.

Example 12.4 - Solution

- With the prefilter approach we can also design bandpass filters centered at $\omega_0 = \frac{\pi}{2}$ and with band edges at $\left(\frac{\pi}{2} - \frac{\omega_{\rm p}}{2}\right)$, $\left(\frac{\pi}{2} + \frac{\omega_{\rm p}}{2}\right)$, $\left(\frac{\pi}{2} - \frac{\omega_{\rm r}}{2}\right)$, and $\left(\frac{\pi}{2} + \frac{\omega_{\rm r}}{2}\right)$. This can be done by noting that such bandpass filters can be obtained from a lowpass filter with band edges at $\omega_{\rm p}$ and $\omega_{\rm r}$, by applying the transformation $z^{-1} \rightarrow -z^{-2}$.

- There are also generalizations of the prefilter approach that allow the design of narrow bandpass filters with central frequency away from $\frac{\pi}{2}$, as well as narrow stopband filters.

- Due to the reduced number of multipliers, filters designed using the prefilter method tend to generate less roundoff noise at the output than minimax filters implemented in direct form. Their sensitivity to coefficient quantization is also reduced when compared to direct-form minimax designs.

Interpolation approach

- FIR filters with narrow passband and transition bands tend to have adjacent multiplier coefficients, representing their impulse responses, with very close magnitude. This means there is a correlation among them that could be exploited to reduce computational complexity.

- Indeed, we could think of removing some samples of the impulse response, by replacing them with zeros, and approximating their values by interpolating the remaining nonzero samples. That is, using the terminology of Chapter 8, we would decimate and then interpolate the filter coefficients. This is the main idea behind the interpolation approach.

Interpolation approach

- Consider an initial filter with frequency and impulse responses given by $H_i(e^{j\omega})$ and $h_i(n)$, respectively. If $h_i(n)$ is interpolated by L, then $(L-1)$ null samples are inserted after each sample of $h_i(n)$, and the resulting sequence $h'_i(n)$ is given by

$$h'_i(n) = \begin{cases} h_i\left(\frac{n}{L}\right), & \text{for } n = kL, \text{ with } k = 0, 1, 2, \ldots \\ 0, & \text{for } n \neq kL \end{cases} \tag{65}$$

- The corresponding frequency response, $H'_i(e^{j\omega})$, is periodic with period $2\pi/L$. For example, Figure 10 illustrates the form of $H'_i(e^{j\omega})$ generated from a lowpass filter with frequency response $H_i(e^{j\omega})$, using $L = 3$.

Interpolation approach

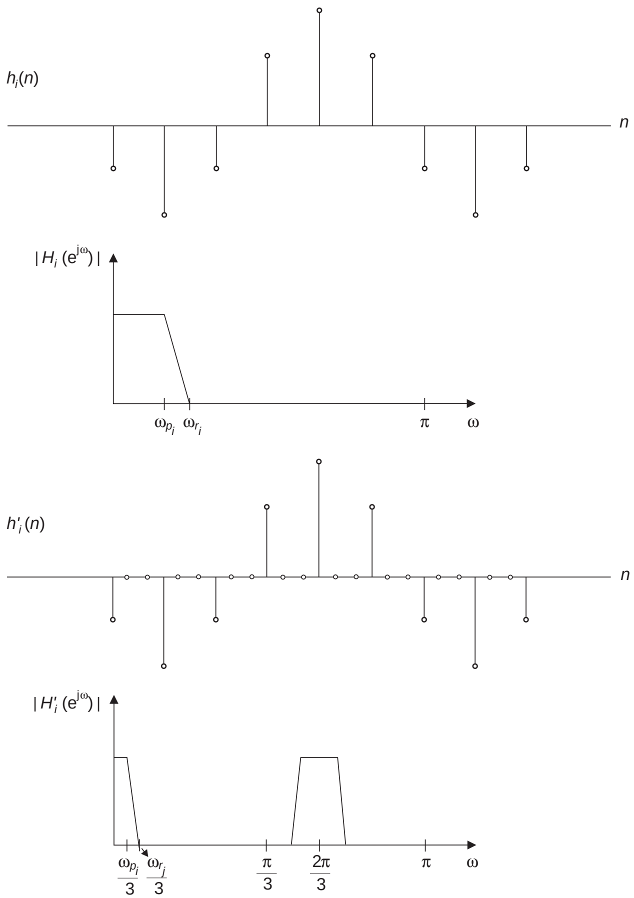


Figure 10: Effects of inserting $(L-1)=2$ zeros in a discrete-time impulse response.

Interpolation approach

- The filter with frequency response $H'_i(e^{j\omega})$, commonly referred to as the interpolated filter, is then connected in cascade with an interpolator $G(e^{j\omega})$, resulting in a transfer function of the form

$$H(z) = H'_i(z)G(z) \tag{66}$$

- The function of the interpolator is to eliminate undesirable bands of $H'_i(z)$ (see Figure 10), while leaving its lowest frequency band unaffected.

- As can be observed, the initial filter has passband and stopband which are L times larger than the ones of the interpolated filter (in Figure 10, $L = 3$). As a consequence, the number of multiplications in the initial filter tends to be approximately L times smaller than the number of multiplications of a filter directly designed with the minimax approach to satisfy the narrowband specifications.

Interpolation approach

- An intuitive explanation for this is that a filter with larger passbands, stopbands, and transition bands is easier to implement than one with narrow bands.
- For lowpass filters, the maximum value of L such that the initial filter satisfies the specifications in the passband and in the stopband is given by

$$L_{\text{max}} = \left\lfloor \frac{\pi}{\omega_r} \right\rfloor \tag{67}$$

where ω_r is the lowest rejection band frequency of the desired filter. This value for L assures that $\omega_{r_i} < \pi$.

Interpolation approach

- For highpass filters, L_{max} is given by

$$L_{\text{max}} = \left\lfloor \frac{\pi}{\pi - \omega_r} \right\rfloor \tag{68}$$

whereas for bandpass filters, L_{max} is the largest L such that $\frac{\pi k}{L} \leq \omega_{r_1} < \omega_{r_2} \leq \frac{\pi(k+1)}{L}$, for some k. In practice, L is chosen smaller than L_{max} in order to relieve the interpolator specifications.

- Naturally, to achieve reduction in the computational requirements of the final filter, the interpolator must be as simple as possible, not requiring too many multiplications.

- For instance, the interpolator can be designed as a cascade of subsections, in which each subsection places zeros on an undesirable passband.

- For the example in Figure 10, if we wish to design a lowpass filter, $G(e^{j\omega})$ should have zeros at $e^{\pm j\frac{2\pi}{3}}$.

Interpolation approach

- Alternatively, we can use the minimax method to design the interpolator, with the passband of $G(e^{j\omega})$ coinciding with the specified passband, and with the stopbands located in the frequency ranges of the undesired passband replicas of the interpolated filter.

- Once the value of L and the interpolator are chosen, the interpolated filter can be designed such that

$$\left.\begin{array}{ll} (1-\delta_p) \leq \left|H_i(e^{j\omega})G(e^{\frac{j\omega}{L}})\right| \leq (1+\delta_p), & \text{for } \omega \in [0, L\omega_p] \\ \left|H_i(e^{j\omega})G(e^{\frac{j\omega}{L}})\right| \leq \delta_r, & \text{for } \omega \in [L\omega_r, \pi] \end{array}\right\} \quad (69)$$

where the minimax method to design optimal FIR filters can be directly used.

Example 12.5

- Design the filter specified in Example 12.1 using the interpolation method.

Example 12.5 - Solution

- Using $L = 2$, we obtain the initial filter of order 20, thus requiring 11 multiplications per output sample, whose coefficients are listed in Table 2. The required interpolator is given by $G(z) = (1 - z^{-1})^4$, and the resulting magnitude response of the cascade of the initial filter and the interpolator is seen in Figure 11.

Example 12.5 - Solution

Table 2: Initial filter coefficients.

$h(0)$ to $h(10)$		
$h(0) = 1.0703\text{E}{-03}$	$h(4) = -2.8131\text{E}{-03}$	$h(8) = 9.5809\text{E}{-03}$
$h(1) = 7.3552\text{E}{-04}$	$h(5) = -3.3483\text{E}{-03}$	$h(9) = 1.4768\text{E}{-02}$
$h(2) = 3.9828\text{E}{-04}$	$h(6) = -1.2690\text{E}{-03}$	$h(10) = 1.6863\text{E}{-02}$
$h(3) = -1.2771\text{E}{-03}$	$h(7) = 3.3882\text{E}{-03}$	

Example 12.5 - Solution

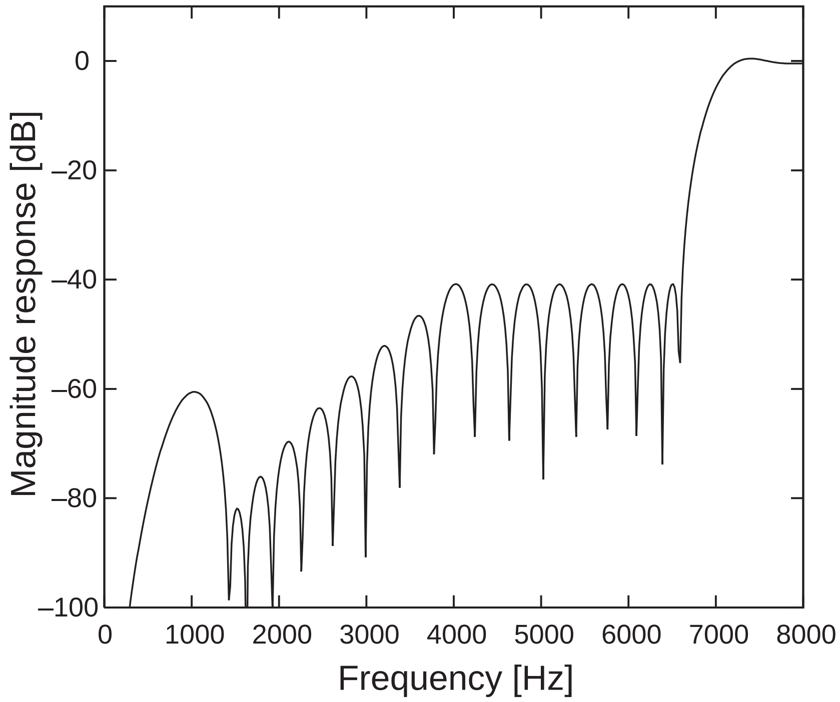


Figure 11: Response of the interpolated model filter in cascade with the interpolator.

Interpolation approach

- It is worth observing that the prefilter and the interpolation methods were initially described as effective methods to design narrowband filters of the types lowpass, highpass, and bandpass.
- However, we can also design both wideband and narrow stopband filters with the interpolation method by noting they can be obtained from a narrowband filter $H(z)$ which is complementary to the desired filter, using

$$H_{FL}(z) = z^{-\frac{M}{2}} - H(z) \tag{70}$$

where M is the even order of $H(z)$.

Frequency response masking approach

- Another interesting application of interpolation appears in the design of wideband sharp cutoff filters using the so-called frequency response masking approach. A brief introduction to it was given in Subsection 8.10.2.

- Such an approach makes use of the concept of complementary filters, which constitute a pair of filters, $H_a(z)$ and $H_c(z)$, whose frequency responses add to a constant delay, that is

$$\left|H_a(e^{j\omega}) + H_c(e^{j\omega})\right| = 1 \tag{71}$$

- If $H_a(z)$ is a linear-phase FIR filter of even order M, its frequency response can be written as

$$H_a(e^{j\omega}) = e^{-j\frac{M}{2}\omega}A(\omega) \tag{72}$$

where $A(\omega)$ is a trigonometric function of ω, as given in Section 5.6.

