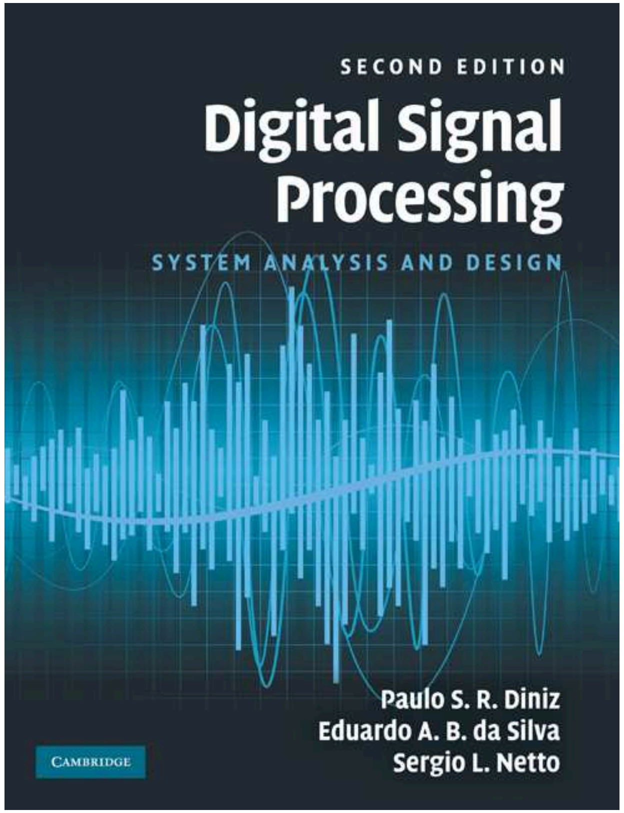
# Filter Banks

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# Contents

# Introduction

- In a number of applications, it is necessary to split a digital signal into several frequency bands.
- After such decomposition, the signal is represented by more samples than in the original stage.
- Systems which decompose and reassemble the signals are generally called filter banks.
- In the following deal with filter banks, showing several ways in which a signal can be decomposed into critically decimated frequency bands, and recovered from them with minimum error.

# Filter banks

- In some applications, such as signal analysis, signal transmission, and signal coding, a digital signal $x(n)$ is decomposed into several frequency bands.

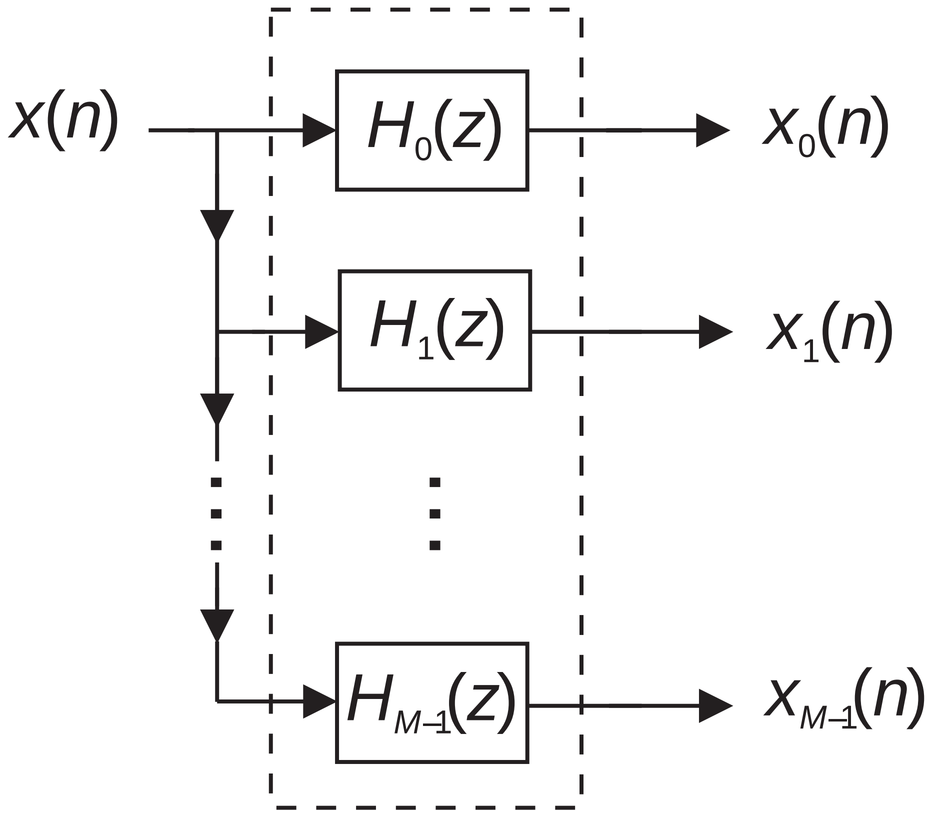


Figure 1: Decomposition of a digital signal into $M$ frequency bands.

# Filter banks

- The signal in each of the bands $x_k(n)$, for $k = 0, 1, \ldots, (M-1)$, has at least the same number of samples as the original signal.
- This implies that after the $M$-band decomposition, the signal is represented with at least $M$ times more samples than the original one.
- In the case where the signal is uniformly split in the frequency domain, each band has bandwidth $M$ times smaller than the one of the original signal.
- The bands $x_k(n)$ can be decimated by a factor of $M$ (critically decimated) without destroying the original.

# Decimation of a bandpass signal

- If the input signal $x(m)$ is lowpass and band-limited to $[-\frac{\pi}{M}, \frac{\pi}{M}]$, the aliasing after decimation by a factor of $M$ can be avoided.
- If before decimation the signal is split into $M$ uniform real frequency bands, the $k$th band will be confined to $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}] \cup [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$.
- This implies that band $k$, for $k \neq 0$, is necessarily not confined to $[-\frac{\pi}{M}, \frac{\pi}{M}]$.
- The spectrum contained in $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}]$ is mapped into $[0, \pi]$, if $k$ is odd, or into $[-\pi, 0]$, if $k$ is even.
- The spectrum contained in the interval $[\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$ is mapped into $[-\pi, 0]$, if $k$ is odd, or into $[0, \pi]$, if $k$ is even.
- Note that that both $\omega = \frac{2l\pi}{M}$ and $\omega = -\frac{2l\pi}{M}$ in the original signal are mapped to $\omega = 0$ in the decimated signal.
- In order to allow proper reconstruction of signals having components of the form $A_l \cos\left(\frac{2l\pi}{M}n\right)$, the ideal filters must have half the passband gain for $\omega = \pm\frac{2l\pi}{M}$.

# Decimation of a bandpass signal

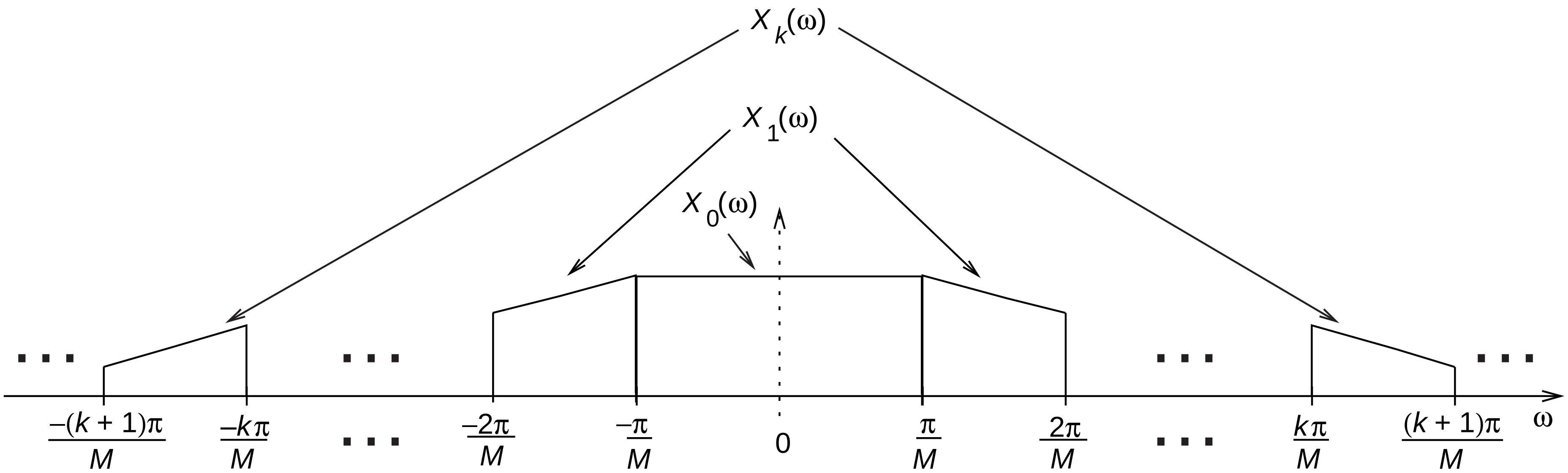


Figure 2: Uniform split of a signal into $M$ real bands.

# Decimation of a bandpass signal

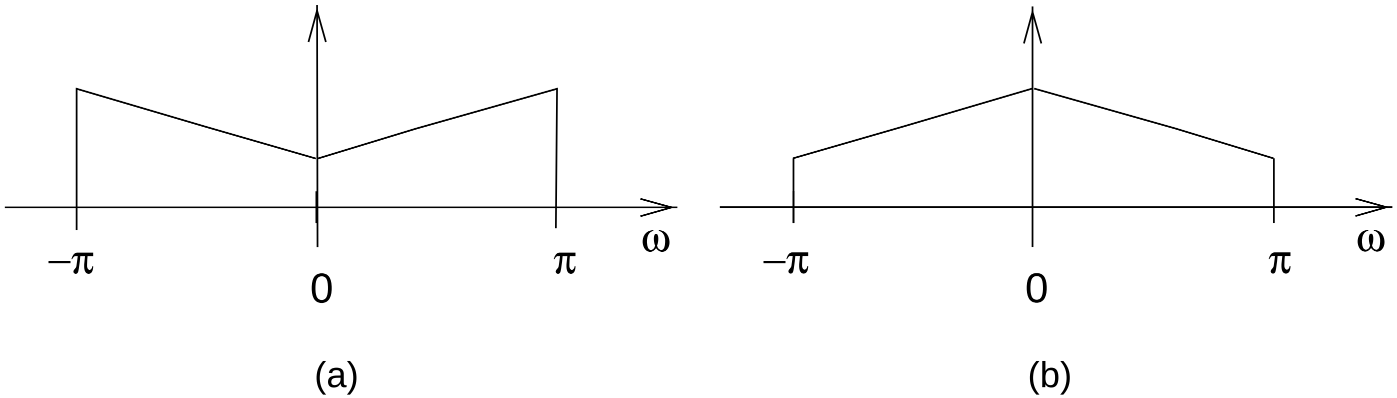


Figure 3: Spectrum of band $k$ decimated by a factor of $M$: (a) $k$ odd; (b) $k$ even.

## Inverse decimation of a bandpass signal

- A bandpass signal can be decimated by $M$ without aliasing, provided that its spectrum is confined to $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}] \cup [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$.
- The original bandpass signal be recovered from its decimated version by an interpolation operation in the bandpass case.

# Inverse decimation of a bandpass signal

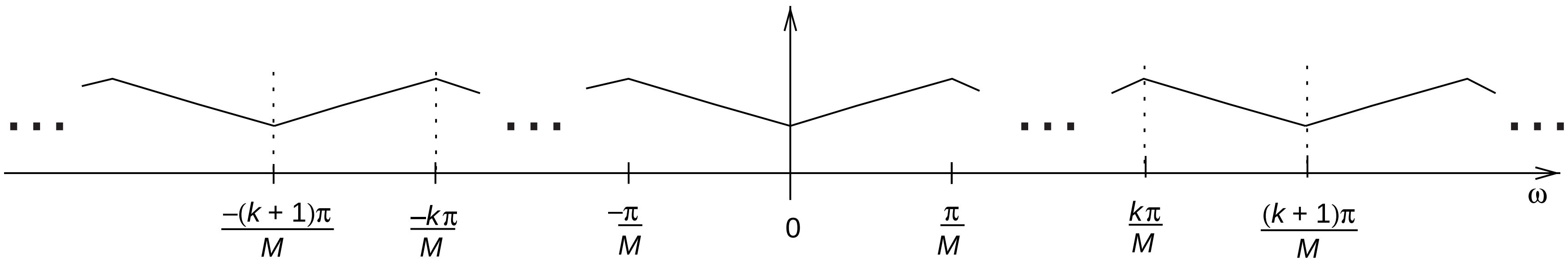


Figure 4: Spectrum of band $k$ after decimation and interpolation by a factor of $M$ for $k$ odd.

# Inverse decimation of a bandpass signal

- To recover band $k$, it suffices to keep the region of the spectrum in Figure 4 within $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}] \cup [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$.
- For $k$ even, the procedure is entirely analogous.
- The process of decimating and interpolating a bandpass signal is similar to the case of a lowpass signal, with the difference that for the bandpass case $H(z)$ must be a bandpass filter with bandwidth $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}] \cup [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$.

## Perfect Reconstruction: Critically decimated $M$-band filter banks

- If a signal $x(m)$ is decomposed into $M$ non-overlapping bandpass channels $B_k$, with $k = 0, 1, \ldots, (M-1)$, such that $\bigcup_{k=0}^{M-1} B_k = [-\pi, \pi)$, then it can be recovered by just summing these $M$ channels.

- Exact recovery of the original signal may not be possible if each channel is decimated by $M$.

- We examined a way to recover the bandpass signal from its decimated version. In fact, all that is needed are interpolations followed by filters with passband $[-\frac{(k+1)\pi}{M}, -\frac{k\pi}{M}] \cup [\frac{k\pi}{M}, \frac{(k+1)\pi}{M}]$.

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# Critically decimated $M$-band filter banks

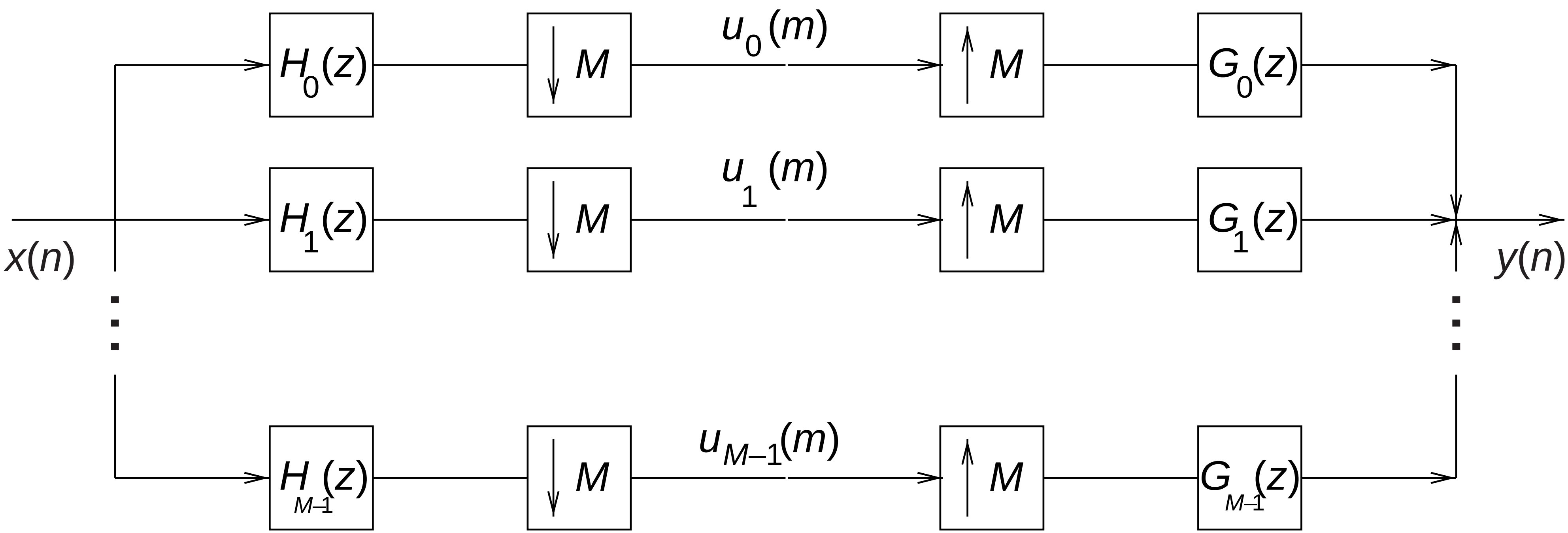


Figure 5: Block diagram of an $M$-band filter bank.

# Critically decimated $\mathcal{M}$-band filter banks

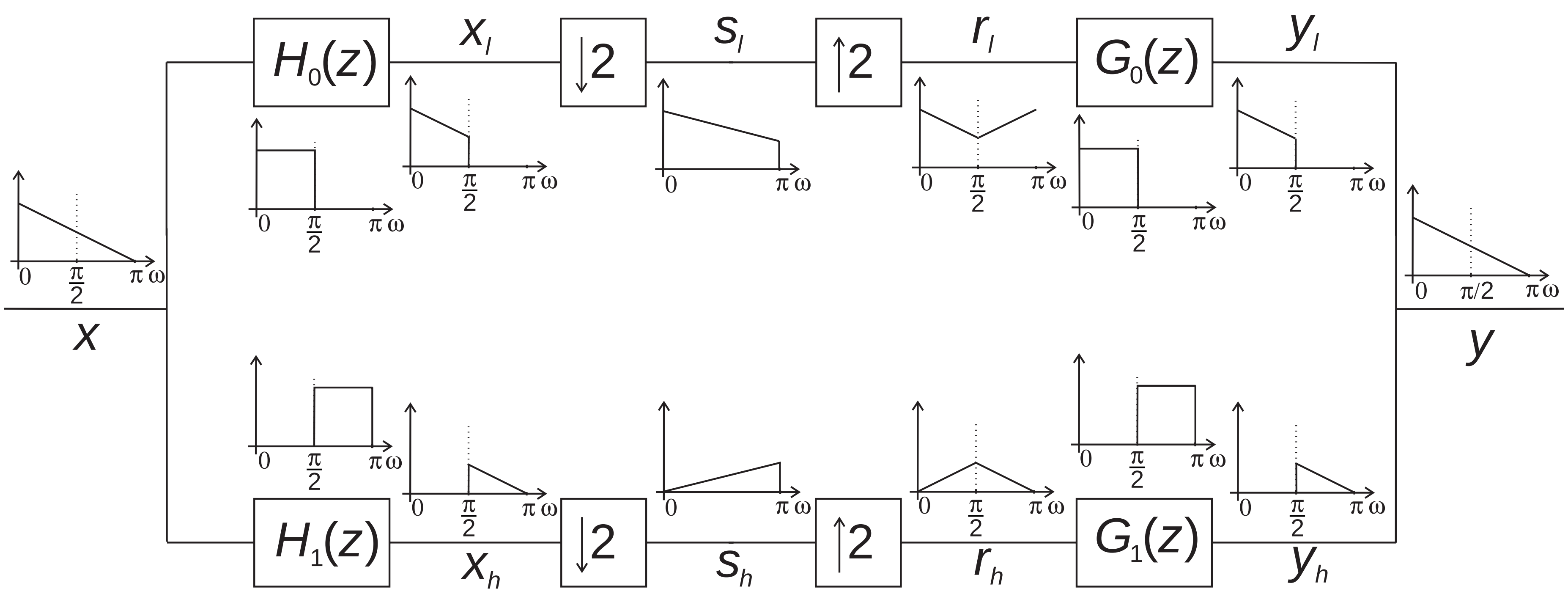


Figure 6: A 2-band perfect reconstruction filter bank using ideal filters.

# Critically decimated $\mathcal{M}$-band filter banks

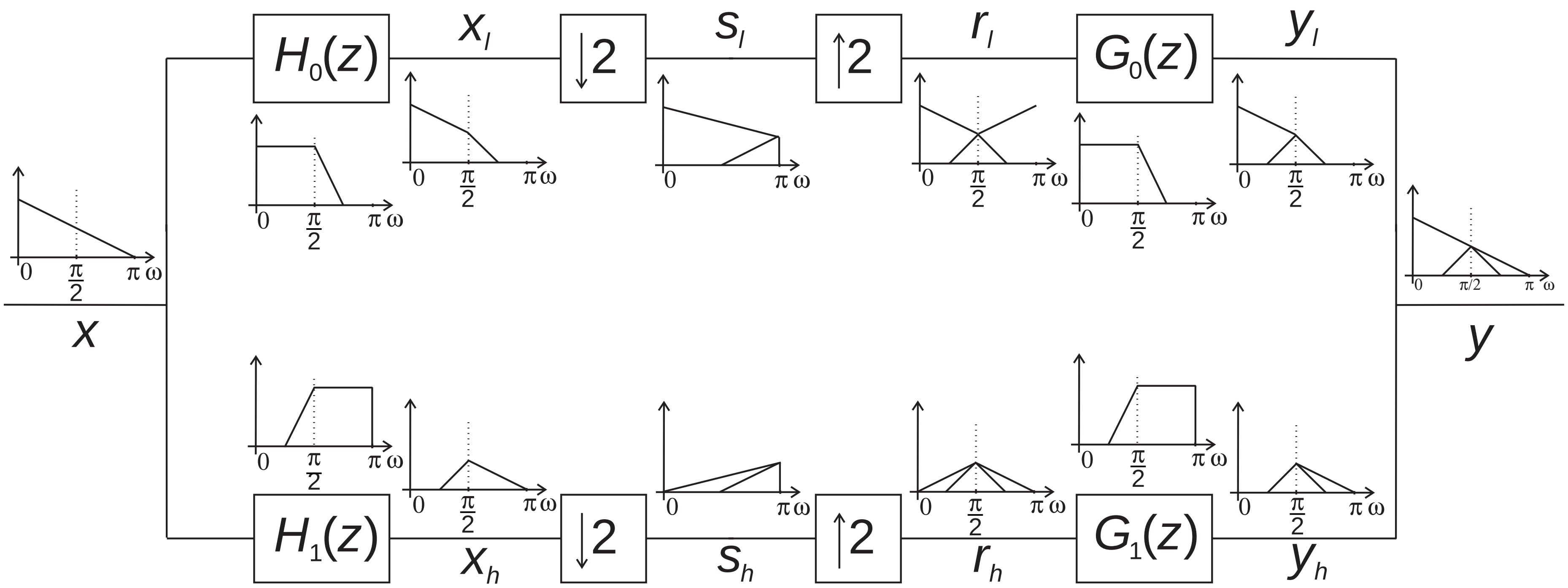


Figure 7: Two-band filter bank using realizable filters.

# Critically decimated $M$-band filter banks

- The filters required for the $M$-band perfect reconstruction filter bank described above are not realizable.
- In a first analysis, the original signal would be only approximately recoverable from its decimated frequency bands.
- One can see that since the filters $H_0(z)$ and $H_1(z)$ are not ideal, the sub-bands $s_l(m)$ and $s_h(m)$ have aliasing.
- The signals $x_l(n)$ and $x_h(n)$ can not be correctly recovered from $s_l(m)$ and $s_h(m)$, respectively.

## Critically decimated $M$-band filter banks

- Nevertheless, one can see that since $y_l(n)$ and $y_h(n)$ are added in order to obtain $y(n)$, the aliased components of $y_l(n)$ can be combined with the ones of $y_h(n)$.

- In principle, there is no reason why these aliased components could not be made to cancel each other, yielding $y(n)$ equal to $x(n)$. In such a case, the original signal could be recovered from its sub-band components.

- In an $M$-band filter bank as shown in Figure 5, the filters $H_k(z)$ and $G_k(z)$ are usually referred to as the analysis and synthesis filters of the filter bank.

# Perfect reconstruction: $M$-band filter banks in terms of polyphase components

- Representing $H_k(z)$ and $G_k(z)$ by their polyphase components,

$$H_k(z) = \sum_{j=0}^{M-1} z^{-j} E_{kj}(z^M) \tag{1}$$

$$G_k(z) = \sum_{j=0}^{M-1} z^{-(M-1-j)} R_{jk}(z^M) \tag{2}$$

where $E_{kj}(z)$ is the $j$th polyphase component of $H_k(z)$, and $R_{jk}(z)$ is the $j$th polyphase component of $G_k(z)$.

## Perfect reconstruction: $M$-band filter banks in terms of polyphase components

- Matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ entries are $E_{ij}(z)$ and $R_{ij}(z)$, for $i, j = 0, 1, \ldots, (M-1)$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \mathbf{E}(z^M) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix} \quad (3)$$

$$\begin{bmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_{M-1}(z) \end{bmatrix} = \mathbf{R}^{\mathrm{T}}(z^M) \begin{bmatrix} z^{-(M-1)} \\ z^{-(M-2)} \\ \vdots \\ 1 \end{bmatrix} \quad (4)$$

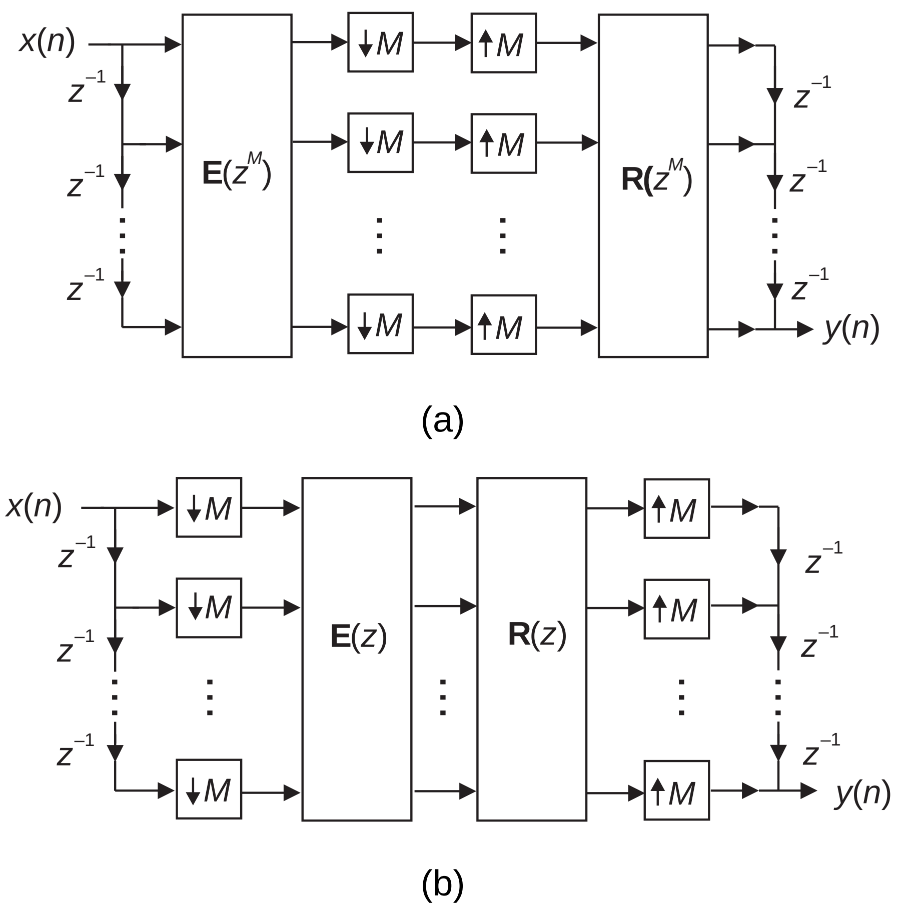


Figure 8: $M$-band filter bank in terms of the polyphase components: (a) before application of the noble identities; (b) after application of the noble identities.

# $M$-band filter banks in terms of polyphase components

- In signal processing it is often advantageous to split a sequence $x(k)$ into several frequency bands prior to processing.

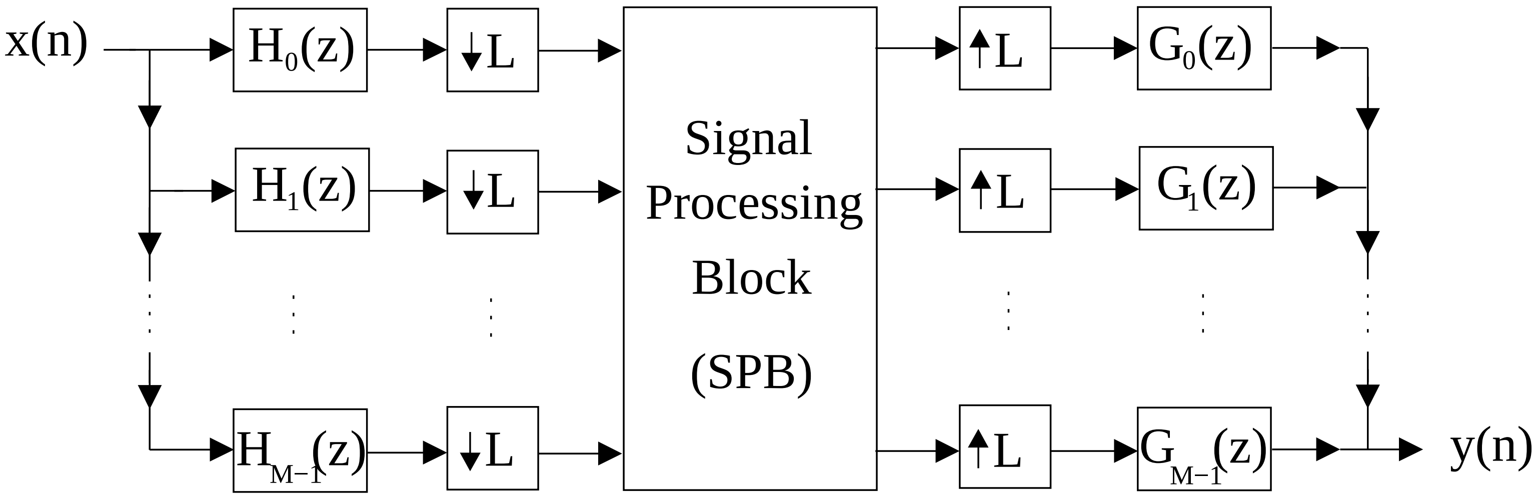


Figure 9: Signal processing in sub-bands.

# $M$-band filter banks in terms of polyphase components

- Since the bandwidth of each analysis filter output is $M$ times smaller than in the original signal, we can decimate each $x_i(k)$ by a factor of $L$ smaller or equal to $M$ and still avoid aliasing.
- For $L \leq M$, it is possible to retain all information contained in the input signal by properly designing the analysis filters in conjunction with the synthesis filters $G_i(z)$, for $i = 0, 1, \ldots, (M-1)$.
- If $L > M$ there is a loss of information due to aliasing which does not allow the recovery of the original signal.
- For $L = M$, we refer to the filter bank as maximally (or critically) decimated.
- For $L < M$, the filter bank is called oversampled (or noncritically sampled) since the set of sub-bands comprises more samples than the input signal.

# Perfect reconstruction $M$-band filter banks

- If $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, the $M$-band filter bank becomes that one shown in Figure 10.

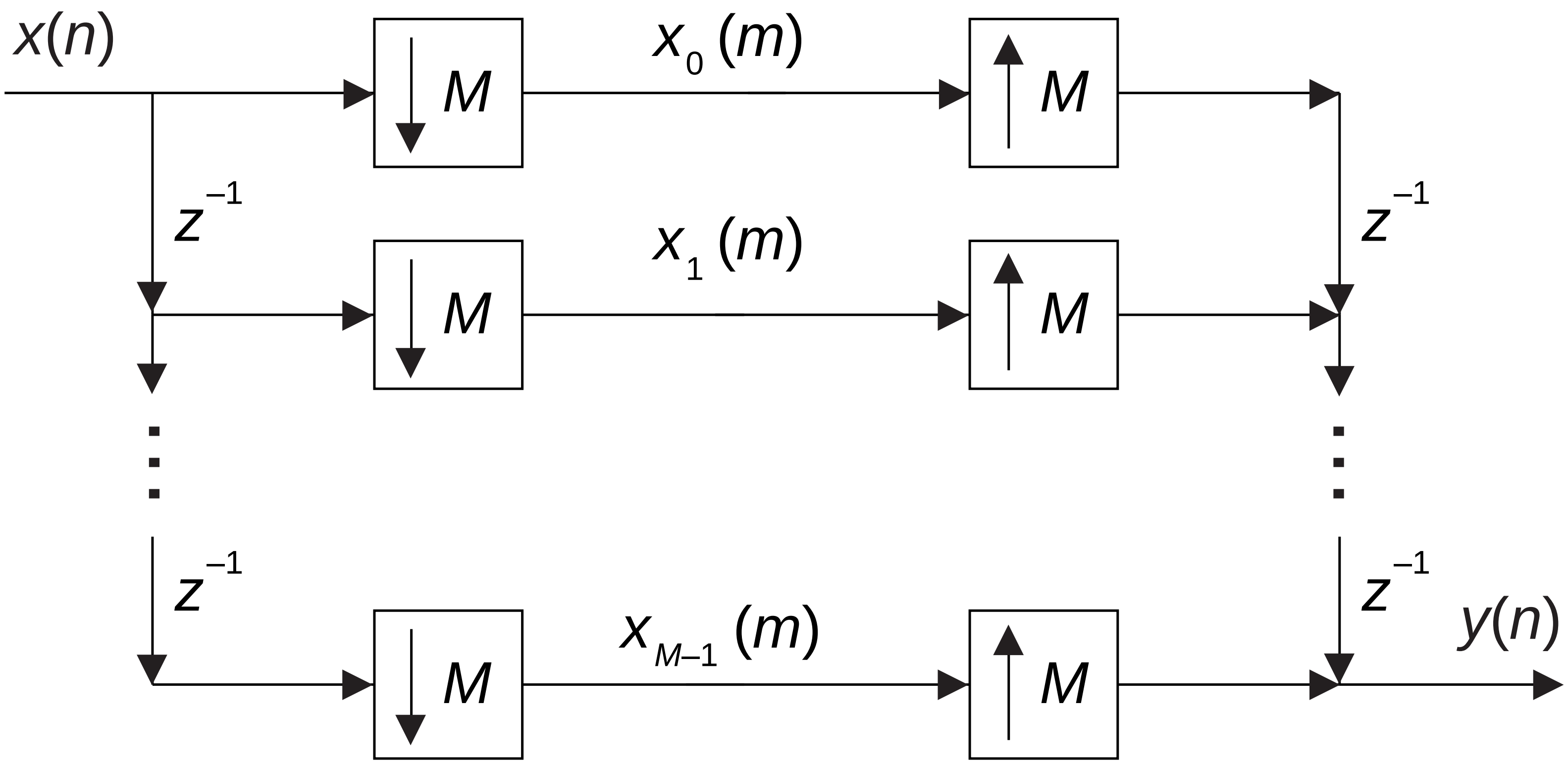


Figure 10: $M$-band filter bank when $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$.

# Perfect reconstruction $M$-band filter banks

- By substituting the decimators and interpolators by the commutator models we arrive at the a scheme which is clearly equivalent to a pure delay.
- Therefore, the condition $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$ guarantees perfect reconstruction for the $M$-band filter bank.
- If $\mathbf{R}(z)\mathbf{E}(z)$ is equal to a pure delay one can still consider that perfect reconstruction holds.
- The weaker condition is sufficient for perfect reconstruction.

$$\mathbf{R}(z)\mathbf{E}(z) = z^{-\Delta}\mathbf{I} \tag{5}$$

- The total delay introduced by a perfect reconstruction filter bank is

$$\Delta_{\text{total}} = M\Delta + M - 1 \tag{6}$$

where $M\Delta$ is the delay originated from the polyphase matrices product and the term $(M-1)$ accounts for the delay introduced by the commutator.

# Perfect reconstruction $M$-band filter banks

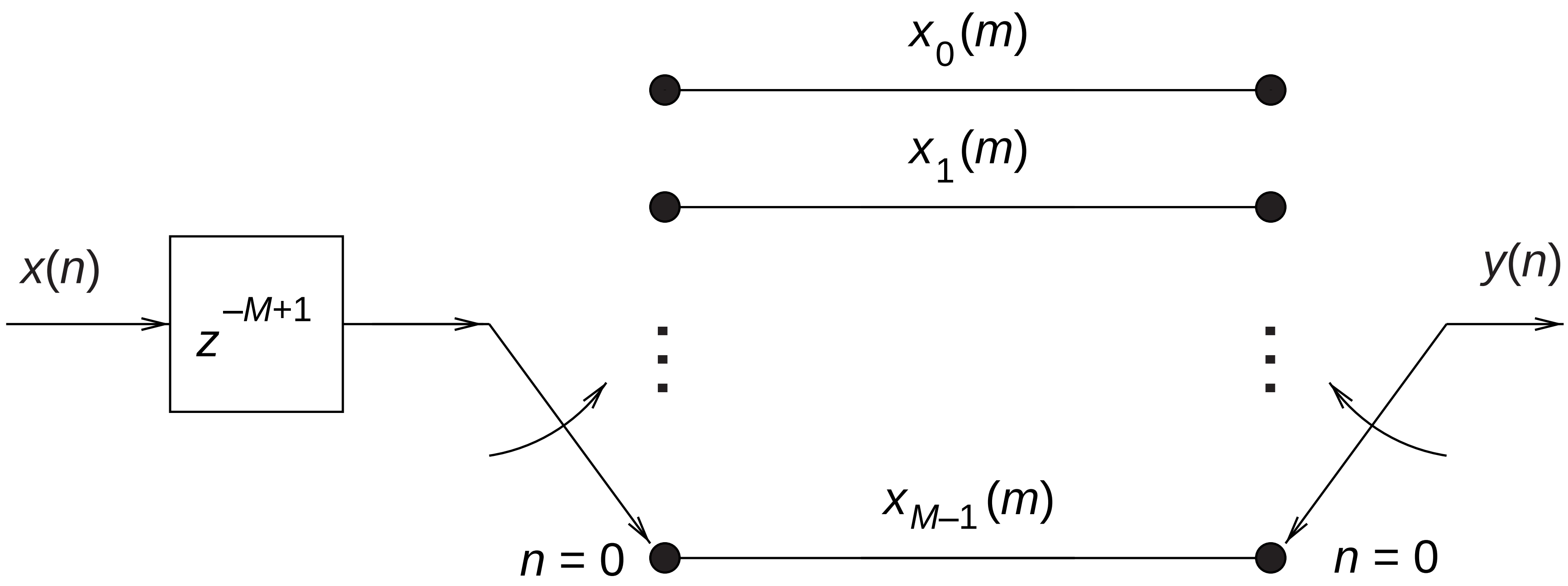


Figure 11: The commutator model of an $M$-band filter bank when $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$ is equivalent to a pure delay.

## Perfect reconstruction $M$-band filter banks

How a simple perfect reconstruction filter bank can be built?

