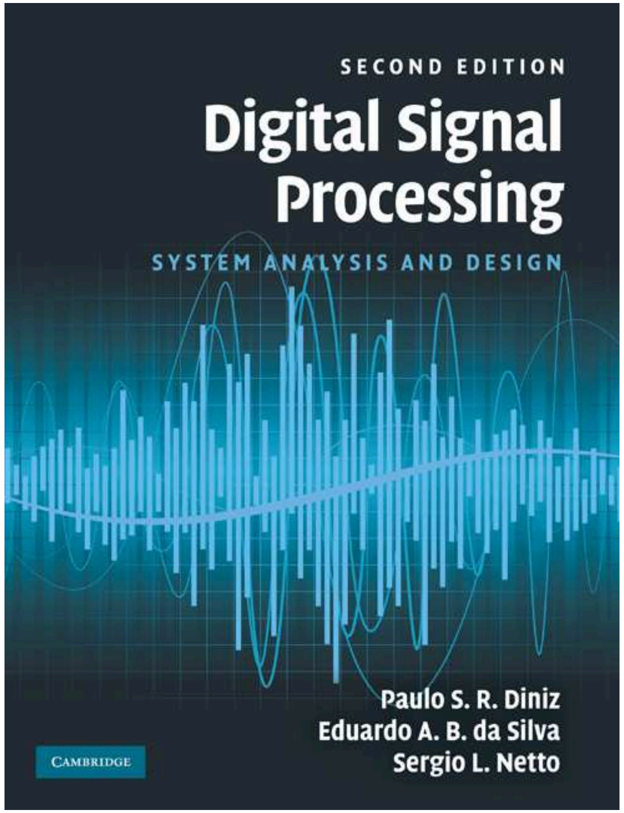
The z and Fourier transforms



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Contents

Introduction

- In Chapter 1, we studied linear time-invariant systems, using both impulse responses and difference equations to characterize them.
- In this chapter, we study another very useful way to characterize discrete-time systems.
- It is linked with the fact that, when an exponential function is input to a linear time-invariant system, its output is an exponential function of the same type, but with a different amplitude.

Introduction

- This can be deduced by considering that a linear time-invariant discrete-time system with impulse response $h(n)$, when excited by an exponential $x(n) = z^n$, produces an output

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) = \sum_{k=-\infty}^{\infty} z^{n-k}h(k) = z^n \sum_{k=-\infty}^{\infty} h(k)z^{-k} \quad (1)$$

that is, the signal at the output is also an exponential z^n, but with an amplitude multiplied by the complex function

$$H(z) = \sum_{k=-\infty}^{\infty} h(k)z^{-k} \quad (2)$$

- In this chapter, we characterize linear time-invariant systems using the quantity $H(z)$ in the above equation, commonly known as the z transform of the discrete-time sequence $h(n)$.

Introduction

- As we will see later in this chapter, with the help of the z transform, linear convolutions can be transformed into simple algebraic equations.
  - The importance of this for discrete-time systems parallels that of the Laplace transform for continuous-time systems.

- The case when z^n is a complex sinusoid with frequency ω, that is, $z = e^{j\omega}$, is of particular importance. In this case, equation (2) becomes

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \qquad (3)$$

which can be represented in polar form as $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\Theta(\omega)}$, yielding, from equation (1), an output signal $y(n)$ such that

$$y(n) = H(e^{j\omega})e^{j\omega n} = |H(e^{j\omega})|e^{j\Theta(\omega)}e^{j\omega n} = |H(e^{j\omega})|e^{j\omega n + j\Theta(\omega)} \qquad (4)$$

Introduction

- This relationship implies that the effect of a linear system characterized by $H(e^{j\omega})$ on a complex sinusoid is to multiply its amplitude by $|H(e^{j\omega})|$ and to add $\Theta(\omega)$ to its phase.
  - For this reason, the descriptions of $|H(e^{j\omega})|$ and $\Theta(\omega)$ as functions of ω are widely used to characterize linear time-invariant systems, and are known as their magnitude and phase responses, respectively.
- The complex function $H(e^{j\omega})$ in equation (4) is also known as the Fourier transform of the discrete-time sequence $h(n)$.
  - The Fourier transform is as important for discrete-time systems as it is for continuous-time systems.

Definition of the z transform

- The z transform of a sequence $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{5}$$

where z is a complex variable. Note that $X(z)$ is only defined for the regions of the complex plane in which the summation on the right converges.

- Very often, the signals we work with start only at $n = 0$, that is, they are nonzero only for $n \geq 0$. Because of that, some textbooks define the z transform as

$$X_U(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \tag{6}$$

which is commonly known as the one-sided z transform, while equation (5) is referred to as the two-sided z transform.

  - Clearly, if the signal $x(n)$ is nonzero for $n < 0$, then the one-sided and two-sided z transforms are different.

Definition of the z transform

- In this text, we work only with the two-sided z transform, which is referred to, without any risk of ambiguity, just as the z transform.
- As mentioned above, the z transform of a sequence exists only for those regions of the complex plane in which the summation in equation (5) converges. The example below clarifies this point.

Definition of the z transform

Example 2.1

Compute the z transform of the sequence $x(n) = Ku(n)$.

Solution

By definition, the z transform of $Ku(n)$ is

$$X(z) = K \sum_{n=0}^{\infty} z^{-n} = K \sum_{n=0}^{\infty} \left(z^{-1}\right)^{n} \tag{7}$$

Thus, $X(z)$ is the sum of a power series which converges only if $|z^{-1}| < 1$. In such a case, $X(z)$ can be expressed as

$$X(z) = \frac{K}{1 - z^{-1}} = \frac{Kz}{z - 1}, \quad |z| > 1 \tag{8}$$

Note that for $|z| < 1$, the nth term of the summation, z^{-n}, tends to infinity as $n \rightarrow \infty$, and therefore $X(z)$ is not defined. For $z = 1$, the summation is also infinite. For $z = -1$, the summation oscillates between 1 and 0. In none of these cases does the z transform converge. $\triangle$

Definition of the z transform

- The z transform of a sequence is a Laurent series in the complex variable z.

$\Rightarrow$ The properties of Laurent series apply directly to the z transform.

- As a general rule, we can apply a result from series theory stating that, given a series of the complex variable z

$$S(z) = \sum_{i=0}^{\infty} f_i(z) \tag{9}$$

such that $|f_i(z)| < \infty$, $i = 0, 1, \ldots$, and given the quantity

$$\alpha(z) = \lim_{n \to \infty} |f_n(z)|^{\frac{1}{n}} \tag{10}$$

then the series converges absolutely if $\alpha(z) < 1$, and diverges if $\alpha(z) > 1$.

  – For $\alpha(z) = 1$, the above procedure tells us nothing about the convergence of the series, which must be investigated by other means.