Frequency response masking approach

- Therefore, the frequency response of the complementary filter must be of the form

$$H_c(e^{j\omega}) = e^{-j\frac{M}{2}\omega}(1 - A(\omega)) \tag{73}$$

and the corresponding transfer functions are such that

$$H_a(z) + H_c(z) = z^{-\frac{M}{2}} \tag{74}$$

- Hence, given the realization of $H_a(z)$, its complementary filter $H_c(z)$ can easily be implemented by subtracting the output of $H_a(z)$ from the $(M/2)$th delayed version of its input, as seen in Figure 12.

Frequency response masking approach

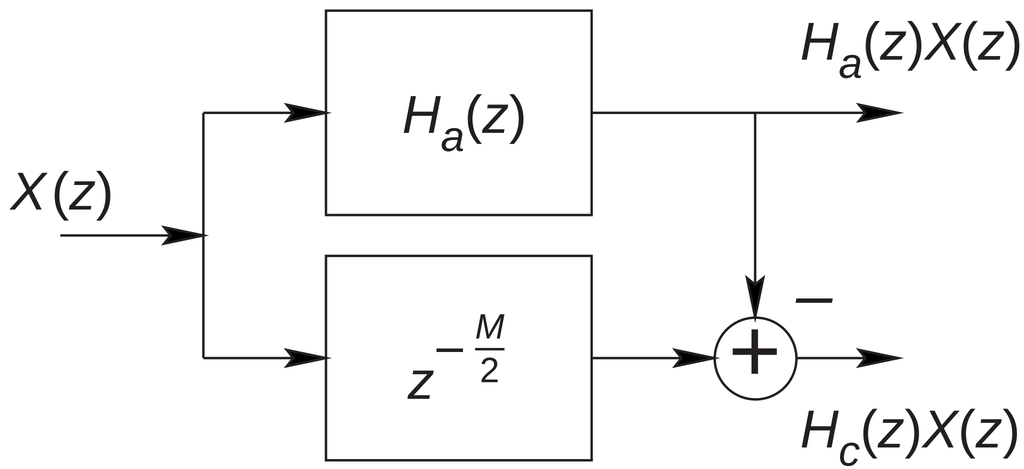


Figure 12: Realization of the complementary filter $H_c(z)$.

Frequency response masking approach

- For an efficient implementation of both filters, the tapped delay line of $\mathrm{H_a}(z)$ can be used to form $\mathrm{H_c}(z)$, as indicated in Figure 13, in which either the symmetry or antisymmetry of $\mathrm{H_a}(z)$ is exploited, as we are assuming that this filter has linear phase.

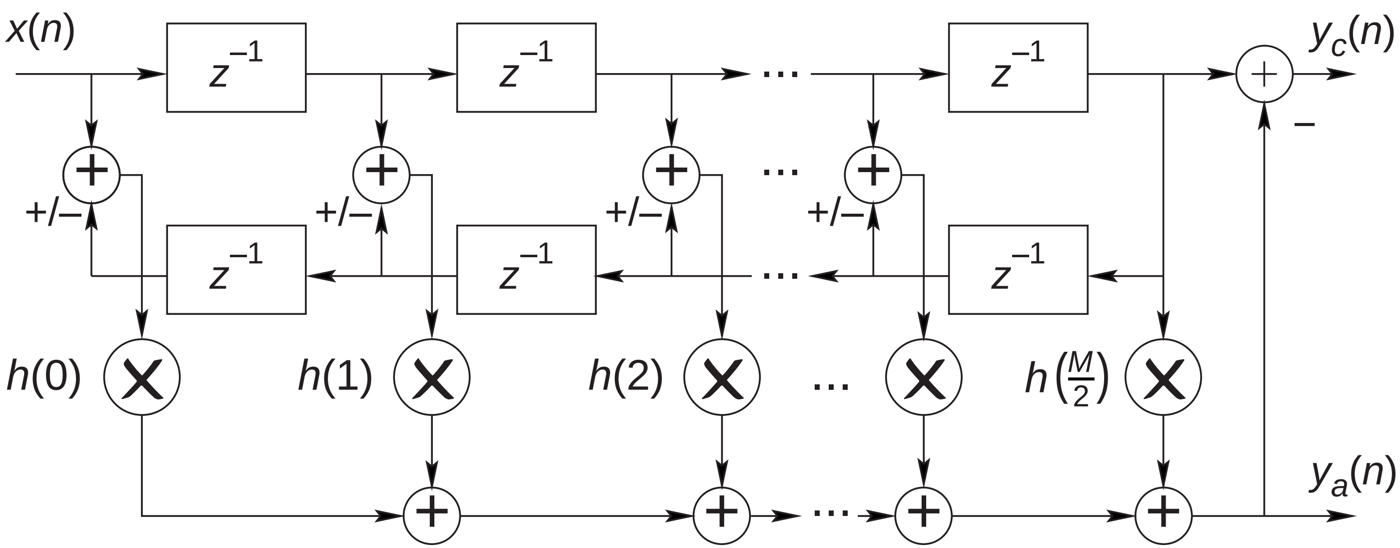


Figure 13: Efficient realization of the complementary filter $\mathrm{H_c}(z)$.

Frequency response masking approach

- The overall structure of the filter designed with the frequency response masking approach is seen in Figure 14.
- The basic idea is to design a wideband lowpass filter and compress its frequency response by using an interpolation operation.
- A complementary filter is obtained, following the development seen above.
- We then use masking filters, $\mathrm{H}_{\mathrm{M}a}(z)$ and $\mathrm{H}_{\mathrm{M}c}(z)$, to eliminate the undesired bands in the interpolated and complementary filters, respectively.
- The corresponding outputs are added together to form the desired lowpass filter.

Frequency response masking approach

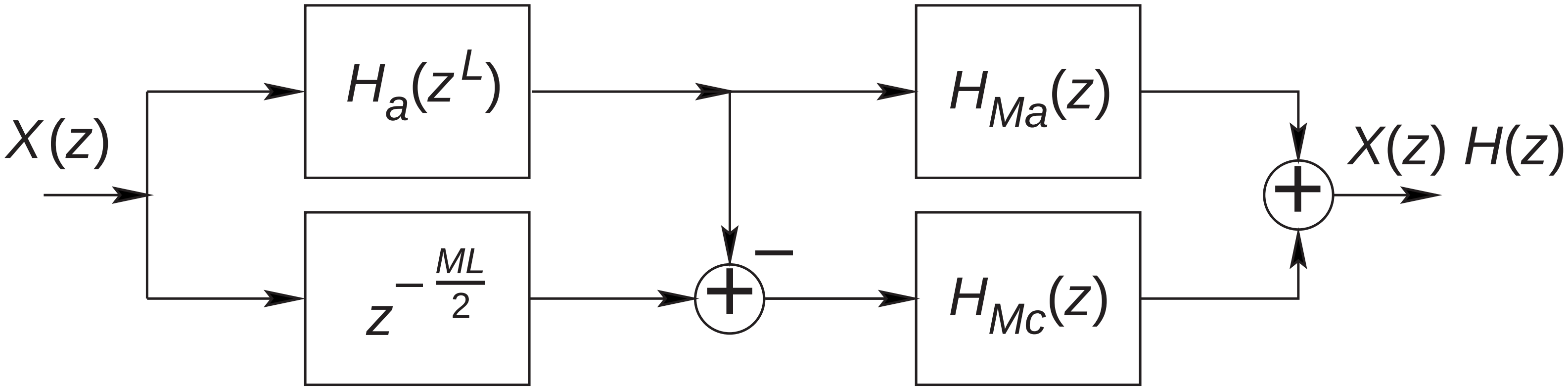


Figure 14: Block diagram of the frequency response masking approach.

Frequency response masking approach

- To understand the overall procedure in the frequency domain, consider Figure 15.
- Suppose that $H_a(z)$ corresponds to a lowpass filter of even order M, designed with the standard minimax approach, with passband edge θ and stopband edge ϕ, as seen in Figure 15a.
- We can then form $H_c(z)$, corresponding to a highpass filter, with θ and ϕ being the respective stopband and passband edges.
- By interpolating both filters by L, two complementary multiband filters are generated, as represented in Figures 15b and 15c, respectively.

Frequency response masking approach

- We can then use two masking filters, $\mathrm{H}_{Ma}(z)$ and $\mathrm{H}_{Mc}(z)$, characterized as in Figures 15d and 15e, to generate the magnitude responses shown in Figures 15f and 15g.
- Adding these two components, the resulting desired filter seen in Figure 15h can have a passband of arbitrary width with a very narrow transition band.
- In Figure 15, the positions of the transition band edges are dictated by the $\mathrm{H}_{Ma}(z)$ masking filter.
- An example of the frequency response masking approach where the transition band edges are determined by the $\mathrm{H}_{Mc}(z)$ masking filter, is shown in Figure 16.

Frequency response masking approach

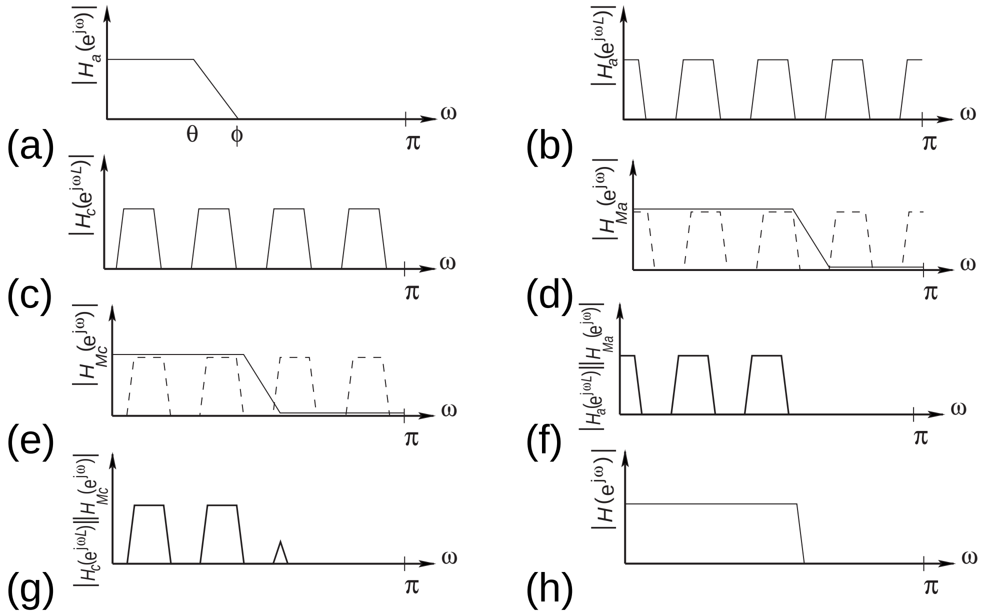


Figure 15: Frequency response masking design of a lowpass filter with the $H_{Ma}(z)$ mask determining the passband: (a) base filter; (b) interpolated filter; (c) complementary to the interpolated filter; (d) masking filter $H_{Ma}(z)$; (e) masking filter $H_{Mc}(z)$; (f) cascade of $H_a(z^L)$ with masking filter $H_{Ma}(z)$; (g) cascade of $H_c(z^L)$ with masking filter $H_{Mc}(z)$; (h) frequency response masking filter.