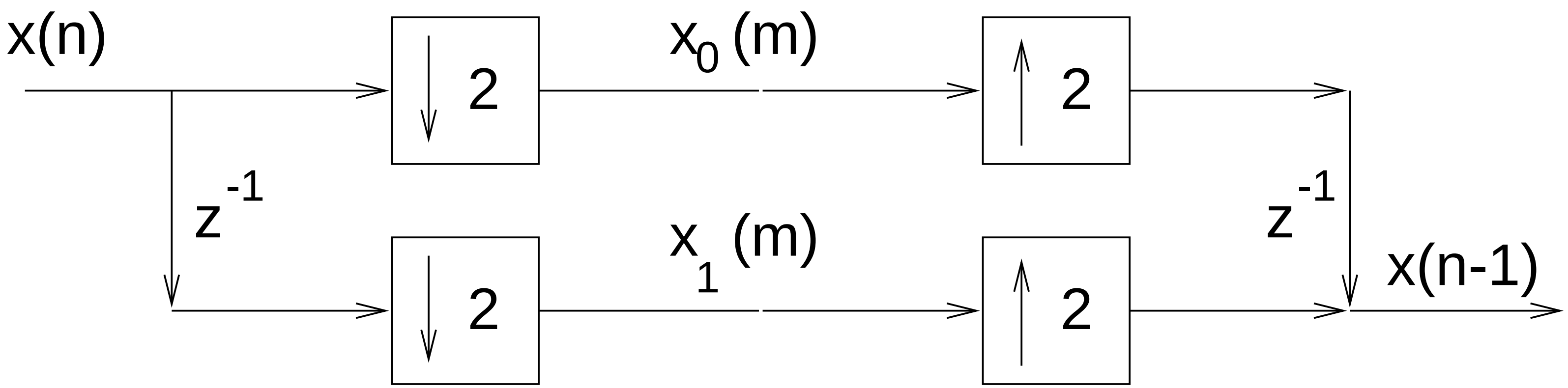


Figure 12: 2-band unit delay.

## Perfect reconstruction $M$-band filter banks

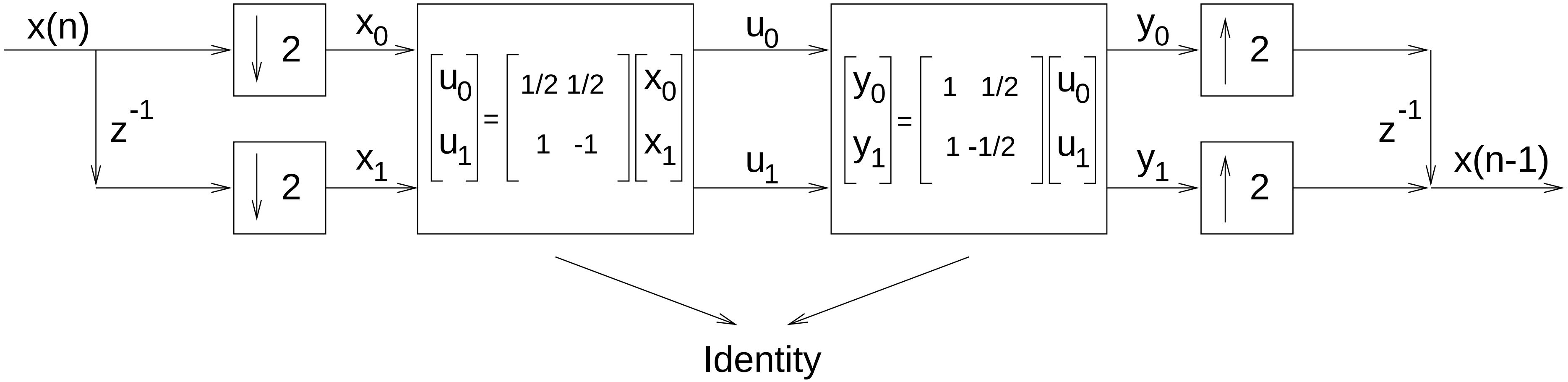


Figure 13: 2-band unit delay, including inverse matrices.

# Perfect reconstruction $M$-band filter banks

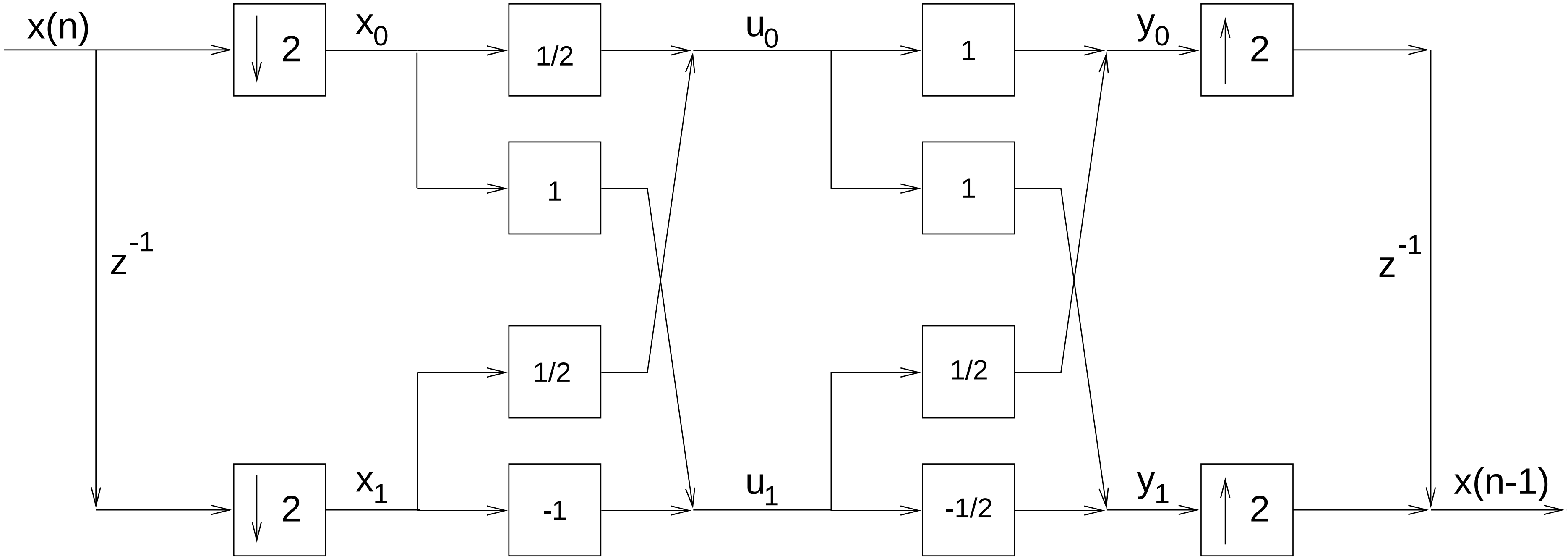


Figure 14: 2-band unit delay, with explicit realization of the matrix products.

# Perfect reconstruction $M$-band filter banks

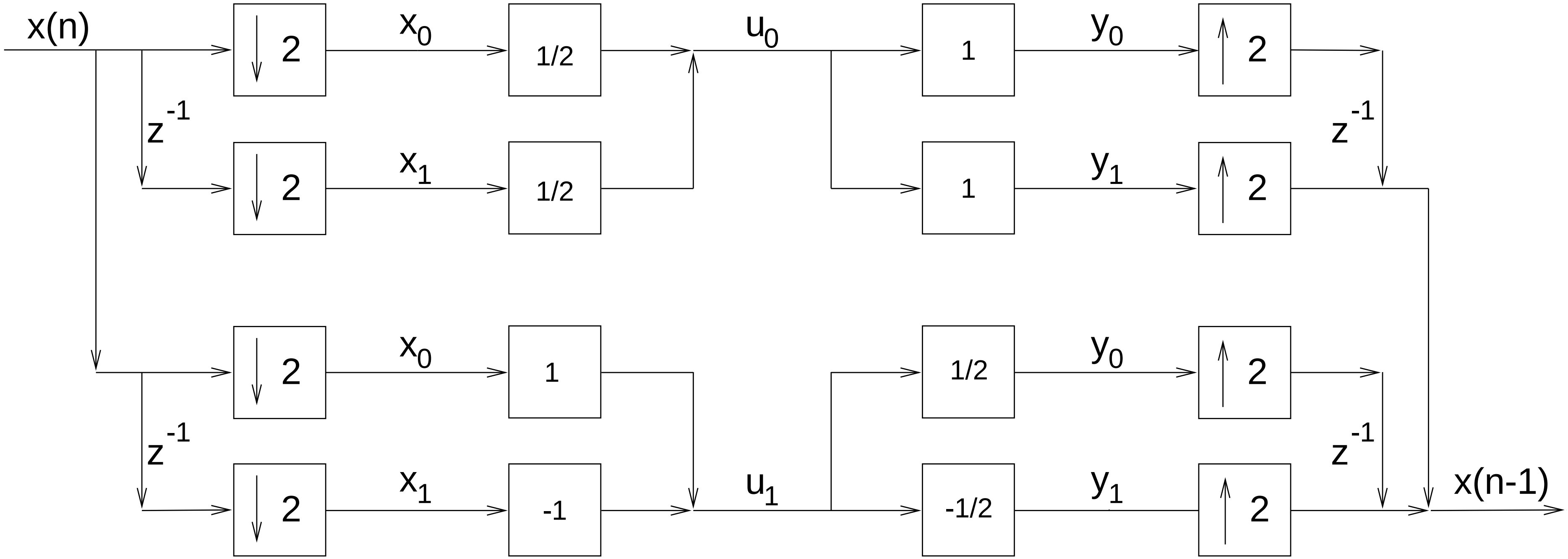


Figure 15: 2-band unit delay, splitting the decimators and interpolators.

## Perfect reconstruction $M$-band filter banks

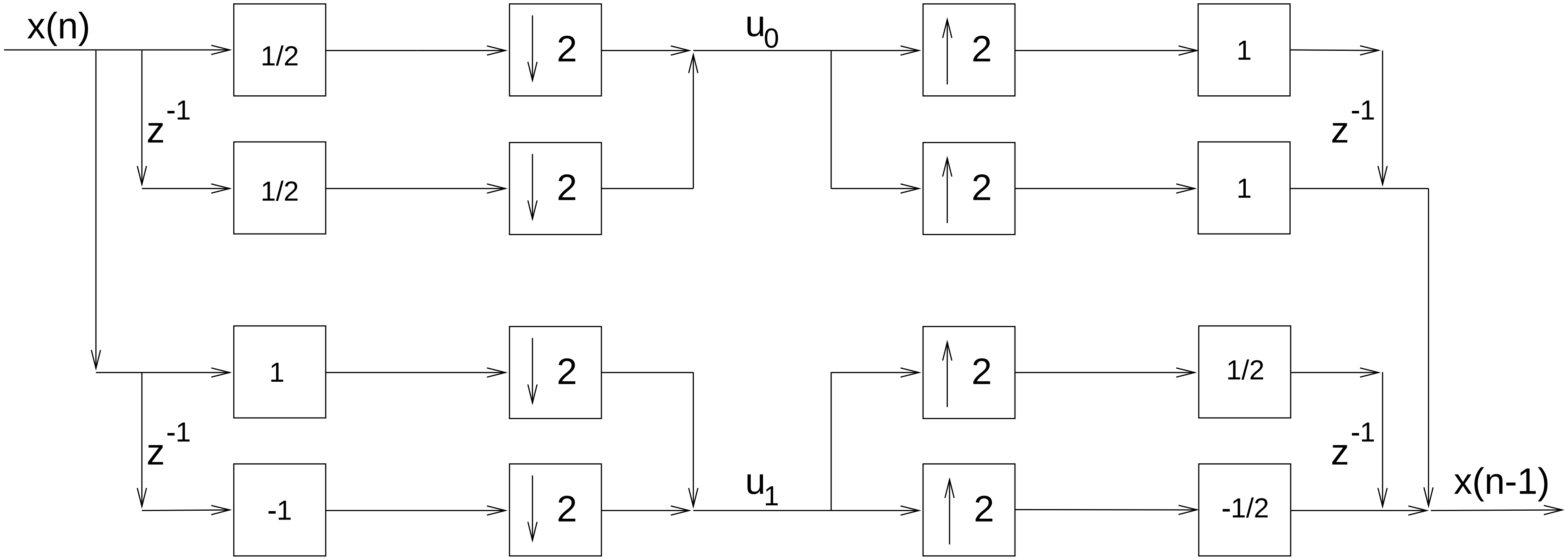


Figure 16: 2-band unit delay, moving the decimators and interpolators.

## Perfect reconstruction $M$-band filter banks

By "merging" the decimators/interpolators we reach the realization of the unit delay as shown in Figure 17. Figure 17 is equivalent to the filter bank in Figure 18.

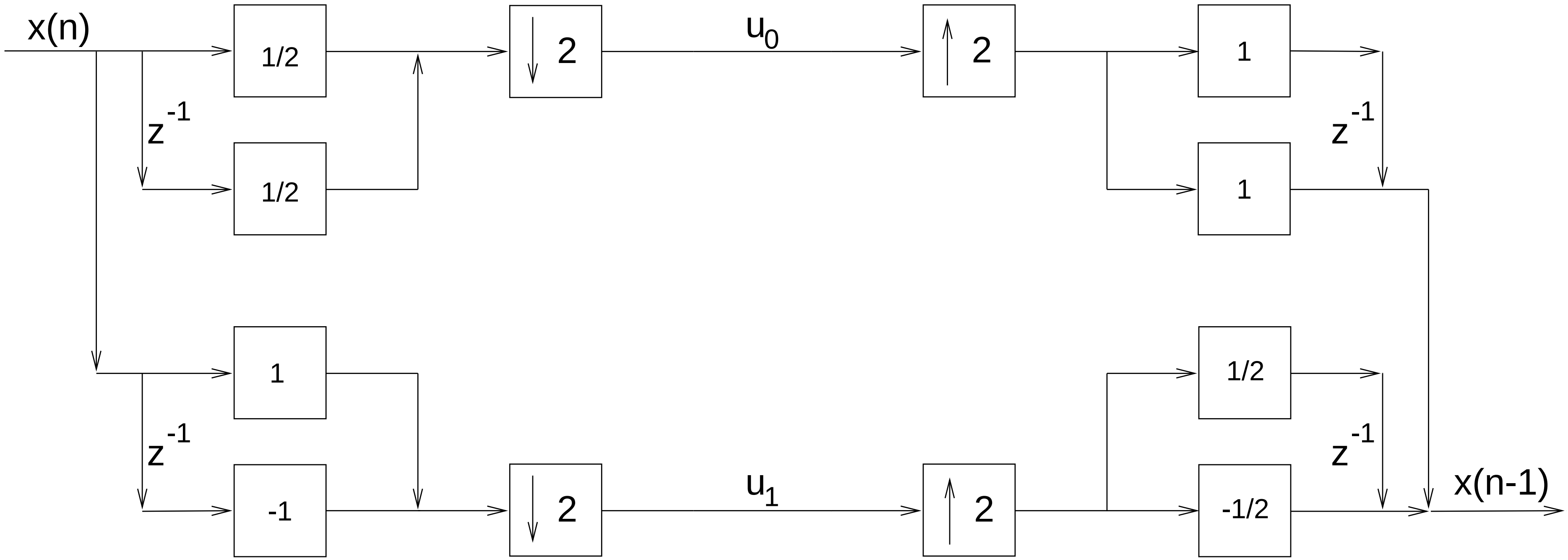


Figure 17: 2-band unit delay, merging the decimators and interpolators.

# Perfect reconstruction $M$-band filter banks

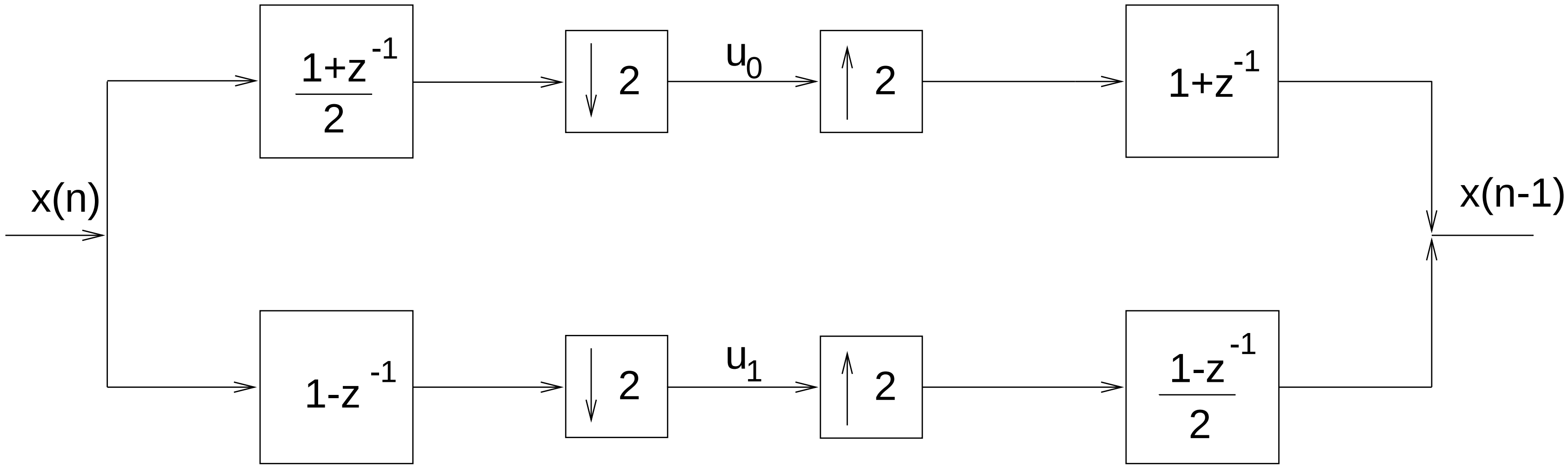


Figure 18: 2-band filter bank with perfect reconstruction.

# $M$-band filter banks in terms of polyphase components

**Example 9.1**

Let $M = 2$, and

$$\mathbf{E}(z) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} \tag{7}$$

$$\mathbf{R}(z) = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} \tag{8}$$

Show that these matrices characterize a perfect reconstruction filter bank, and find the analysis and synthesis filters and their corresponding polyphase components.

# $M$-band filter banks in terms of polyphase components

**Solution**

- Clearly $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$, and the filter bank yields perfect reconstruction. The polyphase components $E_{kj}(z)$ of the analysis filters $H_k(z)$, and $R_{jk}(z)$ of the synthesis filters $G_k(z)$ are then

$$E_{00}(z) = \frac{1}{2}, \qquad E_{01}(z) = \frac{1}{2}, \qquad E_{10}(z) = 1, \qquad E_{11}(z) = -1 \tag{9}$$

$$R_{00}(z) = 1, \qquad R_{01}(z) = \frac{1}{2}, \qquad R_{10}(z) = 1, \qquad R_{11}(z) = -\frac{1}{2} \tag{10}$$

# $M$-band filter banks in terms of polyphase components

- We can find $H_k(z)$, and $G_k(z)$.

$$H_0(z) = \frac{1}{2}(1 + z^{-1}) \tag{11}$$

$$H_1(z) = 1 - z^{-1} \tag{12}$$

$$G_0(z) = 1 + z^{-1} \tag{13}$$

$$G_1(z) = -\frac{1}{2}(1 - z^{-1}) \tag{14}$$

- The magnitude response of $G_k(z)$ is equal to the one of $H_k(z)$ for $k = 0, 1$ except for a gain constant.

- Perfect reconstruction could be achieved with filters that are far from being ideal.

- Each sub-band is highly aliased, still we recover the original signal exactly. $\triangle$

## Perfect reconstruction $M$-band filter banks

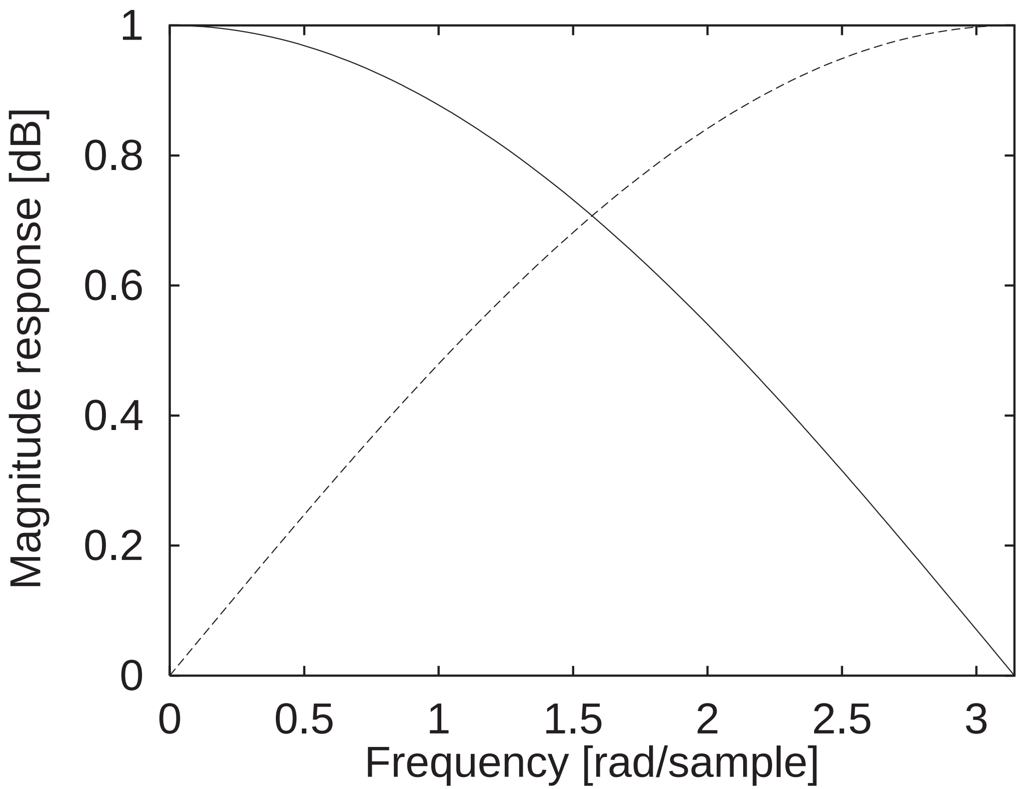


Figure 19: Magnitude responses of the filters described by equations (11) and (12): $H_0(z)$ (solid line); $H_1(z)$ (dashed line).

# Perfect reconstruction $M$-band filter banks

**Example 9.2**

- Repeat Example 33 for the case when

$$\mathbf{E}(z) = \begin{bmatrix} \left(-\frac{1}{8} + \frac{3}{4}z^{-1} - \frac{1}{8}z^{-2}\right) & \left(\frac{1}{4} + \frac{1}{4}z^{-1}\right) \\ \left(\frac{1}{2} + \frac{1}{2}z^{-1}\right) & -1 \end{bmatrix} \tag{15}$$

$$\mathbf{R}(z) = \begin{bmatrix} 1 & \left(\frac{1}{4} + \frac{1}{4}z^{-1}\right) \\ \left(\frac{1}{2} + \frac{1}{2}z^{-1}\right) & \left(\frac{1}{8} - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right) \end{bmatrix} \tag{16}$$

## $M$-band filter banks in terms of polyphase components

**Solution** Since

- $$\mathbf{R}(z)\mathbf{E}(z) = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix} = z^{-1}\mathbf{I} \tag{17}$$

then the filter bank has perfect reconstruction.

# $M$-band filter banks in terms of polyphase components

- From equations (15) and (16), the polyphase components, $E_{kj}(z)$, of the analysis filters $H_k(z)$, and $R_{jk}(z)$ of the synthesis filters $G_k(z)$ are

$$\left.\begin{aligned}
E_{00}(z) &= -\frac{1}{8} + \frac{3}{4}z^{-1} - \frac{1}{8}z^{-2} \\
E_{01}(z) &= \frac{1}{4} + \frac{1}{4}z^{-1} \\
E_{10}(z) &= \frac{1}{2} + \frac{1}{2}z^{-1} \\
E_{11}(z) &= -1
\end{aligned}\right\} \quad (18)$$

## $M$-band filter banks in terms of polyphase components

- $$\left.\begin{aligned} R_{00}(z) &= 1 \\ R_{01}(z) &= \frac{1}{4} + \frac{1}{4}z^{-1} \\ R_{10}(z) &= \frac{1}{2} + \frac{1}{2}z^{-1} \\ R_{11}(z) &= \frac{1}{8} - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2} \end{aligned}\right\} \tag{19}$$

# $M$-band filter banks in terms of polyphase components

- From equations (18) and (1), we can find $H_k(z)$, and, from equations (19) and (2), we can find $G_k(z)$. They are

$$H_0(z) = -\frac{1}{8} + \frac{1}{4}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} - \frac{1}{8}z^{-4} \tag{20}$$

$$H_1(z) = \frac{1}{2} - z^{-1} + \frac{1}{2}z^{-2} \tag{21}$$

$$G_0(z) = \frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2} \tag{22}$$

$$G_1(z) = \frac{1}{8} + \frac{1}{4}z^{-1} - \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{8}z^{-4} \tag{23}$$

The magnitude responses of the analysis filters are depicted in Figure 20.

$\triangle$

# Perfect reconstruction $M$-band filter banks

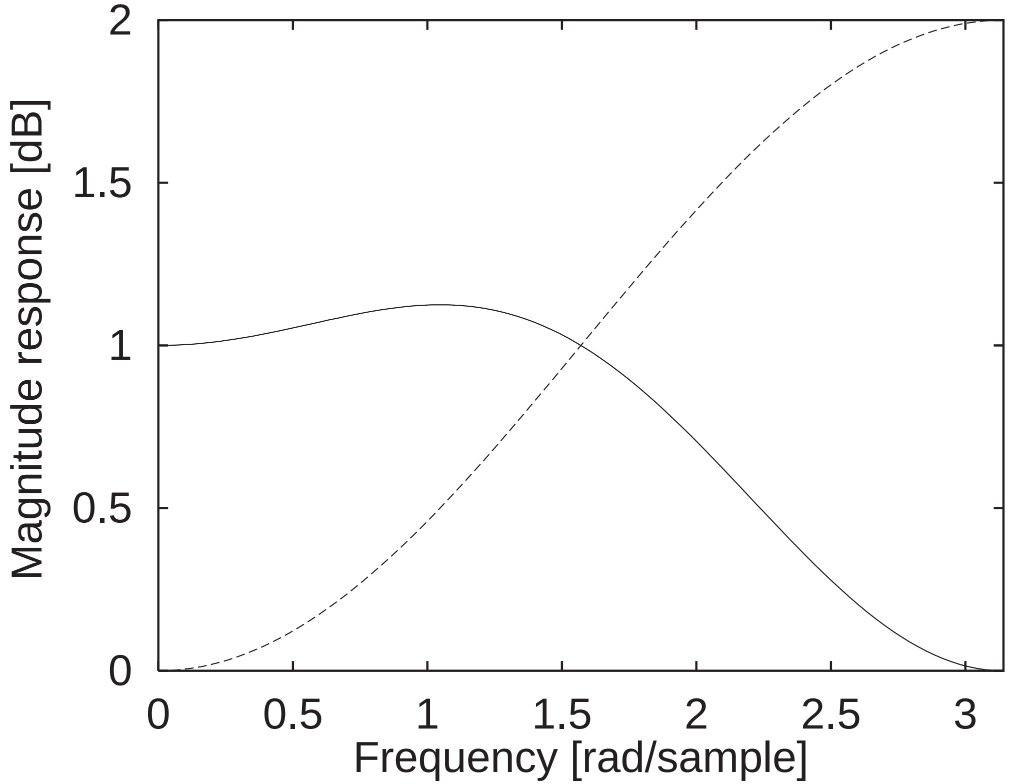


Figure 20: Magnitude responses of the filters described by equations (20) and (21): $H_0(z)$ (solid line); $H_1(z)$ (dashed line).

## Perfect reconstruction $M$-band filter banks

**Example 9.3** If the analysis filters of a perfect reconstruction filter bank are given by

$$\left.\begin{aligned} H_0(z) &= 1 + z^{-1} + \tfrac{1}{2}z^{-2} \\ H_1(z) &= 1 - z^{-1} + \tfrac{1}{2}z^{-2} \end{aligned}\right\} \tag{24}$$

determine its synthesis filters.

**Solution**

- From equation (24), we can write the lowpass and highpass analysis filters as

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 + \frac{1}{2}z^{-2} & 1 \\ 1 + \frac{1}{2}z^{-2} & -1 \end{bmatrix}}_{\mathbf{E}(z^2)} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \tag{25}$$

## Perfect reconstruction $M$-band filter banks

- Then the polyphase analysis matrix is

$$\mathbf{E}(z) = \begin{bmatrix} 1+\frac{1}{2}z^{-1} & 1 \\ 1+\frac{1}{2}z^{-1} & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1+\frac{1}{2}z^{-1} & 0 \\ 0 & 1 \end{bmatrix} \tag{26}$$

# Perfect reconstruction $M$-band filter banks

- Therefore, since the filter has perfect reconstruction, we must have that $\mathbf{R}(z) = z^{-\Delta}\mathbf{E}^{-1}(z)$, which gives

$$\mathbf{R}(z) = z^{-\Delta}\begin{bmatrix} \dfrac{1}{1+\frac{1}{2}z^{-1}} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \frac{z^{-\Delta}}{2}\begin{bmatrix} \dfrac{1}{1+\frac{1}{2}z^{-1}} & \dfrac{1}{1+\frac{1}{2}z^{-1}} \\ 1 & -1 \end{bmatrix} \tag{27}$$

## Perfect reconstruction $M$-band filter banks

- Then, from equation (4), the synthesis filters are

$$\begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \mathbf{R}^{\mathrm{T}}(z) \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix} = \frac{z^{-\Delta}}{2} \begin{bmatrix} \frac{1}{1+\frac{1}{2}z^{-2}} & 1 \\ \frac{1}{1+\frac{1}{2}z^{-2}} & -1 \end{bmatrix} \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix} \tag{28}$$

that is,

$$\left.\begin{aligned} G_0(z) &= z^{-\Delta}\frac{1+z^{-1}+\frac{1}{2}z^{-2}}{2+z^{-2}} \\ G_1(z) &= z^{-\Delta}\frac{-1+z^{-1}-\frac{1}{2}z^{-2}}{2+z^{-2}} \end{aligned}\right\} \tag{29}$$

Note that the FIR solution is not possible. $\triangle$

# Perfect reconstruction $M$-band filter banks

**Example 9.3** Assume the analysis filters of a 3-band perfect reconstruction filter bank are given by

$$\left.\begin{aligned} H_0(z) &= z^{-2} + 6z^{-1} + 4 \\ H_1(z) &= z^{-1} + 2 \\ H_2(z) &= 1 \end{aligned}\right\} \tag{30}$$

Determine its synthesis filters.

# Perfect reconstruction $M$-band filter banks

**Solution**

- From equations (1) and (3), the polyphase description of the given analysis filters is

$$\begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & 6 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{E}(z^3)} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix}$$

- Then

$$\mathbf{E}(z^3) = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{31}$$

# Perfect reconstruction $M$-band filter banks

- We can have perfect reconstruction if $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$. Thus,

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \begin{bmatrix} 4 & 6 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -6 & 8 \end{bmatrix} \tag{32}$$

- From equation (4), it follows that

$$\begin{bmatrix} G_0(z) \\ G_1(z) \\ G_2(z) \end{bmatrix} = \mathbf{R}^{\mathrm{T}}(z^3) \begin{bmatrix} z^{-2} \\ z^{-1} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -2 & 8 \end{bmatrix} \begin{bmatrix} z^{-2} \\ z^{-1} \\ 1 \end{bmatrix} \tag{33}$$

## Perfect reconstruction $M$-band filter banks

- Therefore, the transfer functions of the synthesis sub filters are given by:

$$\left.\begin{aligned} G_0(z) &= 1 \\ G_1(z) &= z^{-1} - 6 \\ G_2(z) &= z^{-2} - 2z^{-1} + 8 \end{aligned}\right\} \qquad (34)$$

We note that the synthesis filters are all FIR. This is only possible because the determinant of the polyphase matrix of the analysis filter is proportional to a pure delay (equal to 1 in this case).

$\triangle$

# Analysis of $M$-band filter banks

- The “signal processing block” could consist, for example, of quantization, filtering, or other types of signal transformations.

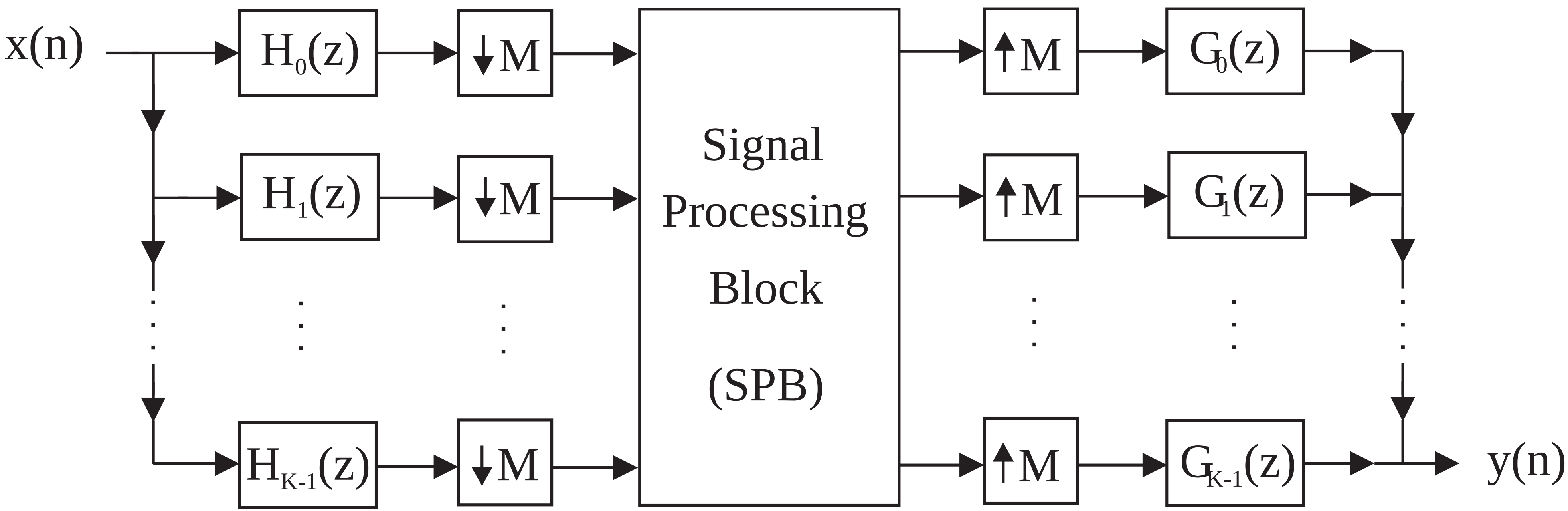


Figure 21: $M$-band filter bank.

# Modulation matrix representation

Using this concept, the analysis of the $M$-band filter bank can be performed in three different, but equivalent ways:

- Using the polyphase decomposition: When the polyphase decomposition is used in the analysis and synthesis banks, the resulting polyphase matrices are very useful to establish design conditions for perfect reconstruction filter banks.

- Using the modulation matrix representation: By representing the sub-band signals in the frequency domain, it is possible to describe the input-output relation of a filter bank. This representation leads to the so called modulation matrix representation. It is particularly effective in exposing the aliasing effects generated by the decimators. Although this formulation is useful to design alias-free filter banks, it is not the easiest formulation for design purposes.

## Modulation matrix representation

- Using time-domain analysis: It represents the input-output relation of the filter banks in the time domain in terms of the impulse responses of the analysis and synthesis sub-filters.
  - It is very effective in exposing the properties of perfect reconstruction filter banks as defining bases of vector spaces.
  - The analysis operations are seen as signal projections onto bases, while the synthesis operations are seen as signal expansions using bases of the vector space.
  - Properties such as orthogonality and biorthogonality come into play and provide useful insights on filter bank analysis and design.

# Modulation matrix representation

- Noting that, the decimated signal $X_d(z)$ is the sum of $X(z^{\frac{1}{M}})$ and its $(M-1)$ aliased components, $X(z^{\frac{1}{M}} e^{-j\frac{2\pi}{M}k})$, for $k = 1, 2, \ldots, (M-1)$, that is

$$X_d(z) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{\frac{1}{M}} e^{-j\frac{2\pi}{M}k}) = \frac{1}{M} \sum_{k=0}^{M-1} X(z^{\frac{1}{M}} W_M^k) \tag{35}$$

where $W_M = e^{-j\frac{2\pi}{M}}$.