Definition of the z transform

- One can justify this by noting that, if $\alpha(z) < 1$, the terms of the series are under an exponential a^n for some $a < 1$, and therefore their sum converges as $n \to \infty$.

- If $|f_i(z)| = \infty$, for some i, then the series is not convergent.
  Also, convergence demands that $\lim_{n\to\infty} |f_n(z)| = 0$.

- The above result can be extended for the case of two-sided series as in the equation below

$$S(z) = \sum_{i=-\infty}^{\infty} f_i(z) \tag{11}$$

if we express $S(z)$ above as the sum of two series $S_1(z)$ and $S_2(z)$ such that

$$S_1(z) = \sum_{i=0}^{\infty} f_i(z); \quad S_2(z) = \sum_{i=-\infty}^{-1} f_i(z) \tag{12}$$

then $S(z)$ converges if the two series $S_1(z)$ and $S_2(z)$ converge.

Definition of the z transform

- Therefore, in this case, we have to compute the two quantities

$$\alpha_1(z) = \lim_{n\to\infty} |f_n(z)|^{\frac{1}{n}} \quad \text{and} \quad \alpha_2(z) = \lim_{n\to-\infty} |f_n(z)|^{\frac{1}{n}} \tag{13}$$

- Naturally, $S(z)$ converges absolutely if $\alpha_1(z) < 1$ and $\alpha_2(z) > 1$.
  - The condition $\alpha_1(z) < 1$ is equivalent to saying that, for $n \to \infty$, the terms of the series are under a^n for some $a < 1$.
  - The condition $\alpha_2(z) > 1$ is equivalent to saying that, for $n \to -\infty$, the terms of the series are under b^n for some $b > 1$.
  - One should note that, for convergence, we must also have $|f_i(z)| < \infty$, for all i.

Definition of the z transform

- Applying these convergence results to the z-transform definition given in equation (5), we conclude that the z transform converges if

$$\alpha_1 = \lim_{n\to\infty} \left|x(n)z^{-n}\right|^{\frac{1}{n}} = \left|z^{-1}\right| \lim_{n\to\infty} |x(n)|^{\frac{1}{n}} < 1 \tag{14}$$

$$\alpha_2 = \lim_{n\to-\infty} \left|x(n)z^{-n}\right|^{\frac{1}{n}} = \left|z^{-1}\right| \lim_{n\to-\infty} |x(n)|^{\frac{1}{n}} > 1 \tag{15}$$

Defining

$$r_1 = \lim_{n\to\infty} |x(n)|^{\frac{1}{n}} \tag{16}$$

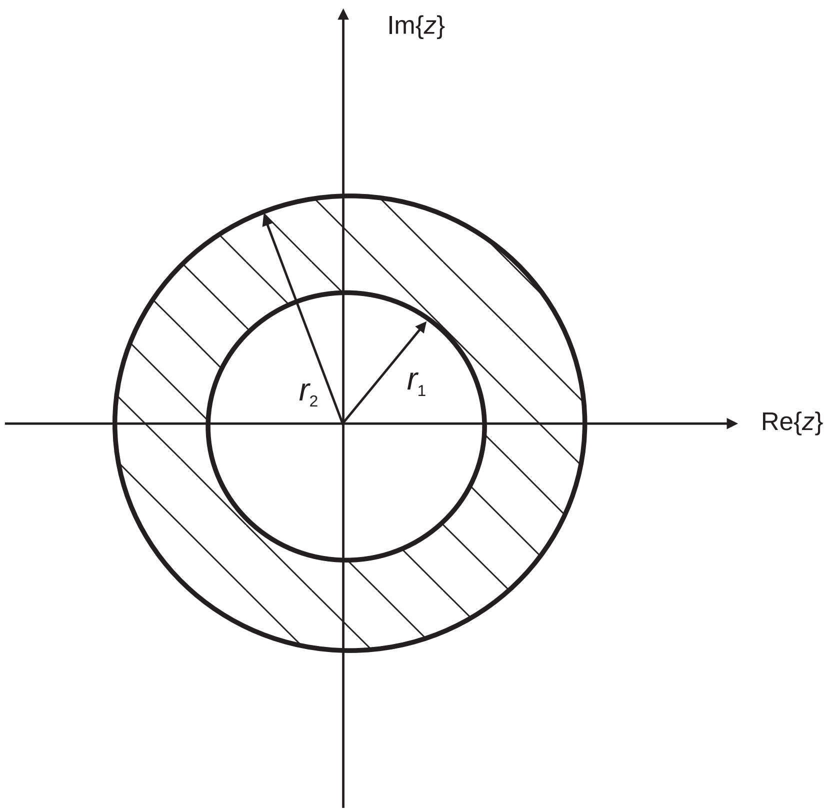
$$r_2 = \lim_{n\to-\infty} |x(n)|^{\frac{1}{n}} \tag{17}$$

then equations (14) and (15) are equivalent to

$$r_1 < |z| < r_2 \tag{18}$$

Definition of the z transform

- That is, the z transform of a sequence exists in an annular region of the complex plane defined by equation (18) and illustrated in the figure below. It is important to note that, for some sequences, $r_1 = 0$ or $r_2 \to \infty$. In these cases, the region of convergence may or may not include $z = 0$ or $|z| = \infty$, respectively.



Definition of the z transform

We now take a closer look at the convergence of z transforms for four important classes of sequences.

- Right-handed, one-sided sequences: These are sequences such that $x(n) = 0$, for $n < n_0$, that is

$$X(z) = \sum_{n=n_0}^{\infty} x(n)z^{-n} \tag{19}$$

In this case, the z transform converges for $|z| > r_1$, where r_1 is given by equation (16). Since $|x(n)z^{-n}|$ must be finite, then, if $n_0 < 0$, the convergence region excludes $|z| = \infty$.

Definition of the z transform

- Left-handed, one-sided sequences: These are sequences such that $x(n) = 0$, for $n > n_0$, that is

$$X(z) = \sum_{n=-\infty}^{n_0} x(n)z^{-n} \tag{20}$$

In this case, the z transform converges for $|z| < r_2$, where r_2 is given by equation (17). Since $|x(n)z^{-n}|$ must be finite, then, if $n_0 > 0$, the convergence region excludes $z = 0$.

- Two-sided sequences: In this case,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{21}$$

and the z transform converges for $r_1 < |z| < r_2$, where r_1 and r_2 are given by equations (16) and (17). Clearly, if $r_1 > r_2$, then the z transform does not exist.

Definition of the z transform

- Finite-length sequences: These are sequences such that $x(n) = 0$, for $n < n_0$ and $n > n_1$, that is

$$X(z) = \sum_{n=n_0}^{n_1} x(n) z^{-n} \tag{22}$$

In such cases, the z transform converges everywhere except at the points such that $|x(n)z^{-n}| = \infty$. This implies that the convergence region excludes the point $z = 0$ if $n_1 > 0$ and $|z| = \infty$ if $n_0 < 0$.