Frequency response masking approach

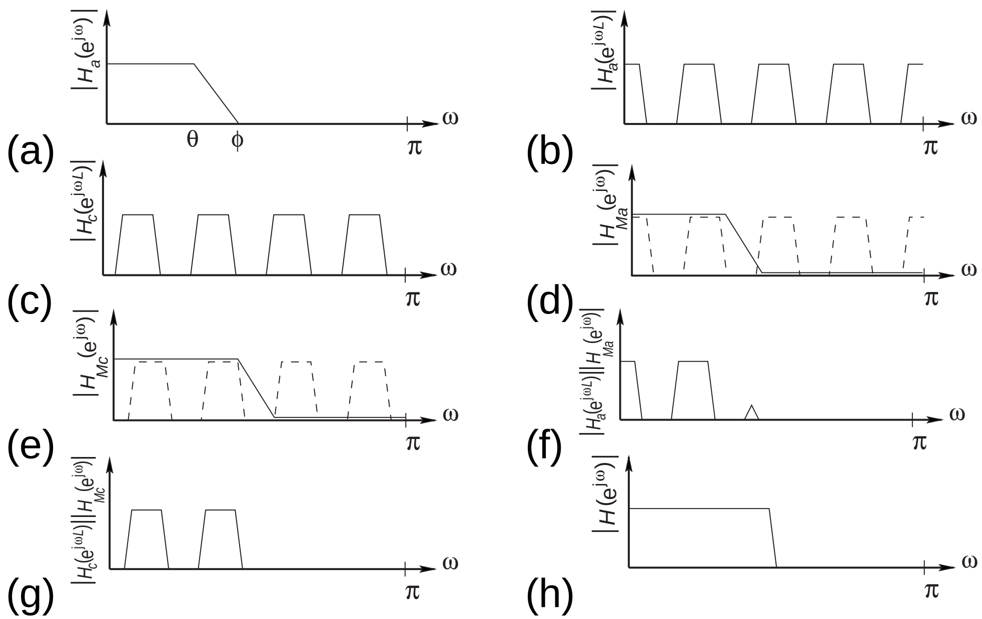


Figure 16: Frequency response masking design of a lowpass filter with the $H_{Mc}(z)$ mask determining the passband: (a) base filter; (b) interpolated filter; (c) complementary to the interpolated filter; (d) masking filter $H_{Ma}(z)$; (e) masking filter $H_{Mc}(z)$; (f) cascade of $H_a(z^L)$ with masking filter $H_{Ma}(z)$; (g) cascade of $H_c(z^L)$ with masking filter $H_{Mc}(z)$; (h) frequency response masking filter.

Frequency response masking approach

- From Figure 14, it is easy to see that the product ML must be even to avoid a half-sample delay. This is commonly satisfied by forcing M to be even, as above, thus freeing the parameter L from any constraint.

- In addition, $H_{Ma}(z)$ and $H_{Mc}(z)$ must have the same group delay, so that they complement each other appropriately in the resulting passband when added together to form the desired filter $H(z)$.

- This means that they must be both of even order or both of odd order, and that a few delays may be appended before and after either $H_{Ma}(z)$ or $H_{Mc}(z)$, if necessary, to equalize their group delays.

- For a complete description of the frequency response masking approach, we must characterize the filters $H_a(z)$, $H_{Ma}(z)$, and $H_{Mc}(z)$.

Frequency response masking approach

- When the resulting magnitude response is determined mainly by the masking filter $H_{Ma}(z)$, as exemplified in Figure 15, then we can conclude that the desired band edges are such that

$$\omega_p = \frac{2m\pi + \theta}{L} \tag{75}$$

$$\omega_r = \frac{2m\pi + \phi}{L} \tag{76}$$

where m is an integer less than L.

Frequency response masking approach

- Therefore, a solution for the values of m, θ, and ϕ, such that $0 < \theta < \phi < \pi$, is given by

$$m = \left\lfloor \frac{\omega_p L}{2\pi} \right\rfloor \tag{77}$$

$$\theta = \omega_p L - 2m\pi \tag{78}$$

$$\phi = \omega_r L - 2m\pi \tag{79}$$

where $\lfloor x \rfloor$ indicates the largest integer less than or equal to x.

Frequency response masking approach

- With these values, from Figure 15, we can determine the band edges for the masking filters as given by

$$\omega_{p,Ma} = \frac{2m\pi + \theta}{L} \tag{80}$$

$$\omega_{r,Ma} = \frac{2(m+1)\pi - \phi}{L} \tag{81}$$

$$\omega_{p,Mc} = \frac{2m\pi - \theta}{L} \tag{82}$$

$$\omega_{r,Mc} = \frac{2m\pi + \phi}{L} \tag{83}$$

where $\omega_{p,Ma}$ and $\omega_{r,Ma}$ are the passband and stopband edges for the $H_{Ma}(z)$ masking filter, respectively, and $\omega_{p,Mc}$ and $\omega_{r,Mc}$ are the passband and stopband edges for the $H_{Mc}(z)$ masking filter, respectively.

Frequency response masking approach

- When $H_{Mc}(z)$ is the dominating masking filter, as seen in Figure 16, we have that

$$\omega_p = \frac{2m\pi - \phi}{L} \tag{84}$$

$$\omega_r = \frac{2m\pi - \theta}{L} \tag{85}$$

and a solution for m, θ, and ϕ, such that $0 < \theta < \phi < \pi$, is given by

$$m = \left\lceil \frac{\omega_r L}{2\pi} \right\rceil \tag{86}$$

$$\theta = 2m\pi - \omega_r L \tag{87}$$

$$\phi = 2m\pi - \omega_p L \tag{88}$$

where $\lceil x \rceil$ indicates the smallest integer greater than or equal to x.

Frequency response masking approach

- In this case, from Figure 16, the band edges for the masking filters are given by

$$\omega_{p,Ma} = \frac{2(m-1)\pi + \phi}{L} \tag{89}$$

$$\omega_{r,Ma} = \frac{2m\pi - \theta}{L} \tag{90}$$

$$\omega_{p,Mc} = \frac{2m\pi - \phi}{L} \tag{91}$$

$$\omega_{r,Mc} = \frac{2m\pi + \theta}{L} \tag{92}$$

Frequency response masking approach

- Given the desired ω_p and ω_r, each value of L may allow one solution such that $\theta < \phi$, either in the form of equations (77)–(79) or of equations (86)–(88).
- In practice, the determination of the best L, that is the one which minimizes the total number of multiplications per output sample, can be done empirically with the aid of the order estimation given in Exercise 5.25.
- The passband ripples and attenuation levels used when designing $H_a(z)$, $H_{Ma}(z)$, and $H_{Mc}(z)$ are determined based on the specifications of the desired filter. As these filters are cascaded, their frequency responses will be added in dB, thus requiring a certain margin to be used in their designs.

Frequency response masking approach

- For the base filter $H_a(z)$, one must keep in mind that its passband ripple δ_p corresponds to the stopband attenuation δ_r of its complementary filter $H_c(z)$, and vice versa.
- Therefore, when designing $H_a(z)$ one should use the smallest value between δ_p and δ_r, incorporating an adequate margin.
- In general, a margin of 50% in the values of the passband ripples and stopband attenuations should be used, as in the example that follows.

Example 12.6

- Design a lowpass filter using the frequency response masking method satisfying the following specifications:

$$\left.\begin{aligned} A_p &= 0.2 \text{ dB} \\ A_r &= 60 \text{ dB} \\ \Omega_p &= 0.6\pi \text{ rad/s} \\ \Omega_r &= 0.61\pi \text{ rad/s} \\ \Omega_s &= 2\pi \text{ rad/s} \end{aligned}\right\} \tag{93}$$

Compare your results with the filter obtained using the standard minimax scheme.

Example 12.6 - Solution

- Table 3 shows the estimated orders for the base filter and the masking filters for several values of the interpolation factor L.
- Although these values are probably slight underestimates, they allow a quick decision as to the value of L that minimizes the total number of multiplications required.
- In this table M_{H_a} is the order of the base filter $H_a(z)$, $M_{H_{Ma}}$ is the order of the masking filter $H_{Ma}(z)$, and $M_{H_{Mc}}$ is the order of the masking filter $H_{Mc}(z)$.

Example 12.6 - Solution

- Also,

$$\Pi = f(M_{H_a}) + f(M_{H_{Ma}}) + f(M_{H_{Mc}}) \tag{94}$$

indicates the total number of multiplications required to implement the overall filter, where

$$f(x) = \begin{cases} \dfrac{x+1}{2}, & \text{if } x \text{ is odd} \\ \dfrac{x}{2} + 1, & \text{if } x \text{ is even} \end{cases} \tag{95}$$

and

$$M = LM_{H_a} + \max\{M_{H_{Ma}}, M_{H_{Mc}}\} \tag{96}$$

is the effective order of the overall filter designed with the frequency response masking approach.

Example 12.6 - Solution

Table 3: Filter characteristics for several values of the interpolation factor L.

L	M_{H_a}	$M_{H_{Ma}}$	$M_{H_{Mc}}$	Π	M
2	368	29	0	200	765
3	246	11	49	155	787
4	186	21	29	120	773
5	150	582	16	377	1332
6	124	29	49	103	793
7	108	28	88	115	844
8	94	147	29	137	899
9	**84**	**61**	**49**	**99**	**817**
10	76	32	582	348	1342
11	68	95	51	109	843

- From this table, we predict that $L = 9$ should yield the most efficient filter with respect to the total number of multiplications per output sample.

Example 12.6 - Solution

- Using $L = 9$ in equations (77)–(92), the corresponding band edges for all filters are given by

$$\left.\begin{aligned} \theta &= 0.5100\pi \\ \phi &= 0.6000\pi \\ \omega_{p,Ma} &= 0.5111\pi \\ \omega_{r,Ma} &= 0.6100\pi \\ \omega_{p,Mc} &= 0.6000\pi \\ \omega_{r,Mc} &= 0.7233\pi \end{aligned}\right\} \tag{97}$$

Example 12.6 - Solution

- From the filter specifications, we have that $\delta_p = 0.0115$ and $\delta_r = 0.001$. Therefore, the value of δ_r with a margin of 50% was used as the passband ripple and the stopband gain for the base filter to generate Table 3, that is

$$\delta_{p,a} = \delta_{r,a} = \min\{\delta_p, \delta_r\} \times 50\% = 0.0005 \tag{98}$$

  corresponding to a passband ripple of 0.0087 dB, and a stopband attenuation of 66.0206 dB.

- As $\delta_{p,a} = \delta_{r,a}$ the relative weights for the passband and stopband in the minimax design of the base filter are both equal to 1.0000.

Example 12.6 - Solution

- For the masking filters, we use

$$\delta_{p,Ma} = \delta_{p,Mc} = \delta_p \times 50\% = 0.005\,75 \tag{99}$$

$$\delta_{r,Ma} = \delta_{r,Mc} = \delta_r \times 50\% = 0.0005 \tag{100}$$

corresponding to a passband ripple of 0.0996 dB, and a stopband attenuation of 66.0206 dB.

- In this case, the relative weights for the minimax design of both masking filters were made equal to 1.0000 and 11.5124 in the passband and stopband, respectively.

Example 12.6 - Solution

- The magnitude responses of the resulting filters for $L = 9$ are depicted in Figures 17 and 18.
- In Figures 17a and 17b, the base filter and its complementary filter are depicted, and Figures 17c and 17d show the corresponding interpolated filters.
- Similarly, Figures 18a and 18b depict the two masking filters, $H_{Ma}(z)$ and $H_{Mc}(z)$, respectively, and Figures 18c and 18d show the results at the outputs of these filters, which are added together to form the desired filter.
- The overall frequency response masking filter is characterized in Figure 19 and presents a passband ripple equal to $A_p = 0.0873$ dB and a stopband attenuation of $A_r = 61.4591$ dB.