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## Modulation matrix representation

- Using the above equation we can express the decimated output of the analysis filters in Figure 5, $U_k(z)$, for $k = 0, 1, \ldots, (M-1)$, as

$$\begin{aligned} U_k(z) &= \frac{1}{M}\sum_{k=0}^{M-1} X(z^{\frac{1}{M}}W_M^k)H_k(z^{\frac{1}{M}}W_M^k) \\ &= \frac{1}{M}\mathbf{x}_m^T(z^{\frac{1}{M}})\begin{bmatrix} H_k(z^{\frac{1}{M}}) \\ H_k(z^{\frac{1}{M}}W_M) \\ \vdots \\ H_k(z^{\frac{1}{M}}W_M^{M-1}) \end{bmatrix} \end{aligned} \tag{36}$$

where

$$\mathbf{x}_m(z^{\frac{1}{M}}) = \begin{bmatrix} X(z^{\frac{1}{M}}) & X(z^{\frac{1}{M}}W_M) & \ldots & X(z^{\frac{1}{M}}W_M^{M-1}) \end{bmatrix}^T \tag{37}$$

# Modulation matrix representation

- Therefore, we can define the auxiliary vector

$$\mathbf{U}^{\mathsf{T}}(z) = \begin{bmatrix} U_0(z) & U_1(z) & \ldots & U_{M-1}(z) \end{bmatrix} = \frac{1}{M}\mathbf{x}_{\mathrm{m}}^{\mathsf{T}}(z^{\frac{1}{M}})\mathbf{H}_{\mathrm{m}}(z^{\frac{1}{M}}) \qquad (38)$$

- With

$$\mathbf{H}_{\mathrm{m}}(z^{\frac{1}{M}}) = \begin{bmatrix} H_0(z^{\frac{1}{M}}) & H_1(z^{\frac{1}{M}}) & \ldots & H_{M-1}(z^{\frac{1}{M}}) \\ H_0(z^{\frac{1}{M}}W_M) & H_1(z^{\frac{1}{M}}W_M) & \ldots & H_{M-1}(z^{\frac{1}{M}}W_M) \\ \vdots & \vdots & \ddots & \vdots \\ H_0(z^{\frac{1}{M}}W_M^{M-1}) & H_1(z^{\frac{1}{M}}W_M^{M-1}) & \ldots & H_{M-1}(z^{\frac{1}{M}}W_M^{M-1}) \end{bmatrix} \qquad (39)$$

# Modulation matrix representation

- Applying the noble identities, we see that the filter bank output as a function of the sub-bands $U_k(z)$ is given by

$$Y(z) = \sum_{k=0}^{M-1} U_k(z^M) G_k(z) = \mathbf{U}^T(z^M)\mathbf{g}(z) \tag{40}$$

- Where

$$\mathbf{g}(z) = \begin{bmatrix} G_0(z) & G_1(z) & \dots & G_{M-1}(z) \end{bmatrix}^T \tag{41}$$

Then, from equations (38) and (40), we can express the input-output relation of an $M$-band filter bank as

$$Y(z) = \frac{1}{M}\mathbf{x}_m^T(z)\mathbf{H}_m(z)\mathbf{g}(z) \tag{42}$$

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# Modulation matrix representation

- Since $Y(z)$ above is a scalar, we have that $\mathbf{x}_m^T(z)\mathbf{H}_m(z)\mathbf{g}(z) = \mathbf{g}^T(z)\mathbf{H}_m^T(z)\mathbf{x}_m(z)$, and then

$$
\begin{aligned}
Y(z) &= \frac{1}{M}\mathbf{g}^T(z)\mathbf{H}_m^T(z)\mathbf{x}_m(z) \\
&= \frac{1}{M}\begin{bmatrix} G_0(z) & G_1(z) & \ldots & G_{M-1}(z) \end{bmatrix} \\
&\quad \times \begin{bmatrix} H_0(z) & H_0(zW_M) & \ldots & H_0(zW_M^{M-1}) \\ H_1(z) & H_1(zW_M) & \ldots & H_1(zW_M^{M-1}) \\ \vdots & \vdots & \ddots & \vdots \\ H_{M-1}(z) & H_{M-1}(zW_M) & \ldots & H_{M-1}(zW_M^{M-1}) \end{bmatrix} \begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix}
\end{aligned}
\tag{43}
$$

# Modulation matrix representation

- In equation (43), often referred to as the modulation matrix representation of the filter bank, if

$$\mathbf{g}^{\mathsf{T}}(z)\mathbf{H}_{\mathrm{m}}^{\mathsf{T}}(z) = \begin{bmatrix} B(z) & 0 & \ldots & 0 \end{bmatrix} \quad (44)$$

- Then aliasing is canceled, since

$$Y(z) = \frac{1}{M}\begin{bmatrix} B(z) & 0 & \ldots & 0 \end{bmatrix}\begin{bmatrix} X(z) \\ X(zW_M) \\ \vdots \\ X(zW_M^{M-1}) \end{bmatrix} = \frac{1}{M}B(z)X(z) \quad (45)$$

- It can also be inferred that if $B(z) = Mcz^{-\Delta}$, then the output of the filter bank is just a delayed version of the input scaled by a constant $c$, that is, the filter bank has perfect reconstruction.

# Modulation matrix representation

**Example 9.5** Find the perfect reconstruction conditions for all 2-band filter banks using the modulation matrix approach.

**Solution**

- For the 2-band case, perfect reconstruction requires, since $W_2 = -1$, that

$$\begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = \begin{bmatrix} 2cz^{-\Delta} & 0 \end{bmatrix} \tag{46}$$

which implies

$$\left.\begin{aligned} H_0(z)G_0(z) + H_1(z)G_1(z) &= 2cz^{-\Delta} \\ H_0(-z)G_0(z) + H_1(-z)G_1(z) &= 0 \end{aligned}\right\} \tag{47}$$

Then the output of the filter bank to be equal to the input delayed by $\Delta$ and scaled by a constant $c$. $\triangle$

## Time-domain analysis

- To carry out the time-domain analysis using the signals in matrix form, we have that the signals $x_k(m)$ at the output of the analysis filters can be expressed as

$$x_k(n) = x(n) * h_k(n) = \sum_{l=-\infty}^{\infty} h_k(n-l)x(l) \tag{48}$$

where $h_k(n)$ denotes the impulse response of the $k$th analysis filter $H_k(z)$, for $k = 0, 1, \ldots, (M-1)$.

# Time-domain analysis

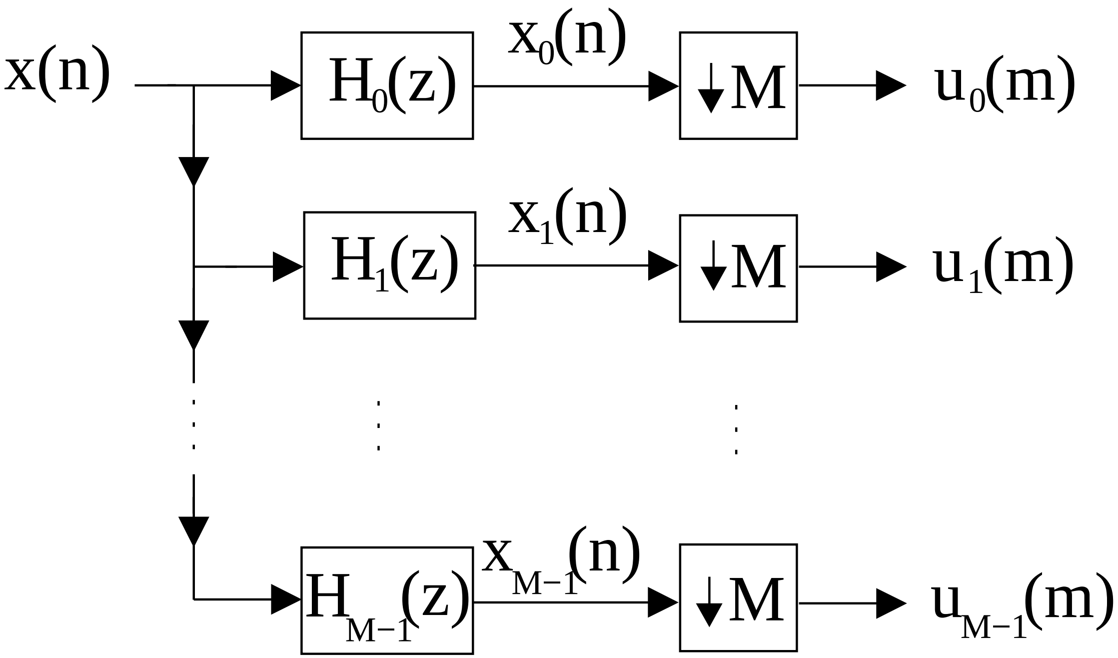


Figure 22: Analysis filter bank.

## Time-domain analysis

- The signals $u_k(m)$ at sub-band $k$ are the signals $x_k(n)$ decimated by a factor of $M$ are

$$u_k(m) = \sum_{n=-\infty}^{\infty} h_k(mM - n)x(n) \quad (49)$$

- Define from the impulse response of the $k$th analysis filter bank $h_k(n)$

$$\tilde{\mathbf{h}}_k(m) = \begin{bmatrix} \ldots & h_k(mM) & h_k(mM-1) & h_k(mM-2) & \ldots & h_k(mM-n) & \ldots \end{bmatrix}^{\mathrm{T}} \quad (50)$$

## Time-domain analysis

- Such that the $n$th entry of $\tilde{\mathbf{h}}_k^T(m)$ is given by

$$[\tilde{\mathbf{h}}_k^T(m)]_n = h_k(mM - n) \tag{51}$$

-

$$u_k(m) = \sum_{n=-\infty}^{\infty} [\tilde{\mathbf{h}}_k(m)]_n x(n) \tag{52}$$

# Time-domain analysis

- The signal in sub-band $k$ can be expressed as

$$u_k(m) = \tilde{\mathbf{h}}_k^T(m)\mathbf{x} \tag{53}$$

- where

$$\mathbf{x} = \begin{bmatrix} \ldots & x(0) & x(1) & x(2) & \ldots & x(n) & \ldots \end{bmatrix}^T \tag{54}$$

- Therefore, the signal in sub-band $k$ is, in matrix form,

$$\mathbf{u}_k = \begin{bmatrix} \vdots \\ u_k(0) \\ u_k(1) \\ u_k(2) \\ \vdots \\ u_k(m) \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \tilde{\mathbf{h}}_k^T(0) \\ \tilde{\mathbf{h}}_k^T(1) \\ \tilde{\mathbf{h}}_k^T(2) \\ \vdots \\ \tilde{\mathbf{h}}_k^T(m) \\ \vdots \end{bmatrix} \mathbf{x} = \mathbf{H}_k \mathbf{x} \tag{55}$$

# Time-domain analysis

- Which can be expanded to

$$
\mathbf{u}_k = \underbrace{\begin{bmatrix}
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & h_k(0) & h_k(-1) & h_k(-2) & \cdots & h_k(-l) & \cdots \\
\cdots & h_k(M) & h_k(M1) & h_k(M-2) & \cdots & h_k(M-l) & \cdots \\
\cdots & h_k(2M) & h_k(2M-1) & h_k(2M-2) & \cdots & h_k(2M-l) & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & h_k(mM) & h_k(mM-1) & h_k(mM-2) & \cdots & h_k(mM-l) & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{bmatrix}}_{\mathbf{H}_k}
\underbrace{\begin{bmatrix}
\vdots \\ x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(n) \\ \vdots
\end{bmatrix}}_{\mathbf{x}}
\tag{56}
$$

# Time-domain analysis

- Now by defining the vector of outputs of the $M$ sub-bands at sample $m$ as

$$\mathbf{u}(m) = \begin{bmatrix} u_0(m) & u_1(m) & \ldots & u_{M-1}(m) \end{bmatrix}^{\mathrm{T}} \tag{57}$$

- We have that, from equation (52),

$$\mathbf{u}(m) = \begin{bmatrix} u_0(m) \\ u_1(m) \\ \vdots \\ u_{M-1}(m) \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{h}}_0^T(m) \\ \tilde{\mathbf{h}}_1^T(m) \\ \vdots \\ \tilde{\mathbf{h}}_{M-1}^T(m) \end{bmatrix} \mathbf{x} \tag{58}$$

## Time-domain analysis

- Equivalently

$$
\mathbf{u}(m) = \underbrace{\begin{bmatrix} \cdots & h_0(mM) & h_0(mM-1) & h_0(mM-2) & \cdots & h_0(mM-l) & \cdots \\ \cdots & h_1(mM) & h_1(mM-1) & h_1(mM-2) & \cdots & h_1(mM-l) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & h_{M-1}(mM) & h_{M-1}(mM-1) & h_{M-1}(mM-2) & \cdots & h_{M-1}(mM-l) & \cdots \end{bmatrix}}_{\mathbf{H}(m)} \underbrace{\begin{bmatrix} \vdots \\ x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(n) \\ \vdots \end{bmatrix}}_{\mathbf{x}}
$$

# Time-domain analysis

- Therefore, the whole time-domain expression can be written as

$$\underbrace{\begin{bmatrix} \vdots \\ \mathbf{u}(0) \\ \mathbf{u}(1) \\ \mathbf{u}(2) \\ \vdots \\ \mathbf{u}(m) \\ \vdots \end{bmatrix}}_{\mathcal{U}} = \underbrace{\begin{bmatrix} \vdots \\ \mathbf{H}(0) \\ \mathbf{H}(1) \\ \mathbf{H}(2) \\ \vdots \\ \mathbf{H}(m) \\ \vdots \end{bmatrix}}_{\mathcal{H}} \mathbf{x} \tag{60}$$

that is,

$$\mathcal{U} = \mathcal{H}\mathbf{x} \tag{61}$$

## Time-domain analysis

- Note that matrix $\mathbf{H}(m)$ consists of matrix $\mathbf{H}(0)$ shifted $mM$ columns to the right.
- Then matrix $\mathcal{H}$ consists of a concatenation of properly shifted submatrices.
- As for the synthesis operation the signals $y_k(n)$ are

$$y_k(n) = \sum_{m=-\infty}^{\infty} g_k(n - mM)u_k(m) \tag{62}$$

## Time-domain analysis

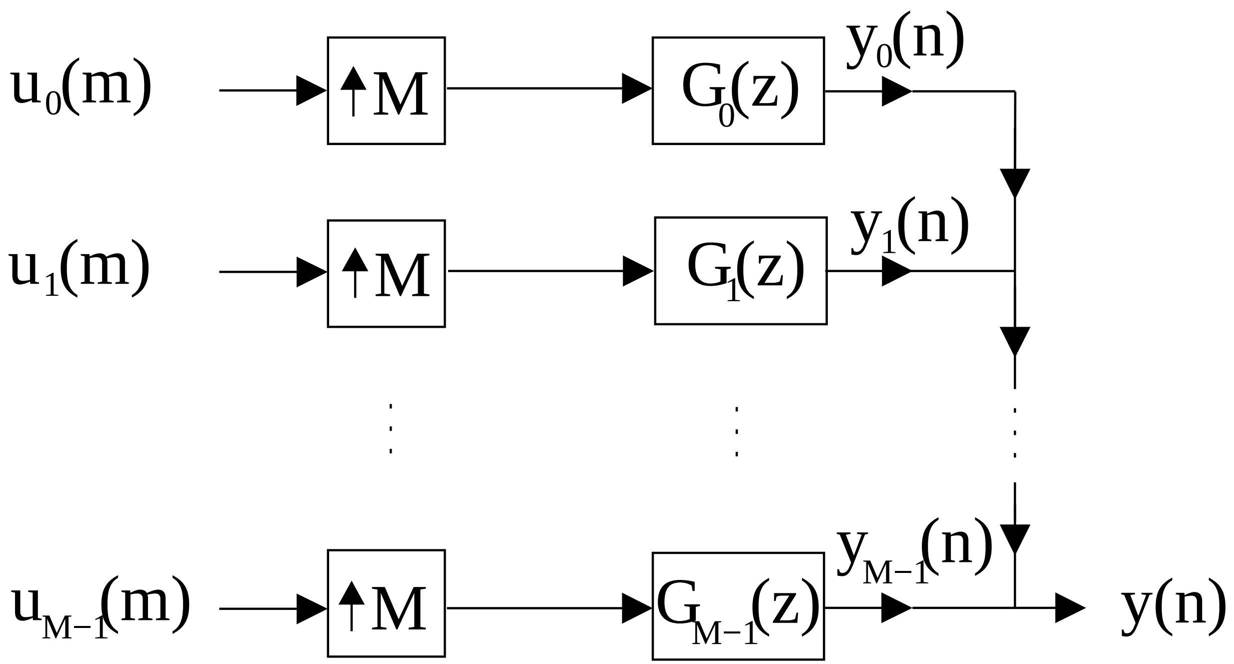


Figure 23: Synthesis filter bank.

# Time-domain analysis

- From the impulse response of the $k$th synthesis filter bank, $g_k(n)$, define a vector corresponding to this impulse response shifted by $mM$ samples as

$$\mathbf{g}_k(m) = \begin{bmatrix} \vdots \\ g_k(-mM) \\ g_k(1-mM) \\ g_k(2-mM) \\ \vdots \\ g_k(n-mM) \\ \vdots \end{bmatrix} \qquad (63)$$

## Time-domain analysis

- Such that $[\mathbf{g}_k(m)]_n = g_k(n - mM)$, equation (62) can be written as

$$y_k(n) = \sum_{m=-\infty}^{\infty} [\mathbf{g}_k(m)]_n u_k(m) \tag{64}$$

## Time-domain analysis

- Thus, in matrix form, we have that

$$\begin{aligned}
\mathbf{y}_k &= \sum_{m=-\infty}^{\infty} \mathbf{g}_k(m) u_k(m) \\
&= \begin{bmatrix} \ldots & \mathbf{g}_k(0) & \mathbf{g}_k(1) & \mathbf{g}_k(2) & \ldots & \mathbf{g}_k(m) & \ldots \end{bmatrix} \begin{bmatrix} \vdots \\ u_k(0) \\ u_k(1) \\ u_k(2) \\ \vdots \\ u_k(m) \\ \vdots \end{bmatrix} \qquad (65)
\end{aligned}$$

Using matrix $\mathcal{U}$ in equation (60), the above equation can be rewritten as

$$\mathbf{y}_k = \begin{bmatrix} \cdots & 0 & \cdots & \mathbf{g}_k(0) & \cdots & 0 & \cdots & \mathbf{g}_k(1) & \cdots & 0 & \cdots & \mathbf{g}_k(m) & \cdots & 0 & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ u_0(0) \\ \vdots \\ u_k(0) \\ \vdots \\ u_{M-1}(0) \\ u_0(1) \\ \vdots \\ u_k(1) \\ \vdots \\ u_{M-1}(1) \\ \vdots \\ u_0(m) \\ \vdots \\ u_k(m) \\ \vdots \\ u_{M-1}(m) \\ \vdots \end{bmatrix} \quad (66)$$

# Time-domain analysis

- Since, from Figure 23,

$$y(n) = \sum_{k=0}^{M-1} y_k(n) \tag{67}$$

- In matrix form we have that

$$\mathbf{y} = \sum_{k=0}^{M-1} \mathbf{y}_k \tag{68}$$

- The above equation together with equation (66) imply that

$$\mathbf{y} = \mathcal{G}\mathcal{U} \tag{69}$$

# Time-domain analysis

Where

$$\mathcal{G} = \begin{bmatrix} \cdots & \mathbf{g}_0(0) & \cdots & \mathbf{g}_{M-1}(0) & \mathbf{g}_0(1) & \cdots & \mathbf{g}_{M-1}(1) & \cdots & \mathbf{g}_0(m) & \cdots & \mathbf{g}_{M-1}(m) & \cdots \end{bmatrix}$$

$$= \begin{bmatrix} \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & g_0(0) & \cdots & g_{M-1}(0) & g_0(-M) & \cdots & g_{M-1}(-M) & \cdots & g_0(-mM) & \cdots & g_{M-1}(-mM) & \cdots \\ \cdots & g_0(1) & \cdots & g_{M-1}(1) & g_0(1-M) & \cdots & g_{M-1}(1-M) & \cdots & g_0(1-mM) & \cdots & g_{M-1}(1-mM) & \cdots \\ \cdots & g_0(2) & \cdots & g_{M-1}(2) & g_0(2-M) & \cdots & g_{M-1}(2-M) & \cdots & g_0(2-mM) & \cdots & g_{M-1}(2-mM) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & g_0(n) & \cdots & g_{M-1}(n) & g_0(n-M) & \cdots & g_{M-1}(n-M) & \cdots & g_0(n-mM) & \cdots & g_{M-1}(n-mM) & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

## Time-domain analysis

- The structure of $\mathcal{G}$ becomes more evident if we define

$$\mathbf{G}(m) = \begin{bmatrix} \mathbf{g}_0(m) & \mathbf{g}_1(m) & \cdots & \mathbf{g}_{M-1}(m) \end{bmatrix}$$

$$= \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ g_0(-mM) & g_1(-mM) & \cdots & g_{M-1}(-mM) \\ g_0(1-mM) & g_1(1-mM) & \cdots & g_{M-1}(1-mM) \\ g_0(2-mM) & g_1(2-mM) & \cdots & g_{M-1}(2-mM) \\ \vdots & \vdots & \vdots & \vdots \\ g_0(n-mM) & g_1(n-mM) & \cdots & g_{M-1}(n-mM) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{71}$$

## Time-domain analysis

- Using this definition, $\mathcal{G}$ in equation (70) can be expressed as

$$\mathcal{G} = \begin{bmatrix} \cdots & \mathbf{G}(0) & \mathbf{G}(1) & \mathbf{G}(2) & \ldots & \mathbf{G}(m) & \cdots \end{bmatrix} \tag{72}$$

and thus equation (69) can be rewritten as

$$\mathbf{y} = \underbrace{\begin{bmatrix} \ldots & \mathbf{G}(0) & \mathbf{G}(1) & \mathbf{G}(2) & \ldots & \mathbf{G}(m) & \ldots \end{bmatrix}}_{\mathcal{G}} \underbrace{\begin{bmatrix} \vdots \\ \mathbf{u}(0) \\ \mathbf{u}(1) \\ \mathbf{u}(2) \\ \vdots \\ \mathbf{u}(m) \\ \vdots \end{bmatrix}}_{\mathcal{U}} \tag{73}$$

# Time-domain analysis

- Note, from equation (71), that matrix $\mathbf{G}(m)$ consists of matrix $\mathbf{G}(0)$ shifted $mM$ rows up. Then, similarly to matrix $\mathcal{H}$ in equation (60), from equation (72), matrix $\mathcal{G}$ consists of a concatenation of properly shifted submatrices.

- In order to summarize what we have seen so far, we can take equations (58), (61), (69), and (73), and express the analysis and synthesis operations in a filter bank as

$$\left.\begin{aligned} \mathbf{u}(m) &= \mathbf{H}(m)\mathbf{x}, \quad \text{for } m = \ldots, 0, 1, 2, \ldots \\ \mathbf{y} &= \sum_{m=-\infty}^{\infty} \mathbf{G}(m)\mathbf{u}(m) \end{aligned}\right\} \tag{74}$$

- or

$$\left.\begin{aligned} \mathcal{U} &= \mathcal{H}\mathbf{x} \\ \mathbf{y} &= \mathcal{G}\mathcal{U} \end{aligned}\right\} \tag{75}$$

## Time-domain analysis

- If the filter bank has perfect reconstruction and zero delay, we have that $\mathbf{x} = \mathbf{y}$.
- Therefore, from equation (75), we have that $\mathcal{H}$ and $\mathcal{G}$ must satisfy the following constraints:

$$\mathcal{G}\mathcal{H} = \mathcal{H}\mathcal{G} = \mathbf{I} \tag{76}$$

- Note that if the filter bank has delay equal to $\Delta$, then $\mathcal{G}\mathcal{H}$ corresponds to a delay of $\Delta$ and $\mathcal{H}\mathcal{G}$ corresponds to an advance of $\Delta$.

# Time-domain analysis

**Example 9.5** For the 2-band perfect reconstruction filter bank from example 37 (equations (20), (21), (22), and (23)),

$$\left.\begin{aligned} H_0(z) &= -\frac{1}{8} + \frac{1}{4}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} - \frac{1}{8}z^{-4} \\ H_1(z) &= \frac{1}{2} - z^{-1} + \frac{1}{2}z^{-2} \end{aligned}\right\} \quad (77)$$

$$\left.\begin{aligned} G_0(z) &= \frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2} \\ G_1(z) &= \frac{1}{8} + \frac{1}{4}z^{-1} - \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{8}z^{-4} \end{aligned}\right\} \quad (78)$$

describe the matrices $\mathcal{H}$ and $\mathcal{G}$.

# Time-domain analysis

## Solution

- The impulse responses of the analysis and synthesis and filters are (only the non-zero samples are shown):

$$\left.\begin{array}{lllll} h_0(0) = -\frac{1}{8}, & h_0(1) = \frac{1}{4}, & h_0(2) = \frac{3}{4}, & h_0(3) = \frac{1}{4}, & h_0(4) = -\frac{1}{8} \\ h_1(0) = \frac{1}{2}, & h_1(1) = -1 & h_1(2) = \frac{1}{2} & & \end{array}\right\} \quad (79)$$

$$\left.\begin{array}{lllll} g_0(0) = \frac{1}{2}, & g_0(1) = 1 & g_0(2) = \frac{1}{2} & & \\ g_1(0) = \frac{1}{8}, & g_1(1) = \frac{1}{4}, & g_1(2) = -\frac{3}{4}, & g_1(3) = \frac{1}{4}, & g_1(4) = \frac{1}{8} \end{array}\right\} \quad (80)$$

- From equations (58) and (60), matrix $\mathcal{H}$ is

$$
\mathcal{H} = \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & h_0(1) & h_0(0) & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & h_1(1) & h_1(0) & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & h_0(3) & h_0(2) & h_0(1) & h_0(0) & 0 & 0 & 0 & \cdots \\
\cdots & 0 & h_1(2) & h_1(1) & h_1(0) & 0 & 0 & 0 & \cdots \\
\cdots & 0 & h_0(4) & h_0(3) & h_0(2) & h_0(1) & h_0(0) & 0 & \cdots \\
\cdots & 0 & 0 & 0 & h_1(2) & h_1(1) & h_1(0) & 0 & \cdots \\
\cdots & 0 & 0 & 0 & h_0(4) & h_0(3) & h_0(2) & h_0(1) & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & h_1(2) & h_1(1) & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
= \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & \frac{1}{4} & -\frac{1}{8} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & -1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{8} & 0 & 0 & 0 & \cdots \\
\cdots & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\
\cdots & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & -\frac{1}{8} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 & \cdots \\
\cdots & 0 & 0 & 0 & -\frac{1}{8} & \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -1 & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \tag{81}
$$

- And from equations (71) and (72), the matrix $\mathcal{G}$ is

$$
\begin{aligned}
\mathcal{G} &= \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & g_0(0) & g_1(0) & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & g_0(1) & g_1(1) & \cdots \\
\cdots & 0 & 0 & 0 & 0 & g_0(0) & g_1(0) & g_0(2) & g_1(2) & \cdots \\
\cdots & 0 & 0 & 0 & 0 & g_0(1) & g_1(1) & 0 & g_1(3) & \cdots \\
\cdots & 0 & 0 & g_0(0) & g_1(0) & g_0(2) & g_1(2) & 0 & g_1(4) & \cdots \\
\cdots & 0 & 0 & g_0(1) & g_1(1) & 0 & g_1(3) & 0 & 0 & \cdots \\
\cdots & g_0(0) & g_1(0) & g_0(2) & g_1(2) & 0 & g_1(4) & 0 & 0 & \cdots \\
\cdots & g_0(1) & g_1(1) & 0 & g_1(3) & 0 & 0 & 0 & 0 & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \\
&= \begin{bmatrix}
\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{2} & -\frac{3}{4} & \cdots \\
\cdots & 0 & 0 & 0 & 0 & 1 & \frac{1}{4} & 0 & \frac{1}{4} & \cdots \\
\cdots & 0 & 0 & \frac{1}{2} & \frac{1}{8} & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{8} & \cdots \\
\cdots & 0 & 0 & 1 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \cdots \\
\cdots & \frac{1}{2} & \frac{1}{8} & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{8} & 0 & 0 & \cdots \\
\cdots & 1 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \cdots \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\end{aligned} \tag{82}
$$

# Orthogonality and biorthogonality in filter banks

- We have seen that the perfect reconstruction condition requires that

$$\mathcal{H}\mathcal{G} = \mathbf{I} \tag{83}$$

- Using the expressions for $\mathcal{H}$ and $\mathcal{G}$ in equations (60) and (72), the above equation becomes

$$\begin{bmatrix} \vdots \\ \mathbf{H}(r) \\ \mathbf{H}(r+1) \\ \vdots \\ \mathbf{H}(r+m) \\ \vdots \end{bmatrix} \begin{bmatrix} \ldots & \mathbf{G}(0) & \mathbf{G}(1) & \ldots & \mathbf{G}(m) & \ldots \end{bmatrix} = \mathbf{I} \tag{84}$$

## Orthogonality and biorthogonality in filter banks

- Note that $r$ is an integer value that is used to emphasize the fact that we are not specifying neither the number of rows nor the index of the first row of the matrix $\mathcal{H}$. This gives

$$\begin{bmatrix} \ddots & \vdots & \vdots & \cdots & \vdots & \cdots \\ \cdots & \mathbf{H}(r)\mathbf{G}(0) & \mathbf{H}(r)\mathbf{G}(1) & \cdots & \mathbf{H}(r)\mathbf{G}(m) & \cdots \\ \cdots & \mathbf{H}(r+1)\mathbf{G}(0) & \mathbf{H}(r+1)\mathbf{G}(1) & \cdots & \mathbf{H}(r+1)\mathbf{G}(m) & \cdots \\ \cdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ \cdots & \mathbf{H}(r+m)\mathbf{G}(0) & \mathbf{H}(r+m)\mathbf{G}(1) & \cdots & \mathbf{H}(r+m)\mathbf{G}(m) & \cdots \\ \cdots & \vdots & \vdots & \cdots & \vdots & \ddots \end{bmatrix} = \mathbf{I} \quad (85)$$

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## Orthogonality and biorthogonality in filter banks

- Since, from equations (58) and (71), $\mathbf{H}(k)$ has $M$ rows and $\mathbf{G}(l)$ has $M$ columns, the above equation implies that there must be an integer $r$ such that

$$\mathbf{H}(r+k)\mathbf{G}(l) = \delta(k-l)\mathbf{I}_M \tag{86}$$

# Orthogonality and biorthogonality in filter banks

- Now, expressing $\mathbf{H}(r+k)$ as function of its rows $\tilde{\mathbf{h}}_i^{\mathsf{T}}(r+k)$ (equation (58)) and $\mathbf{G}(l)$ as function of its columns $\mathbf{g}_j(l)$ (equation (71)) we have that equation (86) becomes

$$
\begin{aligned}
\mathbf{H}(r+k)\mathbf{G}(l) &= \begin{bmatrix} \tilde{\mathbf{h}}_0^{\mathsf{T}}(r+k) \\ \tilde{\mathbf{h}}_1^{\mathsf{T}}(r+k) \\ \vdots \\ \tilde{\mathbf{h}}_{M-1}^{\mathsf{T}}(r+k) \end{bmatrix} \begin{bmatrix} \mathbf{g}_0(l) & \mathbf{g}_1(l) & \ldots & \mathbf{g}_{M-1}(l) \end{bmatrix} \\
&= \begin{bmatrix} \tilde{\mathbf{h}}_0^{\mathsf{T}}(r+k)\mathbf{g}_0(l) & \tilde{\mathbf{h}}_0^{\mathsf{T}}(r+k)\mathbf{g}_1(l) & \cdots & \tilde{\mathbf{h}}_0^{\mathsf{T}}(r+k)\mathbf{g}_{M-1}(l) \\ \tilde{\mathbf{h}}_1^{\mathsf{T}}(r+k)\mathbf{g}_0(l) & \tilde{\mathbf{h}}_1^{\mathsf{T}}(r+k)\mathbf{g}_1(l) & \cdots & \tilde{\mathbf{h}}_1^{\mathsf{T}}(r+k)\mathbf{g}_{M-1}(l) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\mathbf{h}}_{M-1}^{\mathsf{T}}(r+k)\mathbf{g}_0(l) & \tilde{\mathbf{h}}_{M-1}^{\mathsf{T}}(r+k)\mathbf{g}_1(l) & \cdots & \tilde{\mathbf{h}}_{M-1}^{\mathsf{T}}(r+k)\mathbf{g}_{M-1}(l) \end{bmatrix} \\
&= \delta(k-l)\mathbf{I}_M
\end{aligned}
\tag{87}
$$

# Orthogonality and biorthogonality in filter banks

- The above equation is equivalent to

$$\tilde{\mathbf{h}}_i^T(r+k)\mathbf{g}_j(l) = \delta(k-l)\delta(i-j) \tag{88}$$

- This means that for $k \neq l$ or $i \neq j$ the vectors $\tilde{\mathbf{h}}_i^T(r+k)$ and $\mathbf{g}_j^*(l)$ are orthogonal.
- If $k = l$ and $i = j$ their inner product should be 1. Since $\tilde{\mathbf{h}}_i^T(k)$ are the rows of $\mathcal{H}$ and $\mathbf{g}_j(l)$ are the columns of $\mathcal{G}$, then perfect reconstruction implies that the rows of $\mathcal{H}$ are orthogonal to the columns of $\mathcal{G}^*$, except the cases where the $(r+m)$th row of $\mathcal{H}$ should not be orthogonal to the $m$th column of $\mathcal{G}^*$, for all $m$.
- One often refers to the rows of $\mathcal{H}$ and columns of $\mathcal{G}^*$ as being biorthogonal.

## Orthogonality and biorthogonality in filter banks

- In the case when $\mathcal{H} = \mathcal{G}^{*^{\mathsf{T}}}$, there must be an integer $r$ such that $\tilde{\mathbf{h}}_i(r + m) = \mathbf{g}_i^*(m)$, which implies that, for $i = 0, 1, \ldots, (M-1)$ and $m \in \mathbb{Z}$, there is an integer $s$ such that

$$h_i(rM + mM - n - s) = g_i^*(n - mM) \tag{89}$$

where the integer $s$ is used to emphasize the fact that we are not specifying neither the number of columns nor the index of the first column of the matrix $\mathcal{H}$. This gives

$$h_i(rM - n - s) = g_i^*(n) \tag{90}$$

# Orthogonality and biorthogonality in filter banks

- Then equation (88) becomes

$$\mathbf{g}_i^{*^T}(k)\mathbf{g}_j(l) = \delta(k-l)\delta(i-j) \tag{91}$$

which means that the columns of $\mathcal{G}$ (or the rows of $\mathcal{H}$) are orthogonal to each other. In this case, the filter bank is said to be orthogonal, and the perfect reconstruction condition is

$$\mathcal{H}\mathcal{H}^{*^T} = \mathcal{H}^{*^T}\mathcal{H} = \mathbf{I} \tag{92}$$

or

$$\mathcal{G}\mathcal{G}^{*^T} = \mathcal{G}^{*^T}\mathcal{G} = \mathbf{I} \tag{93}$$

and the pair in equation (75) becomes

$$\left.\begin{array}{l} \mathcal{U} = \mathcal{H}\mathbf{x} \\ \mathbf{x} = \mathcal{H}^{*^T}\mathcal{U} \end{array}\right\} \tag{94}$$

# Orthogonality and biorthogonality in filter banks

- We see that an orthogonal perfect reconstruction filter bank can be regarded as a unitary transform that maps a signal **x** into the transform coefficients $\mathcal{U}$.
- The interpretation of an orthogonal filter bank as being equivalent to a unitary transform brings about an interesting insight on biorthogonal filter banks. If

$$\mathcal{U} = \mathcal{H}\mathbf{x} \tag{95}$$

then the $k$th element of $\mathcal{U}$ can be regarded as the inner product between the complex conjugate of the $k$th row of $\mathcal{H}$ and **x**.

# Orthogonality and biorthogonality in filter banks

- Likewise, we can say that if

$$\mathbf{x} = \mathcal{G}\mathcal{U} \tag{96}$$

then $\mathbf{x}$ can be regarded as a linear combination of the columns of $\mathcal{G}$, where the weight of the $k$th column is the $k$th element of $\mathcal{U}$.

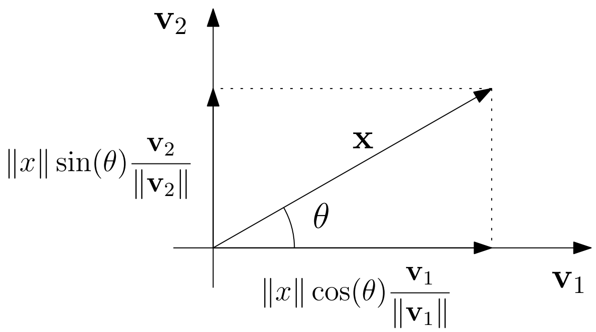
- In a biorthogonal filter bank, the analysis filters projects the input signal on the rows of $\mathcal{H}^*$. The synthesis filter bank takes these projections and uses them to weight the columns of $\mathcal{G}$ in order to recover the input signal.

# Orthogonality and biorthogonality in filter banks

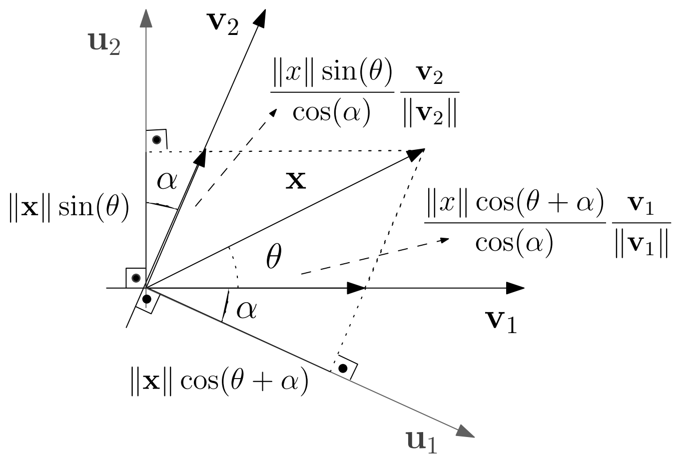
- Given that $\mathbf{v}_1$ and $\mathbf{v}_2$ are unit-norm, orthogonal vectors. One can compute the coordinates (transform) of vector $\mathbf{x}$ by projecting it on $\mathbf{v}_1$ and $\mathbf{v}_2$.

- It can be recovered by the vector addition of the two projections, that is

$$
\begin{aligned}
\mathbf{x} &= \|\mathbf{x}\| \cos(\theta) \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} + \|\mathbf{x}\| \sin(\theta) \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \\
&= \left\langle \mathbf{x}, \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \right\rangle \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} + \left\langle \mathbf{x}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \right\rangle \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \qquad (97)
\end{aligned}
$$




(a)




(b)

Figure 24: Orthogonality and biorthogonality in two dimensions.

## Orthogonality and biorthogonality in filter banks

- One wants to express a vector $\mathbf{x}$ as a linear combination of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ that are not orthogonal.
- This can be made by constructing the parallelogram with sides parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$.
- But a side parallel to $\mathbf{v}_1$ is orthogonal to a vector $\mathbf{u}_2$ that is also orthogonal to $\mathbf{v}_1$, and a side parallel to $\mathbf{v}_2$ is orthogonal to a vector $\mathbf{u}_1$ that is also orthogonal to $\mathbf{v}_2$.

## Orthogonality and biorthogonality in filter banks

- We can see that in order to find the length of the parallelogram side with the same direction as $\mathbf{v}_1$, we project it on $\mathbf{u}_1$ and divide it by the cosine of the angle between $\mathbf{v}_1$ and $\mathbf{u}_1$.
- Similarly, in order to find the length of the parallelogram side with the same direction as $\mathbf{v}_2$, we project it on $\mathbf{u}_2$ and divide it by the cosine of the angle between $\mathbf{v}_2$ and $\mathbf{u}_2$.
- That is, the analysis operation is carried out by the projection on $\mathbf{u}_1$ and $\mathbf{u}_2$, while the synthesis by the linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, where $\mathbf{u}_1$ is orthogonal to $\mathbf{v}_2$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}_1$.