Definition of the z transform

Example 2.2

Compute the z transforms of the following sequences, specifying their region of convergence:

(a) $x(n) = k2^n u(n)$

(b) $x(n) = u(-n+1)$

(c) $x(n) = -k2^n u(-n-1)$

(d) $x(n) = 0.5^n u(n) + 3^n u(-n)$

(e) $x(n) = 4^{-n} u(n) + 5^{-n} u(n+1)$

Definition of the z transform

Solution

(a) $X(z) = \sum_{n=0}^{\infty} k2^n z^{-n}$

This series converges if $|2z^{-1}| < 1$, that is, for $|z| > 2$. In this case, $X(z)$ is the sum of a geometric series, and therefore

$$X(z) = \frac{k}{1 - 2z^{-1}} = \frac{kz}{z - 2}, \quad \text{for } 2 < |z| \leq \infty \tag{23}$$

(b) $X(z) = \sum_{n=-\infty}^{1} z^{-n}$

This series converges if $|z^{-1}| > 1$, that is, for $|z| < 1$. Also, in order for the term z^{-1} to be finite, $|z| \neq 0$. In this case, $X(z)$ is the sum of a geometric series, such that

$$X(z) = \frac{z^{-1}}{1 - z} = \frac{1}{z - z^2}, \quad \text{for } 0 < |z| < 1 \tag{24}$$

(c) $X(z) = \sum_{n=-\infty}^{-1} -k2^n z^{-n}$

This series converges if $|\frac{z}{2}| < 1$, that is, for $|z| < 2$. In this case, $X(z)$ is the sum of a geometric series, such that

$$X(z) = \frac{-k\frac{z}{2}}{1-\frac{z}{2}} = \frac{kz}{z-2}, \quad \text{for } 0 \leq |z| < 2 \tag{25}$$

(d) $X(z) = \sum_{n=0}^{\infty} 0.5^n z^{-n} + \sum_{n=-\infty}^{0} 3^n z^{-n}$

This series converges if $|0.5z^{-1}| < 1$ and $|3z^{-1}| > 1$, that is, for $0.5 < |z| < 3$. In this case, $X(z)$ is the sum of two geometric series, and therefore

$$X(z) = \frac{1}{1-0.5z^{-1}} + \frac{1}{1-\frac{1}{3}z} = \frac{z}{z-0.5} + \frac{3}{3-z}, \quad \text{for } 0.5 < |z| < 3 \tag{26}$$

(e) $X(z) = \sum_{n=0}^{\infty} 4^{-n} z^{-n} + \sum_{n=-1}^{\infty} 5^{-n} z^{-n}$

This series converges if $|\frac{1}{4}z^{-1}| < 1$ and $|\frac{1}{5}z^{-1}| < 1$, that is, for $|z| > \frac{1}{4}$. Also, the term for $n = -1$, $\left(\frac{1}{5}z^{-1}\right)^{-1} = 5z$, is finite only for $|z| < \infty$. In this case, $X(z)$ is the sum of two geometric series, resulting in

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{5z}{1 - \frac{1}{5}z^{-1}} = \frac{4z}{4z-1} + \frac{25z^2}{5z-1}, \quad \text{for } \frac{1}{4} < |z| < \infty \tag{27}$$

- In this example, although the sequences in items a and c are distinct, the expressions for their z transforms are the same, the difference being only in their regions of convergence.
- This highlights the important fact that, in order to completely specify a z transform, its region of convergence must be supplied.

$\triangle$

Definition of the z transform

- In several cases we deal with causal and stable systems. Since for a causal system its impulse response $h(n)$ is zero for $n < n_0$, $n_0 \geq 0$ the we have that a causal system is also BIBO stable if

$$\sum_{n=n_0}^{\infty} |h(n)| < \infty \tag{28}$$

  Applying the series convergence criterion seen above, we have that the system is stable only if

$$\lim_{n\to\infty} |h(n)|^{\frac{1}{n}} = r < 1 \tag{29}$$

- This is equivalent to saying that $H(z)$, the z transform of $h(n)$, converges for $|z| > r$.

$\Rightarrow$ Since, for stability, $r < 1$, then we conclude that the convergence region of the z transform of the impulse response of a stable causal system includes the region outside the unit circle and the unit circle itself (in fact, if $n_0 < 0$, then this region excludes $|z| = \infty$).

Definition of the z transform

A very important case is when $X(z)$ can be expressed as a ratio of polynomials, in the form

$$X(z) = \frac{N(z)}{D(z)} \tag{30}$$

We refer to the roots of $N(z)$ as the zeros of $X(z)$ and to the roots of $D(z)$ as the poles of $X(z)$. More specifically, in this case $X(z)$ can be expressed as

$$X(z) = \frac{N(z)}{\prod_{k=1}^{K}(z - p_k)^{m_k}} \tag{31}$$

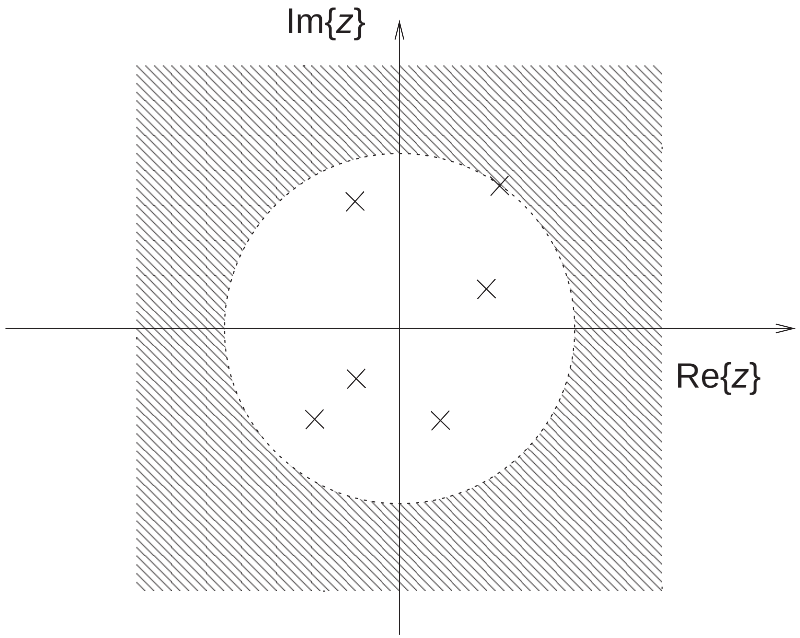
where p_k is a pole of multiplicity m_k, and K is the total number of distinct poles.

- Since $X(z)$ is not defined at its poles, its convergence region must not include them.