Example 12.6 - Solution

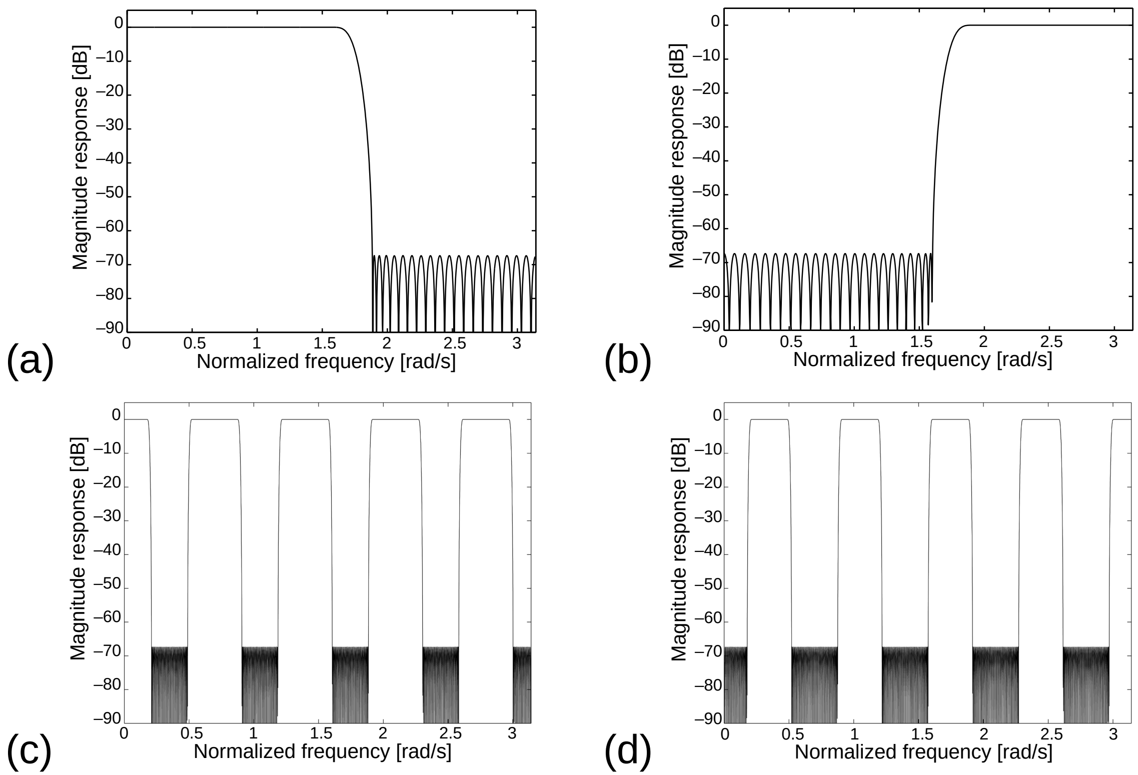


Figure 17: Magnitude responses: (a) base filter $H_a(z)$; (b) complementary to the base filter $H_c(z)$; (c) interpolated base filter $H_a(z^L)$; (d) complementary to the interpolated base filter $H_c(z^L)$.

Example 12.6 - Solution

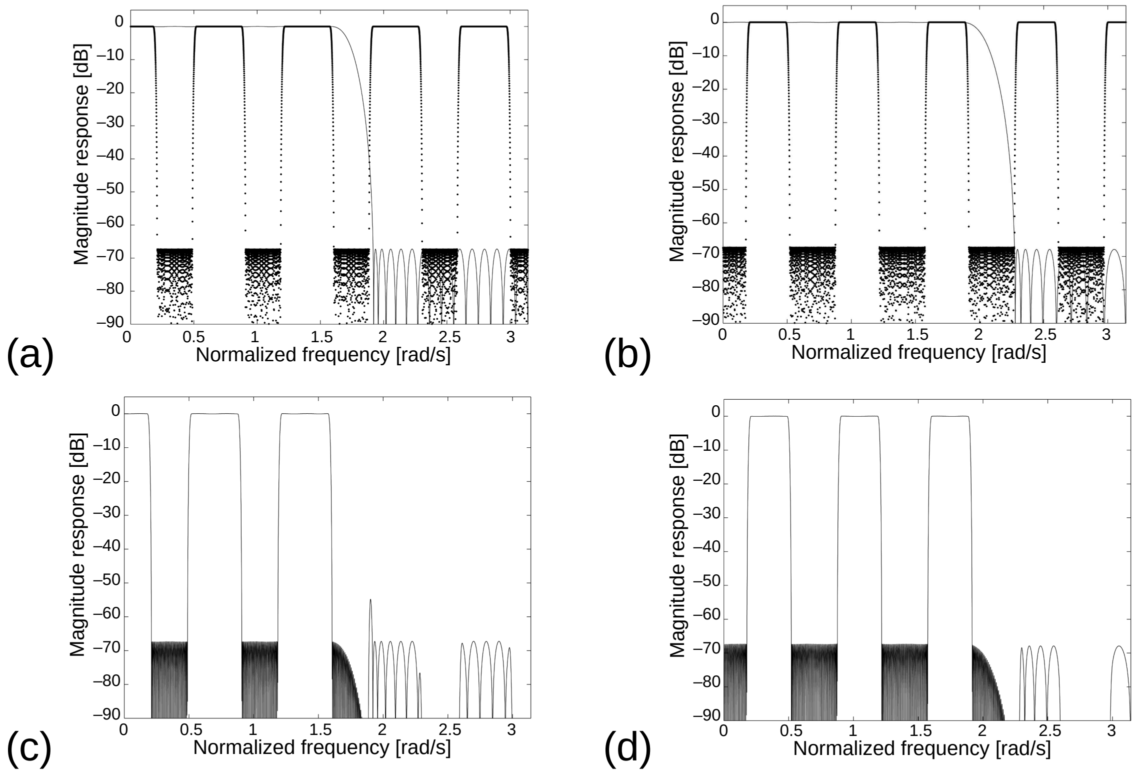


Figure 18: Magnitude responses: (a) masking filter $\mathrm{H}_{\mathrm{M}a}(z)$; (b) masking filter $\mathrm{H}_{\mathrm{M}c}(z)$; (c) combination of $\mathrm{H}_a(z^L)\mathrm{H}_{\mathrm{M}a}(z)$; (d) combination of $\mathrm{H}_c(z^L)\mathrm{H}_{\mathrm{M}c}(z)$.

Example 12.6 - Solution

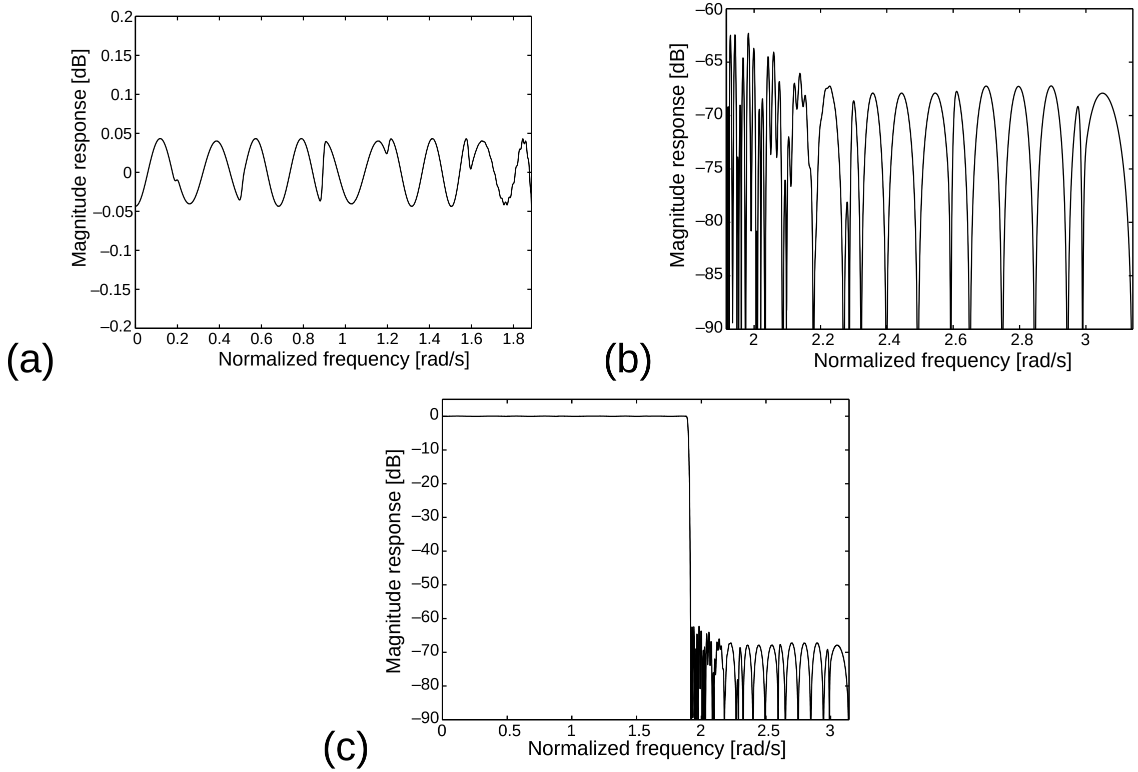


Figure 19: Magnitude response of the frequency response masking filter: (a) passband detail; (b) stopband detail; (c) overall response.

Example 12.6 - Solution

- The resulting minimax filter is of order 504, thus requiring 253 multiplications per output sample. Therefore, in this case, the frequency response masking design represents a saving of about 60% of the number of multiplications required by the standard minimax filter.

Frequency response masking approach

- Example 12.3 above shows that a constant ripple margin throughout the whole frequency range $\omega \in [0, \pi]$ is not required.
- In fact, with the ripple margin of 50% the passband ripple was considerably smaller than necessary, as seen in Figure 19a, and the attenuation was higher than required through most of the stopband, as seen in Figure 19b.
- A detailed analysis of the required margins in each band was performed by Y. C. Lim who concluded that:
  - The ripple margin must be of the order of 50% at the beginning of the stopbands of each masking filter.
  - For the remaining frequency values, the ripple margin can be set around 15–20%.
- It can be verified that such a distribution of the ripple margins results in a more efficient design, yielding an overall filter with a smaller group delay and fewer multiplications per output sample, as illustrated in the following example.

Example 12.7

- Design the lowpass filter specified in Example 12.3 using the frequency response masking method with an efficient assignment of the ripple margin. Compare the results with the filter obtained in Example 12.3.

Example 12.7 - Solution

- The design follows the same procedure as before, except that the relative weights at the beginning of the stopbands of the masking filters is set to 2.5, whereas for the remaining frequency the weight is set to 1.0.
- That corresponds to ripple margins proportional to 50% and 20% respectively, in these two distinct frequency ranges.
- Table 4 shows the filter characteristics for several values of the interpolation factor L.
- As in Example 12.3, the minimum number of multiplications per output sample is obtained when $L = 9$, and the band edges for the base and frequency response masking filters are as given in equation (97).
- In this case, however, only 91 multiplications are required, as opposed to 99 multiplications in Example 12.3.

Example 12.7 - Solution

- Figures 20a and 20b depict the magnitude responses of the two masking filters, $\mathrm{H}_{\mathrm{Ma}}(z)$ and $\mathrm{H}_{\mathrm{Mc}}(z)$, respectively, and Figures 20c and 20d show the magnitude responses which are added together to form the desired filter.
- From these figures, one can clearly see the effects of the more efficient ripple margin distribution.
- The overall frequency response masking filter is characterized in Figure 21, and presents a passband ripple equal to $A_{\mathrm{p}} = 0.1502$ dB and a stopband attenuation of $A_{\mathrm{r}} = 60.5578$ dB. Notice how these values are closer to the specifications than the values of the filter obtained in Example 12.3.

Example 12.7 - Solution

- Table 5 presents the first half of the base filter coefficients before interpolation, and the coefficients of the masking filters $H_{Ma}(z)$ and $H_{Mc}(z)$ are given in Tables 6 and 7, respectively.
- It must be noted that, as stated before, for a smooth composition of the outputs of the masking filters when forming the filter $H(z)$, both these filters must have the same group delay.
- To achieve that in this design example, we must add 5 delays before and 5 delays after the masking filter $H_{Mc}(z)$, which has a smaller number of coefficients.

Example 12.7 - Solution

Table 4: Filter characteristics for several values of the interpolation factor L.