## Orthogonality and biorthogonality in filter banks

- Mathematically, the above operations are expressed as

$$\mathbf{x} \quad = \quad \frac{\|\mathbf{x}\| \cos(\theta + \alpha)}{\cos(\alpha)} \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} + \frac{\|\mathbf{x}\| \sin(\theta)}{\cos(\alpha)} \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \tag{98}$$

- Since

$$\left. \begin{array}{l} \cos(\alpha) = \left\langle \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \right\rangle = \left\langle \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \right\rangle \\ \|\mathbf{x}\| \cos(\alpha + \theta) = \left\langle \mathbf{x}, \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \right\rangle \\ \|\mathbf{x}\| \sin(\theta) = \left\langle \mathbf{x}, \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \right\rangle \end{array} \right\} \tag{99}$$

# Orthogonality and biorthogonality in filter banks

- Equation (98) becomes

$$\begin{aligned}\mathbf{x} &= \frac{\left\langle \mathbf{x}, \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \right\rangle}{\left\langle \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \right\rangle} \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} + \frac{\left\langle \mathbf{x}, \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \right\rangle}{\left\langle \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \right\rangle} \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} \\ &= \frac{\langle \mathbf{x}, \mathbf{u}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{u}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{u}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{u}_2 \rangle} \mathbf{v}_2 \qquad (100)\end{aligned}$$

## Orthogonality and biorthogonality in filter banks

- In the above equation it can be clearly seen that the coordinates in the non-orthogonal basis composed by $\mathbf{v}_1$ and $\mathbf{v}_2$ are given by $\frac{\langle \mathbf{x}, \mathbf{u}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{u}_1 \rangle}$ and $\frac{\langle \mathbf{x}, \mathbf{u}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{u}_2 \rangle}$, where $\mathbf{u}_1$ is orthogonal to $\mathbf{v}_2$ and $\mathbf{u}_2$ is orthogonal to $\mathbf{v}_1$.

- For an orthogonal filter bank we have that, from equation (90), there are integers $r$ and $s$ such that the analysis and synthesis filter banks satisfy

$$g_i(n) = h_i^*(rM - n - s), \quad \text{for} \quad i = 0, 1, \ldots, (M-1) \tag{101}$$

- In the $z$ transform domain, for $i = 0, 1, \ldots, (M-1)$,

$$\begin{aligned}
G_i(z) &= \sum_{n=-\infty}^{\infty} g_i(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} h_i^*(rM - n - s) z^{-n} \\
&= \sum_{t=-\infty}^{\infty} h_i^*(t) z^{t+s-rM} \\
&= z^{-rM+s} \sum_{t=-\infty}^{\infty} h_i^*(t) z^{t} \\
&= z^{-rM+s} \left( \sum_{t=-\infty}^{\infty} h_i(t) (z^*)^{t} \right)^* \\
&= z^{-rM+s} H_i^*((z^*)^{-1}) \qquad (102)
\end{aligned}$$

## Orthogonality in the $z$ transform domain

- In case the filters have real coefficients orthogonality of the filter bank demands that there must be integers $r$ and $s$ such that

$$G_i(z) = z^{-rM+s} H_i(z^{-1}), \quad \text{for} \quad i = 0, 1, \ldots, (M-1) \tag{103}$$

- In the general complex case, equation (102) can be expressed as

$$\begin{bmatrix} G_0(z) \\ G_1(z) \\ \vdots \\ G_{M-1}(z) \end{bmatrix} = z^{-rM+s} \begin{bmatrix} H_0^*((z^*)^{-1}) \\ H_1^*((z^*)^{-1}) \\ \vdots \\ H_{M-1}^*((z^*)^{-1}) \end{bmatrix} \tag{104}$$

- Using the matrix forms of the polyphase decompositions we have that

$$
\begin{aligned}
\mathbf{R}^{\mathsf{T}}(z^{M}) \begin{bmatrix} z^{-(M-1)} \\ z^{-(M-2)} \\ \vdots \\ 1 \end{bmatrix} &= z^{-rM+s} \left\{ \mathbf{E}((z^{*})^{-M}) \begin{bmatrix} 1 \\ z^{*} \\ \vdots \\ (z^{*})^{(M-1)} \end{bmatrix} \right\}^{*} \\
&= z^{-rM+s} \mathbf{E}^{*}((z^{*})^{-M}) \begin{bmatrix} 1 \\ z \\ \vdots \\ z^{M-1} \end{bmatrix} \\
&= z^{(-rM+s+M-1)} \mathbf{E}^{*}((z^{*})^{-M}) \begin{bmatrix} z^{-(M-1)} \\ z^{-(M-2)} \\ \vdots \\ 1 \end{bmatrix}
\end{aligned}
\qquad (105)
$$

## Orthogonality in the $z$ transform domain

- Therefore, for orthogonal filters banks

$$\mathbf{R}^{\mathsf{T}}(z^{M}) = z^{(-rM+s+M-1)}\mathbf{E}^{*}((z^{*})^{-M}) \tag{106}$$

- If we choose a matrix $\mathcal{H}$ such that $s = 1$, the above equation becomes

$$\mathbf{R}^{\mathsf{T}}(z^{M}) = z^{-rM+M}\mathbf{E}^{*}((z^{*})^{-M}) \tag{107}$$

and thus we can write

$$\mathbf{R}^{\mathsf{T}}(z) = z^{-r+1}\mathbf{E}^{*}((z^{*})^{-1}) \tag{108}$$

## Orthogonality in the $z$ transform domain

- Since the perfect reconstruction condition is $\mathbf{R}(z)\mathbf{E}(z) = z^{-\Delta}\mathbf{I}_M$, we have that

$$\mathbf{E}^{*^{\mathsf{T}}}((z^*)^{-1})\mathbf{E}(z) = z^{(r-1-\Delta)}\mathbf{I}_M \qquad (109)$$

- By choosing $\mathcal{H}$ such that $r = \Delta + 1$, a sufficient condition for perfect reconstruction filter bank to be orthogonal is

$$\mathbf{E}^{*^{\mathsf{T}}}((z^*)^{-1})\mathbf{E}(z) = \mathbf{I}_M \qquad (110)$$

# Orthogonality in the $z$ transform domain

- A matrix $\mathbf{E}(z)$ satisfying the above condition is referred to as a paraunitary matrix. Therefore, an orthogonal perfect reconstruction filter bank is also referred to as a paraunitary filter bank. If the filters have real coefficients, equation (110) simplifies to

$$\mathbf{E}^{\mathsf{T}}(z^{-1})\mathbf{E}(z) = \mathbf{I}_M \tag{111}$$

## Transmultiplexers

- If two identical zero-delay $M$-channel perfect reconstruction filter banks are cascaded, we have that the signal corresponding to $u_k(m)$ in one filter bank is identical to the corresponding signal in the other filter bank, for each $k = 0, 1, \ldots, (M-1)$.

- Therefore, one can build a perfect reconstruction transmultiplexer as in Figure 25, which can combine the $M$ signals $u_k(m)$ into one single signal $y(n)$, and then recover the signals $v_k(m)$ that are identical to $u_k(m)$.

- One important application of such transmultiplexers is the multiplexing of $M$ signals so that they can be easily recovered without any cross-interference.

# Transmultiplexers

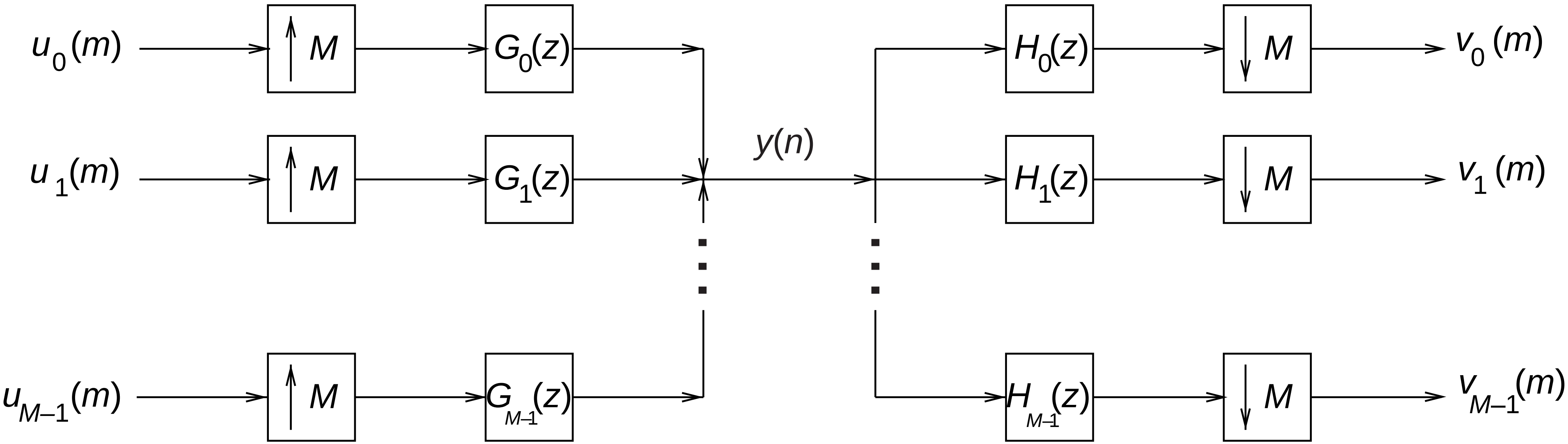



Figure 25: $M$-band transmultiplexer.

# General 2-band perfect reconstruction filter banks

The general 2-band case is shown in Figure 26.

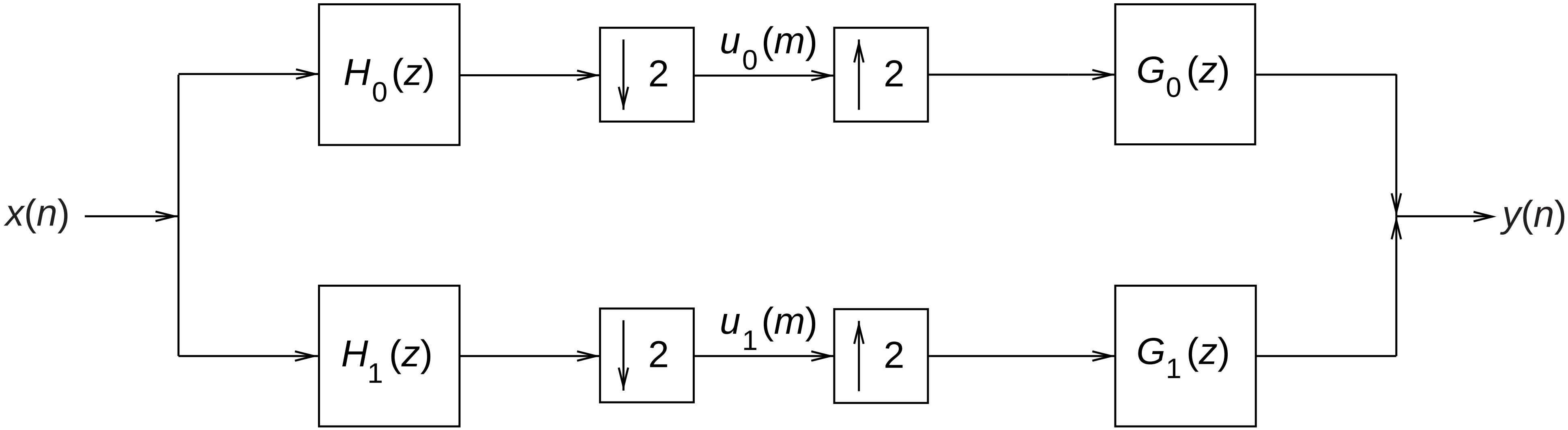



Figure 26: 2-band filter bank.

# General 2-band perfect reconstruction filter banks

- Representing the filters $H_0(z)$, $H_1(z)$, $G_0(z)$, and $G_1(z)$ in terms of their polyphase components

$$H_0(z) = E_{00}(z^2) + z^{-1}E_{01}(z^2) \quad (112)$$

$$H_1(z) = E_{10}(z^2) + z^{-1}E_{11}(z^2) \quad (113)$$

$$G_0(z) = z^{-1}R_{00}(z^2) + R_{10}(z^2) \quad (114)$$

$$G_1(z) = z^{-1}R_{01}(z^2) + R_{11}(z^2) \quad (115)$$

- The matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ in Figure 8b are then

$$\mathbf{E}(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}(z) & E_{11}(z) \end{bmatrix} \quad (116)$$

$$\mathbf{R}(z) = \begin{bmatrix} R_{00}(z) & R_{01}(z) \\ R_{10}(z) & R_{11}(z) \end{bmatrix} \quad (117)$$

## General 2-band perfect reconstruction filter banks

- If $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$, we have perfect reconstruction.
- We see that the output signal $y(n)$ will be equal to $x(n)$ delayed by $(M-1)$ samples, which for a 2-band filter bank is equal to just one sample.
- In the general case, we have $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}z^{-\Delta}$, which makes the output signal of the 2-band filter bank, $y(n)$, equal to $x(n)$ delayed by $(2\Delta + 1)$ samples.
- Therefore, the 2-band filter bank will be equivalent to a delay of $(2\Delta + 1)$ samples if

$$\mathbf{R}(z) = z^{-\Delta}\mathbf{E}^{-1}(z) \tag{118}$$

# General 2-band perfect reconstruction filter banks

- From equations (116) and (117), this implies that

$$\begin{bmatrix} R_{00}(z) & R_{01}(z) \\ R_{10}(z) & R_{11}(z) \end{bmatrix} = \frac{z^{-\Delta}}{E_{00}(z)E_{11}(z) - E_{01}(z)E_{10}(z)} \begin{bmatrix} E_{11}(z) & -E_{01}(z) \\ -E_{10}(z) & E_{00}(z) \end{bmatrix} \tag{119}$$

  This is sufficient for IIR filter bank design, as long as stability constraints are taken into consideration.

- If we want the filters to be FIR, the term in the denominator must be proportional to a pure delay

$$E_{00}(z)E_{11}(z) - E_{01}(z)E_{10}(z) = cz^{-l} \tag{120}$$

## General 2-band perfect reconstruction filter banks

- From equations (112)–(115), we can express the polyphase components in terms of the filters $H_k(z)$ and $G_k(z)$ as

$$\begin{aligned} E_{00}(z^2) &= \frac{H_0(z) + H_0(-z)}{2}, \qquad & E_{01}(z^2) &= \frac{H_0(z) - H_0(-z)}{2z^{-1}} \\ E_{10}(z^2) &= \frac{H_1(z) + H_1(-z)}{2}, \qquad & E_{11}(z^2) &= \frac{H_1(z) - H_1(-z)}{2z^{-1}} \end{aligned} \tag{121}$$

$$\begin{aligned} R_{00}(z^2) &= \frac{G_0(z) - G_0(-z)}{2z^{-1}}, \qquad & R_{10}(z^2) &= \frac{G_0(z) + G_0(-z)}{2} \\ R_{01}(z^2) &= \frac{G_1(z) - G_1(-z)}{2z^{-1}}, \qquad & R_{11}(z^2) &= \frac{G_1(z) + G_1(-z)}{2} \end{aligned} \tag{122}$$

# General 2-band perfect reconstruction filter banks

- Substituting equation (121) into equation (120), we have that

$$H_0(-z)H_1(z) - H_0(z)H_1(-z) = 2cz^{-2l-1} \tag{123}$$

- Now, substituting equation (120) into equation (119), and computing the $G_k(z)$ from equations (114) and (115), we arrive at

$$G_0(z) = -\frac{z^{2(l-\Delta)}}{c}H_1(-z) \tag{124}$$

$$G_1(z) = \frac{z^{2(l-\Delta)}}{c}H_0(-z) \tag{125}$$

- The reader can verify that whatever is the value of $l$, the overall filter bank transfer function consists of a delay of value $\Delta_{\text{total}} = 2\Delta + 1$.

## General 2-band perfect reconstruction filter banks

- Equations (123)–(125) suggest a possible way to design 2-band perfect reconstruction filter banks. The design procedure is as follows:

  (i) Find a polynomial $P(z)$ such that $P(-z) - P(z) = 2cz^{-2l-1}$.

  (ii) Factorize $P(z)$ into two factors, $H_0(z)$ and $H_1(-z)$. Care must be taken to guarantee that $H_0(z)$ and $H_1(-z)$ are lowpass filters.

  (iii) Find $G_0(z)$ and $G_1(z)$ using equations (124) and (125).

# General 2-band perfect reconstruction filter banks

- Some important points should be noted in this case:
  - If the delay, $\Delta$, is zero, some of the filters are certainly noncausal: for negative $l$, either $H_0(z)$ or $H_1(z)$ must be noncausal (see equation (123)); for positive $l$, either $G_0(z)$ or $G_1(z)$ must be noncausal. Therefore, a causal perfect reconstruction filter bank will necessarily have nonzero delay.
  - The magnitude responses $|G_0(e^{j\omega})|$ and $|H_1(e^{j\omega})|$ are mirror images of each other around $\omega = \frac{\pi}{2}$ (equation (124)). The same happens to $|H_0(e^{j\omega})|$ and $|G_1(e^{j\omega})|$ (see equation (125)).

## General 2-band perfect reconstruction filter banks

- If one wants the filter bank to be composed of linear-phase filters, it suffices to find a linear-phase product filter $P(z)$, and make linear-phase factorizations of it (one should remember that if $z_0$ is a zero of a linear phase FIR filter, then $z_0^{-1}$ is also a zero).

- In this case we have some additional constraints on the filters. As we have seen above,

$$P(z) - P(-z) = 2cz^{-2l-1} \tag{126}$$

# General 2-band perfect reconstruction filter banks

- This implies that all the odd-power terms of polynomial $P(z)$ except $z^{-2l-1}$ are null. In addition, a linear-phase $P(z)$ should be either symmetric or antisymmetric, with the central term being $cz^{-2l-1}$. Therefore, if $P(z)$ has more than two terms, and considering that the first term of $P(z)$ is $az^{0}$, then its last term should be $\pm az^{-4l-2}$. Then, its order is $(4l+2)$. Taking this into consideration, we have two cases:

  a) Both filters have even order. In this case, $H_0(z)$ and $H_1(-z)$ should be either Type-I or Type-III filters. However, since a Type-III filter should have zeros at $\omega = 0$ and $\omega = \pi$, and $P(z)$ should be a lowpass filter, then both $H_0(z)$ and $H_1(-z)$ are Type-I filters, and so is $H_1(z)$. In addition, since the sum of their orders is $(4l+2)$, the order of one is a multiple of 4 and the order of the other is not (that is, their orders differ by an odd multiple of 2).

  b) Both filters have odd order. In this case, $H_0(z)$ and $H_1(-z)$ should be either Type-II or Type-IV filters. However, since a Type-IV filter should have a zero at

$\omega = 0$, and $P(z)$ should be a lowpass filter, then both $H_0(z)$ and $H_1(-z)$ should be Type-II filters, and $H_1(z)$ is Type-IV. In addition, if the orders of $H_0(z)$ and $H_1(z)$ are $(2k_0 + 1)$ and $(2k_1 + 1)$, respectively, since the sum of their orders is equal to $(4l + 2)$, we have that

$$(2k_0 + 1) + (2k_1 + 1) \quad = \quad 4l + 2 \quad \Rightarrow \quad k_0 = 2l - k_1 \tag{127}$$

and then the difference of their orders is

$$\begin{aligned}(2k_0 + 1) - (2k_1 + 1) &= 4l - 2k_1 + 1 - 2k_1 - 1 \\ &= 4l - 2k_1 - 2k_1 \\ &= 4(l - k_1)\end{aligned} \tag{128}$$

which is a multiple of 4.

## General 2-band perfect reconstruction filter banks

- In the case that $P(z)$ has only two terms, since it is linear phase, we have that equation (126) implies that

$$P(z) = cz^{-2l-1} \pm cz^{-2r} = cz^{-2r}(z^{-2l+2r-1} \pm 1) \tag{129}$$

- Thus, $P(z)$ has odd order equal to $|2l - 2r + 1|$, and all its zeros on the unity circle. We then have two cases:

  a) $H_0(z)$ has even order and $H_1(z)$ has odd order. Then, $P(z)$ being a lowpass filter implies that $H_0(z)$ is Type-I and $H_1(-z)$ is Type-II, and therefore $H_1(z)$ is Type-IV.

  b) $H_0(z)$ has odd order and $H_1(z)$ has even order. Then, in this case $H_0(z)$ is Type-II and $H_1(-z)$ is Type-I, and therefore $H_1(z)$ is also Type-I.

- This last case is of little practical interest, since the resulting $H_0(z)$ and $H_1(z)$ tend to have poor frequency selectivity.

## General 2-band perfect reconstruction filter banks

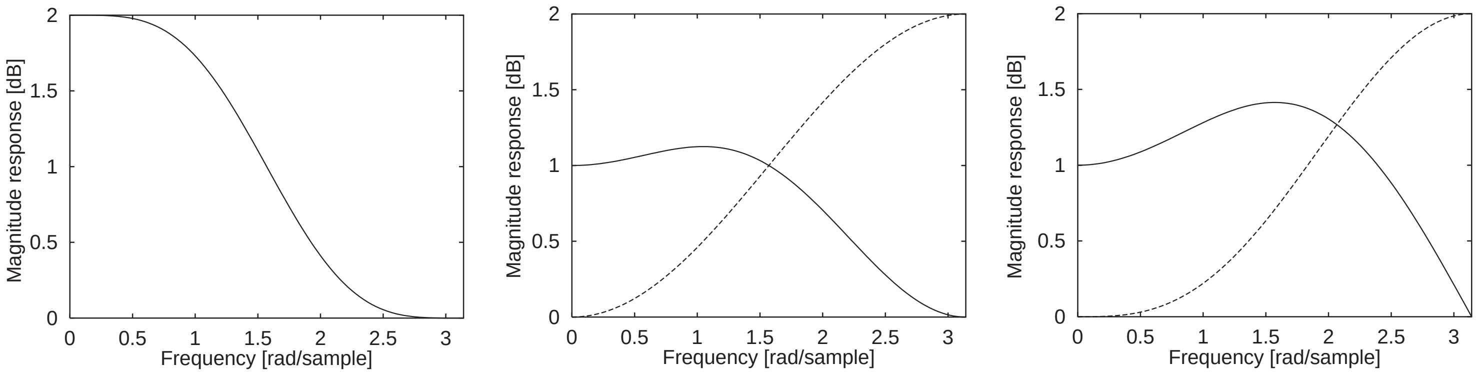
**Example 9.6** A product filter $P(z)$ satisfying $P(z) - P(-z) = 2z^{-2l-1}$ is

$$P(z) = \frac{1}{16}(-1+9z^{-2}+16z^{-3}+9z^{-4}-z^{-6}) = \frac{1}{16}(1+z^{-1})^4(-1+4z^{-1}-z^{-2}) \tag{130}$$

Find two possible factorizations of $P(z)$ and plot the magnitude responses of their corresponding analysis filters.

## General 2-band perfect reconstruction filter banks

**Solution** We can see from the magnitude response that $P(z)$ is a lowpass filter.




(a) (b) (c)

Figure 27: Magnitude responses: (a) $P(z)$ from equation (130); (b) $H_0(z)$ (solid line) and $H_1(z)$ (dashed line) from the factorizations in equations (131) and (132); (c) $H_0(z)$ (solid line) and $H_1(z)$ (dashed line) from the factorizations in equations (135) and (136).

## General 2-band perfect reconstruction filter banks

- One possible factorization of $P(z)$ results in the following filter bank, This filter bank is the popular symmetric short-kernel filter bank:

$$H_0(z) = \frac{1}{8}(-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4}) \tag{131}$$

$$H_1(z) = \frac{1}{2}(1 - 2z^{-1} + z^{-2}) \tag{132}$$

$$G_0(z) = \frac{1}{2}(1 + 2z^{-1} + z^{-2}) \tag{133}$$

$$G_1(z) = \frac{1}{8}(1 + 2z^{-1} - 6z^{-2} + 2z^{-3} + z^{-4}) \tag{134}$$

The magnitude responses of the analysis filters are depicted in Figure 27b.

## General 2-band perfect reconstruction filter banks

- Another possible factorization is as follows:

$$H_0(z) = \frac{1}{4}(-1 + 3z^{-1} + 3z^{-2} - z^{-3}) \quad (135)$$

$$H_1(z) = \frac{1}{4}(1 - 3z^{-1} + 3z^{-2} - z^{-3}) \quad (136)$$

$$G_0(z) = \frac{1}{4}(1 + 3z^{-1} + 3z^{-2} + z^{-3}) \quad (137)$$

$$G_1(z) = \frac{1}{4}(1 + 3z^{-1} - 3z^{-2} - z^{-3}) \quad (138)$$

The corresponding magnitude responses of the analysis filters are depicted in Figure 27c.

$\triangle$

# QMF filter banks

- One of the earliest proposed approaches for the design of 2-band FIR filter banks is the so-called quadrature mirror filter (QMF) bank, where the analysis highpass filter is designed to alternate the signs of the impulse-response samples of the lowpass filter, that is

$$H_1(z) = H_0(-z) \tag{139}$$

- For filters with real coefficients for the analysis filter bank, the magnitude response of the highpass filter, $|H_1(e^{j\omega})|$, is the mirror image of the lowpass filter magnitude response, $|H_0(e^{j\omega})|$, with respect to the quadrature frequency $\frac{\pi}{2}$. Hence the QMF nomenclature.

# QMF filter banks

- QMF filter banks are designed to structurally avoid aliasing while keeping the constraint in equation (139). Filter $H_0(z)$ is then designed so that the filter bank is close enough to achieving perfect reconstruction.

- Its design equations can be derived by starting from the modulation matrix representation in equation (42), for $M = 2$ bands,

$$Y(z) = \frac{1}{2}\begin{bmatrix} X(z) & X(-z) \end{bmatrix} \begin{bmatrix} H_0(z) & H_1(z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} \quad (140)$$

# QMF filter banks

- The aliasing effect is represented by the terms containing $X(-z)$. To avoid aliasing choose the synthesis filters such that

$$G_0(z) = H_1(-z) \tag{141}$$

$$G_1(z) = -H_0(-z) \tag{142}$$

- This choice keeps the filters $G_0(z)$ and $G_1(z)$ as lowpass and highpass filters, respectively, as desired.
- Also, the aliasing is now canceled by the synthesis filters, instead of being totally avoided by the analysis filters, relieving the specifications of the latter filters.
- The overall transfer function of the filter bank is given by

$$H(z) = \frac{1}{2}(H_0(z)G_0(z)+H_1(z)G_1(z)) = \frac{1}{2}(H_0(z)H_1(-z)-H_1(z)H_0(-z)) \tag{143}$$

where in the second equality we employed the aliasing elimination constraint.

# QMF filter banks

- In the original QMF design, the aliasing elimination condition is combined with the alternating-sign choice for the highpass filter of equation (139). In such a case, the overall transfer function is given by

$$H(z) = \frac{1}{2}(H_0^2(z) - H_0^2(-z)) \tag{144}$$

The expression above can be rewritten in a more convenient form by employing the polyphase decomposition of the lowpass filter $H_0(z) = E_{00}(z^2) + z^{-1}E_{01}(z^2)$, as follows:

$$\begin{aligned} H(z) &= \frac{1}{2}\left(H_0(z) + H_0(-z)\right)\left(H_0(z) - H_0(-z)\right) \\ &= 2z^{-1}E_{00}(z^2)E_{01}(z^2) \end{aligned} \tag{145}$$

## QMF filter banks

- As pointed out above, the QMF design approach of 2-band filter banks consists of designing the lowpass filter $H_0(z)$.
- Perfect reconstruction is achievable only if the polyphase components of the lowpass filter (that is, $E_{00}(z)$ and $E_{01}(z)$) are simple delays.
- This constraint limits the selectivity of the generated filters.

## QMF filter banks

- Therefore, for QMF design we usually adopt an approximate solution by choosing $H_0(z)$ to be an $N$th-order FIR linear-phase lowpass filter.
- This eliminates any phase distortion of the overall transfer function $H(z)$, which, in this case, will also have linear phase.
- For either a Type-I or Type-II $N$th-order filter, the filter bank transfer function in equation (144) can then be written as

$$
\begin{aligned}
H(e^{j\omega}) &= \frac{1}{2}\left(B_0^2(\omega)e^{-j\omega N} - B_0^2(\omega-\pi)e^{-j(\omega-\pi)N}\right) \\
&= \frac{e^{-j\omega N}}{2}\left(\left|H_0(e^{j\omega})\right|^2 - e^{j\pi N}\left|H_0(e^{j(\omega-\pi)})\right|^2\right) \\
&= \frac{e^{-j\omega N}}{2}\left(\left|H_0(e^{j\omega})\right|^2 - (-1)^N\left|H_0(e^{j(\omega-\pi)})\right|^2\right) \qquad (146)
\end{aligned}
$$

# QMF filter banks

- From the above equation, we see that, for $N$ even, $H(e^{j\frac{\pi}{2}}) = 0$, which is undesirable. Therefore, for QMF filters the filter order must be odd.

- In this case, equation (146) becomes

$$\begin{aligned} H(e^{j\omega}) &= \frac{e^{-j\omega N}}{2}\left(\left|H_0(e^{j\omega})\right|^2 + \left|H_0(e^{j(\omega-\pi)})\right|^2\right) \\ &= \frac{e^{-j\omega N}}{2}\left(\left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2\right) \end{aligned} \tag{147}$$

## QMF filter banks

- The design procedure goes on to minimize the following objective function using an optimization algorithm:

$$\xi = \delta \int_{\omega_s}^{\pi} \left| H_0(e^{j\omega}) \right|^2 d\omega + (1-\delta) \int_0^{\pi} \left| H(e^{j\omega}) - \frac{e^{-j\omega N}}{2} \right|^2 d\omega \qquad (148)$$

where $\omega_s$ is the stopband edge, usually chosen slightly above $0.5\pi$.

- The parameter $0 < \delta < 1$ provides a tradeoff between the stopband attenuation of the lowpass filter and the amplitude distortion of the filter bank.

## QMF filter banks

- This objective function has local minima.
- A good starting point for the coefficients of the lowpass filter and an adequate nonlinear optimization algorithm lead to good results, that is, filter banks with low amplitude distortions and good selectivity of the filters.
- Usually, a simple window-based design provides a good starting point for the lowpass filter. Overall, the simplicity of the QMF design makes it widely used in practice.
- Johnston was among the first to provide QMF coefficients for several designs. Due to his pioneering work, such QMF filters are usually termed Johnston filter banks.

# QMF filter banks

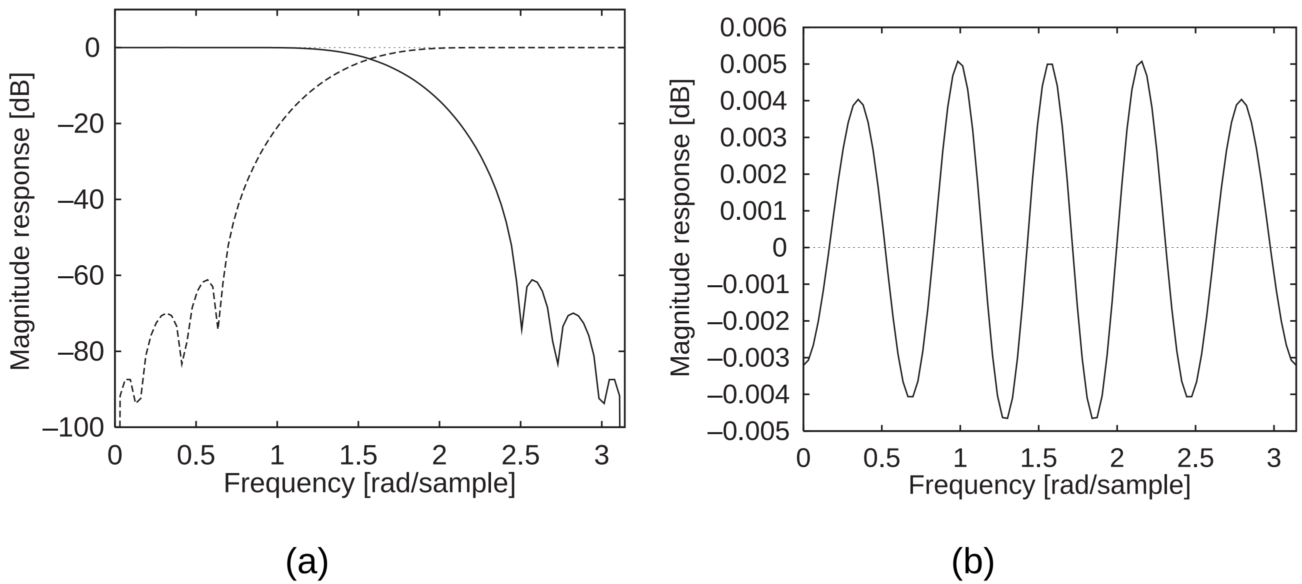



Figure 28: QMF design of order $N = 15$: (a) magnitude responses of the analysis filters (solid line: $H_0(z)$; dashed line: $H_1(z)$); (b) overall magnitude response.

# CQF filter banks

- The QMF design is quite simple. However, the possibility of getting perfect reconstruction is lost except in a few trivial cases.
- By time-reversing the impulse response and alternating the signs of the lowpass filter, one can design perfect reconstruction filter banks with more selective sub-filters.
- The resulting filters became known as the conjugate quadrature filter (CQF) banks.
- The CQF filter bank is an orthogonal filter bank. Its design can then be deduced from the orthogonality condition, which, for the 2-band case, is

$$G_i(z) = z^{-2r+s} H_i(z^{-1}), \quad \text{for} \quad i = 0, 1 \tag{149}$$

# CQF filter banks

- The perfect reconstruction conditions for the 2-band case are given by equations (124) and (125),

$$G_0(z) = -\frac{z^{2(l-\Delta)}}{c} H_1(-z) \tag{150}$$

$$G_1(z) = \frac{z^{2(l-\Delta)}}{c} H_0(-z) \tag{151}$$

where $2\Delta + 1$ is the total delay of the filter bank and $l$ can be any integer.

- Using equation (149) these conditions become

$$z^{-2r+s} H_0(z^{-1}) = -\frac{z^{2(l-\Delta)}}{c} H_1(-z) \tag{152}$$

$$z^{-2r+s} H_1(z^{-1}) = \frac{z^{2(l-\Delta)}}{c} H_0(-z) \tag{153}$$

## CQF filter banks

- From equation (152) we have that

$$(-z)^{2r-s}H_0(-z) = -\frac{(-z)^{-2(l-\Delta)}}{c}H_1(z^{-1}) \tag{154}$$

- Then

$$H_1(z^{-1}) = -c(-z)^{[2(l-\Delta)+2r-s]}H_0(-z) \tag{155}$$

- By replacing $H_1(z^{-1})$ from equation (155) in equation (153) we get

$$z^{-2r+s}\left[-c(-z)^{[2(l-\Delta)+2r-s]}H_0(-z)\right] = \frac{z^{2(l-\Delta)}}{c}H_0(-z) \tag{156}$$

# CQF filter banks

- Equivalently

$$-c^2(-1)^{[2(l-\Delta)+2r-s]}\left[z^{2(l-\Delta)}H_0(-z)\right] = z^{2(l-\Delta)}H_0(-z) \qquad (157)$$

- Then

$$c^2 = -(-1)^{-[2(l-\Delta)+2r-s]} = -(-1)^s \qquad (158)$$

# CQF filter banks

- In order for the 2-band perfect reconstruction filter bank to be orthogonal, we must have $s$ odd and $c = \pm 1$. Then, since for $s$ odd, $[2(l - \Delta) + 2r - s]$ is also odd, equation (155) implies that

$$H_1(z) = cz^{-[2(l-\Delta)+2r-s]} H_0(-z^{-1}) \qquad (159)$$

- In the CQF design one usually makes the following design choices:
  - $c = -1$
  - The order of the lowpass filter is the odd number

$$N = 2(l - \Delta) + 2r - s \qquad (160)$$

## CQF filter banks

- In order for orthogonality to imply the paraunitarity of matrix $\mathbf{E}(z)$, we must have $s = 1$.
- On the other hand, in the CQF bank it is sufficient that $s$ be odd.
- However, from equation (160) we see that for any odd value of $N$, we can find a value of $r$ such that $s = 1$.
- Therefore, we have that for CQF filter banks the polyphase matrix $\mathbf{E}(z)$ is always paraunitary. This is an important property for CQF filter banks.

# CQF filter banks

- With these design choices, we have that the analysis highpass filter is given by

$$H_1(z) \quad = -z^{-N} H_0(-z^{-1}) \tag{161}$$

- Since perfect reconstruction also requires that equation (123) to be valid, we must have

$$\begin{aligned} -2z^{-2l-1} &= H_0(-z)H_1(z) - H_0(z)H_1(-z) \\ &= H_0(-z)(-z^{-N})H_0(-z^{-1}) - H_0(z)[-(-z)^{-N}]H_0(z^{-1}) \\ &= -z^{-N}\left[H_0(-z)H_0(-z^{-1}) + H_0(z)H_0(z^{-1})\right] \end{aligned} \tag{162}$$

## CQF filter banks

- By defining

$$P(z) = H_0(z)H_0(z^{-1}) \tag{163}$$

- The perfect reconstruction condition in equation (162) becomes

$$P(z) + P(-z) = 2z^{N-2l-1} \tag{164}$$

## CQF filter banks

- From the definition of $P(z)$ in equation (163), we see that $P(z) = P(z^{-1})$. Therefore, from equation (164) we have

$$
\begin{aligned}
2z^{N-2l-1} &= P(z) + P(-z) \\
&= P(z^{-1}) + P(-z^{-1}) \\
&= 2z^{-N+2l+1}
\end{aligned}
\tag{165}
$$

  which implies that we should choose $l$ such that $N = 2l + 1$.

- This condition makes the PR condition to be

$$
P(z) + P(-z) = 2 \tag{166}
$$

# CQF filter banks

- The synthesis filters from equations (150) and (151) become

$$G_0(z) = z^{[2(l-\Delta)-N]} H_0(z^{-1}) = z^{-(2\Delta+1)} H_0(z^{-1}) \tag{167}$$

$$G_1(z) = -z^{2(l-\Delta)} H_0(-z) = -z^{(N-2\Delta-1)} H_0(-z) \tag{168}$$

- If $p(n)$ is the inverse $z$ transform of $P(z)$, equation (166) is equivalent to

$$p(n)[1 + (-1)^n] = 2\delta(n) \tag{169}$$

# CQF filter banks

- The design procedure for CQF filter banks consists of the following steps:

  (i) Noting that $p(n) = 0$ for $n$ even, except for $n = 0$, we start by designing a half-band filter, such that $\frac{\omega_p + \omega_r}{2} = \frac{\pi}{2}$, with order $2N$ and the same ripple $\delta_{hb}$ in the passband and stopband. The half-band filter will have null samples on its impulse response for every even $n$ except for $n = 0$. Such a filter can be designed using the standard Chebyshev approach for FIR filters as follows:

  (a) Design a zero-delay Hilbert transformer $H_h(z)$ with order $2N$. Since its order is even it must be a Type III FIR filter. Its ripple must be smaller than $\pm\frac{\delta_{hb}}{2}$ in its passband and its transition bandwidth around $\omega = 0$ and $\omega = \pi$ should be $\omega_r$.