Definition of the z transform

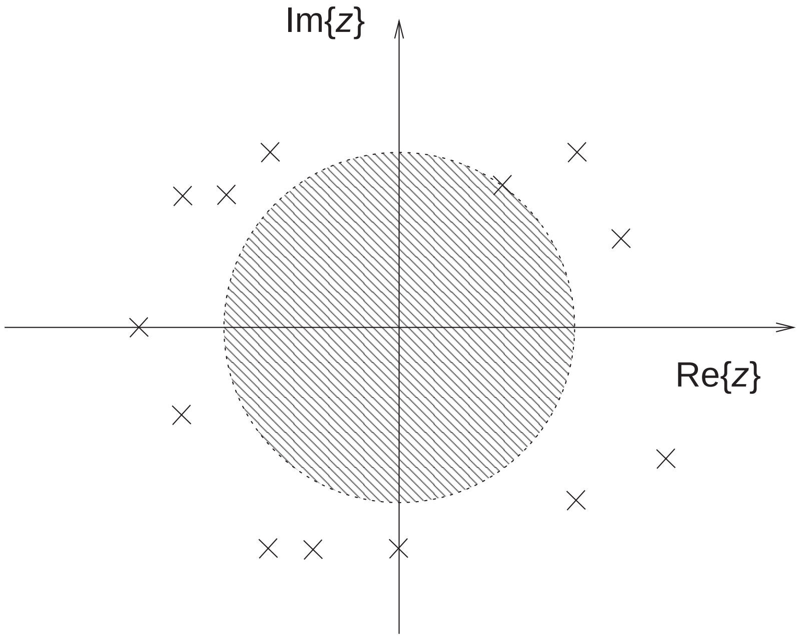
Therefore, given $X(z)$ as in equation (31), there is an easy way of determining its convergence region, depending on the type of sequence $x(n)$:

- Right-handed, one-sided sequences: The convergence region of $X(z)$ is $|z| > r_1$. Since $X(z)$ is not convergent at its poles, then its poles must be inside the circle $|z| = r_1$ (except for poles at $|z| = \infty$), and $r_1 = \max\limits_{1 \leq k \leq K} \{|p_k|\}$. This is illustrated below.



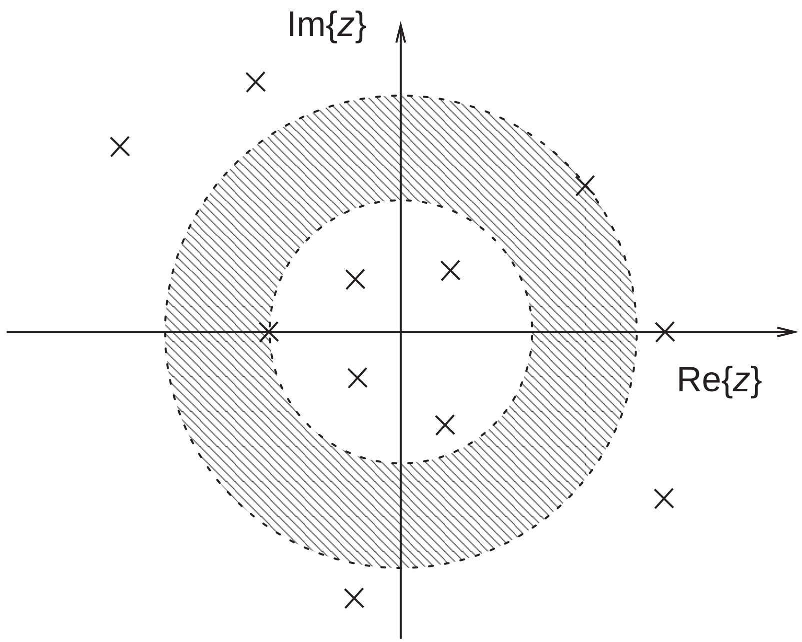
Definition of the z transform

- Left-handed, one-sided sequences: The convergence region of $X(z)$ is $|z| < r_2$. Therefore, its poles must be outside the circle $|z| = r_2$ (except for poles at $z = 0$), and $r_2 = \min_{1 \leq k \leq K} \{|p_k|\}$. This is illustrated below.



Definition of the z transform

- Two-sided sequences: The convergence region of $X(z)$ is $r_1 < |z| < r_2$, and therefore some of its poles are inside the circle $|z| = r_1$ and some outside the circle $|z| = r_2$. In this case, the convergence region needs to be further specified. This is illustrated below.



Inverse z transform

Very often one needs to determine which sequence corresponds to a given z transform. A formula for the inverse z transform can be obtained from the residue theorem, which we state next.

RESIDUE THEOREM

Let $X(z)$ be a complex function that is analytic inside a closed contour C, including the contour itself, except in a finite number of singular points p_n inside C. In this case, the following equality holds:

$$\oint_C X(z)dz = 2\pi j \sum_{k=1}^{K} \underset{z=p_k}{\mathit{res}} \{X(z)\} \tag{32}$$

with the integral evaluated counterclockwise around C.

Inverse z transform

If p_k *is a pole of multiplicity* m_k *of* $X(z)$*, that is, if* $X(z)$ *can be written as*

$$X(z) = \frac{P_k(z)}{(z - p_k)^{m_k}} \tag{33}$$

where $P_k(z)$ *is analytic at* $z = p_k$*, then the residue of* $X(z)$ *with respect to* p_k *is given by*

$$\underset{z=p_k}{res}\{X(z)\} = \frac{1}{(m_k - 1)!} \left. \frac{d^{(m_k-1)}[(z-p_k)^{m_k} X(z)]}{dz^{m_k-1}} \right|_{z=p_k} \tag{34}$$

$\diamond$

Inverse z transform

Using the above theorem, one can show that, if C is a counterclockwise closed contour, encircling the origin of the z plane, then

$$\frac{1}{2\pi j}\oint_C z^{n-1}dz = \begin{cases} 0, & \text{for } n \neq 0 \\ 1, & \text{for } n = 0 \end{cases} \tag{35}$$

and then we can derive that the inverse z transform of $X(z)$ is given by

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \tag{36}$$

where C is a closed counterclockwise contour in the convergence region of $X(z)$.

Inverse z transform

PROOF

Since

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{37}$$

by expressing $x(n)$ using the inverse z transform as in equation (36), then changing the order of the integration and summation, we have that

$$\begin{aligned} \frac{1}{2\pi j} \oint_C X(z) z^{m-1} dz &= \frac{1}{2\pi j} \oint_C \sum_{n=-\infty}^{\infty} x(n) z^{-n+m-1} dz \\ &= \frac{1}{2\pi j} \sum_{n=-\infty}^{\infty} x(n) \oint_C z^{-n+m-1} dz \\ &= x(m) \end{aligned} \tag{38}$$

$\square$

In the remainder of this section, we describe techniques for the computation of the inverse z transform in several practical cases.