L	M_{H_a}	$M_{H_{Ma}}$	$M_{H_{Mc}}$	Π	M
2	342	26	0	186	710
3	228	10	44	144	728
4	172	20	26	112	714
5	138	528	14	343	1218
6	116	26	44	96	740
7	100	26	80	106	780
8	88	134	26	127	838
9	**78**	**55**	**45**	**91**	**757**
10	70	30	528	317	1228
11	64	86	46	101	790

Example 12.7 - Solution

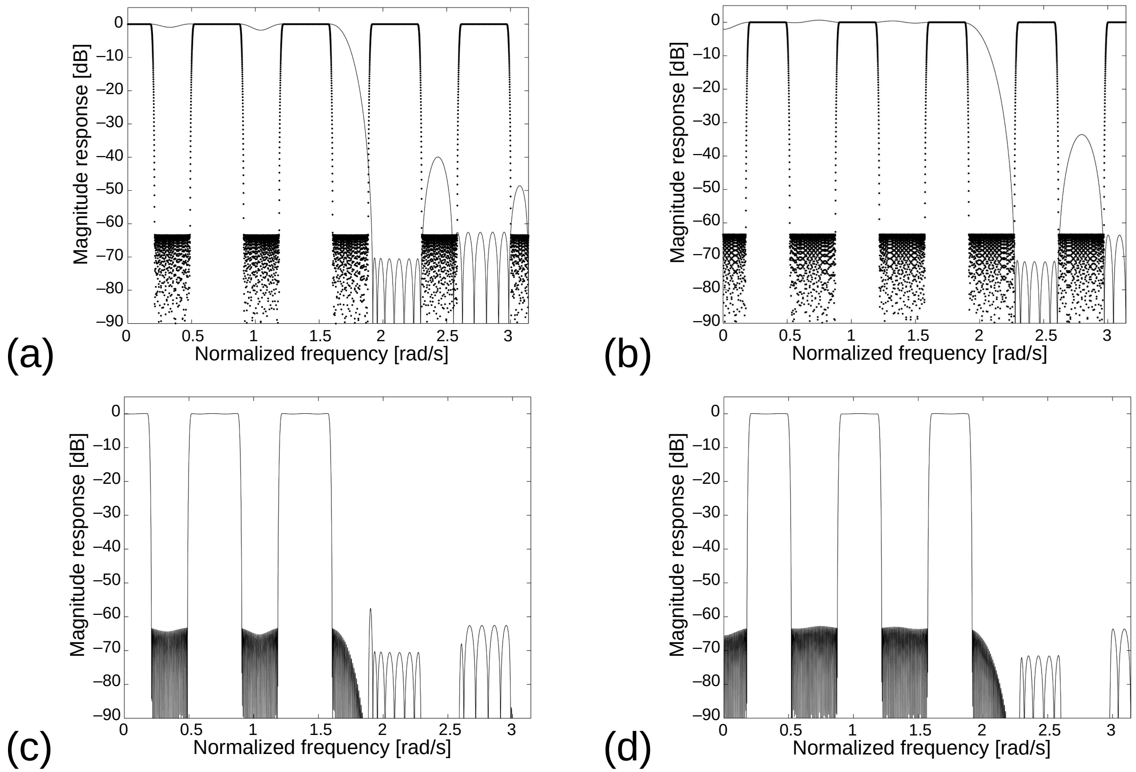


Figure 20: Magnitude responses: (a) masking filter $H_{Ma}(z)$; (b) masking filter $H_{Mc}(z)$; (c) combination of $H_a(z^L)H_{Ma}(z)$; (d) combination of $H_c(z^L)H_{Mc}(z)$.

Example 12.7 - Solution

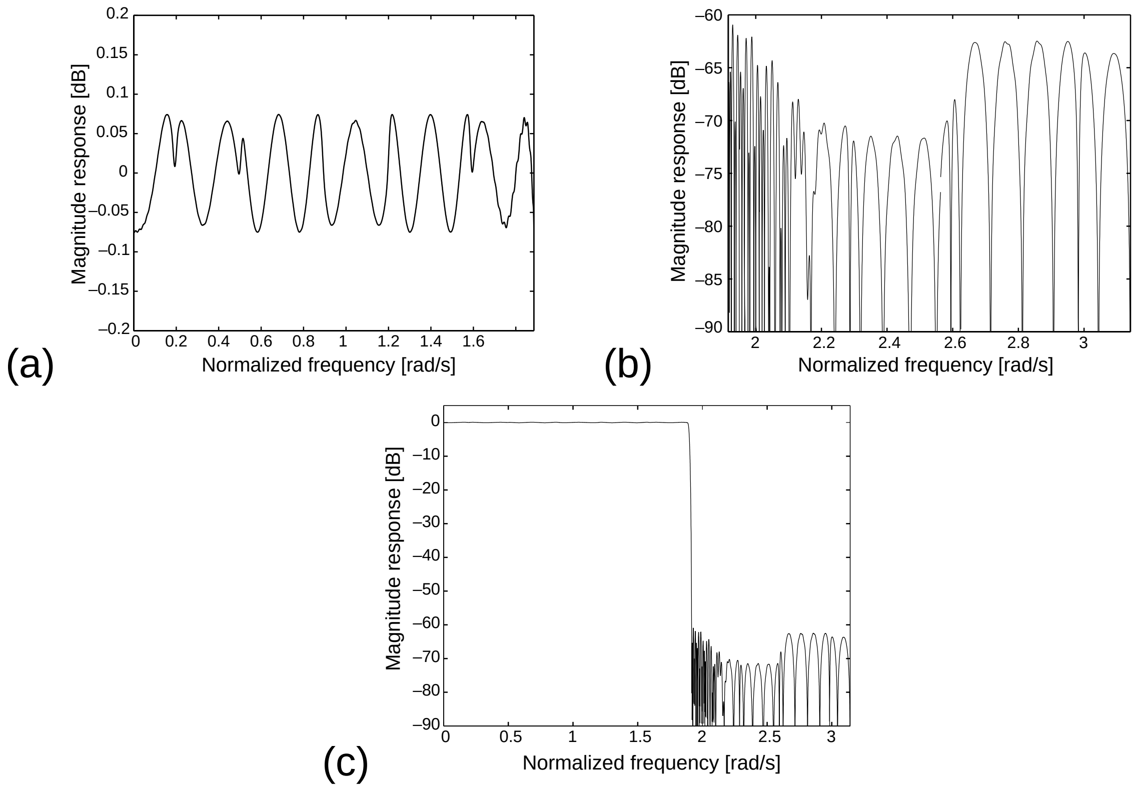


Figure 21: Magnitude response of the frequency response masking filter: (a) passband detail; (b) stopband detail; (c) overall response.

Example 12.7 - Solution

Table 5: Base filter $H_a(z)$ coefficients.

$h_a(0)$ to $h_a(39)$					
$h_a(0)$	$= -3.7728E{-}04$	$h_a(14)$	$= -1.7275E{-}03$	$h_a(28)$	$= 7.7616E{-}03$
$h_a(1)$	$= -2.7253E{-}04$	$h_a(15)$	$= -4.3174E{-}03$	$h_a(29)$	$= -2.6954E{-}02$
$h_a(2)$	$= 6.7027E{-}04$	$h_a(16)$	$= 3.9192E{-}03$	$h_a(30)$	$= 5.0566E{-}04$
$h_a(3)$	$= -1.1222E{-}04$	$h_a(17)$	$= 4.0239E{-}03$	$h_a(31)$	$= 3.5429E{-}02$
$h_a(4)$	$= -8.2895E{-}04$	$h_a(18)$	$= -6.5698E{-}03$	$h_a(32)$	$= -1.4927E{-}02$
$h_a(5)$	$= 4.1263E{-}04$	$h_a(19)$	$= -2.5752E{-}03$	$h_a(33)$	$= -4.3213E{-}02$
$h_a(6)$	$= 1.1137E{-}03$	$h_a(20)$	$= 9.3182E{-}03$	$h_a(34)$	$= 3.9811E{-}02$
$h_a(7)$	$= -1.0911E{-}03$	$h_a(21)$	$= -3.4385E{-}04$	$h_a(35)$	$= 4.9491E{-}02$
$h_a(8)$	$= -1.1058E{-}03$	$h_a(22)$	$= -1.1608E{-}02$	$h_a(36)$	$= -9.0919E{-}02$
$h_a(9)$	$= 1.9480E{-}03$	$h_a(23)$	$= 4.9074E{-}03$	$h_a(37)$	$= -5.3569E{-}02$
$h_a(10)$	$= 7.4658E{-}04$	$h_a(24)$	$= 1.2712E{-}02$	$h_a(38)$	$= 3.1310E{-}01$
$h_a(11)$	$= -2.9427E{-}03$	$h_a(25)$	$= -1.1084E{-}02$	$h_a(39)$	$= 5.5498E{-}01$
$h_a(12)$	$= 1.7063E{-}04$	$h_a(26)$	$= -1.1761E{-}02$		
$h_a(13)$	$= 3.8315E{-}03$	$h_a(27)$	$= 1.8604E{-}02$		

Example 12.7 - Solution

Table 6: Masking filter $H_{Ma}(z)$ coefficients.

$h_{Ma}(0)$ to $h_{Ma}(27)$		
$h_{Ma}(0) = 3.9894\text{E}{-}03$	$h_{Ma}(10) = -1.5993\text{E}{-}02$	$h_{Ma}(20) = 1.8066\text{E}{-}02$
$h_{Ma}(1) = 5.7991\text{E}{-}03$	$h_{Ma}(11) = -4.4088\text{E}{-}03$	$h_{Ma}(21) = -4.8343\text{E}{-}02$
$h_{Ma}(2) = 9.2771\text{E}{-}05$	$h_{Ma}(12) = 1.6123\text{E}{-}02$	$h_{Ma}(22) = -1.2214\text{E}{-}02$
$h_{Ma}(3) = -6.1430\text{E}{-}03$	$h_{Ma}(13) = 4.5664\text{E}{-}03$	$h_{Ma}(23) = 6.7391\text{E}{-}02$
$h_{Ma}(4) = -2.5059\text{E}{-}03$	$h_{Ma}(14) = -1.5292\text{E}{-}02$	$h_{Ma}(24) = -1.3277\text{E}{-}02$
$h_{Ma}(5) = 3.1213\text{E}{-}03$	$h_{Ma}(15) = 1.7599\text{E}{-}03$	$h_{Ma}(25) = -1.1247\text{E}{-}01$
$h_{Ma}(6) = -8.6700\text{E}{-}04$	$h_{Ma}(16) = 1.5389\text{E}{-}02$	$h_{Ma}(26) = 1.0537\text{E}{-}01$
$h_{Ma}(7) = -3.8008\text{E}{-}03$	$h_{Ma}(17) = -1.1324\text{E}{-}02$	$h_{Ma}(27) = 4.7184\text{E}{-}01$
$h_{Ma}(8) = 2.1950\text{E}{-}03$	$h_{Ma}(18) = -7.2774\text{E}{-}03$	
$h_{Ma}(9) = -3.8907\text{E}{-}03$	$h_{Ma}(19) = 3.7826\text{E}{-}02$	

Example 12.7 - Solution

Table 7: Masking filter $H_{Mc}(z)$ coefficients.

$h_{Mc}(0)$ to $h_{Mc}(22)$		
$h_{Mc}(0) = 1.9735\text{E}{-04}$	$h_{Mc}(8) = -1.3031\text{E}{-02}$	$h_{Mc}(16) = 2.3092\text{E}{-02}$
$h_{Mc}(1) = -7.0044\text{E}{-03}$	$h_{Mc}(9) = 3.5921\text{E}{-03}$	$h_{Mc}(17) = -5.8850\text{E}{-02}$
$h_{Mc}(2) = -7.3774\text{E}{-03}$	$h_{Mc}(10) = -4.7280\text{E}{-03}$	$h_{Mc}(18) = 7.3208\text{E}{-03}$
$h_{Mc}(3) = 1.9310\text{E}{-03}$	$h_{Mc}(11) = -2.5730\text{E}{-02}$	$h_{Mc}(19) = 5.5313\text{E}{-02}$
$h_{Mc}(4) = -3.0938\text{E}{-04}$	$h_{Mc}(12) = 6.5528\text{E}{-03}$	$h_{Mc}(20) = -1.2326\text{E}{-01}$
$h_{Mc}(5) = -7.1047\text{E}{-03}$	$h_{Mc}(13) = 1.1745\text{E}{-02}$	$h_{Mc}(21) = 9.7698\text{E}{-03}$
$h_{Mc}(6) = 3.0039\text{E}{-03}$	$h_{Mc}(14) = -3.2147\text{E}{-02}$	$h_{Mc}(22) = 5.4017\text{E}{-01}$
$h_{Mc}(7) = 5.8004\text{E}{-04}$	$h_{Mc}(15) = 7.8385\text{E}{-03}$	

Frequency response masking approach

- So far, we have discussed the use of the frequency response masking filters to design wideband lowpass filters.