  (b) From the Hilbert transformer, create the filter $P(z)$.

$$P(z) = 1 + \frac{\delta_{hb}}{2} - jH_h(-jz) \tag{170}$$

  Note that the term $-jH_h(-jz)$ corresponds to a zero-delay filter with gain equal to $\left(1 \pm \frac{\delta_{hb}}{2}\right)$, for $0 \leq |\omega| \leq \frac{\pi}{2}$, and gain equal to $-\left(1 \pm \frac{\delta_{hb}}{2}\right)$, for $\frac{\pi}{2} \leq |\omega| \leq \pi$. By summing $\left(1 + \frac{\delta_{hb}}{2}\right)$ to its frequency response we obtain a zero-delay lowpass filter with gain $2$ and ripple $\delta_{hb}$ in the passband and a non-negative stopband gain ranging from zero to $\delta_{hb}$. This is necessary since $P(e^{j\omega})$ must be non-negative for all $\omega$, as it corresponds to the modulus squared of $H_0(e^{j\omega})$. As a rule of thumb, to simplify the design procedure, the stopband attenuation in dB of the half-band filter should be at least twice the desired stopband attenuation plus $6$ dB.

(ii) An approach is to decompose $P(z) = H_0(z)H_0(z^{-1})$ such that $H_0(z)$ has either near linear phase or has minimum phase. To obtain near linear phase, one can select the zeros of $H_0(z)$ to be alternately from inside and outside the unit circle as the frequency is increased. Minimum phase is obtained when all zeros are either inside or on the unit circle of the $z$ plane.

# CQF filter banks

- If we wanted the filter bank to be composed of linear-phase filters, find a linear-phase product filter $P(z)$ and make linear-phase factorizations of it.
- The only linear-phase 2-band filter banks that satisfy equation (169) are the ones composed of trivial linear-phase filters such as the ones described by equations (11)–(14).
- There is no point in looking for linear-phase factorizations of $P(z)$. This is why in step (ii) above the usual approach is to look for either minimum phase or near linear-phase factorizations of $P(z)$.

# CQF filter banks

**Example 9.7** Design a perfect reconstruction filter bank for which the lowpass analysis filter is given by:

$$H(z) = -z^{-3} + z^{-2} + z^{-1} + 1 \tag{171}$$

**Solution**

- In this example there is no particular constraint imposed to the highpass analysis filters, so that we can look for a CQF design.
- The CQF design is only possible if equations (163) and (166) are valid. Since

$$\begin{aligned} H(z)H(z^{-1}) &= (-z^{-3} + z^{-2} + z^{-1} + 1)(-z^{3} + z^{2} + z + 1) \\ &= -z^{3} + z + 4 + z^{-1} - z^{-3} \end{aligned} \tag{172}$$

  we have that

$$H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 8 \tag{173}$$

## CQF filter banks

- Therefore, if we choose

$$H_0(z) = \frac{1}{2}H(z) \tag{174}$$

we have that $H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2$, and a CQF design with $H_0(z)$ equal to its lowpass analysis filter is possible. For this choice of $H_0(z)$, the order of the CQF bank is $N = 3$.

# CQF filter banks

- Supposing an overall delay of $2\Delta + 1 = 3$ samples, the filter bank becomes, from equations (161), (167), and (168),

$$\left.\begin{array}{rclcl} H_0(z) & = & \frac{1}{2}H(z) & = & \frac{1}{2}(-z^{-3}+z^{-2}+z^{-1}+1) \\ H_1(z) & = & -z^{-3}H_0(-z^{-1}) & = & \frac{1}{2}(-z^{-3}+z^{-2}-z^{-1}-1) \\ G_0(z) & = & z^{-3}H_0(z^{-1}) & = & \frac{1}{2}(z^{-3}+z^{-2}+z^{-1}-1) \\ G_1(z) & = & -H_0(-z) & = & \frac{1}{2}(-z^{-3}-z^{-2}+z^{-1}-1) \end{array}\right\} \quad (175)$$

## CQF filter banks

- An alternative way of designing a CQF filter bank is using the fact that the analysis polyphase matrix is paraunitary.
- Using $H_0(z)$ and $H_1(z)$ from equation (175), we see that, from equation (121),

$$\left.\begin{aligned} E_{00}(z^2) &= \frac{1}{2}(z^{-2}+1) \\ E_{01}(z^2) &= \frac{1}{2}(-z^{-2}+1) \\ E_{10}(z^2) &= \frac{1}{2}(z^{-2}-1) \\ E_{11}(z^2) &= \frac{1}{2}(-z^{-2}-1) \end{aligned}\right\} \tag{176}$$

# CQF filter banks

- Then the analysis polyphase matrix $\mathbf{E}(z)$ is

$$\mathbf{E}(z) = \frac{1}{2}\begin{bmatrix} 1+z^{-1} & 1-z^{-1} \\ -1+z^{-1} & -1-z^{-1} \end{bmatrix} \tag{177}$$

such that

$$\mathbf{E}^{*^{\mathrm{T}}}\left((z^{*})^{-1}\right) = \frac{1}{2}\begin{bmatrix} 1+z & -1+z \\ 1-z & -1-z \end{bmatrix} \tag{178}$$

# CQF filter banks

- Therefore,

$$\mathbf{E}^{*^{\mathsf{T}}}((z^*)^{-1})\mathbf{E}(z) = \frac{1}{4}\begin{bmatrix} 1+z & -1+z \\ 1-z & -1-z \end{bmatrix}\begin{bmatrix} 1+z^{-1} & 1-z^{-1} \\ -1+z^{-1} & -1-z^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{179}$$

  and the matrix $\mathbf{E}(z)$ is paraunitary, which implies that it is possible to have a CQF filter with the given analysis filters.

- From equation (118)

$$\mathbf{R}(z) = z^{-\Delta}\mathbf{E}^{-1}(z) \tag{180}$$

  where $\Delta = 1$, since the overall delay $(2\Delta + 1)$ in this example was chosen to be equal to the filter order $N = 3$.

# CQF filter banks

- Thus, the paraunitarity of $\mathbf{E}(z)$ implies that

$$\mathbf{R}(z) = z^{-1}\mathbf{E}^{*^{\mathsf{T}}}((z^{*})^{-1}) = \frac{1}{2}\begin{bmatrix} z^{-1}+1 & -z^{-1}+1 \\ z^{-1}-1 & -z^{-1}-1 \end{bmatrix} \tag{181}$$

- The synthesis filters are

$$\begin{aligned} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} &= \mathbf{R}^{\mathsf{T}}(z^2)\begin{bmatrix} z^{-1} \\ 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} z^{-2}+1 & z^{-2}-1 \\ -z^{-2}+1 & -z^{-2}-1 \end{bmatrix}\begin{bmatrix} z^{-1} \\ 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} z^{-3}+z^{-2}+z^{-1}-1 \\ -z^{-3}-z^{-2}+z^{-1}-1 \end{bmatrix} \end{aligned} \tag{182}$$

# Block transforms

- Perhaps the most popular example of $M$-band perfect reconstruction filter banks is given by the block transforms.
- For instance, the discrete cosine transform (DCT) does essentially the same job as a filter bank: It divides a signal into several frequency components.
- The main difference is that, given a length-$N$ signal, the DCT divides it into $N$ frequency channels, whereas a filter bank divides it into $M$ channels, with $M < N$.
- In many applications, one wants to divide a length-$N$ signal into $J$ blocks, each having length $M$, and separately apply the transform to each one.
- This is done, for example, in the MPEG2 standard employed in digital video transmission.

# Block transforms

- In it, instead of transmitting the individual pixels of a video sequence, each frame is first divided into $8 \times 8$ blocks. Then a two-dimensional DCT is applied to each block, and the DCT coefficients are transmitted instead.

- Consider a signal $x(n)$, for $n = 0, 1, \ldots, (N-1)$, divided into $J$ blocks $B_j$, with $j = 0, 1, \ldots, (J-1)$, each of size $M$. Block $B_j$ then consists of the signal $x_j(m)$, given by

$$x_j(m) = x(jM + m) \tag{183}$$

for $j = 0, 1, \ldots, (J-1)$ and $m = 0, 1, \ldots, (M-1)$.

# Block transforms

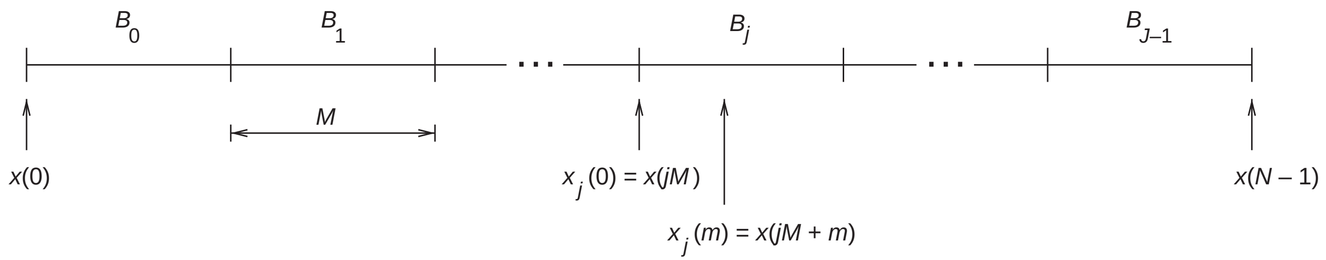



Figure 29: Division of a length-$N$ signal into $J$ non-overlapping blocks of length $M$.

## Block transforms

- Suppose that $y_j(k)$, for $k = 0, 1, \ldots, (M-1)$ and $j = 0, 1, \ldots, (J-1)$, is the transform of a block $x_j(m)$, where the direct and inverse transforms are described by

$$y_j(k) = \sum_{m=0}^{M-1} c_k(m) x_j(m) \tag{184}$$

$$x_j(m) = \sum_{k=0}^{M-1} c_k^*(m) y_j(k) \tag{185}$$

where $c_k^*(m)$, for $m = 0, 1, \ldots, (M-1)$, is the $k$th basis function of the transform, or, alternatively, the $k$th row of the transform matrix.

- We can then regroup the transforms of the blocks sequentially according to $k$, that is, group all the $y_j(0)$, all the $y_j(1)$, and so on.

## Block transforms

- This is equivalent to creating $M$ signals $u_k(j) = y_j(k)$, for $j = 0, 1, \ldots, (J-1)$ and $k = 0, 1, \ldots, (M-1)$.
- Since $x_j(m) = x(Mj + m)$, from equations (184) and (185), we have that

$$u_k(j) = \sum_{m=0}^{M-1} c_k(m) x(Mj + m) \tag{186}$$

$$x(Mj + m) = \sum_{k=0}^{M-1} c_k^*(m) u_k(j) \tag{187}$$

# Block transforms

- We can then interpret $u_k(j)$ as the convolution, sampled at the points $Mj$, of $x(n)$ with $c_k(-n)$. This is the same as filtering $x(n)$ with a filter of impulse response $c_k(-n)$, and decimating its output by $M$.
- If we define $u_k^{(i)}(j)$ as the signal $u_k(j)$ interpolated by $M$, we have that $u_k^{(i)}(Mj+n) = 0$, for $n = 1, 2, \ldots, (M-1)$. This implies that

$$c_k^*(m)u_k(j) = \sum_{n=0}^{M-1} c_k^*(m-n)u_k^{(i)}(Mj+n) \tag{188}$$

- Such an expression can be interpreted as $c_k^*(m)u_k(j)$ being the result of interpolating $u_k(j)$ by $M$, and filtering it with a filter of impulse response $c_k^*(m)$.

## Block transforms

- Substituting equation (188) in equation (187), we arrive at the following expression:

$$x(Mj+m) = \sum_{k=0}^{M-1} \sum_{n=0}^{M-1} c_k^*(m-n) u_k^{(i)}(Mj+n) \tag{189}$$

- A block transform is equivalent to a perfect reconstruction filter bank having the impulse responses of band $k$ analysis and synthesis filters equal to $c_k(-m)$ and $c_k^*(m)$, respectively.

# Block transforms

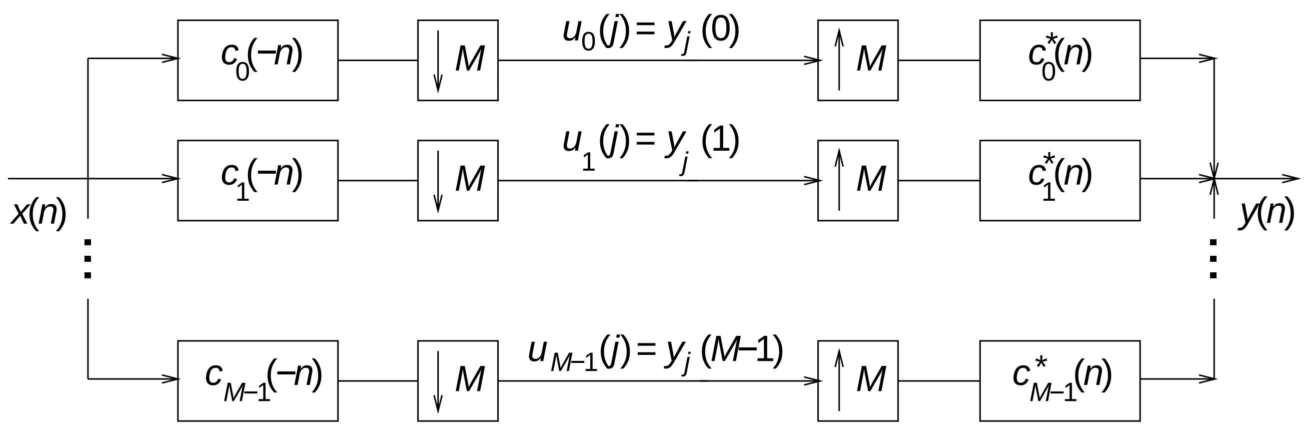



Figure 30: Interpretation of direct and inverse block transforms as a perfect reconstruction filter bank.

## Block transforms

**Example 9.8** The coefficients of the length-$M$ DCT are given by

$$c_k(m) = \alpha(k) \cos\left[\frac{\pi(2m+1)k}{2M}\right], \quad \text{for } m = 0, 1, \ldots, (M-1) \tag{190}$$

where $\alpha(0) = \sqrt{\frac{1}{M}}$ and $\alpha(k) = \sqrt{\frac{2}{M}}$, for $k = 1, 2, \ldots, (M-1)$. Plot the impulse response and the magnitude response of the analysis filters corresponding to the length-$10$ DCT. In addition, determine whether the filter bank has linear phase or not.

**Solution**

- As can be seen from Figure 30, the impulse responses $h_k(n)$ of the analysis filters for the length-10 DCT filter bank are given by

$$h_k(n) = c_k(-n) = \alpha(k) \cos\left[\frac{\pi(1-2n)k}{20}\right], \quad \text{for } k, n = 0, 1, \ldots, 9 \tag{191}$$

# Block transforms

- These impulse responses are depicted in Figure 31 for each band $k$, and the corresponding magnitude responses are depicted in Figure 32.
- Also, from equation (191), we can see that $c_k(m) = (-1)^k c_k(9 - m)$, and therefore the filter bank has linear phase.
- This also implies that the magnitude responses of the analysis and synthesis filters are the same.

$\triangle$

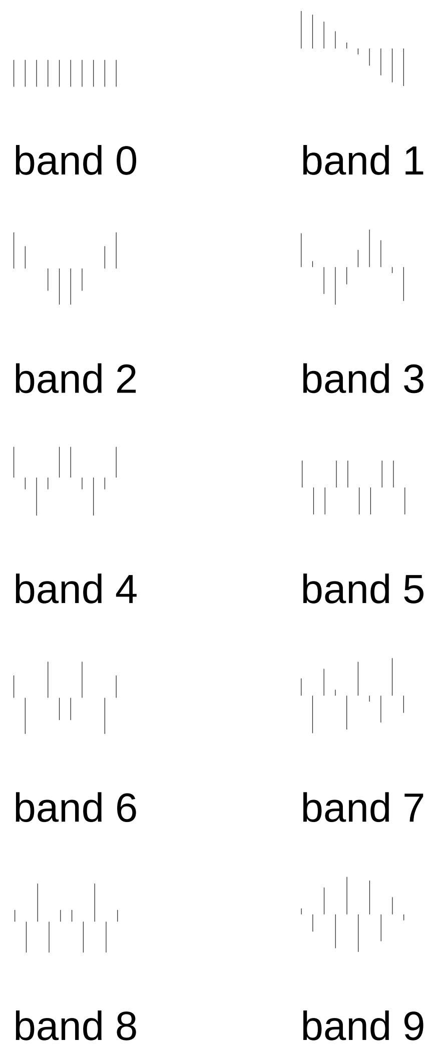



Figure 31: Impulse responses of the filters of the 10-band DCT.

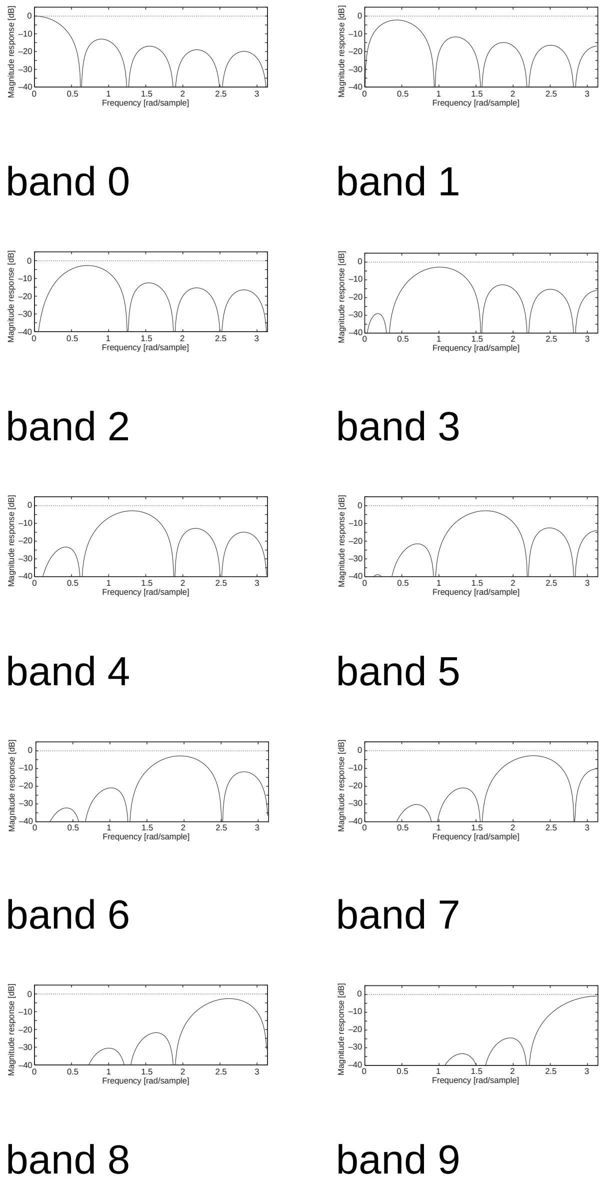



Figure 32: Magnitude responses of the filters of the 10-band DCT.

## Block transforms

- An alternative way to describe the block transforms as filter banks is to redraw them as depicted in Figure 33.
- Note that in this case the $M$-sample block slides along the signal.
- However, since the transform is computed for non-overlapping blocks, then the outputs of the analysis bank of
- Figure 33 should be decimated by $M$. This yields the representation of the direct and inverse transforms as the causal filter bank in Figure 34 (note that there is a delay of $(M-1)$ samples in this causal implementation of the transform).

# Block transforms

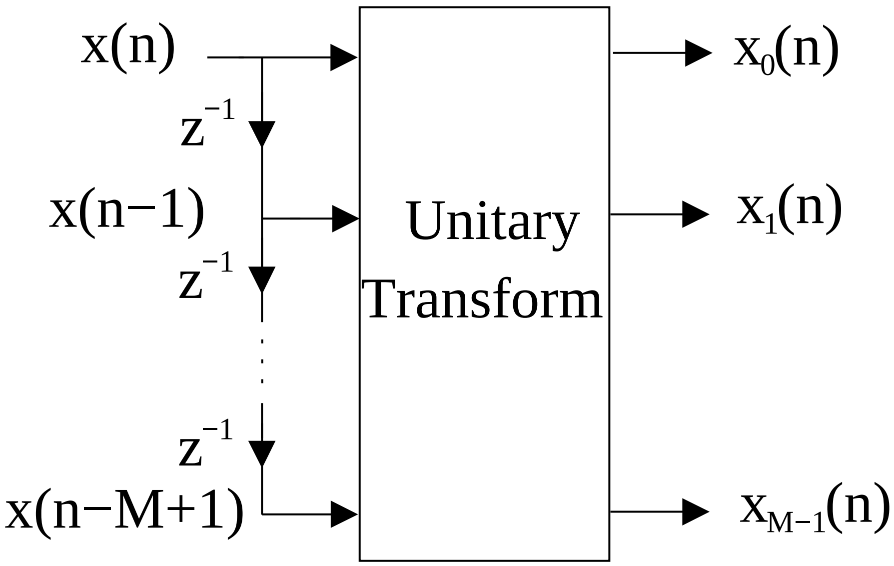



Figure 33: Unitary transform analysis filter bank.

# Block transforms

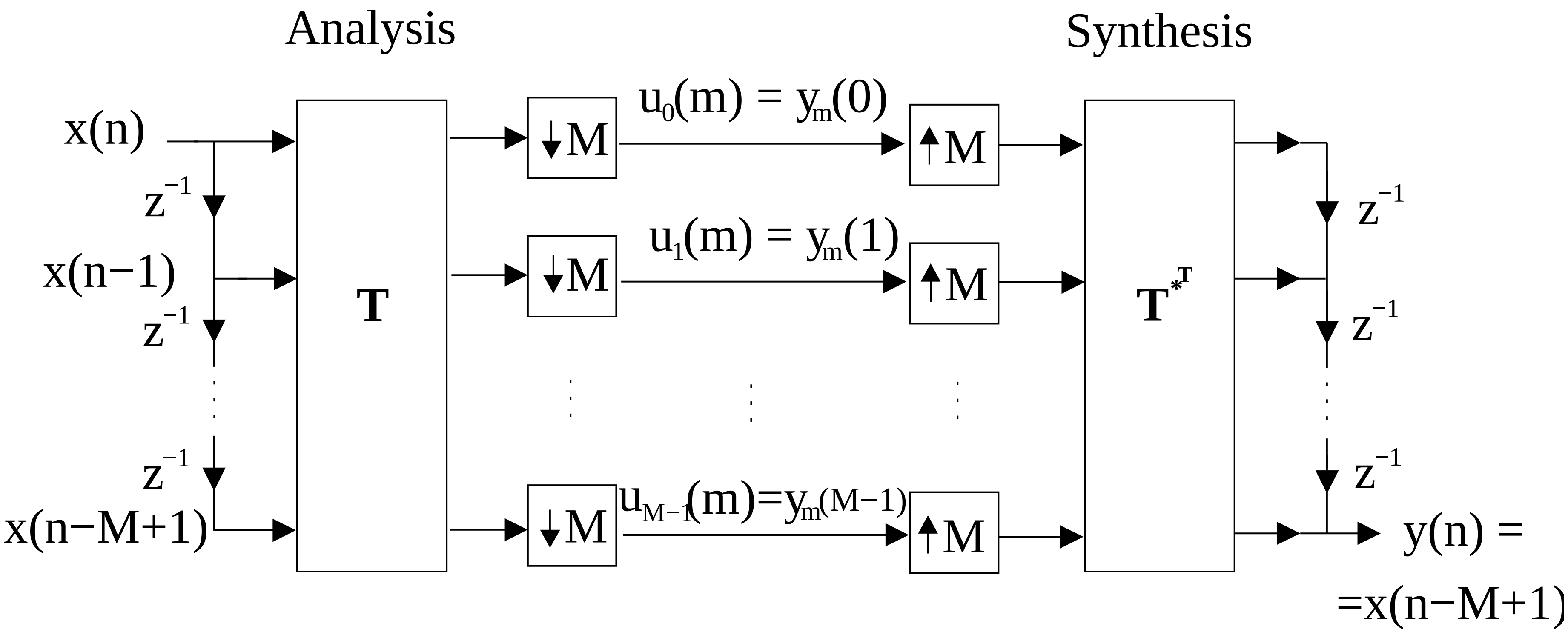



Figure 34: Unitary transform filter bank.

# Block transforms

- By referring to Figure 8b, we see that the transform matrix $\mathbf{T}$ corresponds to the polyphase matrices as follows:

$$\left.\begin{array}{rcl} \mathbf{E}(z^M) & = & \mathbf{T} \\ \mathbf{R}(z^M) & = & \mathbf{T}^{*^{\mathsf{T}}} \end{array}\right\} \tag{192}$$

- We conclude that the polyphase components of the filter bank corresponding to a signal transformation are constant.

- Note that the transform matrix $\mathbf{T}$ is such that $\mathbf{T}_{km} = c_k(M-1-m)$.

- Then, we have that

$$\left.\begin{array}{rcl} E_{km}(z^M) & = & c_k(M-1-m) \\ R_{mk}(z^M) & = & c_k^*(M-1-m) \end{array}\right\} \tag{193}$$

## Block transforms

- Thus, from equations (1) and (2), we conclude that

$$\left.\begin{aligned} h_k(m) &= c_k(M-1-m) \\ g_k(m) &= c_k^*(m) \end{aligned}\right\} \qquad (194)$$

- Note that in the above equation, the analysis bank is delayed by $(M-1)$ samples in comparison with the one shown in Figure 30. This explains the delay of $(M-1)$ samples in Figure 34.

# Cosine-modulated filter banks

- The cosine-modulated filter banks (CMFB) are an attractive choice for the design and implementation of filter banks with a large number of sub-bands. Their main features are:
  - Simple design procedure, consisting of generating a lowpass prototype whose impulse response satisfies some constraints required to achieve perfect reconstruction.
  - Low cost of implementation measured in terms of multiplication count, since the resulting analysis and synthesis filter banks rely on a type of DCT, which is amenable to fast implementation and can share the prototype implementation cost with each sub-filter.

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## Cosine-modulated filter banks

- In the (CMFB) design, we begin by finding a linear-phase prototype lowpass filter $H(z)$ of order $N$, with passband edge $\omega_p = (\frac{\pi}{2M} - \rho)$ and stopband edge $\omega_r = (\frac{\pi}{2M} + \rho)$, where $2\rho$ is the width of the transition band.
- We assume that the length $N + 1$ is an even multiple of the number $M$ of sub-bands, that is, $N = (2LM - 1)$. Although the actual length of the prototype can be arbitrary, this assumption greatly simplifies our analysis.

## Cosine-modulated filter banks

- Given the prototype filter, we can generate cosine-modulated versions of it in order to obtain the analysis and synthesis filter banks as follows:

$$h_m(n) = 2h(n)\cos\left[(2m+1)\frac{\pi}{2M}\left(n - \frac{N}{2}\right) + (-1)^m\frac{\pi}{4}\right] \tag{195}$$

$$g_m(n) = 2h(n)\cos\left[(2m+1)\frac{\pi}{2M}\left(n - \frac{N}{2}\right) - (-1)^m\frac{\pi}{4}\right] \tag{196}$$

for $n = 0, 1, \ldots, N$ and $m = 0, 1, \ldots, (M-1)$, with $N = (2LM - 1)$.

- The term that multiplies $h(n)$ is related to the $(m, n)$ entry $c_{m,n}$ of an $(M \times 2LM)$ DCT-type matrix **C**, given by

$$c_{m,n} = 2\cos\left[(2m+1)\frac{\pi}{2M}\left(n - \frac{N}{2}\right) + (-1)^m\frac{\pi}{4}\right] \tag{197}$$

# Cosine-modulated filter banks

- The prototype filter can be decomposed into $2M$ polyphase components as follows

$$H(z) = \sum_{l=0}^{L-1} \sum_{j=0}^{2M-1} h(2lM + j) z^{-(2lM+j)} = \sum_{j=0}^{2M-1} z^{-j} E_j(z^{2M}) \qquad (198)$$

where $E_j(z) = \sum_{l=0}^{L-1} h(2lM + j) z^{-l}$ are the polyphase components of $H(z)$.

- With this formulation, the analysis filter bank can be described as

$$\begin{aligned} H_m(z) &= \sum_{n=0}^{N} h_m(n) z^{-n} \\ &= \sum_{n=0}^{2LM-1} c_{m,n} h(n) z^{-n} \\ &= \sum_{l=0}^{L-1} \sum_{j=0}^{2M-1} c_{m,2lM+j} h(2lM + j) z^{-(2lM+j)} \end{aligned} \qquad (199)$$

## Cosine-modulated filter banks

- This expression can be further simplified, if we explore the following property:

$$\cos\left\{(2m+1)\frac{\pi}{2M}\left[(n+2kM)-\frac{N}{2}\right]+\phi\right\}$$
$$= (-1)^k \cos\left[(2m+1)\frac{\pi}{2M}\left(n-\frac{N}{2}\right)+\phi\right] \quad (200)$$

- Which leads to

$$c_{m,n+2kM} = (-1)^k c_{m,n} \quad (201)$$

- Substituting $j$ for $n$, and $l$ for $k$, we get

$$c_{m,j+2lM} = (-1)^l c_{m,j} \quad (202)$$

## Cosine-modulated filter banks

- With this relation, and after some manipulation, we can rewrite equation (199) as

$$H_m(z) = \sum_{j=0}^{2M-1} c_{m,j} z^{-j} \sum_{l=0}^{L-1} (-1)^l h(2lM+j) z^{-2lM} = \sum_{j=0}^{2M-1} c_{m,j} z^{-j} E_j(-z^{2M}) \tag{203}$$

- Which can be rewritten as

$$\mathbf{e}(z) = \begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} E_0(-z^{2M}) \\ z^{-1}E_1(-z^{2M}) \\ \vdots \\ z^{-(2M-1)}E_{2M-1}(-z^{2M}) \end{bmatrix} \tag{204}$$

where $\mathbf{C}_1$ and $\mathbf{C}_2$ are $M \times M$ matrices whose $(m, j)$ elements are $c_{m,j}$ and $c_{m,j+M}$, respectively, for $m, j = 0, 1, \ldots, (M-1)$.

- Defining $\mathbf{d}(z) = [1 \;\; z^{-1} \;\; \ldots \;\; z^{-M+1}]^{\mathsf{T}}$,

$$\mathbf{e}(z) = \begin{bmatrix}\mathbf{C}_1 & \mathbf{C}_2\end{bmatrix} \begin{bmatrix} E_0(-z^{2M}) & & & \mathbf{0} \\ & E_1(-z^{2M}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{2M-1}(-z^{2M}) \end{bmatrix} \begin{bmatrix} \mathbf{d}(z) \\ z^{-M}\mathbf{d}(z) \end{bmatrix}$$

$$= \left\{ \mathbf{C}_1 \begin{bmatrix} E_0(-z^{2M}) & & & \mathbf{0} \\ & E_1(-z^{2M}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{M-1}(-z^{2M}) \end{bmatrix} \right.$$

$$\left. + \; z^{-M}\mathbf{C}_2 \begin{bmatrix} E_M(-z^{2M}) & & & \mathbf{0} \\ & E_{M+1}(-z^{2M}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{2M-1}(-z^{2M}) \end{bmatrix} \right\} \mathbf{d}(z)$$

$$= \mathbf{E}(z^M)\mathbf{d}(z) \tag{205}$$

## Cosine-modulated filter banks

- To achieve perfect reconstruction in a filter bank with $M$ bands, we should have $\mathbf{E}(z)\mathbf{R}(z) = \mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}z^{-\Delta}$.
- It is well known that the polyphase matrix of the analysis filter bank can be designed to be paraunitary or lossless, that is, $\mathbf{E}^{\mathrm{T}}(z^{-1})\mathbf{E}(z) = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix of dimension $M$, supposing that the filter-bank coefficients are real.
- In this case, the synthesis filters can be easily obtained from the analysis filter bank using either equation (196) or

$$\mathbf{R}(z) = z^{-\Delta}\mathbf{E}^{-1}(z) = z^{-\Delta}\mathbf{E}^{\mathrm{T}}(z^{-1}) \tag{206}$$

# Cosine-modulated filter banks

- It only remains to show how the prototype filter can be constrained so that the polyphase matrix of the analysis filter bank becomes paraunitary.

- The desired result is the following:

  **Property** The polyphase matrix of the analysis filter bank becomes paraunitary, for a real coefficient prototype filter, if and only if

$$E_j(z^{-1})E_j(z) + E_{j+M}(z^{-1})E_{j+M}(z) = \frac{1}{2M} \tag{207}$$

  for $j = 0, 1, \ldots, (M-1)$.

- If the prototype filter has linear phase, these constraints can be reduced by half, because they are required only for $j = 0, 1, \ldots, \frac{M-1}{2}$, in case $M$ is odd, and for $j = 0, 1, \ldots, \frac{M}{2} - 1$, in case $M$ is even.

## Cosine-modulated filter banks

- **Outline**

We need the following properties related to the matrices $\mathbf{C}_1$ and $\mathbf{C}_2$:

$$\mathbf{C}_1^T\mathbf{C}_1 = 2M[\mathbf{I} + (-1)^{L-1}\mathbf{J}] \tag{208}$$

$$\mathbf{C}_2^T\mathbf{C}_2 = 2M[\mathbf{I} - (-1)^{L-1}\mathbf{J}] \tag{209}$$

$$\mathbf{C}_1^T\mathbf{C}_2 = \mathbf{C}_2^T\mathbf{C}_1 = \mathbf{0} \tag{210}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{J}$ is the reverse identity matrix, and $\mathbf{0}$ is a matrix with all elements equal to zero.

## Cosine-modulated filter banks

- With equations (208)–(210), it follows

$$
\begin{aligned}
&\mathbf{E}^{\mathrm{T}}(z^{-1})\mathbf{E}(z) = \\
&\begin{bmatrix} E_0(-z^{-2}) & & & \mathbf{0} \\ & E_1(-z^{-2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{M-1}(-z^{-2}) \end{bmatrix} \mathbf{C}_1^{\mathrm{T}}\mathbf{C}_1 \begin{bmatrix} E_0(-z^{2}) & & & \mathbf{0} \\ & E_1(-z^{2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{M-1}(-z^{2}) \end{bmatrix} \\
&+ \begin{bmatrix} E_M(-z^{-2}) & & & \mathbf{0} \\ & E_{M+1}(-z^{-2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{2M-1}(-z^{-2}) \end{bmatrix} \mathbf{C}_2^{\mathrm{T}}\mathbf{C}_2 \begin{bmatrix} E_M(-z^{2}) & & & \mathbf{0} \\ & E_{M+1}(-z^{2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{2M-1}(-z^{2}) \end{bmatrix}
\end{aligned}
\tag{211}
$$

# Cosine-modulated filter banks

- Since the prototype is a linear-phase filter, it can be shown that

$$\begin{bmatrix} E_0(-z^{-2}) & & & \mathbf{0} \\ & E_1(-z^{-2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{M-1}(-z^{-2}) \end{bmatrix} \mathbf{J} \begin{bmatrix} E_0(-z^{2}) & & & \mathbf{0} \\ & E_1(-z^{2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{M-1}(-z^{2}) \end{bmatrix}$$
$$= \begin{bmatrix} E_M(-z^{-2}) & & & \mathbf{0} \\ & E_{M+1}(-z^{-2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{2M-1}(-z^{-2}) \end{bmatrix} \mathbf{J} \begin{bmatrix} E_M(-z^{2}) & & & \mathbf{0} \\ & E_{M+1}(-z^{2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{2M-1}(-z^{2}) \end{bmatrix} \tag{212}$$

# Cosine-modulated filter banks

- This result allows some simplification, after we apply the expressions for $\mathbf{C}_1^T\mathbf{C}_1$ and $\mathbf{C}_2^T\mathbf{C}_2$, yielding

$$
\mathbf{E}^T(z^{-1})\mathbf{E}(z) = 2M\left\{\begin{bmatrix} E_0(-z^{-2}) & & & \mathbf{0} \\ & E_1(-z^{-2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{M-1}(-z^{-2}) \end{bmatrix}\begin{bmatrix} E_0(-z^{2}) & & & \mathbf{0} \\ & E_1(-z^{2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{M-1}(-z^{2}) \end{bmatrix}\right.
$$

$$
\left. + \begin{bmatrix} E_M(-z^{-2}) & & & \mathbf{0} \\ & E_{M+1}(-z^{-2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{2M-1}(-z^{-2}) \end{bmatrix}\begin{bmatrix} E_M(-z^{2}) & & & \mathbf{0} \\ & E_{M+1}(-z^{2}) & & \\ & & \ddots & \\ \mathbf{0} & & & E_{2M-1}(-z^{2}) \end{bmatrix}\right\}
$$

  If the matrix above is equal to the identity matrix, we achieve perfect reconstruction.

- This is equivalent to requiring that polyphase components of the prototype filter are pairwise power complementary.

# Cosine-modulated filter banks

- Equation (204) suggests the structure of Figure 35 for the implementation of the CMFB.