Inverse z transform - Computation based on residue theorem

Whenever $X(z)$ is a ratio of polynomials, the residue theorem can be used very efficiently to compute the inverse z transform. In this case, equation (36) becomes

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz = \sum_{p_k \text{ encircled by } C} \operatorname*{res}_{z=p_k}\left\{X(z)z^{n-1}\right\} \tag{39}$$

where

$$X(z)z^{n-1} = \frac{N(z)}{\prod_{k=1}^{K}(z-p_k)^{m_k}} \tag{40}$$

- Note that not all poles p_k (with multiplicity m_k) of $X(z)z^{n-1}$ enter the summation in equation (39). It should contain only the poles that are encircled by the contour C.
- It is also important to note that the contour C must be contained in the convergence region of $X(z)$.
- In addition, in order to compute $x(n)$, for $n \leq 0$, one must consider the residues of the poles of $X(z)z^{n-1}$ at the origin.

Inverse z transform - Computation based on residue theorem

Example 2.3

Determine the inverse z transform of

$$X(z) = \frac{z^2}{(z-0.2)(z+0.8)} \tag{41}$$

considering that it represents the z transform of the impulse response of a causal system.

Inverse z transform - Computation based on residue theorem

Solution

$X(z) = \frac{z^2}{(z-0.2)(z+0.8)}$

- In order to completely specify a z transform, its region of convergence must be supplied.
- In this example, since the system is causal, we have that its impulse response is right handed and one sided.
  - Therefore, as seen earlier in this section, the convergence region of the z transform is characterized by $|z| > r_1$.
  - This implies that its poles are inside the circle $|z| = r_1$ and therefore $r_1 = \max_{1 \leq k \leq K} \{|p_k|\} = 0.8$.

Inverse z transform - Computation based on residue theorem

- We then need to compute

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz = \frac{1}{2\pi j}\oint_C \frac{z^{n+1}}{(z-0.2)(z+0.8)}dz \qquad (42)$$

  where C is any closed contour in the convergence region of $X(z)$, that is, encircling the poles $z = 0.2$ and $z = -0.8$ (as well as the poles at $z = 0$, for $n \leq -2$).

- Since we want to use the residue theorem, there are two distinct cases.
  - For $n \geq -1$, there are two poles inside C, at $z = 0.2$ and $z = -0.8$;
  - For $n \leq -2$, there are three poles inside C, at $z = 0.2$, $z = -0.8$, and $z = 0$.

Inverse z transform - Computation based on residue theorem

Therefore, we have that:

- For $n \geq -1$, equation (39) leads to

$$\begin{aligned} x(n) &= \underset{z=0.2}{\text{res}}\left\{\frac{z^{n+1}}{(z-0.2)(z+0.8)}\right\} + \underset{z=-0.8}{\text{res}}\left\{\frac{z^{n+1}}{(z-0.2)(z+0.8)}\right\} \\ &= \underset{z=0.2}{\text{res}}\left\{\frac{P_1(z)}{z-0.2}\right\} + \underset{z=-0.8}{\text{res}}\left\{\frac{P_2(z)}{z+0.8}\right\} \end{aligned} \tag{43}$$

where

$$P_1(z) = \frac{z^{n+1}}{z+0.8}; \quad P_2(z) = \frac{z^{n+1}}{z-0.2} \tag{44}$$

Inverse z transform - Computation based on residue theorem

From equation (34),

$$\operatorname*{res}_{z=0.2}\left\{\frac{z^{n+1}}{(z-0.2)(z+0.8)}\right\} = P_1(0.2) = (0.2)^{n+1} \tag{45}$$

$$\operatorname*{res}_{z=-0.8}\left\{\frac{z^{n+1}}{(z-0.2)(z+0.8)}\right\} = P_2(-0.8) = -(-0.8)^{n+1} \tag{46}$$

and then

$$x(n) = (0.2)^{n+1} - (-0.8)^{n+1}, \text{ for } n \geq -1 \tag{47}$$

Inverse z transform - Computation based on residue theorem

- For $n \leq -2$, we also have a pole of multiplicity $(-n-1)$ at $z=0$. Therefore, we have to add the residue at $z=0$ to the two residues in equation (47), such that

$$\begin{aligned} x(n) &= (0.2)^{n+1} - (-0.8)^{n+1} + \underset{z=0}{\text{res}} \left\{ \frac{z^{n+1}}{(z-0.2)(z+0.8)} \right\} \\ &= (0.2)^{n+1} - (-0.8)^{n+1} + \underset{z=0}{\text{res}} \left\{ P_3(z) z^{n+1} \right\} \end{aligned} \tag{48}$$

where

$$P_3(z) = \frac{1}{(z-0.2)(z+0.8)} \tag{49}$$

Inverse z transform - Computation based on residue theorem

From equation (34), since the pole $z = 0$ has multiplicity $m_k = (-n-1)$, we have that

$$\begin{aligned}
\underset{z=0}{\text{res}} \left\{P_3(z)z^{n+1}\right\} &= \frac{1}{(-n-2)!} \left.\frac{\mathrm{d}^{(-n-2)} P_3(z)}{\mathrm{d}z^{(-n-2)}}\right|_{z=0} \\
&= \frac{1}{(-n-2)!} \frac{\mathrm{d}^{(-n-2)}}{\mathrm{d}z^{(-n-2)}} \left.\left\{\frac{1}{(z-0.2)(z+0.8)}\right\}\right|_{z=0} \\
&= \left.\left\{\frac{(-1)^{-n-2}}{(z-0.2)^{-n-1}} - \frac{(-1)^{-n-2}}{(z+0.8)^{-n-1}}\right\}\right|_{z=0} \\
&= (-1)^{-n-2}\left[(-0.2)^{n+1} - (0.8)^{n+1}\right] \\
&= -(0.2)^{n+1} + (-0.8)^{n+1}
\end{aligned} \tag{50}$$

Substituting the above result into equation (48), we have that

$$x(n) = (0.2)^{n+1} - (-0.8)^{n+1} - (0.2)^{n+1} + (-0.8)^{n+1} = 0, \ \text{for } n \leq -2 \tag{51}$$

Inverse z transform - Computation based on residue theorem

From equations (47) and (51), we then have that

$$x(n) = \left[(0.2)^{n+1} - (-0.8)^{n+1}\right]u(n+1) \tag{52}$$

$\triangle$

- From what we have seen in the above example, the computation of residues for the case of multiple poles at $z = 0$ involves computation of nth-order derivatives, which can very often become quite involved.
- Fortunately, these cases can be easily solved by means of a simple trick, which we describe next.

Inverse z transform - Computation based on residue theorem

When the integral in

$$X(z) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \tag{53}$$

involves computation of residues of multiple poles at $z = 0$, we make the change of variable $z = \frac{1}{v}$.