- The design of narrowband lowpass filters can also be performed by considering that only one masking filter is necessary. Usually, we consider the branch formed by the base filter $\mathrm{H}_a(z)$ and its corresponding masking filter $\mathrm{H}_{\mathrm{M}a}(z)$, greatly reducing the overall complexity of the designed filter.

- In such cases, the frequency response masking approach becomes similar to the prefilter and interpolation approaches seen in previous subsections. The design of highpass filters can be inferred from the design for lowpass filters, or can be performed using the concept of complementary filters seen in the beginning of this subsection.

- The design of bandpass and bandstop filters with reduced arithmetic complexity is addressed in the next subsection.

Quadrature approach

- In this subsection, a method for designing symmetric bandpass and bandstop filters is introduced. For narrowband filters, the so-called quadrature approach uses an FIR prototype of the form:

$$H_p(z) = H_a(z^L)H_M(z) \tag{101}$$

  where $H_a(z)$ is the base filter and $H_M(z)$ is the masking filter or interpolator, which attenuates the undesired spectral images of the passband of $H_a(z^L)$, commonly referred to as the shaping filter.

- Such a prototype can be designed using prefilter, interpolation, or simplified one-branch frequency response masking approaches seen above.

- The main idea of the quadrature approach is to shift the frequency response of the base filter to the desired central frequency ω_o, and then apply the masking filter (interpolator) to eliminate any other undesired passbands.

Quadrature approach

- Consider a linear-phase lowpass filter $H_a(z)$ with impulse response $h_a(n)$, such that

$$H_a(z) = \sum_{n=0}^{M} h_a(n) z^{-n} \tag{102}$$

- Let the passband ripple and stopband gain be equal to δ'_p and δ'_r, and the passband and stopband edges be ω'_p and ω'_r, respectively.

- If $h_a(n)$ is interpolated by a factor L, and the resulting sequence is multiplied by $e^{j\omega_o n}$, we generate an auxiliary $H_1(z^L)$ as

$$H_1(z^L) = \sum_{n=0}^{M} h_a(n) e^{j\omega_o n} z^{-nL} \tag{103}$$

Quadrature approach

- This implies that the passband of $H_a(z)$ is squeezed by a factor of L, and becomes centered at ω_o. Analogously, using the interpolation operation followed by the modulating sequence $e^{-j\omega_o n}$, we have another auxiliary function such that

$$H_2(z^L) = \sum_{n=0}^{M} h_a(n) e^{-j\omega_o n} z^{-nL} \tag{104}$$

  with the corresponding squeezed passband centered at $-\omega_o$.

- We can then use two masking filters, $H_{M1}(z)$ and $H_{M2}(z)$, appropriately centered at ω_o and $-\omega_o$ to eliminate the undesired bands in $H_1(z^L)$ and $H_2(z^L)$, respectively.

- Clearly, although the two-branch overall bandpass filter will have real coefficients, each branch in this case will present complex coefficients.

- To overcome this problem, first note that $H_1(z^L)$ and $H_2(z^L)$ have complex conjugate coefficients.

Quadrature approach

- If we design $H_{M1}(z)$ and $H_{M2}(z)$ so that their coefficients are complex conjugates of each other, it is easy to verify that

$$
\begin{aligned}
H_1(z^L)H_{M1}(z) &= \left(H_{1,R}(z^L) + jH_{1,I}(z^L)\right)\left(H_{M1,R}(z) + jH_{M1,I}(z)\right) \\
&= \left(H_{1,R}(z^L)H_{M1,R}(z) - H_{1,I}(z^L)H_{M1,I}(z)\right) \\
&\quad +j\left(H_{1,R}(z^L)H_{M1,I}(z) + H_{1,I}(z^L)H_{M1,R}(z)\right) \quad (105) \\
H_2(z^L)H_{M2}(z) &= \left(H_{2,R}(z^L) + jH_{2,I}(z^L)\right)\left(H_{M2,R}(z) + jH_{M2,I}(z)\right) \\
&= \left(H_{1,R}(z^L) - jH_{1,I}(z^L)\right)\left(H_{M1,R}(z) - jH_{M1,I}(z)\right) \\
&= \left(H_{1,R}(z^L)H_{M1,R}(z) - H_{1,I}(z^L)H_{M1,I}(z)\right) \\
&\quad -j\left(H_{1,R}(z^L)H_{M1,I}(z) + H_{1,I}(z^L)H_{M1,R}(z)\right) \quad (106)
\end{aligned}
$$

where the subscripts R and I indicate the parts of the corresponding transfer function with real and imaginary coefficients, respectively.

- Therefore,

$$H_1(z^L)H_{M1}(z)+H_2(z^L)H_{M2}(z) = 2\left(H_{1,R}(z^L)H_{M1,R}(z)-H_{1,I}(z^L)H_{M1,I}(z)\right) \tag{107}$$

and the structure seen in Figure 22 can be used for the real implementation of the quadrature approach for narrowband filters.

- Disregarding the effects of the masking filters, the resulting quadrature filter is characterized by

$$\left.\begin{aligned}
\delta_p &= \delta'_p + \delta'_r \\
\delta_r &= 2\delta'_r \\
\omega_{r_1} &= \omega_o - \omega'_r \\
\omega_{p_1} &= \omega_o - \omega'_p \\
\omega_{p_2} &= \omega_o + \omega'_p \\
\omega_{r_2} &= \omega_o + \omega'_r
\end{aligned}\right\} \tag{108}$$

Quadrature approach

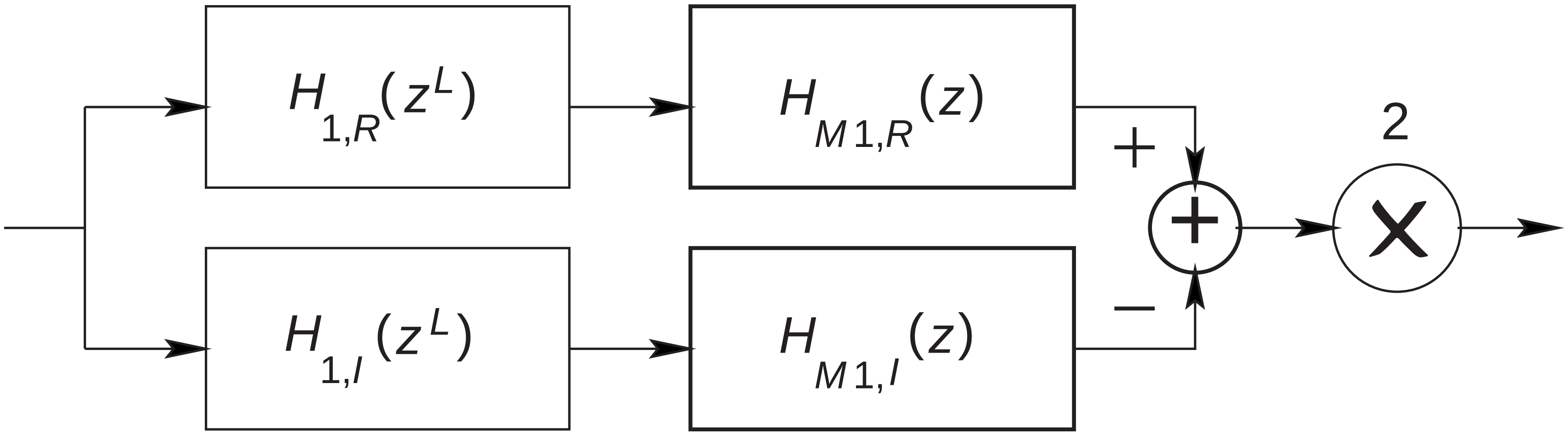


Figure 22: Block diagram of the quadrature approach for narrowband filters.

Quadrature approach

- For wideband filters, the prototype filter should be designed with the frequency response masking approach.
- In that case, we have two complete masking filters, and the quadrature implementation involving solely real filters is seen in Figure 23, with $H_1(z^L)$ as defined in equation (103), and $H_{Ma}(z)$ and $H_{Mc}(z)$ corresponding to the two masking filters, appropriately centered at ω_o and $-\omega_o$, respectively.
- For bandstop filters, we may start with a highpass prototype and apply the quadrature design, or design a bandpass filter and then determine its complementary filter.

Quadrature approach

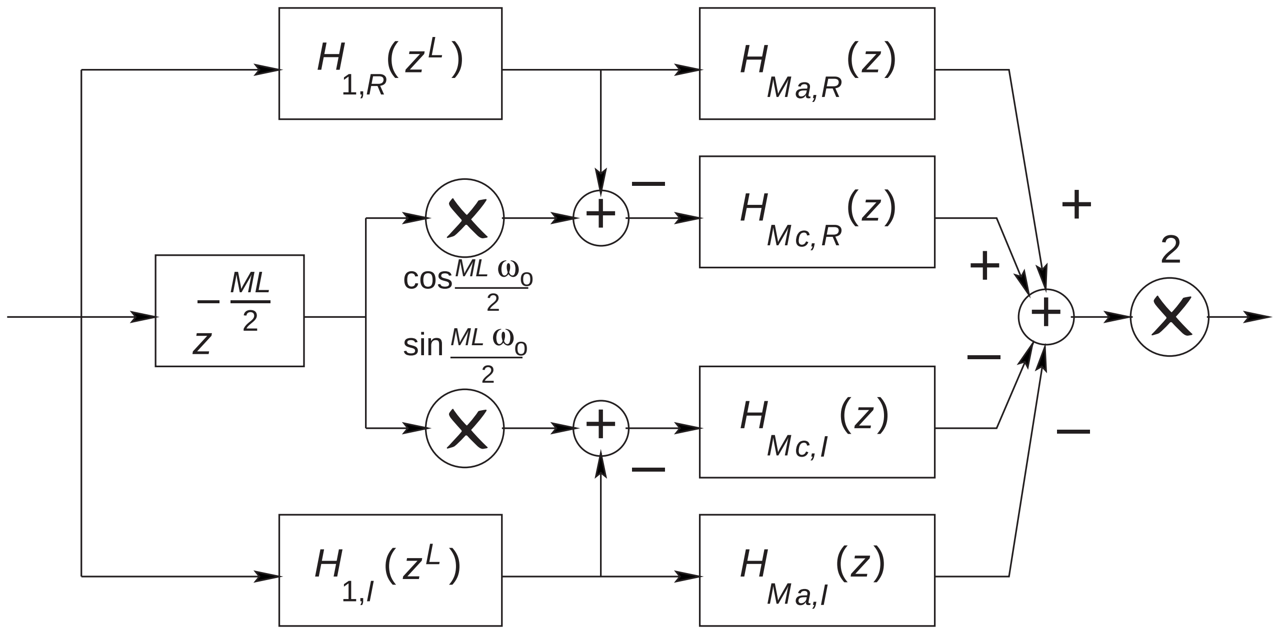


Figure 23: Block diagram of the quadrature approach for wideband filters.

Example 12.8

- Design a bandpass filter using the quadrature method satisfying the following specifications:

$$\left.\begin{aligned} A_p &= 0.2 \text{ dB} \\ A_r &= 40 \text{ dB} \\ \Omega_{r_1} &= 0.09\pi \text{ rad/s} \\ \Omega_{p_1} &= 0.1\pi \text{ rad/s} \\ \Omega_{p_2} &= 0.7\pi \text{ rad/s} \\ \Omega_{r_2} &= 0.71\pi \text{ rad/s} \\ \Omega_s &= 2\pi \text{ rad/s} \end{aligned}\right\} \quad (109)$$

Example 12.8 - Solution

- Given the bandpass specifications, the lowpass prototype filter must have a passband half the size of the desired passband width, and a transition bandwidth equal to the minimum transition bandwidth for the bandpass filter.