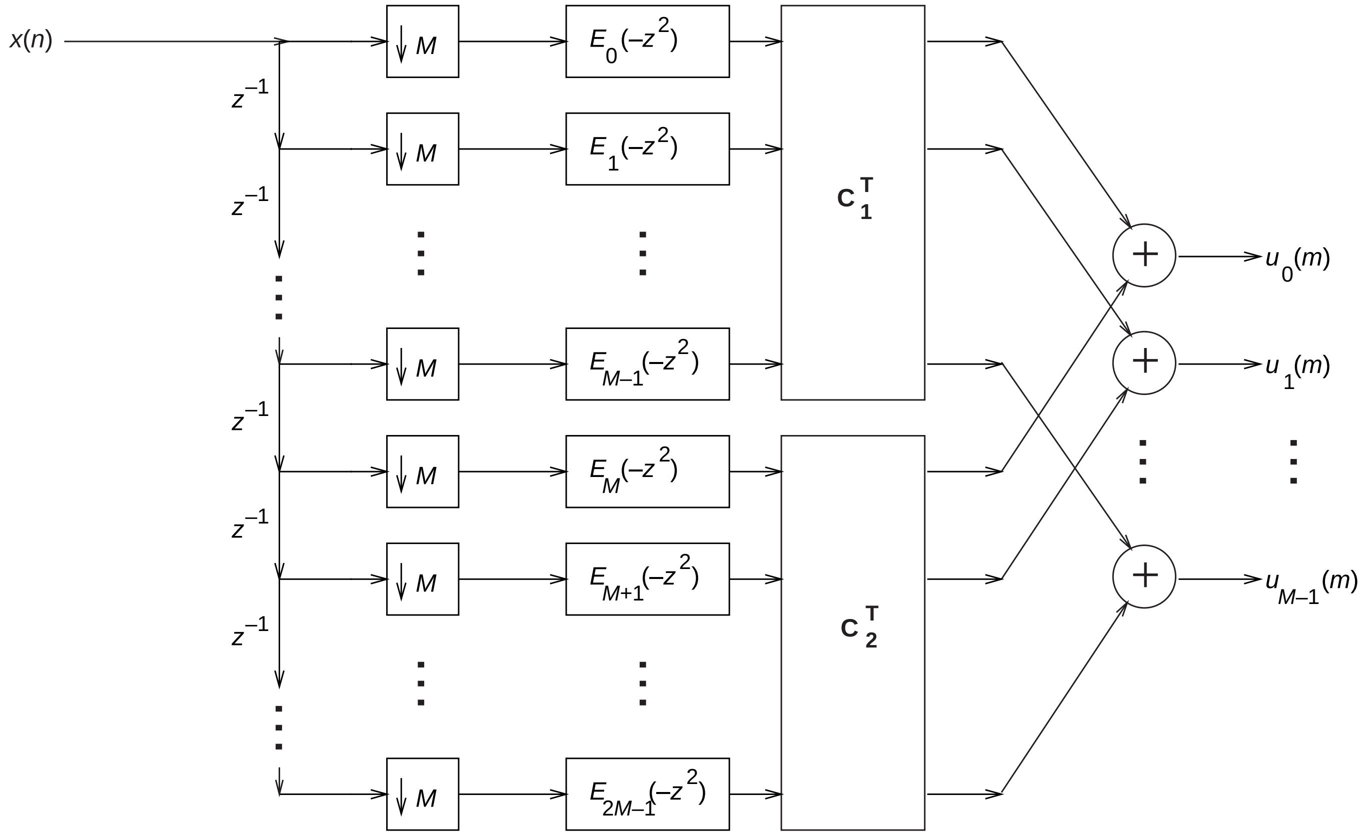


Figure 35: Cosine-modulated filter bank.

# Cosine-modulated filter banks

- This structure can be implemented using the DCT-IV, which is represented by the $\mathbf{C}_M^{\mathrm{IV}}$ matrix, by noting that the matrices $\mathbf{C}_1$ and $\mathbf{C}_2$ can be expressed as follows:

$$\mathbf{C}_1 = \sqrt{M}(-1)^{\frac{L}{2}}\mathbf{C}_M^{\mathrm{IV}}(\mathbf{I}-\mathbf{J}) \tag{214}$$

$$\mathbf{C}_2 = -\sqrt{M}(-1)^{\frac{L}{2}}\mathbf{C}_M^{\mathrm{IV}}(\mathbf{I}+\mathbf{J}) \tag{215}$$

- For $L$ even, and

$$\mathbf{C}_1 = \sqrt{M}(-1)^{\frac{L-1}{2}}\mathbf{C}_M^{\mathrm{IV}}(\mathbf{I}+\mathbf{J}) \tag{216}$$

$$\mathbf{C}_2 = \sqrt{M}(-1)^{\frac{L-1}{2}}\mathbf{C}_M^{\mathrm{IV}}(\mathbf{I}-\mathbf{J}) \tag{217}$$

- For $L$ odd, where

$$\{\mathbf{C}_M^{\mathrm{IV}}\}_{m,n} = \sqrt{\frac{2}{M}}\cos\left[(2m+1)\left(n+\frac{1}{2}\right)\frac{\pi}{2M}\right] \tag{218}$$

## Cosine-modulated filter banks

- Equations (214)–(217) can be put in the form:

$$\mathbf{C}_1 = \sqrt{M}(-1)^{\lfloor \frac{L}{2} \rfloor}\mathbf{C}_M^{\mathrm{IV}}[\mathbf{I} - (-1)^L\mathbf{J}] \tag{219}$$

$$\mathbf{C}_2 = -\sqrt{M}(-1)^{\lfloor \frac{L}{2} \rfloor}\mathbf{C}_M^{\mathrm{IV}}[(-1)^L\mathbf{I} + \mathbf{J}] \tag{220}$$

  where $\lfloor x \rfloor$ represents the largest integer smaller than or equal to $x$.

- From equation (204) and the above equations, the structure in Figure 36 follows. Such a structure can benefit from the fast implementation algorithms for the DCT-IV.

# Cosine-modulated filter banks

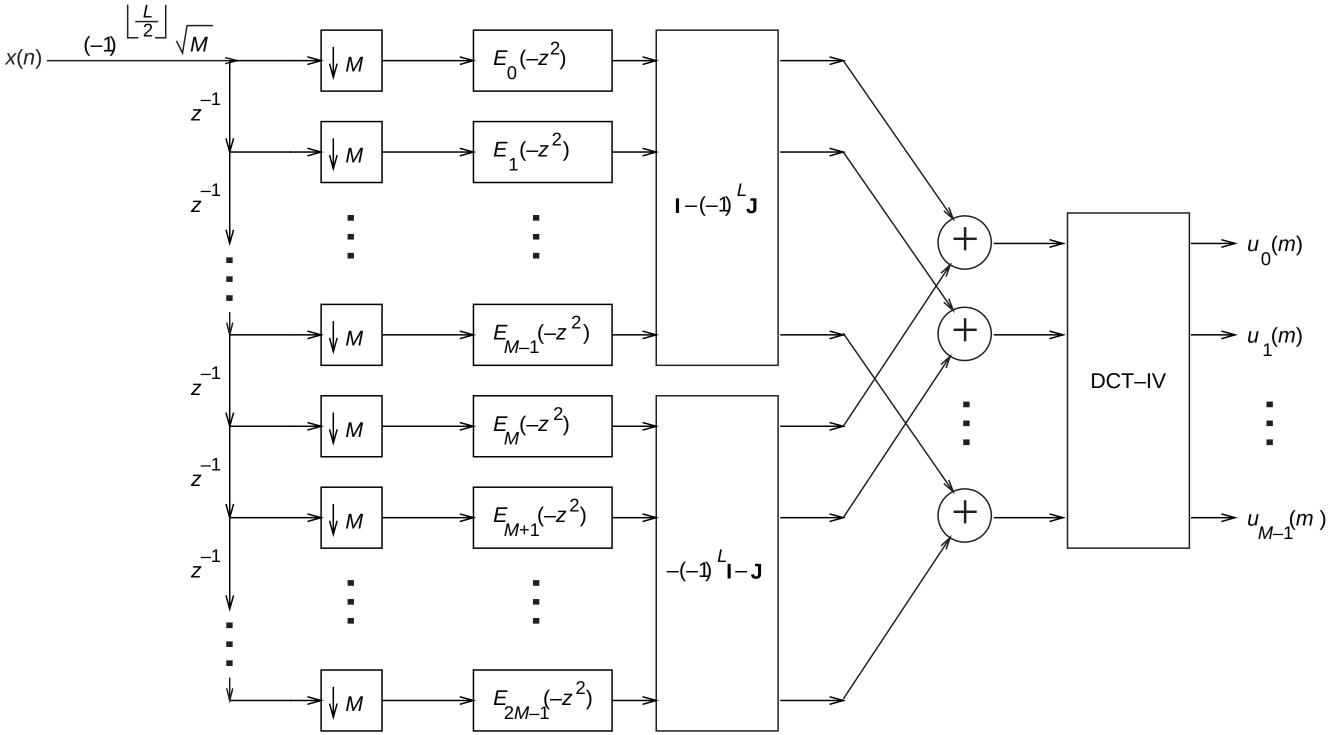


Figure 36: Implementation of the cosine-modulated filter bank using the DCT-IV.

# The optimization problem in the design of cosine-modulated filter banks

- The procedure to design the prototype filter requires the definition of an appropriate objective function imposing not only the frequency selectivity shape but also the perfect reconstruction constraints.

- Since the prototype filter of a cosine-modulated filter bank requires the design of a lowpass filter, the overall objective function should include a term defined as

$$\min_{\mathbf{h}} \{E_p(\omega)\} = \min_{\mathbf{h}} \left\{ \int_{\Omega_r}^{\pi} |H(e^{j\omega})|^p \, d\omega \right\} \qquad (221)$$

where $H(e^{j\omega})$ is the frequency response of the prototype filter, **h** is the prototype filter coefficient vector, $\Omega_r$ is the stopband frequency edge, and usually $p = 2$ or $p = \infty$.

## The optimization problem in the design of cosine-modulated filter banks

- The prototype filter coefficients should be designed by minimizing a modified objective function $\hat{E}_p(\omega)$, that combines the original objective function $E_p(\omega)$ with a weighted set of constraints, described by

$$\hat{E}_p(\omega) = E_p(\omega) + \boldsymbol{\lambda}^T \mathbf{c}(\mathbf{h}) \tag{222}$$

- Where $\boldsymbol{\lambda}$ is the vector of constraint weights and $\mathbf{c}(\mathbf{h})$ is the vector enforcing the constraints.

- The vectors are defined as

$$\boldsymbol{\lambda} = [\lambda_0 \; \lambda_1 \; \ldots \; \lambda_{N_c-1}]^T \tag{223}$$

$$\mathbf{c}(\mathbf{h}) = [c_0(\mathbf{h}) \; c_1(\mathbf{h}) \ldots c_{N_c-1}(\mathbf{h})]^T \tag{224}$$

where $N_c$ is the total number of constraints.

## The optimization problem in the design of cosine-modulated filter banks

- The constraints of equation (207) can be described in the time domain using a MATLAB notation as follows

$$\mathtt{conv}(\hat{\mathbf{e}}_j, \hat{\mathbf{e}}_j \mathbf{J}) + \mathtt{conv}(\hat{\mathbf{e}}_{j+M}, \hat{\mathbf{e}}_{j+M} \mathbf{J}) - \frac{1}{2M} \mathtt{dt}(i) = 0 \tag{225}$$

for $j = 0, 1, \ldots, (M-1)$.

- Where $\hat{\mathbf{e}}_j$ is a row vector containing the $L = \frac{N+1}{2M}$ coefficients of the polyphase component $E_j(z)$, $\mathbf{J}$ is the antidiagonal matrix, and $\mathtt{dt}(i)$ in the present discussion represents a sequence in MATLAB with $(L-1)$ zeros followed by a unit impulse at $i = 0$ and $(L-1)$ additional zeros, that is

$$\mathtt{dt}(i) = [\mathtt{zeros}(L-1, 1);\ 1;\ \mathtt{zeros}(L-1, 1)]; \tag{226}$$

- This set of constraints should be incorporated in the optimization process in the appropriate manner.

# The optimization problem in the design of cosine-modulated filter banks

- This optimization problem can be solved using quadratic programming (QP) which may require information on the first and second derivatives of $E_p(\omega)$ to simplify its implementation and speed the convergence.
- The constraint weights should be chosen beforehand when using a QP optimization method.
- Another solution is to employ a sequential quadratic programming (SQP) algorithm, which optimally sets the weights of the constraints based on the method of Lagrange multipliers with the Kuhn-Tucker conditions.

## The optimization problem in the design of cosine-modulated filter banks

**Example 9.9** Design a filter bank with $M = 10$ sub-bands using the cosine-modulated method with $L = 3$.

**Solution**

- For the prototype linear-phase design, we employ the least-squares design method, using as the objective the minimization of the filter stopband energy. This has been obtained by sampling the error function only over the frequencies $\omega_i > \omega_p$. The perfect reconstruction restrictions in equations (207) and (225) have been dealt with using the method.

- The length of the resulting prototype filter is $(N + 1) = 2LM = 60$, and the minimum stopband attenuation obtained was $A_r \approx 40$ dB, as shown in Figure 37. Its impulse response is depicted in Figure 38.

## The optimization problem in the design of cosine-modulated filter banks

- Let us verify a subset of the constraints of equation (225) for the case $j = 0$, where

$$\begin{aligned} \hat{\mathbf{e}}_0 &= [h(0) \;\; h(20) \;\; h(40)] \\ &= [-8.1483\text{E}{-}04 \;\; 2.5850\text{E}{-}02 \;\; 2.0509\text{E}{-}02] \end{aligned} \tag{227}$$

$$\begin{aligned} \hat{\mathbf{e}}_{10} &= [h(10) \;\; h(30) \;\; h(50)] \\ &= [-4.0655\text{E}{-}03 \;\; 6.2266\text{E}{-}02 \;\; -4.1105\text{E}{-}03] \end{aligned} \tag{228}$$

## The optimization problem in the design of cosine-modulated filter banks

- Which yield

$$\begin{aligned}\mathrm{conv}(\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_0\mathbf{J}) + \mathrm{conv}(\hat{\mathbf{e}}_{10}, \hat{\mathbf{e}}_{10}\mathbf{J}) &= [0 \;\; 0 \;\; 0.05 \;\; 0 \;\; 0]^{\mathrm{T}} \\ &= \frac{1}{2M}\mathrm{dt}(i) \\ &= 0.05\mathrm{dt}(i) \qquad (229)\end{aligned}$$

which is the expected result.

# Cosine-modulated filter banks

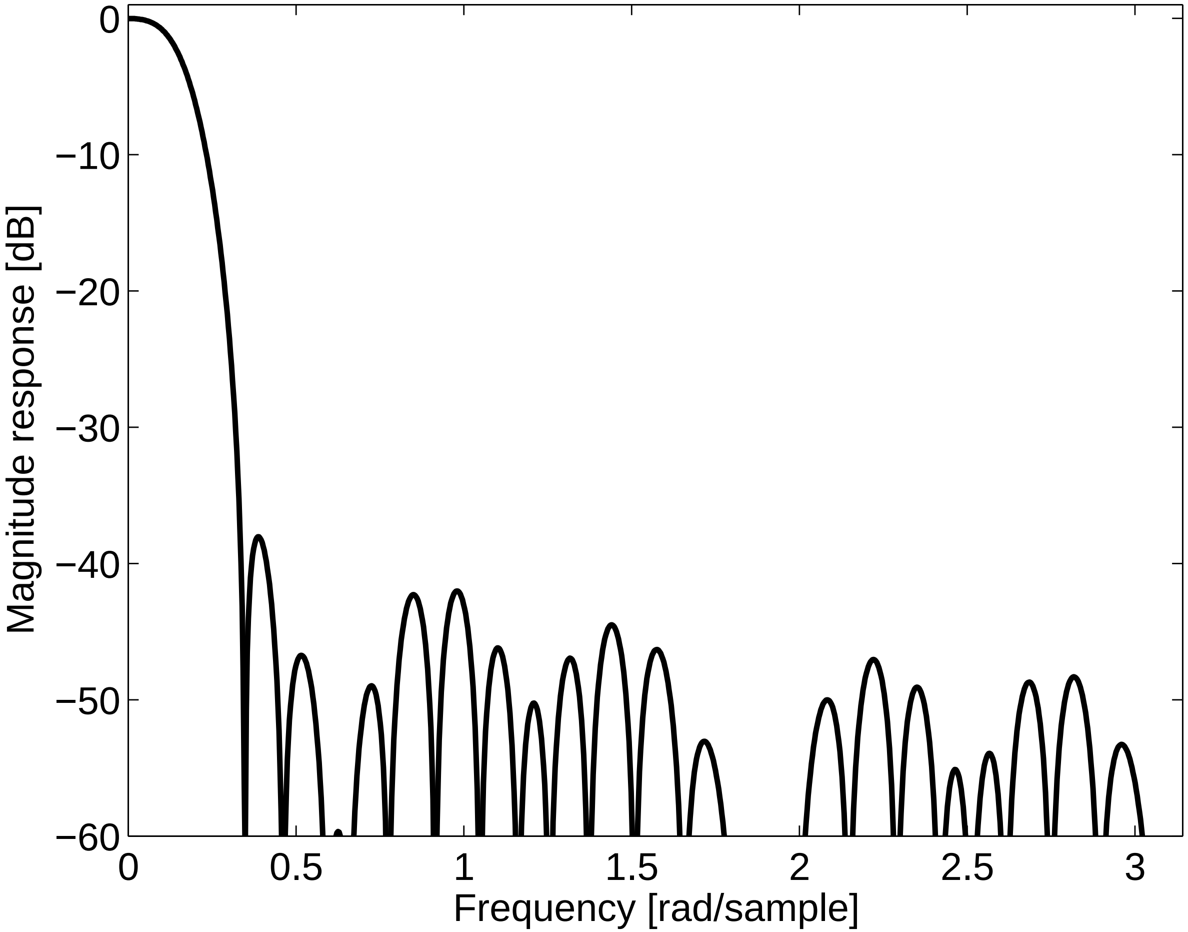


Figure 37: Magnitude response of the prototype filter for cosine-modulated filter bank.

# Cosine-modulated filter banks

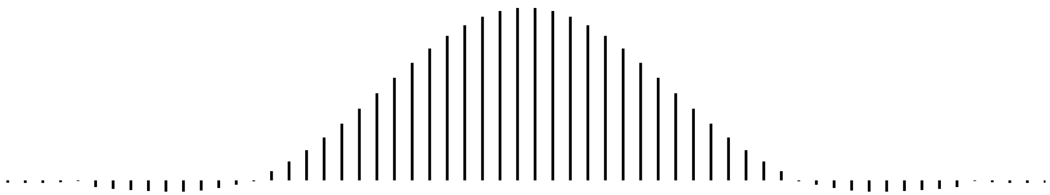

Figure 38: Impulse response of the prototype filter.

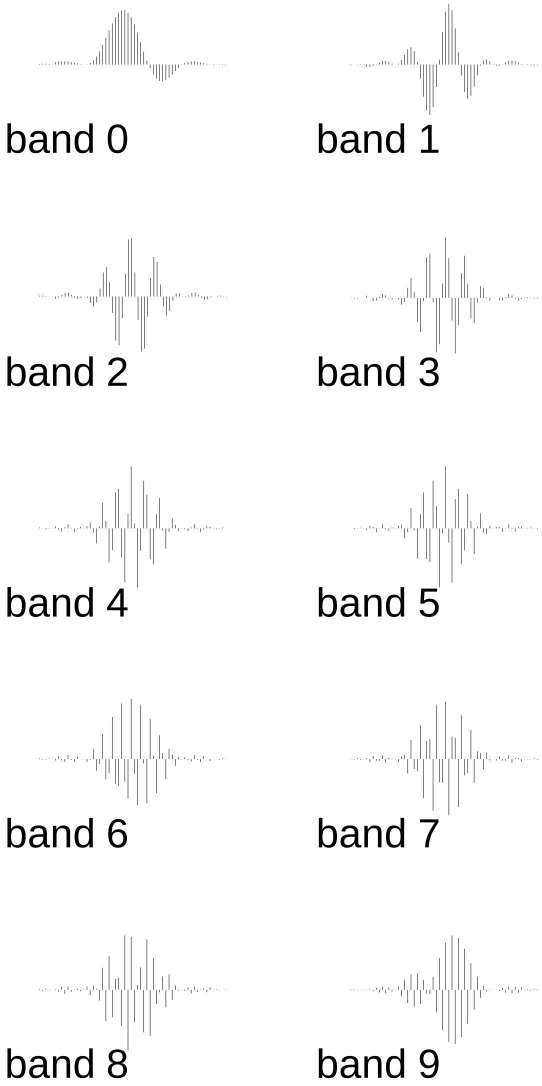


Figure 39: Impulse responses, analysis filters of a 10-band CMFB length 60.

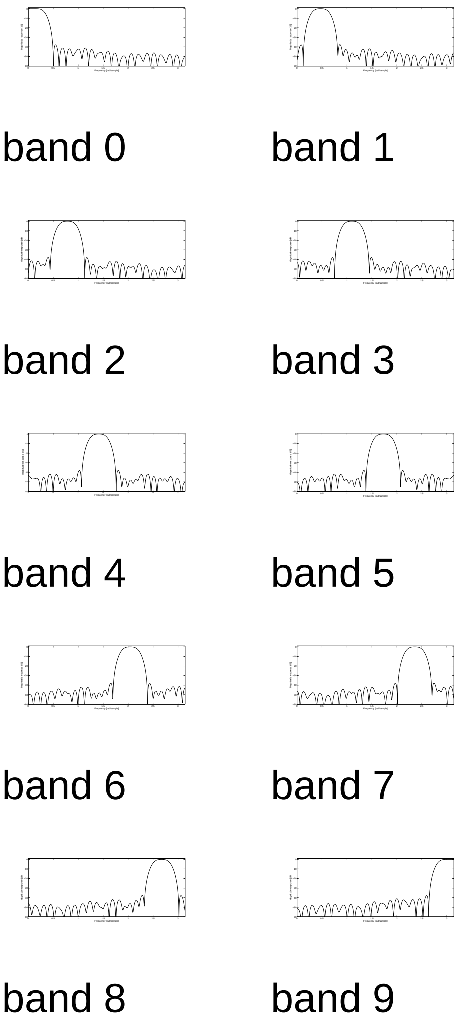


Figure 40: Magnitude responses of the analysis filters of a 10-band cosine-modulated filter bank of length 60.

# Lapped transforms

- The lapped orthogonal transforms (LOT) were originally proposed to be block transforms whose basis functions extended beyond the block boundaries.
- Its main goal was to reduce blocking effects, usually present under quantization of the block transform coefficients.
- Blocking effects are discontinuities that appear across the block boundaries.
- They occur because each block is transformed and quantized independently of the others, and this type of distortion is particularly annoying in images.
- LOT-based filter banks are very attractive because they lead to linear-phase analysis filters and have fast implementation.

# Lapped transforms

- The LOT-based filter banks are members of the family of paraunitary FIR perfect reconstruction filter banks with linear phase.
- The term LOT applies to the cases where the analysis filters have length $2M$.
- Generalizations of the LOT to longer analysis and synthesis filters (length $LM$) are available. They are known as the extended lapped transforms (ELT) and the generalized LOT (GenLOT).
- The ELT is a cosine-modulated filter bank and does not produce linear-phase analysis filters.
- The GenLOT is a good choice when long analysis filters with high selectivity are required together with linear phase.

# Lapped transforms

- The LOT analysis filters are given by

$$\mathbf{e}(z) = \begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(2M-1)} \end{bmatrix} \tag{230}$$

# Lapped transforms

- Or

$$\mathbf{e}(z) = \begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = (\hat{\mathbf{C}}_1 + z^{-M}\hat{\mathbf{C}}_2)\mathbf{d}(z) = \mathbf{E}(z^M)\mathbf{d}(z) \tag{231}$$

where $\hat{\mathbf{C}}_1$ and $\hat{\mathbf{C}}_2$ are $M \times M$ DCT-type matrices, and $\mathbf{E}(z)$ is the polyphase matrix of the analysis filter bank. Note that if $\hat{\mathbf{C}}_1$ is applied to a length-$M$ block, then due to the term $z^{-M}$, matrix $\hat{\mathbf{C}}_2$ is applied to the previous length-$M$ block.

## Lapped transforms

- It is as if two adjacent blocks are needed to compute a transform of a given block.
- There is an overlap between blocks in the computation of this transform.
- The perfect reconstruction condition with paraunitary polyphase matrices is generated if

$$\mathbf{R}(z) = z^{-\Delta}\mathbf{E}^{-1}(z) = z^{-\Delta}\mathbf{E}^{\mathrm{T}}(z^{-1}) \tag{232}$$

**Property**

The polyphase matrix of the analysis filter bank becomes paraunitary, for a real coefficient prototype filter, if the following conditions are satisfied:

$$\hat{\mathbf{C}}_1\hat{\mathbf{C}}_1^{\mathrm{T}} + \hat{\mathbf{C}}_2\hat{\mathbf{C}}_2^{\mathrm{T}} = \mathbf{I} \tag{233}$$

$$\hat{\mathbf{C}}_1\hat{\mathbf{C}}_2^{\mathrm{T}} = \hat{\mathbf{C}}_2\hat{\mathbf{C}}_1^{\mathrm{T}} = \mathbf{0} \tag{234}$$

# Lapped transforms

- **Proof**: From equation (231), we have that

$$\mathbf{E}(z) = \hat{\mathbf{C}}_1 + z^{-1}\hat{\mathbf{C}}_2 \tag{235}$$

- Perfect reconstruction requires that $\mathbf{E}(z)\mathbf{E}^{\mathrm{T}}(z^{-1}) = \mathbf{I}$. Therefore,

$$(\hat{\mathbf{C}}_1 + z^{-1}\hat{\mathbf{C}}_2)(\hat{\mathbf{C}}_1^{\mathrm{T}} + z\hat{\mathbf{C}}_2^{\mathrm{T}}) = \hat{\mathbf{C}}_1\hat{\mathbf{C}}_1^{\mathrm{T}} + \hat{\mathbf{C}}_2\hat{\mathbf{C}}_2^{\mathrm{T}} + z\hat{\mathbf{C}}_1\hat{\mathbf{C}}_2^{\mathrm{T}} + z^{-1}\hat{\mathbf{C}}_2\hat{\mathbf{C}}_1^{\mathrm{T}} = \mathbf{I} \tag{236}$$

and then equations (233) and (234) immediately follow.

$\square$

## Lapped transforms

- The rows of $\hat{\mathbf{C}} = \begin{bmatrix} \hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 \end{bmatrix}$ are orthogonal.
- The rows of $\hat{\mathbf{C}}_1$ are orthogonal to the rows of $\hat{\mathbf{C}}_2$, which is the same as saying that the overlapping tails of the basis functions of the LOT are orthogonal.
- A simple construction for the matrices above, based on the DCT, and leading to linear-phase filters, results from choosing

$$\hat{\mathbf{C}}_1 = \frac{1}{2}\begin{bmatrix} \mathbf{C}_e - \mathbf{C}_o \\ \mathbf{C}_e - \mathbf{C}_o \end{bmatrix} \tag{237}$$

$$\hat{\mathbf{C}}_2 = \frac{1}{2}\begin{bmatrix} (\mathbf{C}_e - \mathbf{C}_o)\mathbf{J} \\ -(\mathbf{C}_e - \mathbf{C}_o)\mathbf{J} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathbf{C}_e + \mathbf{C}_o \\ -\mathbf{C}_e - \mathbf{C}_o \end{bmatrix} \tag{238}$$

where $\mathbf{C}_e$ and $\mathbf{C}_o$ are $\frac{M}{2} \times M$ matrices consisting of the even and odd DCT basis of length $M$.

# Lapped transforms

- Observe that $\mathbf{C}_e\mathbf{J} = \mathbf{C}_e$ and $\mathbf{C}_o\mathbf{J} = -\mathbf{C}_o$, because the even basis functions are symmetric and the odd ones are antisymmetric.

- The choice satisfies the relations (233) and (234). With this, we can build an initial LOT whose polyphase matrix, as in equation (235), is given by

$$\mathbf{E}(z) = \frac{1}{2}\begin{bmatrix} \mathbf{C}_e - \mathbf{C}_o + z^{-1}(\mathbf{C}_e - \mathbf{C}_o)\mathbf{J} \\ \mathbf{C}_e - \mathbf{C}_o - z^{-1}(\mathbf{C}_e - \mathbf{C}_o)\mathbf{J} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathbf{I} & z^{-1}\mathbf{I} \\ \mathbf{I} & -z^{-1}\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{C}_e \\ \mathbf{C}_o \end{bmatrix} \tag{239}$$

# Lapped transforms

- This expression suggests the structure of Figure 41 for the implementation of the LOT filter bank.
- This structure consists of the implementation of the polyphase components of the prototype filter using a DCT-based matrix, followed by several orthogonal matrices $\mathbf{T}_0, \mathbf{T}_1, \ldots, \mathbf{T}_{\frac{M}{2}-2}$, which are discussed next.
- Actually, we can pre-multiply the right-hand term in equation (239) by an orthogonal matrix $\mathbf{L}_1$, and still keep the perfect reconstruction conditions.
- The polyphase matrix is then given by

$$\mathbf{E}(z) = \frac{1}{2}\mathbf{L}_1 \begin{bmatrix} \mathbf{I} & z^{-1}\mathbf{I} \\ \mathbf{I} & -z^{-1}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{C}_e \\ \mathbf{C}_o \end{bmatrix} \tag{240}$$

# Lapped transforms

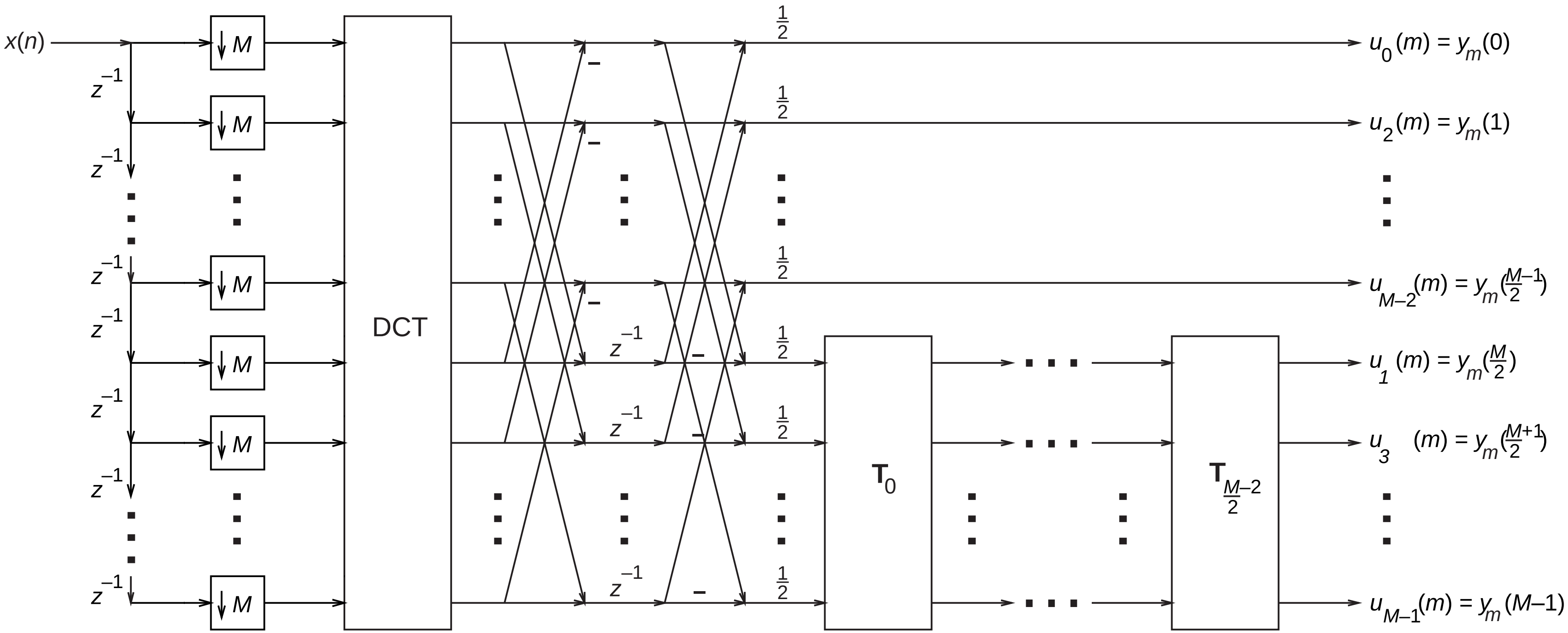


Figure 41: Implementation of the lapped orthogonal transform.

## Lapped transforms

- The basic construction of the LOT presented equivalent to the one proposed in that utilizes a block transform formulation to generate lapped transforms.
- The basic construction of the LOT presented equivalent to the one Note that in the block transform formulation, equation (230) is equivalent to a block transform whose transform matrix $\mathbf{L}_{\mathrm{LOT}}$ is equal to $\left[\hat{\mathbf{C}}_1\ \hat{\mathbf{C}}_2\right]$.
- Since the transform matrix $\mathbf{L}_{\mathrm{LOT}}$ has dimensions $M \times 2M$, in order to compute the transform coefficients $y_j(k)$ of a block $B_j$, one needs the samples $x_j(m)$ and $x_{j+1}(m)$ of the blocks $B_j$ and $B_{j+1}$, respectively.

## Lapped transforms

- The term “lapped transform” comes from the fact that the block $B_{j+1}$ is needed in the computation of both $y_j(k)$ and $y_{j+1}(k)$, that is, the transforms of the two blocks overlap.

- This is expressed formally by the following equation:

$$
\begin{aligned}
\begin{bmatrix} y_j(0) \\ \vdots \\ y_j(M-1) \end{bmatrix} &= \mathbf{L}_{\mathrm{LOT}} \begin{bmatrix} x_j(0) \\ \vdots \\ x_j(M-1) \\ x_{j+1}(0) \\ \vdots \\ x_{j+1}(M-1) \end{bmatrix} \\
&= \begin{bmatrix} \hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 \end{bmatrix} \begin{bmatrix} x_j(0) \\ \vdots \\ x_j(M-1) \\ x_{j+1}(0) \\ \vdots \\ x_{j+1}(M-1) \end{bmatrix} \\
&= \hat{\mathbf{C}}_1 \begin{bmatrix} x_j(0) \\ \vdots \\ x_j(M-1) \end{bmatrix} + \hat{\mathbf{C}}_2 \begin{bmatrix} x_{j+1}(0) \\ \vdots \\ x_{j+1}(M-1) \end{bmatrix}
\end{aligned}
\tag{241}
$$

## Lapped transforms

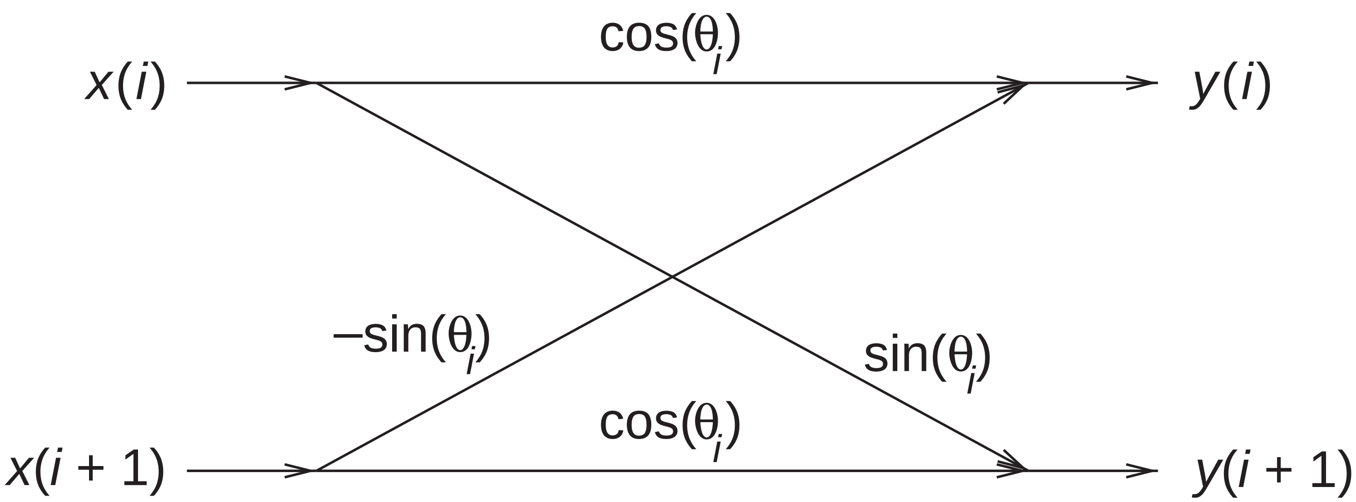


Figure 42: Implementation of multiplication by $\mathbf{T}_i$.

# Lapped transforms

- Malvar starts with an orthogonal matrix based on the DCT having the following form:

$$\mathbf{L}_0 = \frac{1}{2}\begin{bmatrix} \mathbf{C}_e - \mathbf{C}_o & (\mathbf{C}_e - \mathbf{C}_o)\mathbf{J} \\ \mathbf{C}_e - \mathbf{C}_o & -(\mathbf{C}_e - \mathbf{C}_o)\mathbf{J} \end{bmatrix} \tag{242}$$

- The first half of the basis functions are symmetric, whereas the second half are antisymmetric, thus keeping the phase linear, as desired.

# Lapped transforms

- The choice of $\mathbf{L}_0$ based on the DCT is the key to generating a fast implementation algorithm.

- Starting with $\mathbf{L}_0$, we can generate a family of more selective analysis filters in the following form:

$$\mathbf{L}_{\mathrm{LOT}} = \begin{bmatrix} \hat{\mathbf{C}}_1 & \hat{\mathbf{C}}_2 \end{bmatrix} = \mathbf{L}_1 \mathbf{L}_0 \tag{243}$$

  where the matrix $\mathbf{L}_1$ should be orthogonal and should also be amenable to fast implementation.

# Lapped transforms

- Usually, the matrix $\mathbf{L}_1$ is of the form

$$\mathbf{L}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 \end{bmatrix} \tag{244}$$

where $\mathbf{L}_2$ is a square matrix of dimension $\frac{M}{2}$ consisting of a set of plane rotations.

- More specifically,

$$\mathbf{L}_2 = \mathbf{T}_{\frac{M}{2}-2} \ldots \mathbf{T}_1 \mathbf{T}_0 \tag{245}$$

# Lapped transforms

- Where

$$\mathbf{T}_i = \begin{bmatrix} \mathbf{I}_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}(\theta_i) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{\frac{M}{2}-2-i} \end{bmatrix} \tag{246}$$

- And

$$\mathbf{Y}(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \tag{247}$$

The rotation angles $\theta_i$ are submitted to optimization aimed at maximizing the coding gain, when using the filter bank in sub-band coders, or at improving the selectivity of the analysis and synthesis filters.

# Lapped transforms

- The rows of matrix $\mathbf{L}_{\mathrm{LOT}}$ are not organized in frequency order – in fact, the first $\frac{M}{2}$ rows correspond to the even bands, and the last $\frac{M}{2}$ rows correspond to the odd bands.

- A simple fast LOT implementation not allowing any type of optimization consists of, instead of implementing $\mathbf{L}_2$ as a cascade of rotations $\mathbf{T}_i$, implementing $\mathbf{L}_2$ as a cascade of square matrices of dimension $\hat{M} = \frac{M}{2}$ comprised of DCTs Type II and Type IV.

# Lapped transforms

- The elements of which are

$$c_{l,n} = \alpha_{\hat{M}}(l)\sqrt{\frac{2}{\hat{M}}}\cos\left[(2l+1)\frac{\pi}{2\hat{M}}(n)\right] \tag{248}$$

- And

$$c_{l,n} = \sqrt{\frac{2}{\hat{M}}}\cos\left[(2l+1)\frac{\pi}{2\hat{M}}\left(n+\frac{1}{2}\right)\right] \tag{249}$$

respectively, where $\alpha_{\hat{M}}(l) = 1/\sqrt{2}$, for $l = 0$, or $l = \hat{M}$, and $\alpha_{\hat{M}}(l) = 1$ otherwise. The implementation is fast because the DCTs of Types II and IV have fast algorithms.