- If the poles of $X(z)$ are located at $z = p_i$, then the poles of $X\left(\frac{1}{v}\right)$ are located at $v = \frac{1}{p_i}$.
- Also, if $X(z)$ converges for $r_1 < |z| < r_2$, then $X\left(\frac{1}{v}\right)$ converges for $\frac{1}{r_2} < |v| < \frac{1}{r_1}$.
- The integral in equation (36) then becomes

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz = -\frac{1}{2\pi j} \oint_{C'} X\left(\frac{1}{v}\right) v^{-n-1} dv \tag{54}$$

Inverse z transform - Computation based on residue theorem

- Note that, if the contour C is traversed in the counterclockwise direction by z then the contour C' is traversed in the clockwise direction by v.
  - Substituting the contour C' by an equal contour C'' that is traversed in the counterclockwise direction, the sign of the integral is reversed, and the above equation becomes

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz = \frac{1}{2\pi j}\oint_{C''} X\left(\frac{1}{v}\right)v^{-n-1}dv \qquad (55)$$

- If $X(z)z^{n-1}$ has multiple poles at the origin, then $X(\frac{1}{v})v^{-n-1}$ has multiple poles at $|z| = \infty$, which are now outside the closed contour C''.

$\Rightarrow$ Therefore, the computation of the integral on the right-hand side of equation (55) avoids the computation of nth-order derivatives.

- This fact is illustrated by Example 2.4 below, which revisits the computation of the inverse z transform in Example 2.3.

Inverse z transform - Computation based on residue theorem

Example 2.4

Compute the inverse z transform of $X(z)$ in Example 2.3, for $n \leq -2$, using the residue theorem by employing the change of variables in equation (55).

Solution

If we make the change of variables $z = \frac{1}{v}$, equation (42) becomes

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{z^{n+1}}{(z-0.2)(z+0.8)}\,dz = \frac{1}{2\pi j}\oint_{C''} \frac{v^{-n-1}}{(1-0.2v)(1+0.8v)}\,dv \qquad (56)$$

The convergence region of the integrand on the right is $|v| < \frac{1}{0.8}$, and therefore, for $n \leq -2$, no poles are inside the closed contour C''. Then, from equation (39), we conclude that

$$x(n) = 0, \quad \text{for } n \leq -2 \qquad (57)$$

which is the result of Example 2.3 obtained in a straightforward manner. $\triangle$

Inverse z transform - Computation based on partial-fraction expansions

- Using the residue theorem, one can show that the inverse z transform of

$$X(z) = \frac{1}{(z - z_0)^k} \tag{58}$$

if its convergence region is $|z| > |z_0|$, is the one-sided, right-handed sequence

$$x(n) = \frac{(n-1)!}{(n-k)!(k-1)!} z_0^{n-k} u(n-k) = \binom{n-1}{k-1} z_0^{n-k} u(n-k) \tag{59}$$

- If the convergence region of the z transform in equation (58) is $|z| < |z_0|$, its inverse z transform is the one-sided, left-handed sequence

$$x(n) = -\frac{(n-1)!}{(n-k)!(k-1)!} z_0^{n-k} u(-n+k-1) = -\binom{n-1}{k-1} z_0^{n-k} u(-n+k-1) \tag{60}$$

- Using the above relations, it becomes straightforward to compute the inverse z transform of any function $X(z)$ that can be expressed as a ratio of polynomials provided that we first compute a partial-fraction expansion of $X(z)$.

Inverse z transform - Computation based on partial-fraction expansions

- If $X(z) = \frac{N(z)}{D(z)}$ has K distinct poles p_k, for $k = 1, 2, \ldots, K$, each of multiplicity m_k, then the partial-fraction expansion of $X(z)$ is as follows:

$$X(z) = \sum_{l=0}^{M-L} g_l z^l + \sum_{k=1}^{K} \sum_{i=1}^{m_k} \frac{c_{ki}}{(z - p_k)^i} \tag{61}$$

  where M and L are the degrees of the numerator and denominator of $X(z)$, respectively.

  - The coefficients g_l, for $l = 0, 1, \ldots, (M - L)$, can be obtained from the quotient of the polynomials $N(z)$ and $D(z)$ as follows

$$X(z) = \frac{N(z)}{D(z)} = \sum_{l=0}^{M-L} g_l z^l + \frac{C(z)}{D(z)} \tag{62}$$

    where the degree of $C(z)$ is smaller than the degree of $D(z)$. Clearly, if $M < L$, then $g_l = 0$, for all l.

– The coefficients c_{ki} are

$$c_{ki} = \frac{1}{(m_k - i)!} \left. \frac{d^{(m_k - i)}[(z - p_k)^{m_k} X(z)]}{dz^{m_k - i}} \right|_{z=p_k} \tag{63}$$

– In the case of a simple pole, c_{k1} is given by

$$c_{k1} = (z - p_k)X(z)|_{z=p_k} \tag{64}$$

- Since the z transform is linear and the inverse z transform of each of the terms $\frac{c_{ki}}{(z-p_k)^i}$ can be computed using either equation (59) or equation (60) (depending on whether the pole is inside or outside the region of convergence of $X(z)$), then the inverse z transform follows directly from equation (61).

Inverse z transform - Computation based on partial-fraction expansions

Example 2.5

Solve Example 2.3 using the partial-fraction expansion of $X(z)$.

Inverse z transform - Computation based on partial-fraction expansions

Solution

We form

$$X(z) = \frac{z^2}{(z-0.2)(z+0.8)} = g_0 + \frac{c_1}{z-0.2} + \frac{c_2}{z+0.8} \tag{65}$$

where

$$g_0 = \lim_{|z|\to\infty} X(z) = 1 \tag{66}$$

and, using equation (34), we find that

$$c_1 = \left.\frac{z^2}{z+0.8}\right|_{z=0.2} = (0.2)^2 \tag{67}$$

$$c_2 = \left.\frac{z^2}{z-0.2}\right|_{z=-0.8} = -(0.8)^2 \tag{68}$$

Inverse z transform - Computation based on partial-fraction expansions

such that

$$X(z) = 1 + \frac{(0.2)^2}{z - 0.2} - \frac{(0.8)^2}{z + 0.8} \tag{69}$$

- Since $X(z)$ is the z transform of the impulse response of a causal system, then we have that the terms in the above equation correspond to a right-handed, one-sided power series.
- Thus, the inverse z transforms of each term are:

Inverse z transform - Computation based on partial-fraction expansions

$$\mathcal{Z}^{-1}\{1\} = \delta(n) \tag{70}$$

$$\begin{aligned}
\mathcal{Z}^{-1}\left\{\frac{(0.2)^2}{z-0.2}\right\} &= (0.2)^2\mathcal{Z}^{-1}\left\{\frac{1}{z-0.2}\right\} \\
&= (0.2)^2\binom{n-1}{0}(0.2)^{n-1}u(n-1) \\
&= (0.2)^{n+1}u(n-1)
\end{aligned} \tag{71}$$