- For the passband ripple and stopband attenuation, the values specified for the bandpass filter can be used with a margin of about 40%. Therefore, in this example, the lowpass prototype is characterized by

$$\left.\begin{aligned}
\delta'_p &= 0.0115 \times 40\% = 0.0046 \\
\delta'_r &= 0.01 \times 40\% = 0.004 \\
\omega'_p &= \frac{\omega_{p_2} - \omega_{p_1}}{2} = 0.3\pi \\
\omega'_r &= \omega'_p + \min\{(\omega_{p_1} - \omega_{r_1}), (\omega_{r_2} - \omega_{p_2})\} = 0.31\pi
\end{aligned}\right\} \qquad (110)$$

Example 12.8 - Solution

- This filter can be designed using the frequency response masking approach with an efficient ripple margin assignment, seen in the previous subsection. In this case, the interpolation factor that minimizes the total number of multiplications is $L = 8$ and the corresponding filter characteristics are given in Table 8, with the resulting magnitude response in Figure 24.

Table 8: Filter characteristics for the interpolation factor $L = 8$.

L	M_{H_a}	$M_{H_{Ma}}$	$M_{H_{Mc}}$	Π	M
8	**58**	**34**	**42**	**70**	**506**

Example 12.8 - Solution

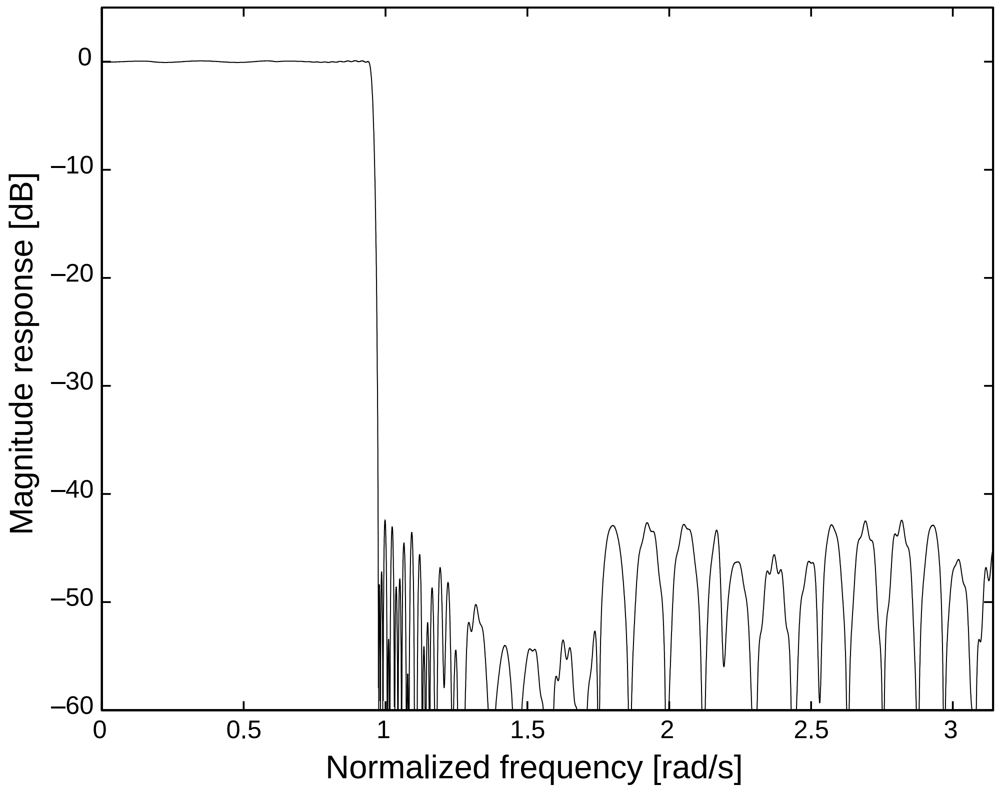


Figure 24: Lowpass prototype designed with the frequency response masking approach for the quadrature design of a bandpass filter.

Example 12.8 - Solution

- The resulting bandpass filter using the quadrature method is shown in Figure 25.
- For the complete quadrature realization, the total number of multiplications is 140, twice the number of multiplications necessary for the prototype lowpass filter. For this example, the minimax filter would be of order 384, thus requiring 193 multiplications per output sample. Therefore, in this case, the quadrature design represents a saving of about 30% of the number of multiplications required by the standard minimax filter.

Example 12.8 - Solution

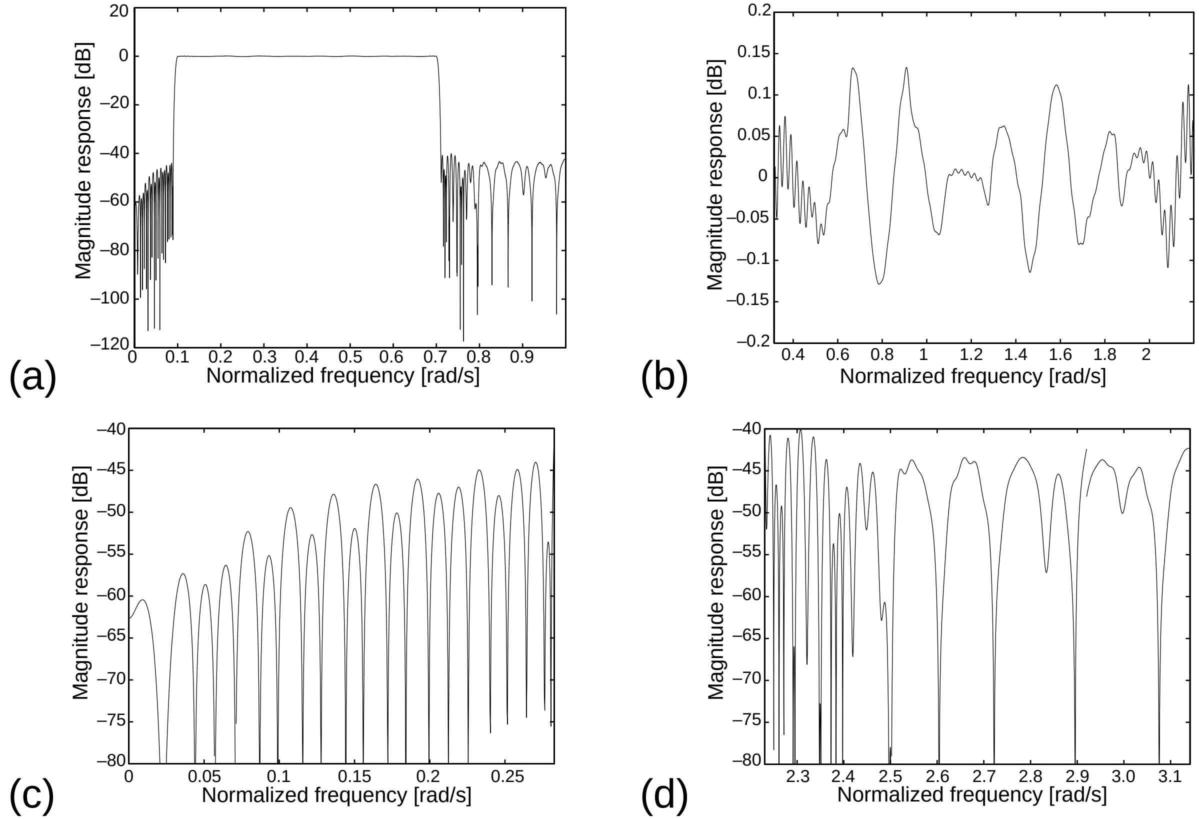


Figure 25: Magnitude responses of the bandpass filter designed with the quadrature approach: (a) overall filter; (b) passband detail; (c) lower stopband detail; (d) upper stopband detail.

Do-it-yourself: Efficient FIR structures

- **Experiment 12.1:** A base-band telephone speech signal occupies the frequency range around 300–3600 Hz. Let us emulate such a signal as a sum of four sinusoids as given by

```
Fs = 40000; Ts = 1/Fs; time = 0:Ts:(1-Ts);
f1 = 300; f2 = 1000; f3 = 2500; f4 = 3600;
s1 = sin(2*pi*f1*time); s2 = sin(2*pi*f2*time);
s3 = sin(2*pi*f3*time); s4 = sin(2*pi*f4*time);
x = s1 + s2 + s3 + s4;
```

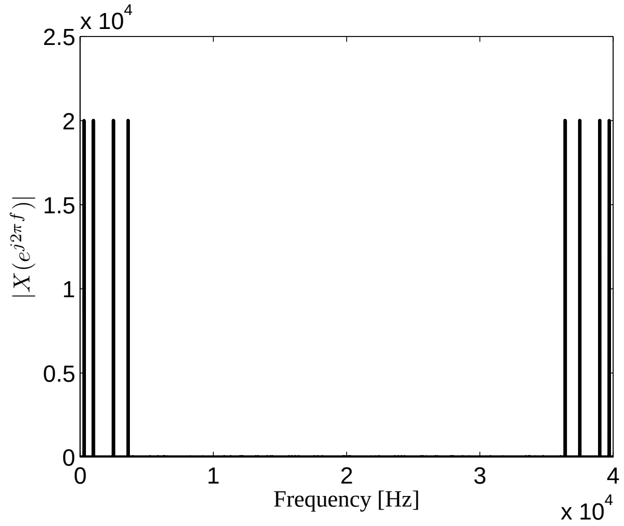
- Using the modulation theorem, we can shift the spectrum of `x` of `fc` by multiplying this signal by a cosine function, that is

```
fc = 10000;
xDSB = x.*cos(2*pi*fc*time);
```

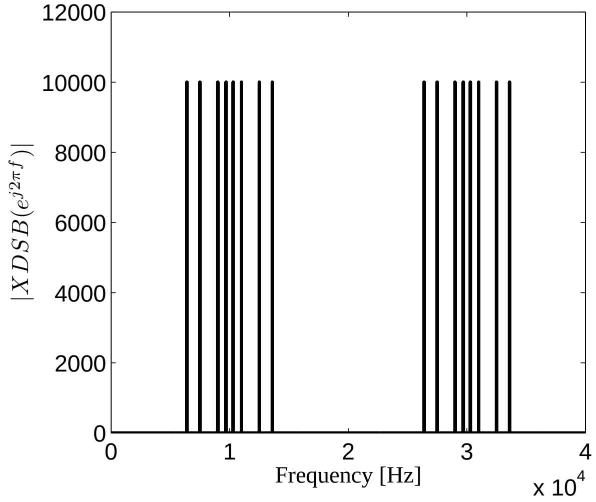
Do-it-yourself: Efficient FIR structures

- In this way, several speech signals can fit in a single communication channel by using distinct values of `fc` for each one. The spectral representations of `x` before and after modulation are seen in Figure 26.
- It this figure, one clearly notices that the spectrum of `xDSB` occupies twice the band of the spectrum of `x`, thus originating the double sideband (DSB) nomenclature for that signal.

Do-it-yourself: Efficient FIR structures



(a)



(b)

Figure 26: Spectra of emulated speech signal: (a) base-band x; (b) modulated xDSB.

Do-it-yourself: Efficient FIR structures

- For a single speech signal, doubling the band does not seem much of a problem, as the additional 4 kHz can be easily dealt with in most communications systems.
- However, when you think of millions of users of the telephone system, this doubling effect can generate an unwelcome overload on the given channel. Therefore, we consider here the elimination of the so-called upper part of the spectrum of `xDSB`, within 10.3–13.6 kHz.
- In this illustrative experiment, one can design a Chebyshev filter using `firpm` command in MATLAB, for instance, using the frequency range between $\omega_{\rm p} = 2\pi 9700$ rad/s and $\omega_{\rm r} = 2\pi 10300$ rad/s as the filter transition band.