# Lapped transforms

**Example 9.10** Design a filter bank with $M = 10$ sub-bands using the fast LOT.

**Solution**

- The length of the analysis and synthesis filters should be $(N+1) = 2M = 20$. Therefore, we use as a basis for this design the DCT matrix of order $M = 10$, whose basis functions are given by equation (190) in Example 166 as

$$c_k(m) = \alpha(k) \cos\left[\frac{\pi(2m+1)k}{20}\right] \tag{250}$$

where $\alpha(0) = \sqrt{\frac{1}{20}}$ and $\alpha(k) = \sqrt{\frac{1}{10}}$, for $k = 1, 2, \ldots, 9$

# Lapped transforms

- Therefore, the $\frac{M}{2} \times M$ matrices $\mathbf{C}_e$ and $\mathbf{C}_o$ are

$$\mathbf{C}_e = \begin{bmatrix} c_0(0) & c_0(1) & c_0(2) & c_0(3) & c_0(4) & c_0(5) & c_0(6) & c_0(7) & c_0(8) & c_0(9) \\ c_2(0) & c_2(1) & c_2(2) & c_2(3) & c_2(4) & c_2(5) & c_2(6) & c_2(7) & c_2(8) & c_2(9) \\ c_4(0) & c_4(1) & c_4(2) & c_4(3) & c_4(4) & c_4(5) & c_4(6) & c_4(7) & c_4(8) & c_4(9) \\ c_6(0) & c_6(1) & c_6(2) & c_6(3) & c_6(4) & c_6(5) & c_6(6) & c_6(7) & c_6(8) & c_6(9) \\ c_8(0) & c_8(1) & c_8(2) & c_8(3) & c_8(4) & c_8(5) & c_8(6) & c_8(7) & c_8(8) & c_8(9) \end{bmatrix} \tag{251}$$

$$\mathbf{C}_o = \begin{bmatrix} c_1(0) & c_1(1) & c_1(2) & c_1(3) & c_1(4) & c_1(5) & c_1(6) & c_1(7) & c_1(8) & c_1(9) \\ c_3(0) & c_3(1) & c_3(2) & c_3(3) & c_3(4) & c_3(5) & c_3(6) & c_3(7) & c_3(8) & c_3(9) \\ c_5(0) & c_5(1) & c_5(2) & c_5(3) & c_5(4) & c_5(5) & c_5(6) & c_5(7) & c_5(8) & c_5(9) \\ c_7(0) & c_7(1) & c_7(2) & c_7(3) & c_7(4) & c_7(5) & c_7(6) & c_7(7) & c_7(8) & c_7(9) \\ c_9(0) & c_9(1) & c_9(2) & c_9(3) & c_9(4) & c_9(5) & c_9(6) & c_9(7) & c_9(8) & c_9(9) \end{bmatrix} \tag{252}$$

## Lapped transforms

- For the fast LOT design, we use a factorable $\mathbf{L}_1$ matrix composed of a cascade of a $\mathbf{C}^{\mathrm{II}}$ transform and a transposed $\mathbf{C}^{\mathrm{IV}}$ transform on its lower part, to allow fast implementation, that is,

$$\mathbf{L}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{\mathrm{II}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^{\mathrm{IV}^{\mathrm{T}}} \end{bmatrix} \tag{253}$$

where $\mathbf{C}^{\mathrm{II}}$ and $\mathbf{C}^{\mathrm{IV}}$ are square matrices of dimension $\frac{M}{2} = 5$.

# Lapped transforms

- The impulse responses of the resulting analysis filters, given by equation (230), with $\widehat{\mathbf{C}}_1$ and $\widehat{\mathbf{C}}_2$ as defined in equations (237) and (238), respectively, are shown in Figure 43. The coefficients of the analysis filters are given by the rows of $\mathbf{L}_{\mathrm{LOT}}$, which is defined in equation (243).

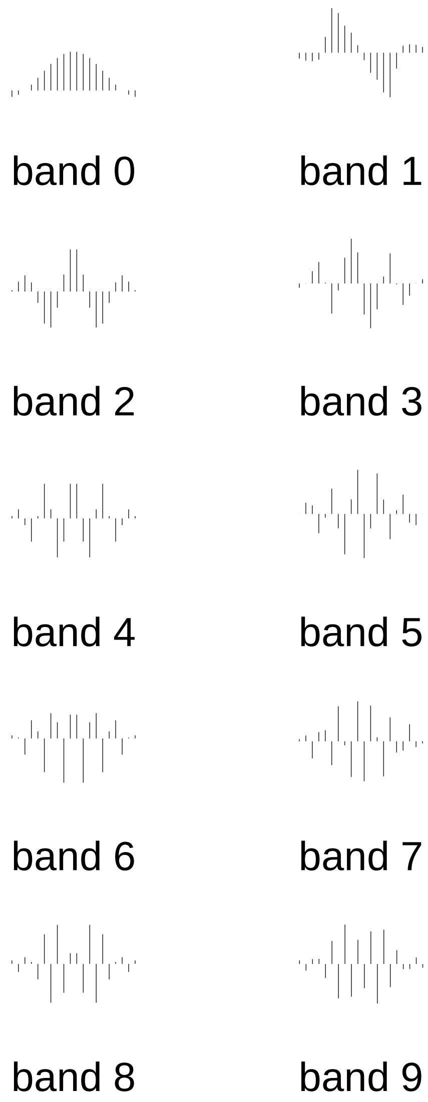


Figure 43: Impulse responses of the filters of a 10-band fast LOT.

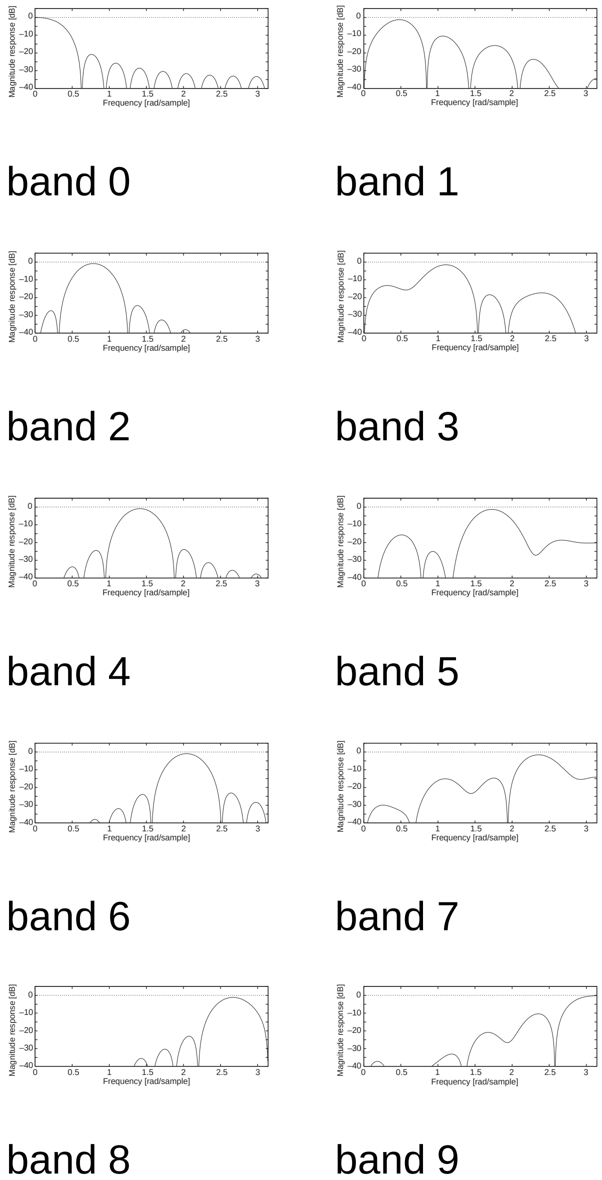


Figure 44: Magnitude responses of the filters of a 10-band fast LOT.

## Fast algorithms and biorthogonal LOT

- We now present a general construction of a fast algorithm for the LOT. Defining two matrices $\widehat{\mathbf{C}}_3$ and $\widehat{\mathbf{C}}_4$ such that

$$\widehat{\mathbf{C}}_1 = \widehat{\mathbf{C}}_3\widehat{\mathbf{C}}_4 \qquad (254)$$

$$\widehat{\mathbf{C}}_2 = (\mathbf{I} - \widehat{\mathbf{C}}_3)\widehat{\mathbf{C}}_4 \qquad (255)$$

- It is straightforward to show using equation (235) that the polyphase components of the analysis filter can be written as

$$\mathbf{E}(z) = [\widehat{\mathbf{C}}_3 + z^{-1}(\mathbf{I} - \widehat{\mathbf{C}}_3)]\widehat{\mathbf{C}}_4 \qquad (256)$$

# Fast algorithms and biorthogonal LOT

- The initial solution for the LOT matrix discussed previously can be analyzed in the light of this general formulation.

- The matrices of the polyphase description above corresponding to the LOT matrix of equation (242) are given by

$$\hat{\mathbf{C}}_3 = \frac{1}{2}\begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{I} \end{bmatrix} \tag{257}$$

$$\hat{\mathbf{C}}_4 = \frac{1}{2}\begin{bmatrix} \mathbf{C}_e - \mathbf{C}_o + (\mathbf{C}_e - \mathbf{C}_o)\mathbf{J} \\ \mathbf{C}_e - \mathbf{C}_o - (\mathbf{C}_e - \mathbf{C}_o)\mathbf{J} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_e \\ -\mathbf{C}_o \end{bmatrix} \tag{258}$$

# Fast algorithms and biorthogonal LOT

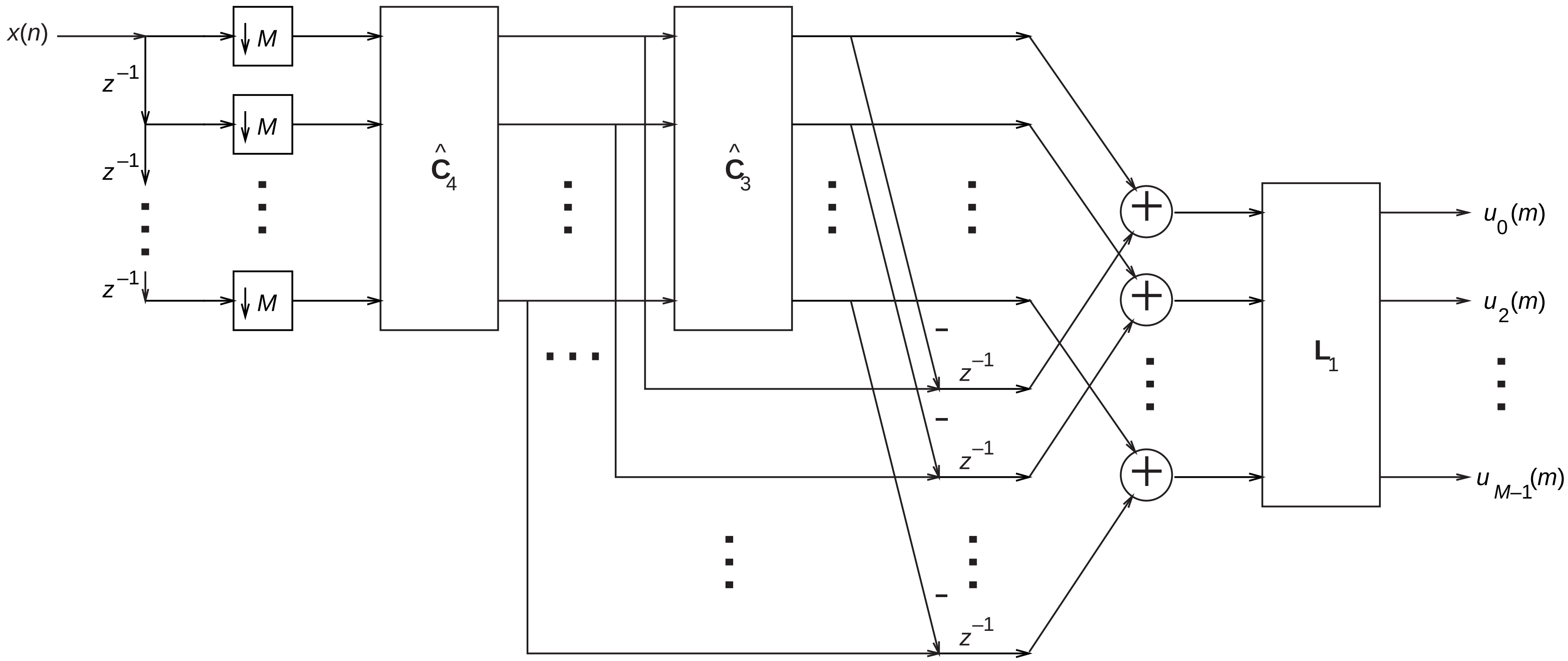


Figure 45: Implementation of the lapped orthogonal transform according to the general formulation in equation (256).

## Fast algorithms and biorthogonal LOT

- As long there are fast implementation algorithms for $\widehat{\mathbf{C}}_3$ and $\widehat{\mathbf{C}}_4$, the LOT will have a fast algorithm.

- Biorthogonal lapped transforms can be constructed using the formulation of equation (256), if $\widehat{\mathbf{C}}_3$ is chosen such that $\widehat{\mathbf{C}}_3\widehat{\mathbf{C}}_3 = \widehat{\mathbf{C}}_3$, matrix $\widehat{\mathbf{C}}_4$ is nonsingular and nonorthogonal, and the polyphase matrix $\mathbf{R}(z)$ is such that

$$\mathbf{R}(z) = \widehat{\mathbf{C}}_4^{-1}[z^{-1}\widehat{\mathbf{C}}_3 + (\mathbf{I} - \widehat{\mathbf{C}}_3)] \tag{259}$$

# Fast algorithms and biorthogonal LOT

**Example 9.11** Show the 2-band lapped-transform structure that realizes the filter bank with lowpass analysis filter

$$H_0(z) = -z^{-3} + 3z^{-2} + 3z^{-1} - 1 \tag{260}$$

specifying the value of each coefficient in that structure. Determine the corresponding highpass analysis filter.

**Solution**

- Since $M = 2$, the DCT matrix has the following form

$$\begin{bmatrix} \mathbf{C_e} \\ \mathbf{C_o} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos(\frac{\pi}{4}) & \cos(\frac{3\pi}{4}) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{261}$$

## Fast algorithms and biorthogonal LOT

- Such that

$$\mathbf{C}_e = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}; \quad \mathbf{C}_\mathrm{o} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \tag{262}$$

- Since the lapped-transform filter bank has two bands, and the desired structure must have linear phase, then there is no nontrivial orthogonal solution for this particular case, that is, $\mathbf{E}^\mathsf{T}(z^{-1}) \neq \mathbf{E}^{-1}(z)$.

## Fast algorithms and biorthogonal LOT

- One option is to look for biorthogonal solutions as given by equations (256) and (259), whereby $\widehat{\mathbf{C}}_4$ should be chosen as a general nonorthogonal $2 \times 2$ matrix.
- Another option is to use an orthogonal $\widehat{\mathbf{C}}_4$ and use a nonorthogonal matrix **P** in the input of Figure 45 in order to achieve perfect reconstruction.

# Fast algorithms and biorthogonal LOT

- In this case, we can choose $\hat{\mathbf{C}}_3$ and $\hat{\mathbf{C}}_4$ according to equations (257) and (258), that is

$$\hat{\mathbf{C}}_3 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{263}$$

$$\hat{\mathbf{C}}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{264}$$

- Such that

$$\hat{\mathbf{C}}_3\hat{\mathbf{C}}_3 = \frac{1}{4}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \hat{\mathbf{C}}_3 \tag{265}$$

## Fast algorithms and biorthogonal LOT

- In order to achieve a biorthogonal solution, matrix

$$\hat{\mathbf{P}} = \begin{bmatrix} \hat{p}_{00} & \hat{p}_{01} \\ \hat{p}_{10} & \hat{p}_{11} \end{bmatrix} \tag{266}$$

is placed before $\hat{\mathbf{C}}_4$ in the block diagram of the analysis filter, post-multiplying $\hat{\mathbf{C}}_4$ as follows:

$$\begin{aligned}
\mathbf{E}(z) &= \left[\hat{\mathbf{C}}_3 + z^{-1}(\mathbf{I} - \hat{\mathbf{C}}_3)\right]\hat{\mathbf{C}}_4\hat{\mathbf{P}} \\
&= \left(\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + z^{-1}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \hat{p}_{00} & \hat{p}_{01} \\ \hat{p}_{10} & \hat{p}_{11} \end{bmatrix}\right) \\
&= \frac{1}{2\sqrt{2}}\left(\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} + z^{-1}\begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}\right)\begin{bmatrix} \hat{p}_{00} & \hat{p}_{01} \\ \hat{p}_{10} & \hat{p}_{11} \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} z^{-1} & 1 \\ -z^{-1} & 1 \end{bmatrix}\begin{bmatrix} \hat{p}_{00} & \hat{p}_{01} \\ \hat{p}_{10} & \hat{p}_{11} \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} \hat{p}_{00}z^{-1} + \hat{p}_{10} & \hat{p}_{01}z^{-1} + \hat{p}_{11} \\ -\hat{p}_{00}z^{-1} + \hat{p}_{10} & -\hat{p}_{01}z^{-1} + \hat{p}_{11} \end{bmatrix}
\end{aligned} \tag{267}$$

## Fast algorithms and biorthogonal LOT

- Since the polyphase components of $H_0(z)$ should be $E_{00}(z) = -1 + 3z^{-1}$ and $E_{01}(z) = 3 - z^{-1}$, the above equation implies that

$$\frac{1}{\sqrt{2}}\begin{bmatrix} \hat{p}_{00}z^{-1} + \hat{p}_{10} & \hat{p}_{01}z^{-1} + \hat{p}_{11} \\ -\hat{p}_{00}z^{-1} + \hat{p}_{10} & -\hat{p}_{01}z^{-1} + \hat{p}_{11} \end{bmatrix} = \begin{bmatrix} -1 + 3z^{-1} & 3 - z^{-1} \\ E_{10}(z) & E_{11}(z) \end{bmatrix} \quad (268)$$

- So that by choosing $\hat{p}_{00} = 3\sqrt{2} = \hat{p}_{11}$ and $\hat{p}_{01} = -\sqrt{2} = \hat{p}_{10}$, it follows that $E_{10}(z) = -1 - 3z^{-1}$ and $E_{11}(z) = 3 + z^{-1}$.

- As result, the transfer function of the highpass analysis filter is given by

$$H_1(z) = -1 + 3z^{-1} - 3z^{-2} + z^{-3} \quad (269)$$

# Lapped transform implementation.

The resulting structure is depicted in Figure 46.

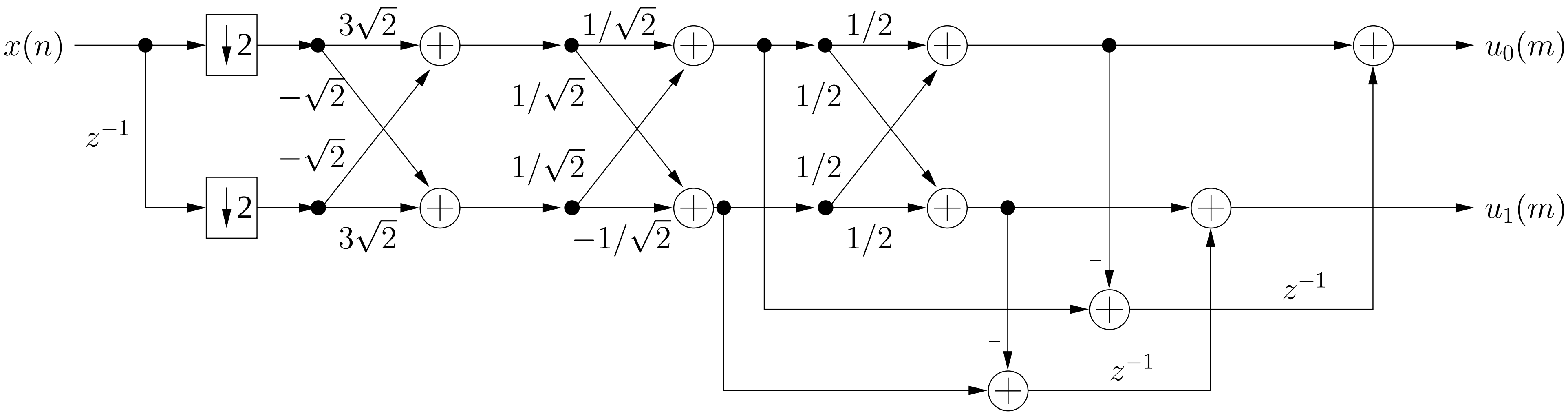


Figure 46: Lapped transform implementation.

△

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# Generalized LOT

- The generalized lapped transforms (GenLOT) can also be constructed if the polyphase matrices are designed as follows

$$\mathbf{E}(z) = \prod_{j=L}^{1} \left\{ \begin{bmatrix} \mathbf{L}_{3,j} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{2,j} \end{bmatrix} \left[\hat{\mathbf{C}}_{3,j} + z^{-1}(\mathbf{I} - \hat{\mathbf{C}}_{3,j})\right] \right\} \hat{\mathbf{C}}_4 \quad (270)$$

$$\mathbf{R}(z) = \hat{\mathbf{C}}_4^{-1} \prod_{j=1}^{L} \left\{ \left[z^{-1}\hat{\mathbf{C}}_{3,j} + (\mathbf{I} - \hat{\mathbf{C}}_{3,j})\right] \begin{bmatrix} \mathbf{L}_{3,j}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}_{2,j}^{-1} \end{bmatrix} \right\} \quad (271)$$

- Biorthogonal GenLOTs are obtained if $\hat{\mathbf{C}}_{3,j}$ is chosen such that $\hat{\mathbf{C}}_{3,j}\hat{\mathbf{C}}_{3,j} = \hat{\mathbf{C}}_{3,j}$.

# Generalized LOT

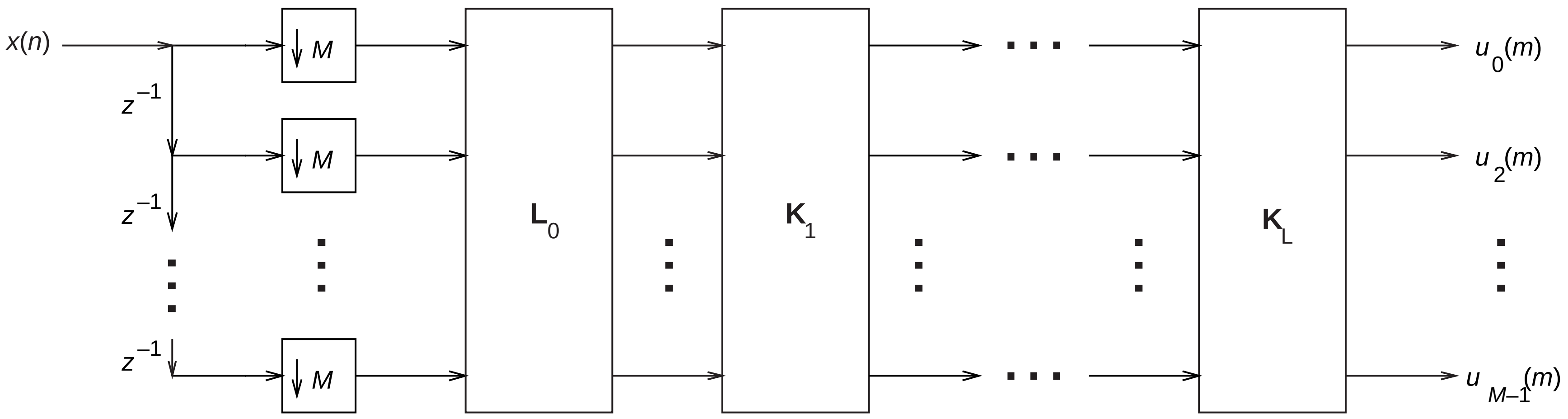


Figure 47: Implementation of the GenLOT.

# Generalized LOT

- To obtain an orthogonal GenLOT it is further required that $\hat{\mathbf{C}}_4^{\mathrm{T}}\hat{\mathbf{C}}_4 = \mathbf{I}$, $\mathbf{L}_{3,j}^{\mathrm{T}}\mathbf{L}_{3,j} = \mathbf{I}$, and $\mathbf{L}_{2,j}^{\mathrm{T}}\mathbf{L}_{2,j} = \mathbf{I}$.
- If we choose $\hat{\mathbf{C}}_3$ and $\hat{\mathbf{C}}_4$ as in equations (257) and (258), we have a valid GenLOT possessing a fast algorithm of the following form:

$$\begin{aligned}
\mathbf{E}(z) &= \prod_{j=L}^{1}\left(\frac{1}{2}\begin{bmatrix}\mathbf{L}_{3,j} & \mathbf{0}\\ \mathbf{0} & \mathbf{L}_{2,j}\end{bmatrix}\begin{bmatrix}\mathbf{I}+z^{-1}\mathbf{I} & \mathbf{I}-z^{-1}\mathbf{I}\\ \mathbf{I}-z^{-1}\mathbf{I} & \mathbf{I}+z^{-1}\mathbf{I}\end{bmatrix}\right)\hat{\mathbf{C}}_4\\
&= \prod_{j=L}^{1}\left(\frac{1}{2}\begin{bmatrix}\mathbf{L}_{3,j} & \mathbf{0}\\ \mathbf{0} & \mathbf{L}_{2,j}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{I}\\ \mathbf{I} & -\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & z^{-1}\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{I}\\ \mathbf{I} & -\mathbf{I}\end{bmatrix}\right)\hat{\mathbf{C}}_4\\
&= \prod_{j=L}^{1}\left(\mathbf{K}_j\right)\hat{\mathbf{C}}_4
\end{aligned} \tag{272}$$

## Generalized LOT

- If, in order to have a fast algorithm, we fix $\widehat{\mathbf{C}}_4$ as in equation (258), the degrees of freedom to design the filter bank are the choices of matrices $\mathbf{L}_{3,j}$ and $\mathbf{L}_{2,j}$, which are constrained to real and orthogonal matrices, to allow a valid GenLOT.
- The effectiveness in terms of computational complexity is highly dependent on how fast we can compute these matrices.
- Equation (272) suggests the structure of Figure 47 for the implementation of the GenLOT filter bank. This structure consists of the implementation of the matrix $\mathbf{L}_0$ in cascade with a set of similar building blocks denoted as $\mathbf{K}_j$. The structure for the implementation of each $\mathbf{K}_j$ is depicted in Figure 48.

# Generalized LOT

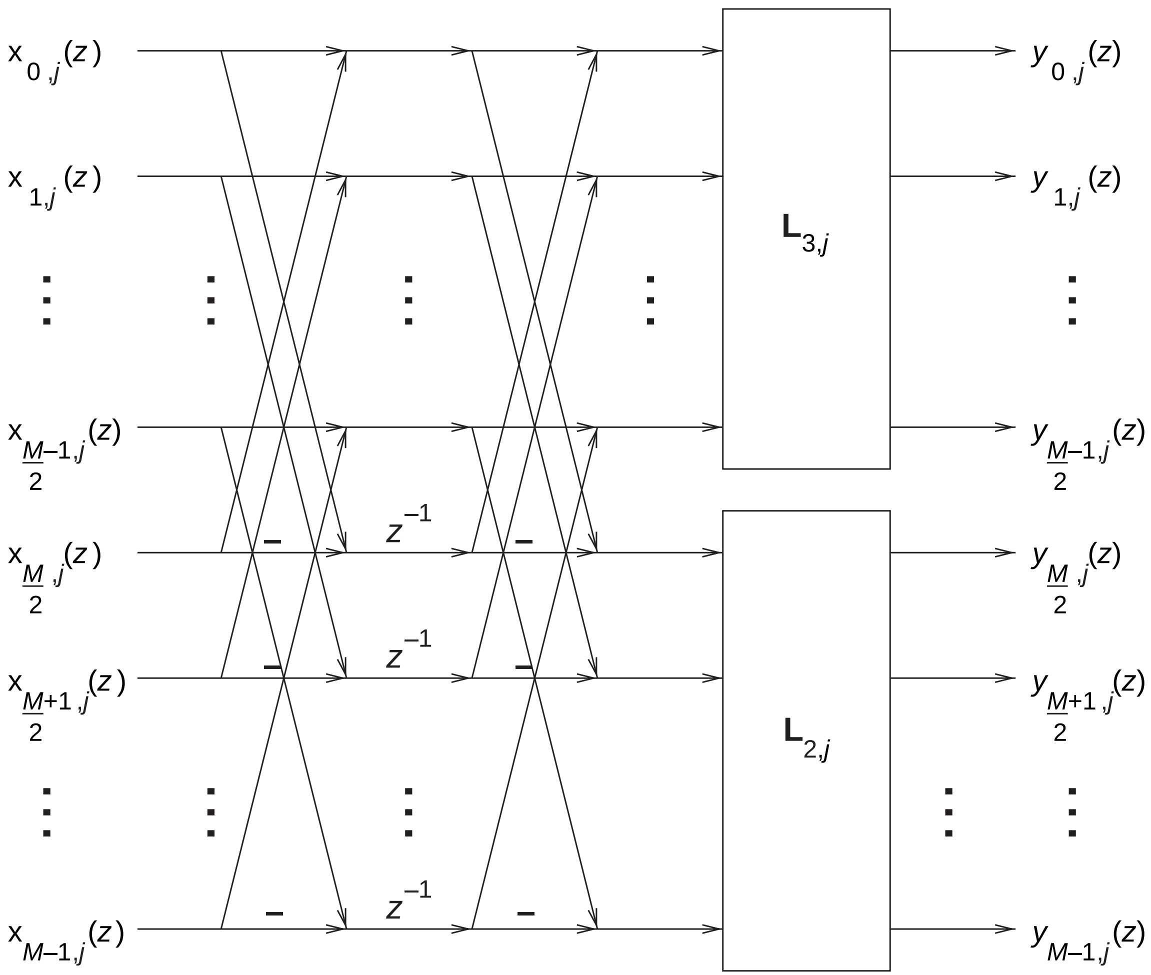


Figure 48: Implementation of the building blocks $\mathbf{K}_j$ of the GenLOT.

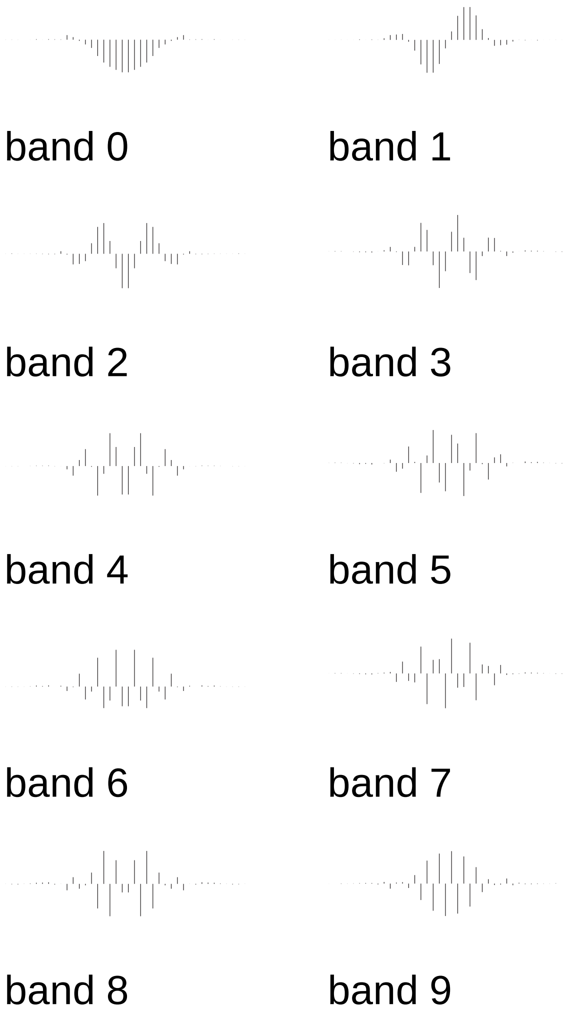


Figure 49: Impulse responses of the analysis filters of the 10-band GenLOT of length 40.

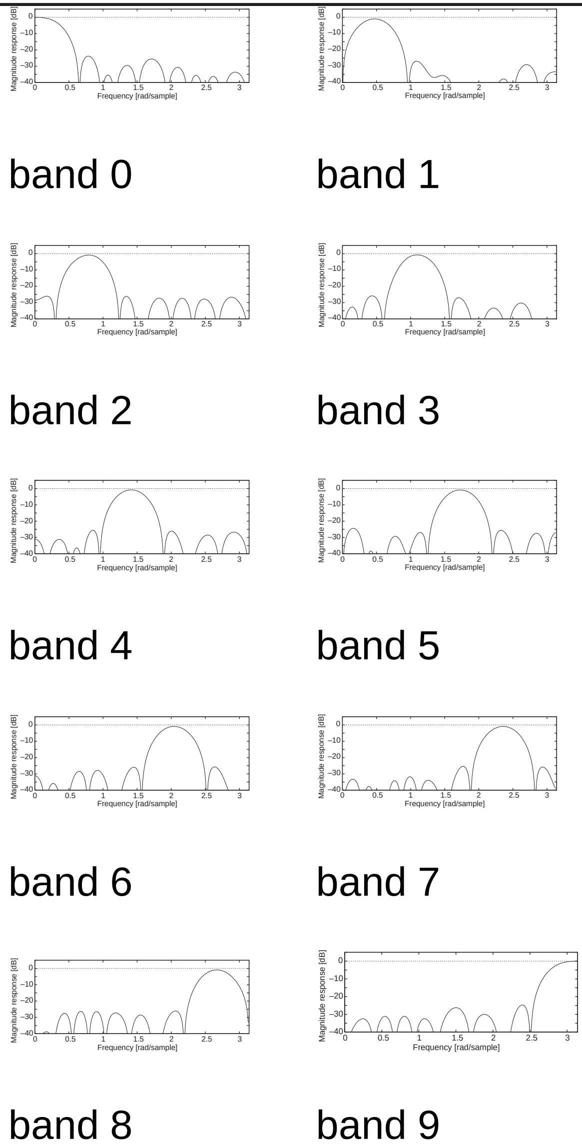


Figure 50: Magnitude responses of the analysis filters of the 10-band GenLOT of length 40.

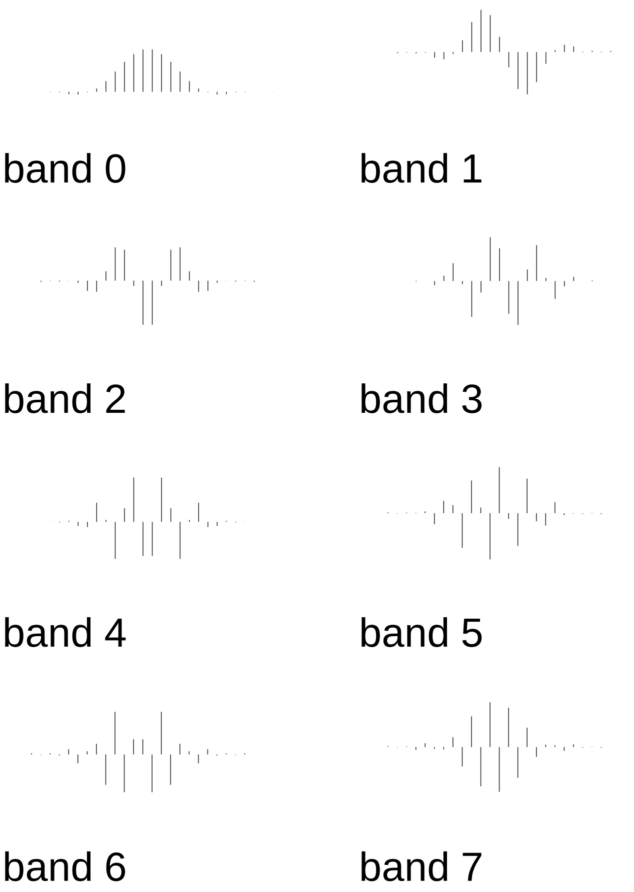


Figure 51: Impulse responses of the analysis filters of the 8-band lapped biorthogonal transform filter bank of length 32.

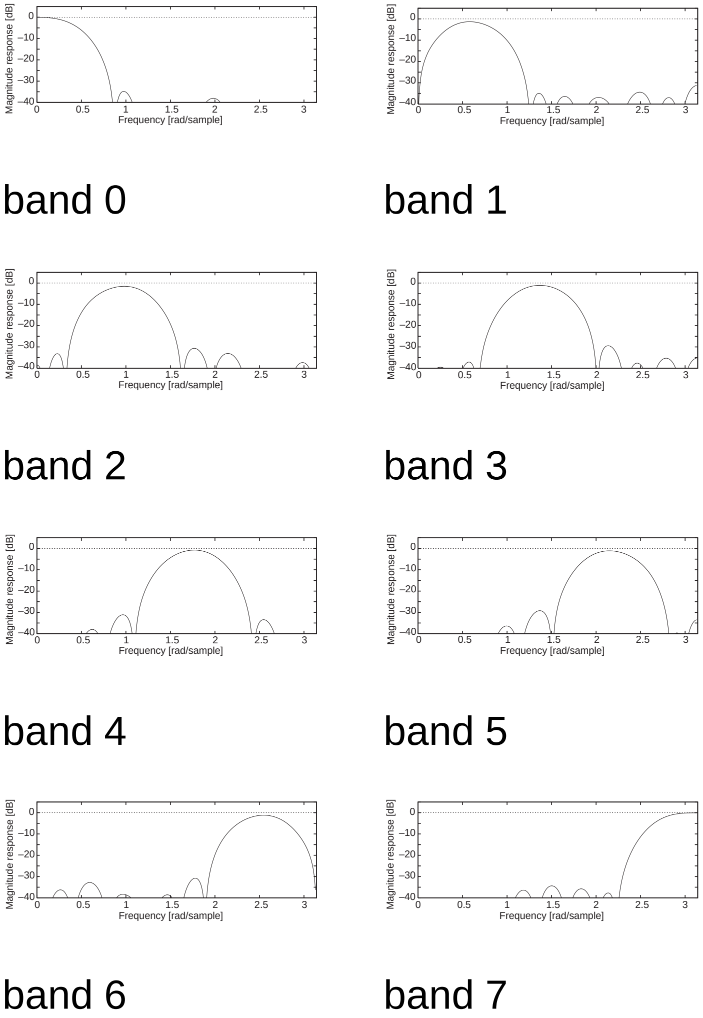


Figure 52: Magnitude responses of the analysis filters of the 8-band lapped biorthogonal transform filter bank of length 32.