Inverse z transform - Computation based on partial-fraction expansions

$$
\begin{aligned}
\mathcal{Z}^{-1}\left\{\frac{-(0.8)^2}{z+0.8}\right\} &= -(0.8)^2\mathcal{Z}^{-1}\left\{\frac{1}{z+0.8}\right\} \\
&= -(0.8)^2\binom{n-1}{0}(-0.8)^{n-1}u(n-1) \\
&= -(-0.8)^{n+1}u(n-1) \qquad (72)
\end{aligned}
$$

Summing the three terms above (equations (70) to (72)), we have that the inverse z transform of $X(z)$ is

$$
\begin{aligned}
x(n) &= \delta(n) + (0.2)^{n+1}u(n-1) - (-0.8)^{n+1}u(n-1) \\
&= (0.2)^{n+1}u(n) - (-0.8)^{n+1}u(n) \qquad (73)
\end{aligned}
$$

△

Inverse z transform - Computation based on partial-fraction expansions

Example 2.6

Compute the right-handed, one-sided inverse z transform of

$$X(z) = \frac{1}{z^2 - 3z + 3} \tag{74}$$

Inverse z transform - Computation based on partial-fraction expansions

Solution

$X(z) = \frac{1}{z^2-3z+3}$

Applying partial fraction expansion to $X(z)$, we have that

$$X(z) = \frac{1}{(z-\sqrt{3}e^{j\frac{\pi}{6}})(z-\sqrt{3}e^{-j\frac{\pi}{6}})} = \frac{A}{z-\sqrt{3}e^{j\frac{\pi}{6}}} + \frac{B}{z-\sqrt{3}e^{-j\frac{\pi}{6}}} \quad (75)$$

where

$$A = \left.\frac{1}{z-\sqrt{3}e^{-j\frac{\pi}{6}}}\right|_{z=\sqrt{3}e^{j\frac{\pi}{6}}} = \frac{1}{\sqrt{3}e^{j\frac{\pi}{6}}-\sqrt{3}e^{-j\frac{\pi}{6}}} = \frac{1}{2j\sqrt{3}\sin\frac{\pi}{6}} = \frac{1}{j\sqrt{3}} \quad (76)$$

$$B = \left.\frac{1}{z-\sqrt{3}e^{j\frac{\pi}{6}}}\right|_{z=\sqrt{3}e^{-j\frac{\pi}{6}}} = \frac{1}{\sqrt{3}e^{-j\frac{\pi}{6}}-\sqrt{3}e^{j\frac{\pi}{6}}} = \frac{1}{-2j\sqrt{3}\sin\frac{\pi}{6}} = -\frac{1}{j\sqrt{3}} \quad (77)$$

and thus

$$X(z) = \frac{1}{j\sqrt{3}}\left[\frac{1}{z-\sqrt{3}e^{j\frac{\pi}{6}}} - \frac{1}{z-\sqrt{3}e^{-j\frac{\pi}{6}}}\right] \quad (78)$$

Inverse z transform - Computation based on partial-fraction expansions

From equation (59), we have that

$$\begin{aligned}
x(n) &= \frac{1}{j\sqrt{3}}\left[(\sqrt{3}e^{j\frac{\pi}{6}})^{n-1} - (\sqrt{3}e^{-j\frac{\pi}{6}})^{n-1}\right]u(n-1) \\
&= \frac{1}{j\sqrt{3}}\left[(\sqrt{3})^{n-1}e^{j(n-1)\frac{\pi}{6}} - (\sqrt{3})^{n-1}e^{-j(n-1)\frac{\pi}{6}}\right]u(n-1) \\
&= \frac{1}{j\sqrt{3}}(\sqrt{3})^{n-1}2j\sin\left[(n-1)\frac{\pi}{6}\right]u(n-1) \\
&= 2(\sqrt{3})^{n-2}\sin\left[(n-1)\frac{\pi}{6}\right]u(n-1) \qquad (79)
\end{aligned}$$

$\triangle$

Inverse z transform - Computation based on polynomial division

- Given $X(z) = \frac{N(z)}{D(z)}$, we can perform long division on the polynomials $N(z)$ and $D(z)$, and obtain that the coefficient of z^k corresponds to the value of $x(n)$ at $n = k$.
- One should note that this is possible only in the case of one-sided sequences.
  - If the sequence is right handed, then the polynomials should be a function of z.
  - If the sequence is left handed, the polynomials should be a function of z^{-1}.
- This is made clear in Examples 2.6 and 2.7.

Inverse z transform - Computation based on polynomial division

Example 2.7

Solve Example 2.3 using polynomial division.

Solution

Since $X(z)$ is the z transform of a right-handed, one-sided (causal) sequence, we can express it as the ratio of polynomials in z, that is

$$X(z) = \frac{z^2}{(z-0.2)(z+0.8)} = \frac{z^2}{z^2+0.6z-0.16} \tag{80}$$

Inverse z transform - Computation based on polynomial division

Then the division becomes

$$
\begin{array}{l|l}
z^2 + 0.6z - 0.16 & z^2 \\
\hline
1 - 0.6z^{-1} + 0.52z^{-2} - 0.408z^{-3} + \cdots & -z^2 - 0.6z + 0.16 \\
\cline{2-2}
 & \quad -0.6z + 0.16 \\
\cline{2-2}
 & \quad 0.6z + 0.36 - 0.096z^{-1} \\
\cline{2-2}
 & \qquad 0.52 - 0.096z^{-1} \\
\cline{2-2}
 & \qquad -0.52 - 0.312z^{-1} + 0.0832z^{-2} \\
\cline{2-2}
 & \qquad\quad -0.408z^{-1} + 0.0832z^{-2} \\
 & \qquad\qquad \vdots
\end{array}
$$

Inverse z transform - Computation based on polynomial division

and therefore

$$X(z) = 1 + (-0.6)z^{-1} + (0.52)z^{-2} + (-0.408)z^{-3} + \cdots \tag{81}$$

This is the same as saying that

$$x(n) = \begin{cases} 0, & \text{for } n < 0 \\ 1, -0.6, 0.52, -0.408, \ldots & \text{for } n = 0, 1, 2, \ldots \end{cases} \tag{82}$$

- The main difficulty with this method is to find a closed-form expression for $x(n)$.
- In the above case, we can check that indeed the above sequence corresponds to equation (52).

$\triangle$

Inverse z transform - Computation based on polynomial division

Example 2.8

Find the inverse z transform of $X(z)$ in Example 2.3 using polynomial division and supposing that the sequence $x(n)$ is left handed and one sided.