Do-it-yourself: Efficient FIR structures

- However, in more demanding cases, of higher sampling frequency or even narrower transition bands, an efficient FIR structure must be employed, as exemplified here.

- A frequency response masking filter for this application can be designed in MATLAB such as

```
wp = (fc-f1)*2*pi/Fs; wr = (fc+f1)*2*pi/Fs;
```

- Using $\mathrm{L} = 5$, the $\mathrm{H}_{\mathrm{Ma}}(z)$ masking filter defines the transition band, and therefore, from equations (77)–(79), we have that

```
L = 5; m = floor(wp*L/(2*pi));
theta = wp*L-m*2*pi; phi = wr*L-m*2*pi;
```

Do-it-yourself: Efficient FIR structures

- It must be emphasized that different specifications may force the reader to employ equations (86)–(88) instead.
- Setting the passband ripple and attenuation levels to $A_\mathrm{p} = 0.1$ dB and $A_\mathrm{r} = 50$ dB, respectively, we get

```
Ap = 0.1; delta_p = (10^(Ap/20)-1)/(10^(Ap/20)+1);
Ar = 50; delta_r = 10^(-Ar/20);
```

  and then the `firpmord` and `firpm` commands can be used in tandem to design the FRM base filter, as given by

```
Fvec_b = [theta phi]./pi;
[M,f_b,m_b,w_b] = firpmord(Fvec_b,[1 0],[delta_p
delta_r]);
hb = firpm(M,f_b,m_b,w_b);
```

Do-it-yourself: Efficient FIR structures

- In this stage, as discussed above, one must remember to enforce an even base-filter order `M`, increasing it by 1 if necessary. In the script above, `M`=32 and no order increment is required.

- The interpolated base filter can be formed as

```
hbL = [hb; zeros(L-1,M+1)];
hbL = reshape(hbL,1,L*(M+1));
```

  and the corresponding complementary filter as

```
hbLc = -hbL;
hbLc(M*L/2 + 1) = 1-hbL(M*L/2 + 1);
```

Do-it-yourself: Efficient FIR structures

- The positive masking filter, as specified in equations (80) and (81), is designed as

```
wp_p = (2*m*pi+theta)/L; wr_p = (2*(m+1)*pi-phi)/L;
Fvec_p = [wp_p wr_p]./pi;
[M_p,f_p,m_p,w_p] = firpmord(Fvec_p,[1 0],[delta_p
delta_r]);
hp = firpm(M_p,f_p,m_p,w_p);
```

- And the negative masking filter, as given by equations (82) and (81), may be determined as

```
wp_n = (2*m*pi-theta)/L; wr_n = (2*m*pi+phi)/L;
Fvec_n = [wp_n wr_n]./pi;
[M_n,f_n,m_n,w_n] = firpmord(Fvec_n,[1 0],[delta_p
delta_r]);
hn = firpm(M_n,f_n,m_n,w_n);
```

Do-it-yourself: Efficient FIR structures

- The magnitude responses for the interpolated base filter `hbL`, complementary filter `hbLc`, positive masking filter `hp`, and negative masking filter `hn` are shown in Figure 27 for the entire 0–F_s frequency range.

- The impulse response for the general frequency response masking filter can be obtained as

```
hFRM = conv(hbL,hp)+conv(hbLc,hm);
```

  which corresponds to the magnitude response shown in Figure 28.

Do-it-yourself: Efficient FIR structures

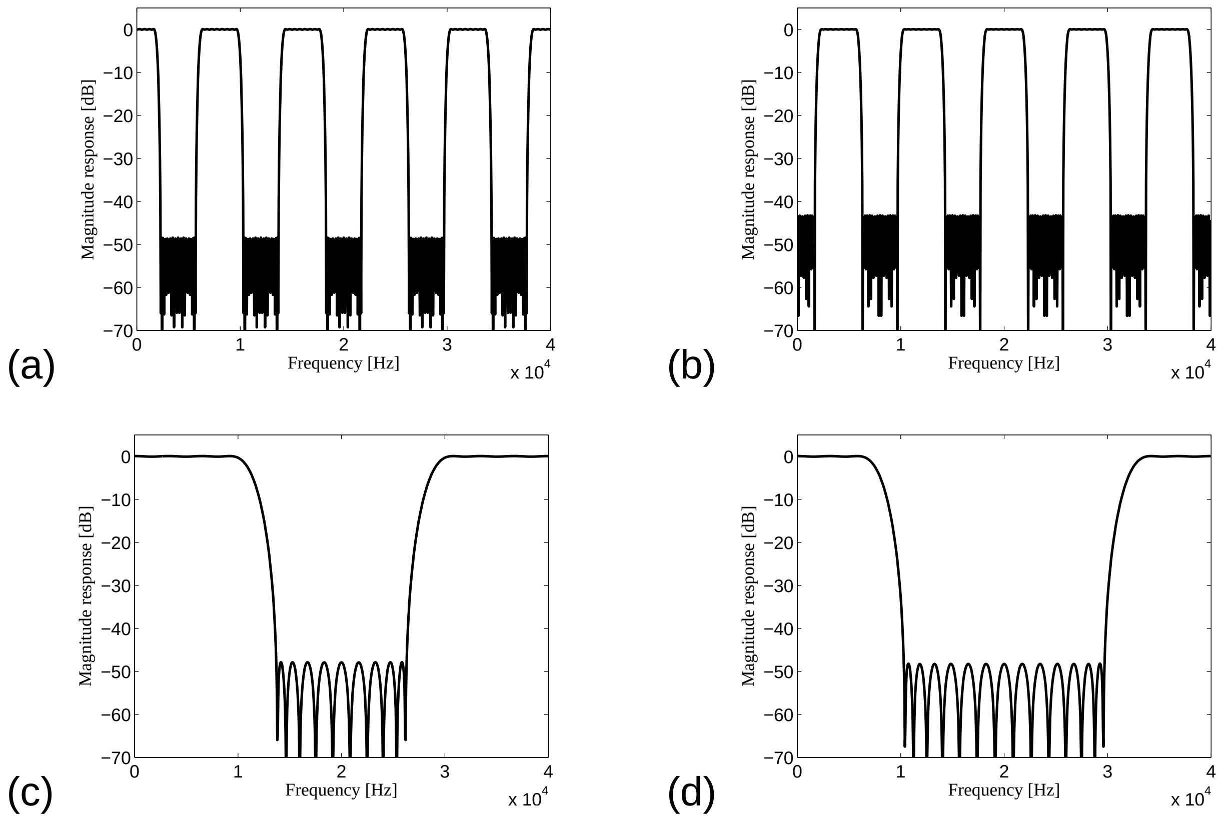


Figure 27: Magnitude responses of FRM subfilters: (a) interpolated base filter hbL; (b) complementary filter hbLc; (c) positive masking filter hp; (d) negative masking filter hn.

Do-it-yourself: Efficient FIR structures

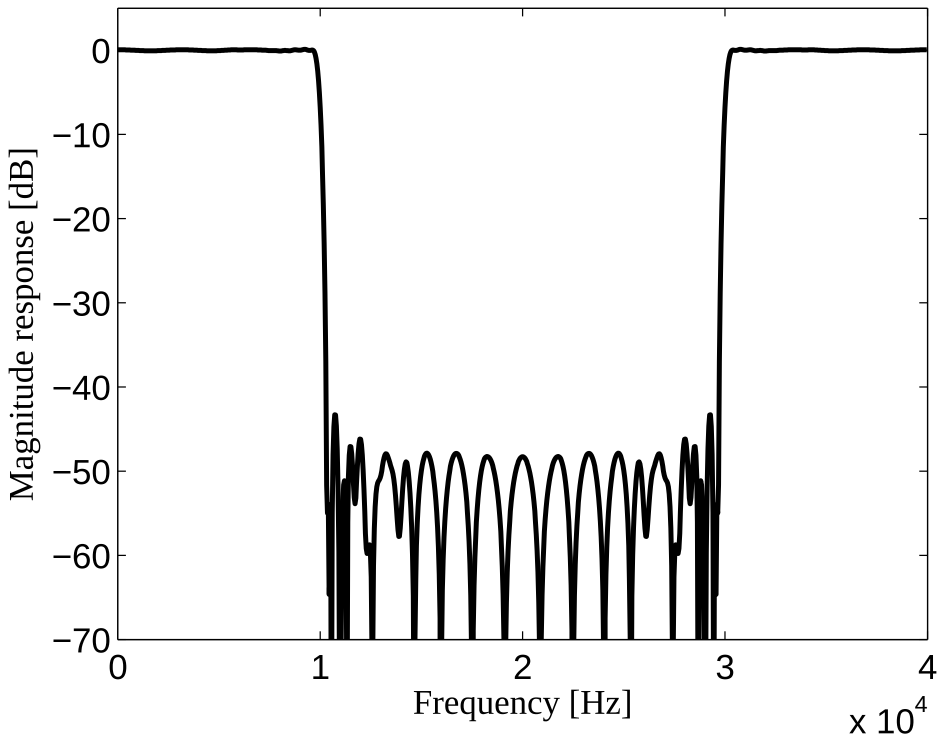


Figure 28: Magnitude response of frequency response masking filter.

Do-it-yourself: Efficient FIR structures

- Applying the `xDSB` to the input of the frequency response masking filter designed above yields the single sideband (SSB) `xSSB` modulated signal

  ```
  xSSB = filter(hFRM,1,xDSB);
  ```

  which presents the same bandwidth as the original base-band signal `x`, as depicted in Figure 29.

- In practical real-time applications, `hFRM` should not be employed to perform the desired filtering operation, since it does not benefit from the modular frequency response masking internal structure.

- In such case, signal processing should be based directly on the individual `hbL`, `hp`, and `hm` filters to reduce the computational effort in determining each output sample.

Do-it-yourself: Efficient FIR structures

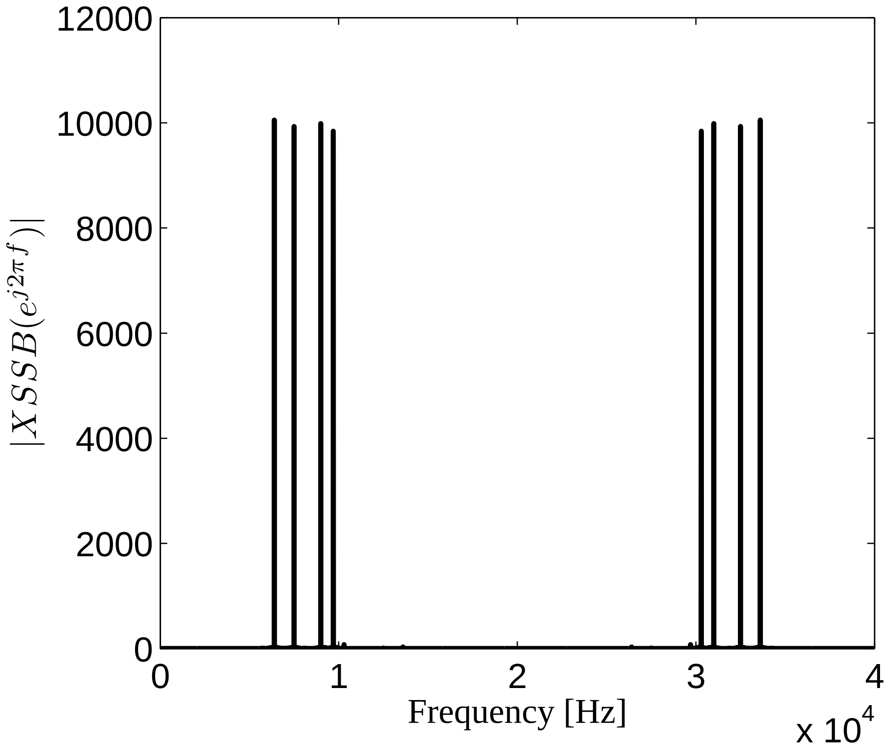


Figure 29: Spectrum of xSSB signal.