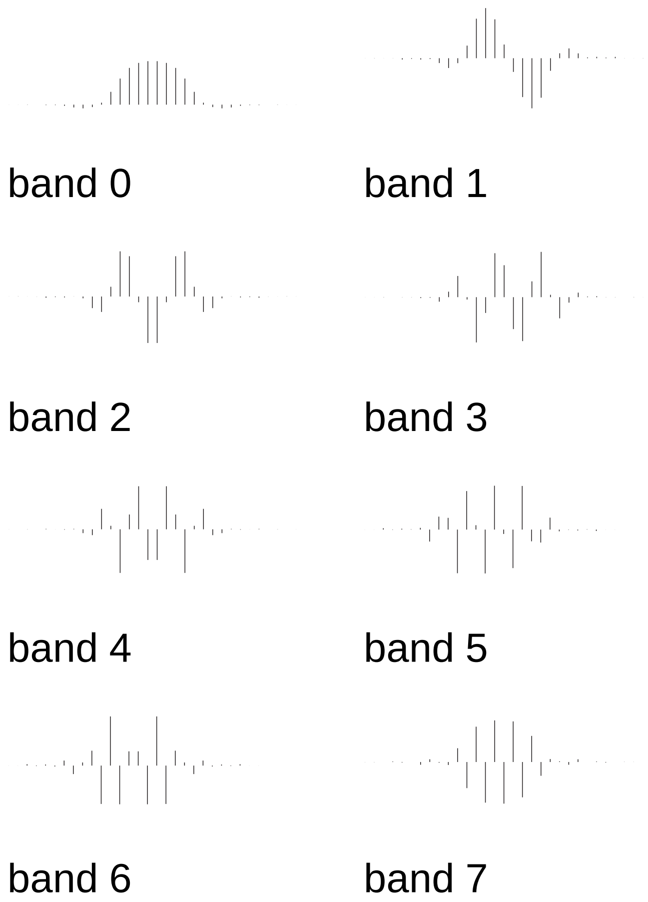


Figure 53: Impulse responses of the synthesis filters of the 8-band lapped biorthogonal transform filter bank of length 32.

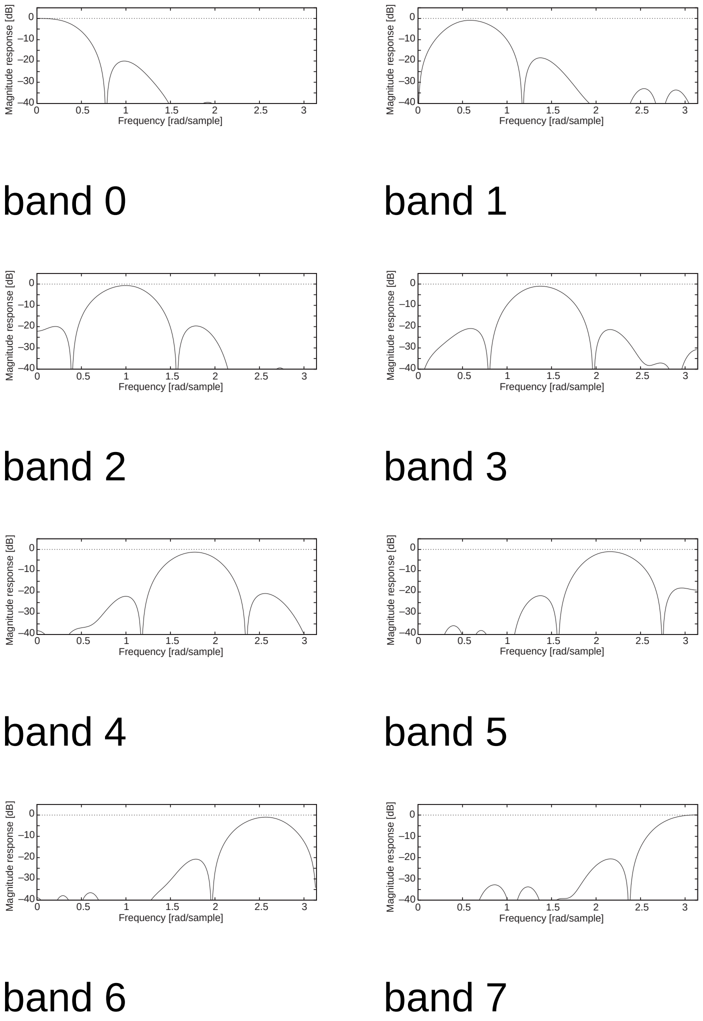


Figure 54: Magnitude responses of the synthesis filters of the 8-band lapped biorthogonal transform filter bank of length 32.

## Do-it-yourself: Filter banks

**Experiment 9.1:**

- Here we will design a third order CQF filter bank. We start by designing the product filter $P(z)$ using a Hilbert transformer as in equation (170)

$$P(z) = \frac{1}{2}\left(1 + \frac{\delta_{hb}}{2} - jH_h(-jz)\right) \tag{273}$$

- The impulse response of an ideal Hilbert transformer is

$$h(n) = \begin{cases} 0, & \text{for } n = 0 \\ \frac{1}{\pi n}\left[1 - (-1)^n\right], & \text{for } n \neq 0 \end{cases} \tag{274}$$

## Do-it-yourself: Filter banks

- Since the CQF filter bank should have order 3, then the product filter $P(z)$ should have order 6, and so should the Hilbert transformer.

- Applying a rectangular window to the ideal impulse response, the $z$ transform of a sixth order Hilbert transformer is

$$H_h(z) = \frac{2}{\pi}\left(-\frac{1}{3}z^3 - z + z^{-1} + \frac{1}{3}z^{-3}\right) \tag{275}$$

- We can plot its magnitude response using the MATLAB commands

```
hh = (2/pi)*[-1/3 0 -1 0 1 0 1/3];
[Hh,w]= freqz(hh);
plot(w,abs(Hh));
```

which yield Figure 55.

# Do-it-yourself: Filter banks

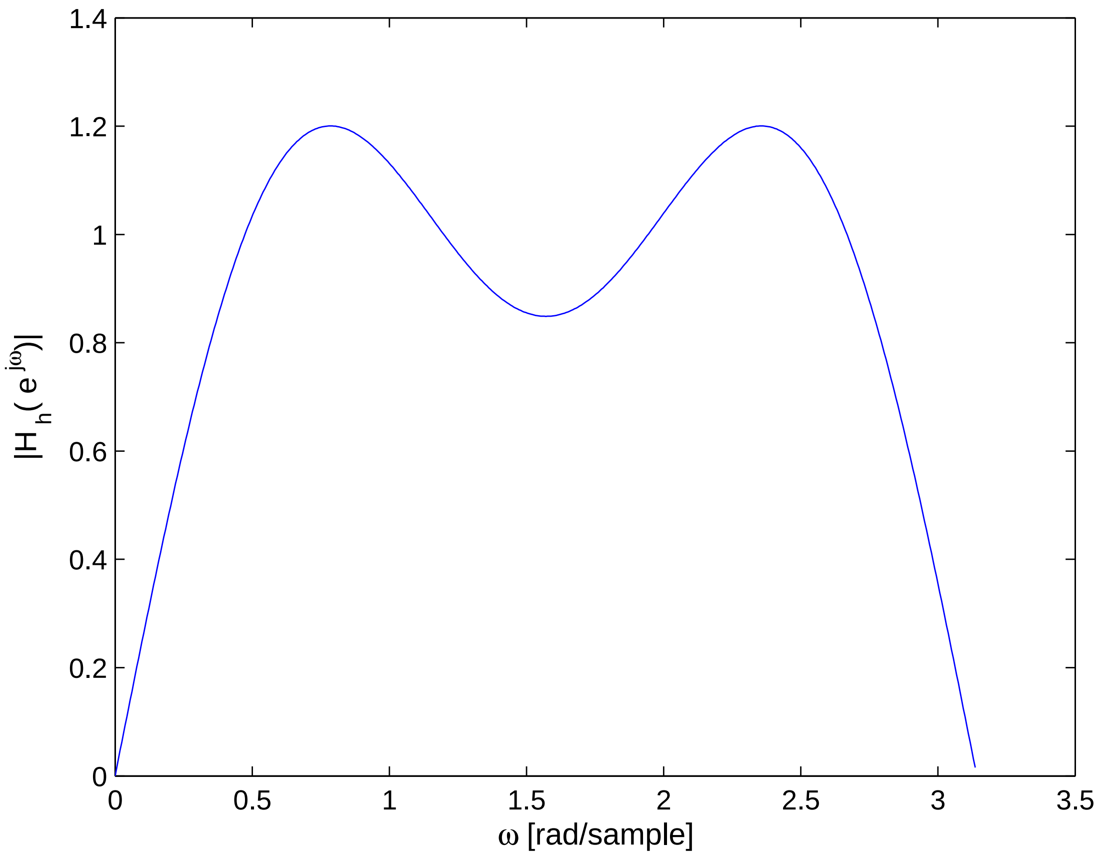


Figure 55: Magnitude response of the order 6 Hilbert transformer from Experiment .1.

## Do-it-yourself: Filter banks

- We observe from Figure 55 that the magnitude response of the Hilbert transformer has two maxima.

- We can compute them in MATLAB using the command

```
maxHh = max(abs(Hh));
freqmax = w(find(abs(Hh)==maxHh))/pi;
```

  that gives for the maxima the value $\texttt{masHh} = 1.2004$ at the frequencies $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

- Alternatively, we can compute them analytically from $H_h(e^{j\omega})$ as

$$\begin{aligned} H_h(e^{j\omega}) &= \frac{2}{\pi}\left(-\frac{1}{3}e^{j3\omega} - e^{j\omega} + e^{-j\omega} + \frac{1}{3}e^{-j3\omega}\right) \\ &= -\frac{4j}{\pi}\left(\frac{1}{3}\sin 3\omega + \sin\omega\right) \end{aligned} \quad (276)$$

# Do-it-yourself: Filter banks

- At the maxima, we should have

$$\begin{aligned}
\frac{dH_h(e^{j\omega})}{d\omega} &= -\frac{4j}{\pi}\left(\cos 3\omega + \cos\omega\right) \\
&= -\frac{4j}{\pi}\left(4\cos^3\omega - 3\cos\omega + \cos\omega\right) \\
&= -\frac{4j}{\pi}\cos\omega\left(4\cos^2\omega - 2\right) \\
&= 0
\end{aligned} \tag{277}$$

- Then

$$\begin{cases} \cos\omega = 0 \\ \cos\omega = \pm\dfrac{\sqrt{2}}{2} \end{cases} \Rightarrow \begin{cases} \omega = \dfrac{\pi}{2} + k\pi, & k \in \mathbb{Z} \\ \omega = \pm\dfrac{\pi}{4} + k\pi, & k \in \mathbb{Z} \end{cases} \tag{278}$$

# Do-it-yourself: Filter banks

- Since,

$$\left.\begin{array}{rcl} H_h(e^{j\frac{\pi}{2}}) & = & -\frac{4j}{\pi}\left(\frac{1}{3}\sin\frac{3\pi}{2}+\sin\frac{\pi}{2}\right)=-\frac{8j}{3\pi} \\ H_h(e^{j\frac{\pi}{4}}) & = & -\frac{4j}{\pi}\left(\frac{1}{3}\sin\frac{3\pi}{4}+\sin\frac{\pi}{4}\right)=-\frac{8\sqrt{2}j}{3\pi} \\ H_h(e^{j\frac{3\pi}{4}}) & = & -\frac{4j}{\pi}\left(\frac{1}{3}\sin\frac{9\pi}{4}+\sin\frac{3\pi}{4}\right)=-\frac{8\sqrt{2}j}{3\pi} \end{array}\right\} \tag{279}$$

and thus we confirm that the maxima of $|H_h(e^{j\omega})|$ occur at $\omega=\frac{\pi}{4}$ and $\omega=\frac{3\pi}{4}$, having the value of $\frac{8\sqrt{2}}{3\pi}\approx 1.2004$.

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# Do-it-yourself: Filter banks

- Since $\mathrm{P}(e^{j\omega})$ should be non-negative, we have that the value of $\frac{\delta_{hb}}{2}$ in equation (273) should be such that

$$1+\frac{\delta_{hb}}{2}=\frac{8\sqrt{2}}{3\pi} \tag{280}$$

- With this value, the product filter $\mathrm{P}(z)$ becomes

$$\begin{aligned}
\mathrm{P}(z) &= \frac{1}{2}\left(\frac{8\sqrt{2}}{3\pi}-j\mathrm{H_h}(-jz)\right)\\
&= \frac{4\sqrt{2}}{3\pi}-j\frac{1}{\pi}\left[-\frac{1}{3}(-jz)^3-(-jz)+(-jz)^{-1}+\frac{1}{3}(-jz)^{-3}\right]\\
&= \frac{1}{3\pi}\left(-z^3+3z+4\sqrt{2}+3z^{-1}-z^{-3}\right)
\end{aligned} \tag{281}$$

## Do-it-yourself: Filter banks

- However, for this product filter $P(z)$, we have that

$$P(z) + P(-z) = \frac{8\sqrt{2}}{3\pi} \neq 2 \tag{282}$$

- Since for the CQF filter bank we must have, from equation (166), that $P(z) + P(-z) = 2$, then the product filter must be normalized by $\frac{3\pi}{4\sqrt{2}}$, yielding

$$P(z) = \frac{1}{4\sqrt{2}}\left(-z^3 + 3z + 4\sqrt{2} + 3z^{-1} - z^{-3}\right) \tag{283}$$

- The corresponding frequency response $P(e^{j\omega})$ can be plotted in MATLAB as follows (note that, since $P(z)$ is symmetric, then $P(e^{j\omega})$ is real):

```
p = (1/4*sqrt(2))*[-1 0 3 4*sqrt(2) 3 0 -1];
[P,w]= freqz(p);
plot(w,abs(P));
```

as shown in Figure 56.

# Do-it-yourself: Filter banks

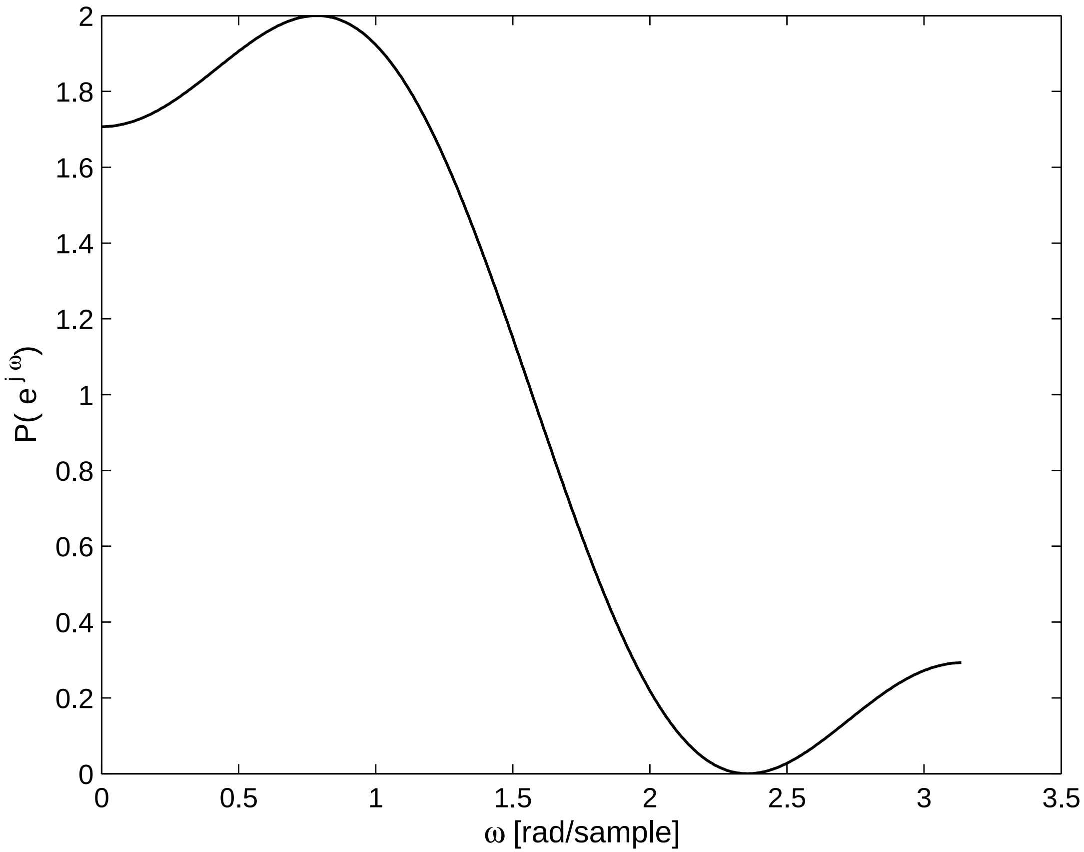


Figure 56: Plot of the frequency response of product filter $P(e^{j\omega})$ from Experiment .1.

## Do-it-yourself: Filter banks

- Now we must factorize $P(z)$ into $H_0(z)H_0(z^{-1})$.
- One way to do that is to find the zeros of $P(z)$ using the functions `tf2zpk` and `zplane` from MATLAB

```
p = (1/(4*sqrt(2)))*[-1 0 3 4*sqrt(2) 3 0 -1];
[zi pi k] = tf2zpk(p,1)
```

which indicate that we have a double zero at $e^{j\frac{3\pi}{4}}$, another double zero at $e^{-j\frac{3\pi}{4}}$, a zero at $(\sqrt{2}+1)$ and a zero at $(\sqrt{2}-1)$.

# Do-it-yourself: Filter banks

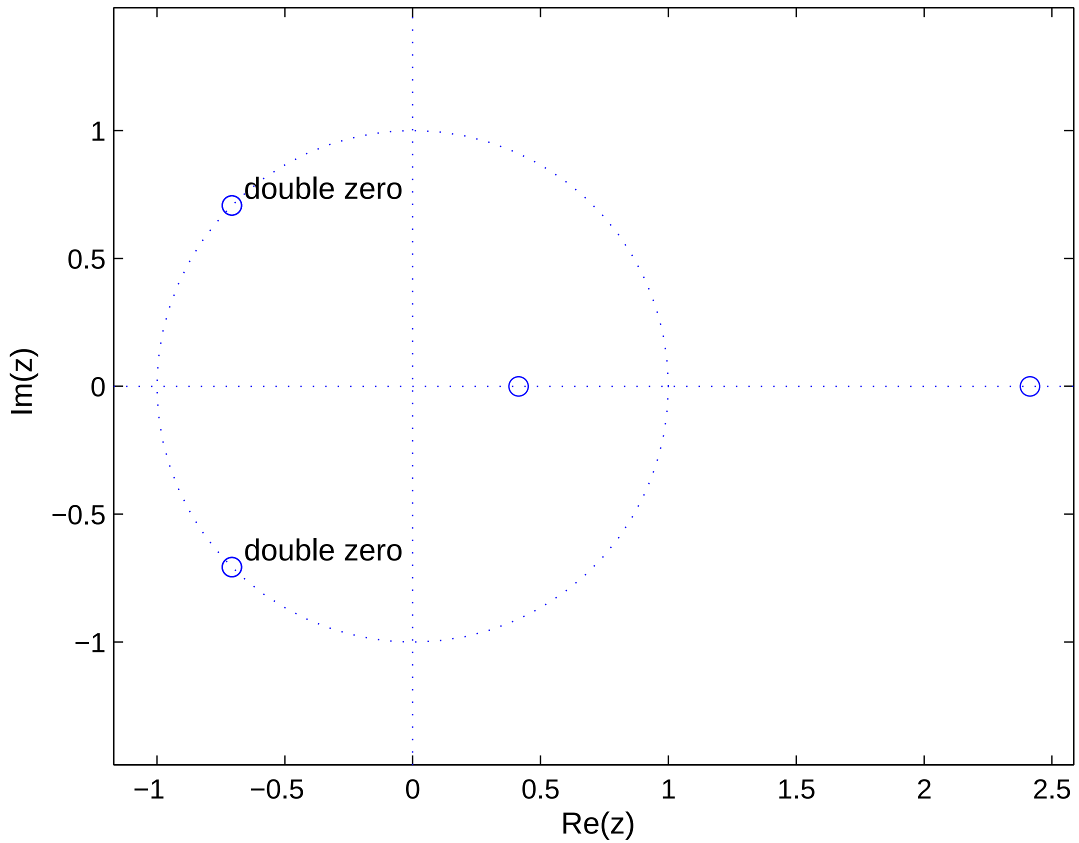


Figure 57: Zeros of the product filter $P(z)$ from Experiment .1.

## Do-it-yourself: Filter banks

- Hence, $P(z)$ can be expressed as

$$P(z) = -\frac{1}{4\sqrt{2}} z^{-3} (z - e^{j\frac{3\pi}{4}})^2 (z - e^{-j\frac{3\pi}{4}})^2 (z - \sqrt{2} - 1)(z - \sqrt{2} + 1) \tag{284}$$

- Grouping the complex conjugate poles together, we have

$$\begin{aligned}
P(z) &= -\frac{1}{4\sqrt{2}} z^{-3} (z^2 + \sqrt{2}z + 1)^2 (z - \sqrt{2} - 1)(z - \sqrt{2} + 1) \\
&= -\frac{1}{4\sqrt{2}} z^{-1} (1 + \sqrt{2}z^{-1} + z^{-2})(z^2 + \sqrt{2}z + 1)[1 - (\sqrt{2} + 1)z^{-1}](z - \sqrt{2} + 1) \\
&= -\frac{1 - \sqrt{2}}{4\sqrt{2}} (1 + \sqrt{2}z^{-1} + z^{-2})(1 + \sqrt{2}z + z^2)[1 - (\sqrt{2} + 1)z^{-1}] \left(1 + \frac{z}{1 - \sqrt{2}}\right) \\
&= \frac{\sqrt{2} - 1}{4\sqrt{2}} (1 + \sqrt{2}z^{-1} + z^{-2})(1 + \sqrt{2}z + z^2)[1 - (\sqrt{2} + 1)z^{-1}][1 - (1 + \sqrt{2})z]
\end{aligned} \tag{285}$$

# Do-it-yourself: Filter banks

- Since $P(z) = H_0(z)H_0(z^{-1})$, we can choose

$$\begin{aligned} H_0(z) &= \sqrt{\frac{\sqrt{2}-1}{4\sqrt{2}}}(1+\sqrt{2}z^{-1}+z^{-2})[1-(\sqrt{2}+1)z^{-1}] \\ &= \sqrt{\frac{\sqrt{2}-1}{4\sqrt{2}}}\left[1-z^{-1}-(1+\sqrt{2})z^{-2}-(1+\sqrt{2})z^{-3}\right] \end{aligned} \quad (286)$$

## Do-it-yourself: Filter banks

- Supposing an overall delay of $(2\Delta + 1) = N = 3$ samples, the highpass analysis, lowpass synthesis, and highpass synthesis filters are, from equations (161), (167), and (168),

$$\left.\begin{aligned} H_1(z) &= -z^{-3}H_0(-z^{-1}) \\ G_0(z) &= z^{-3}H_0(z^{-1}) \\ G_1(z) &= -H_0(-z) \end{aligned}\right\} \tag{287}$$

- Which correspond to a CQF filter bank described by

$$\left.\begin{aligned}
H_0(z) &= \sqrt{\frac{\sqrt{2}-1}{4\sqrt{2}}}\left[1 - z^{-1} - (1+\sqrt{2})z^{-2} - (1+\sqrt{2})z^{-3}\right] \\
H_1(z) &= \sqrt{\frac{\sqrt{2}-1}{4\sqrt{2}}}\left[-(1+\sqrt{2}) + (1+\sqrt{2})z^{-1} - z^{-2} - z^{-3}\right] \\
G_0(z) &= \sqrt{\frac{\sqrt{2}-1}{4\sqrt{2}}}\left[-(1+\sqrt{2}) - (1+\sqrt{2})z^{-1} - z^{-2} + z^{-3}\right] \\
G_1(z) &= \sqrt{\frac{\sqrt{2}-1}{4\sqrt{2}}}\left[-1 - z^{-1} + (1+\sqrt{2})z^{-2} - (1+\sqrt{2})z^{-3}\right]
\end{aligned}\right\} \quad (288)$$

# Do-it-yourself: Filter banks

- The magnitude responses of the lowpass and highpass filters for the analysis and synthesis banks can be plotted using the MATLAB commands below.

```
c = sqrt((sqrt(2)-1)/(4*sqrt(2)));
h0 = c*[1 -1 -(1+sqrt(2)) -(1+sqrt(2))];
[H0,w]= freqz(h0);
plot(w,abs(H0)); hold;
h1 = c*[-(1+sqrt(2)) (1+sqrt(2)) -1 -1];
[H1,w]= freqz(h1);
plot(w,abs(H1));
```

## Do-it-yourself: Filter banks

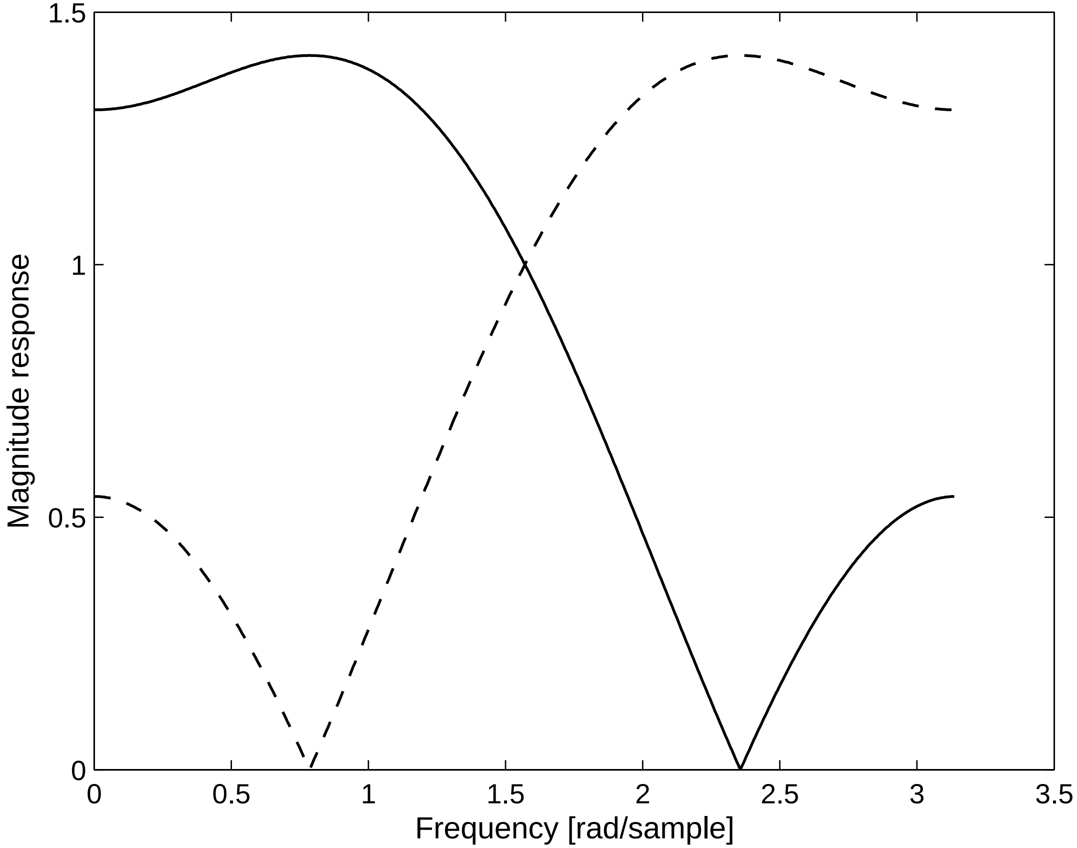


Figure 58: Magnitude responses of the lowpass and highpass filters of the analysis and synthesis banks from Experiment .1: $\mathrm{H}_0(z)$ (solid line); $\mathrm{H}_1(z)$ (dashed line).

## Do-it-yourself: Filter banks

**Experiment 9.2:**

Design a filter bank with $M = 10$ sub-bands using the LOT.

- Since the LOT to be designed has $M = 10$, the matrix $\mathbf{L}_0$ can be built using equation (242) with the matrices $\mathbf{C}_e$ and $\mathbf{C}_o$. A MATLAB script to compute $\mathbf{L}_0$ is

```
C = dctmtx(10);
Ce = C([1:2:9],:); Co = C([2:2:10],:);
I = eye(10); J = fliplr(I);
L0 = 0.5*[Ce-Co (Ce-Co)*J; Ce-Co -(Ce-Co)*J];
```

## Do-it-yourself: Filter banks

- To complete the design, we have to compute $\mathbf{L}_{\text{LOT}} = \mathbf{L}_1 \mathbf{L}_0$. Hence, we must find the matrix $\mathbf{L}_1$, which is an orthogonal matrix as given in equation (245), where $\mathbf{L}_2$ is determined by $(\frac{M}{2} - 1) = 4$ rotation angles $\theta_i$, for $i = 0, 1, 2, 3$ as follows:

$$\mathbf{L}_2 = \mathbf{T}_3 \mathbf{T}_2 \mathbf{T}_1 \mathbf{T}_0 \tag{289}$$

## Do-it-yourself: Filter banks

- Where

$$\mathbf{T}_i = \begin{bmatrix} \mathbf{I}_i & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}(\theta_i) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{3-i} \end{bmatrix}, \quad i = 0, 1, 2, 3 \tag{290}$$

- And

$$\mathbf{Y}(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix} \tag{291}$$

# Do-it-yourself: Filter banks

- The MATLAB code to compute $\mathbf{L}_{\mathrm{LOT}}$ given $\mathbf{L}_0$ and the rotation angles $\theta_0$, $\theta_1$, $\theta_2$, and $\theta_3$ in a vector `t` is

```
function y = LOT(t,L0)
Y = zeros(2,2,3); T = zeros(5,5,3); L2 = eye(5);
for i=1:4
   Y(:,:,i) = [cos(t(i)), -sin(t(i)) ; sin(t(i)),
cos(t(i))];
   T(:,:,i) = [eye(i-1), zeros(i-1,2),
zeros(i-1,4-i);
        zeros(2,i-1), Y(:,:,i), zeros(2,4-i);
        zeros(4-i,i-1), zeros(4-i,2), eye(4-i)];
   L2 = T(:,:,i)*L2;
end;
L1 = [eye(5), zeros(5,5); zeros(5,5), L2];
y = L1*L0;
```

## Do-it-yourself: Filter banks

- Therefore, these $4$ rotation angles should be such that a certain optimization criterion is satisfied.
- It can be the summation of the energy in the stopband of all bands.
- In this experiment we choose them such that the maximum energy compaction on the transform coefficients is obtained, that is, most energy concentrates on the smallest number of transform coefficients.
- It is assumed that the input signal is an autoregressive (AR) process generated by filtering a white noise with a first-order lowpass digital filter with a single pole at $z = 0.9$.

## Do-it-yourself: Filter banks

- We have that if a WSS process $\{X\}$ with autocorrelation $R_X(n)$ is input to a linear system with impulse response $h(n)$, then the autocorrelation $R_Y(n)$ at its output is

$$R_Y(n) = \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} R_X(n-k)h(k+r)h(r) \tag{292}$$

## Do-it-yourself: Filter banks

- Therefore, since the PSD of white noise is $R_X(n) = \sigma^2\delta(n)$, and the impulse response of a stable digital filter with a pole at $z = \rho$ is $h(n) = \rho^n u(n)$, we have that the PSD of an AR process $\{W\}$ with a pole at $z = \rho$ is

$$\begin{aligned} R_W(n) &= \sum_{k=-\infty}^{\infty}\sum_{r=-\infty}^{\infty} \sigma^2\delta(n-k)h(k+r)h(r) \\ &= \sigma^2 \sum_{r=-\infty}^{\infty} h(n+r)h(r) = \sigma^2 \sum_{r=-\infty}^{\infty} \rho^{n+r}u(n+r)\rho^r u(r) \\ &= \sigma^2\rho^n \sum_{r=-\infty}^{\infty} \rho^{2r}u(n+r)u(r) = \sigma^2\rho^n \sum_{r=\max\{-n,0\}}^{\infty} \rho^{2r} \\ &= \sigma^2 \frac{\rho^{n+2\max\{-n,0\}}}{1-\rho^2} = \frac{\sigma^2}{1-\rho^2}\rho^{|n|} \end{aligned} \tag{293}$$

# Do-it-yourself: Filter banks

- Therefore, the autocorrelation of the output of the $k$th analysis filter $h_k(n)$, when an AR process with a pole at $z = \rho$ is input to it, from equations (292) and (293), is

$$\begin{aligned} R_{Y_k}(n) &= \sum_{l=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} R_W(n-l) h_k(l+r) h_k(r) \\ &= \sum_{l=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \frac{\sigma^2}{1-\rho^2} \rho^{|n-l|} h_k(l+r) h_k(r) \\ &= \frac{\sigma^2}{1-\rho^2} \sum_{s=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \rho^{|n-s+r|} h_k(s) h_k(r) \end{aligned} \tag{294}$$

- Consequently, its variance is

$$\sigma_{Y_k}^2 = R_{Y_k}(0) = \frac{\sigma^2}{1-\rho^2} \sum_{s=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \rho^{|s-r|} h_k(s) h_k(r) \tag{295}$$

# Do-it-yourself: Filter banks

- For a stationary process, its variance is equal to the variance of its decimated output. Since the analysis filters have length $M = 20$ and $\rho = 0.9$, we have that the variance of the decimated output of the $k$th analysis filter is

$$\sigma_k^2 = \frac{\sigma^2}{1-\rho^2} \sum_{s=0}^{19} \sum_{r=0}^{19} 0.9^{|s-r|} h_k(s) h_k(r) \tag{296}$$

- A good measure of the energy concentration at the output of an $M$-band orthogonal filter bank is given by the coding gain

$$G = \frac{\frac{1}{M} \sum_{k=0}^{M-1} \sigma_k^2}{\prod_{k=0}^{M-1} \left(\sigma_k^2\right)^{\frac{1}{M}}} \tag{297}$$

## Do-it-yourself: Filter banks

- A MATLAB code for computing the coding gain of a given LOT matrix $\mathbf{L}_{\text{LOT}}$ when the AR pole is equal to `rho` is

```
function y = CG(Llot,rho)
sigma = zeros(1,10);
for k=1:10,
   for s=1:20,
      for r=1:20,
         sigma(k) = sigma(k) +
rho^(abs(s-r))*Llot(k,s)*Llot(k,r);
      end;
   end;
end;
y = (sum(sigma.^2/10)/(prod(sigma.^2)^(1/10));
```

## Do-it-yourself: Filter banks

- We should find the vector t$= [\theta_0, \theta_1, \theta_2, \theta_3]$ that maximizes the coding gain, that is, provides the maximum value of `CG(LOT(t,L0),0.9)`.
- This can be done by using either the optimization routine `fminsearch` or `fminunc` from the MATLAB Optimization toolbox. The MATLAB code can be as follows:

```
function [coding_gain,tf] = LOTtest(t)
tf = fminsearch(@LotCG,t);
coding_gain = 1/LotCG(tf);
```

## Do-it-yourself: Filter banks

- Running the routines above for `rho=0.9` with the initial point `t = [0 0 0 0]`, we get

  `coding_gain = 133.7581`

  `tf = [0.4648 0.5926 -1.1191 -0.0912]`

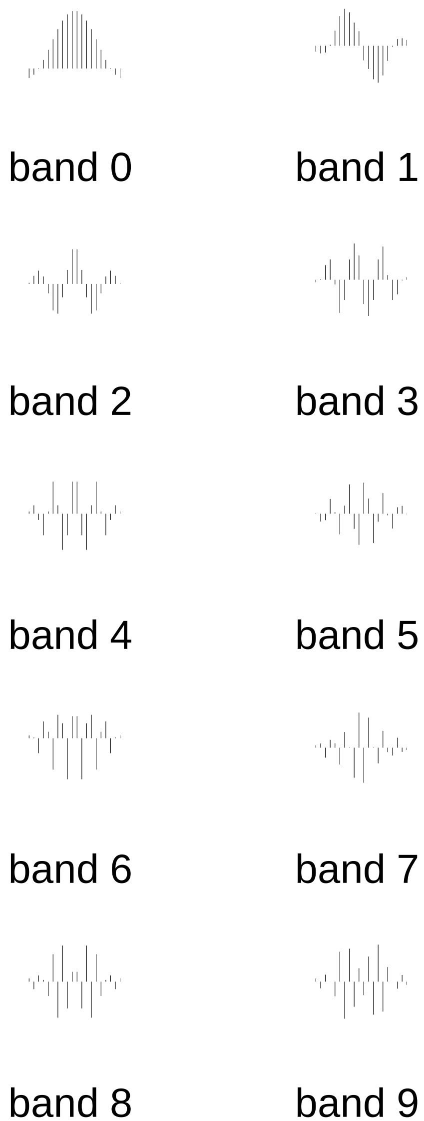


Figure 59: Impulse responses of the filters of a 10-band LOT.

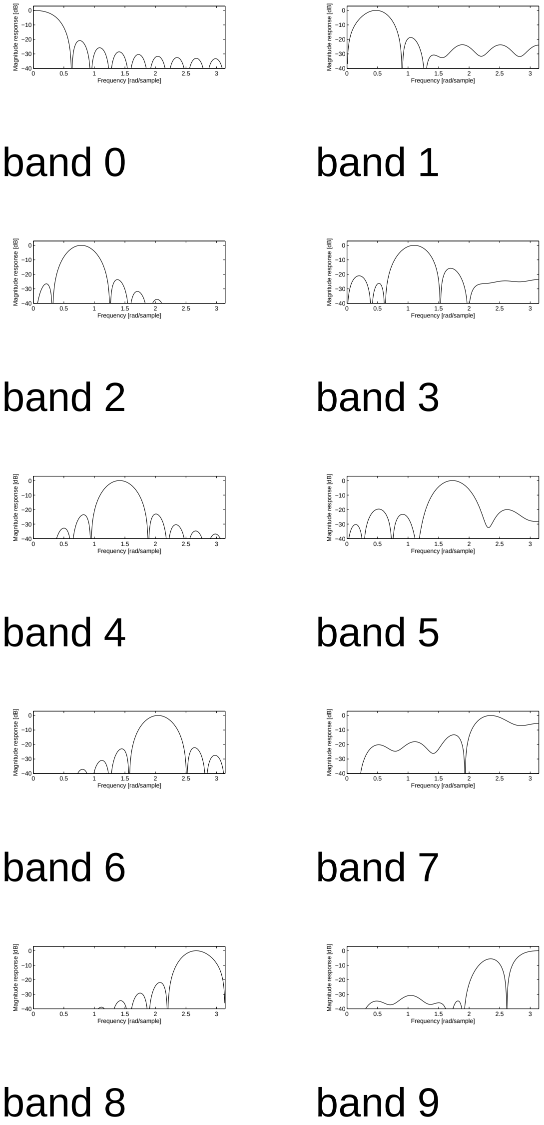


Figure 60: Magnitude responses of the filters of a 10-band LOT.