Solution

Since $X(z)$ is the z transform of a left-handed, one-sided sequence, we should express it as

$$X(z) = \frac{z^2}{(z-0.2)(z+0.8)} = \frac{1}{-0.16z^{-2} + 0.6z^{-1} + 1} \tag{83}$$

Then the division becomes

$$
\begin{array}{l|l}
-0.16z^{-2} + 0.6z^{-1} + 1 & 1 \\
\hline
-6.25z^2 - 23.4375z^3 - 126.953\,125z^4 - \ldots & -1 + 3.75z + 6.25z^2 \\
\cline{2-2}
 & \quad 3.75z + 6.25z^2 \\
\cline{2-2}
 & -3.75z + 14.0625z^2 + 23.4375z^3 \\
\cline{2-2}
 & \qquad 20.3125z^2 + 23.4375z^3 \\
 & \qquad\qquad \vdots
\end{array}
$$

yielding

$$X(z) = -6.25z^2 - 23.4375z^3 - 126.953\,125z^4 - \cdots \tag{84}$$

implying that

$$x(n) = \begin{cases} \ldots, -126.953\,125, -23.4375, -6.25, & \text{for } n = \ldots, -4, -3, -2 \\ 0, & \text{for } n > -2 \end{cases} \tag{85}$$

$\triangle$

Inverse z transform - Computation based on series expansion

- When a z transform is not expressed as a ratio of polynomials, we can try to perform its inversion using a Taylor series expansion around either $z^{-1} = 0$ or $z = 0$, depending on whether the convergence region includes $|z| = \infty$ or $z = 0$.
  - For right-handed, one-sided sequences, we use the expansion of $X(z)$ using the variable z^{-1} around $z^{-1} = 0$. The Taylor series expansion of $F(x)$ around $x = 0$ is given by

$$\begin{aligned} F(x) &= F(0) + x \left.\frac{dF}{dx}\right|_{x=0} + \frac{x^2}{2!} \left.\frac{d^2F}{dx^2}\right|_{x=0} + \frac{x^3}{3!} \left.\frac{d^3F}{dx^3}\right|_{x=0} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \left.\frac{d^nF}{dx^n}\right|_{x=0} \end{aligned} \tag{86}$$

- If we make $x = z^{-1}$, then the expansion above has the form of a z transform of a right-handed, one-sided sequence.

Inverse z transform - Computation based on series expansion

Example 2.9

Find the inverse z transform of

$$X(z) = \ln\left(\frac{1}{1 - z^{-1}}\right) \tag{87}$$

Consider the sequence as right handed and one sided.

Inverse z transform - Computation based on series expansion

Solution

Expanding $X(z)$ as in equation (86), using z^{-1} as the variable, we have that

$$X(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \tag{88}$$

One can see that the above series is convergent for $|z| > 1$ since, from equation (14),

$$\lim_{n\to\infty} \left| \frac{z^{-n}}{n} \right|^{\frac{1}{n}} = z^{-1} \lim_{n\to\infty} \left| \frac{1}{n} \right|^{\frac{1}{n}} = z^{-1} \tag{89}$$

Therefore the inverse z transform of $X(z)$ is, by inspection,

$$x(n) = \frac{1}{n} u(n-1) \tag{90}$$

$\triangle$

Inverse z transform - Computation based on series expansion

Example 2.10

(a) Calculate the right-hand unilateral inverse z transform related to the function described below:

$$X(z) = \text{arctan}(z^{-1}) \tag{91}$$

knowing that

$$\frac{d^k \,\text{arctan}(x)}{dx^k}(0) = \begin{cases} 0, & k = 2l \\ (-1)^{\frac{k-1}{2}} k!, & k = 2l+1 \end{cases} \tag{92}$$

(b) Could the resulting sequence represent the impulse response of a stable system? Why?

Inverse z transform - Computation based on series expansion

a) Given the series definition in equation (86) and equation (92), the series for the arctan function can be expressed as

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{(-1)^l x^{(2l+1)}}{2l+1} + \cdots \tag{93}$$

and thus

$$\arctan(z^{-1}) = z^{-1} - \frac{z^{-3}}{3} + \frac{z^{-5}}{5} + \cdots + \frac{(-1)^l z^{-(2l+1)}}{2l+1} \cdots \tag{94}$$

As a result the corresponding time-domain sequence is given by

$$u(k) = \begin{cases} 0, & k = 2l \\ \dfrac{(-1)^{\frac{k-1}{2}}}{k}, & k = 2l+1 \end{cases} \tag{95}$$

b) For a sequence $h(n)$ to represent an impulse of a stable system, then it should be absolutely summable. From equation (29) this is equivalent to

$$\lim_{n\to\infty} |h(n)|^{\frac{1}{n}} < 1 \tag{96}$$

In our case, from equation (95),

$$\lim_{k\to\infty} |u(k)|^{\frac{1}{k}} = \lim_{k\to\infty} \left|\frac{(-1)^{\frac{k-1}{2}}}{k}\right|^{\frac{1}{k}} = \lim_{k\to\infty} \left|\frac{1}{k}\right|^{\frac{1}{k}} = 1 \tag{97}$$

- In this case, this stability test is inconclusive.
- However,

$$\sum_{k=1}^{\infty} \frac{1}{k} = \sum_{l=1}^{\infty} \left(\frac{1}{2l-1} + \frac{1}{2l}\right) < 2\sum_{l=1}^{\infty} \frac{1}{2l-1}$$

- Therefore, since $\sum_{k=1}^{\infty} \frac{1}{k}$ is not bounded, then $\sum_{n=0}^{\infty} |h(n)| = \sum_{l=1}^{\infty} \frac{1}{2l-1}$ is also not bounded, and the system is not stable.

$\triangle$

Properties of the z transform

1. Linearity:

Given two sequences $x_1(n)$ and $x_2(n)$ and two arbitrary constants k_1 and k_2 such that $x(n) = k_1x_1(n) + k_2x_2(n)$, then

$$X(z) = k_1X_1(z) + k_2X_2(z) \tag{98}$$

where the region of convergence of $X(z)$ is the intersection of the regions of convergence of $X_1(z)$ and $X_2(z)$.

Proof:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} (k_1x_1(n) + k_2x_2(n))z^{-n} \\ &= k_1 \sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + k_2 \sum_{n=-\infty}^{\infty} x_2(n)z^{-n} \\ &= k_1X_1(z) + k_2X_2(z) \end{aligned} \tag{99}$$

$\square$

Properties of the z transform

2. Time-reversal:

$$x(-n) \longleftrightarrow X(z^{-1}) \tag{100}$$

where, if the region of convergence of $X(z)$ is $r_1 < |z| < r_2$, then the region of convergence of $\mathcal{Z}\{x(-n)\}$ is $\frac{1}{r_2} < |z| < \frac{1}{r_1}$.

Proof:

$$\mathcal{Z}\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)z^{-n} = \sum_{m=-\infty}^{\infty} x(m)z^{m} = \sum_{m=-\infty}^{\infty} x(m)(z^{-1})^{-m} = X(z^{-1}) \tag{101}$$

implying that the region of convergence of $\mathcal{Z}\{x(-n)\}$ is $r_1 < |z^{-1}| < r_2$, which is equivalent to $\frac{1}{r_2} < |z| < \frac{1}{r_1}$. □

Properties of the z transform

3. Time-shift theorem

$$x(n+l) \longleftrightarrow z^{l}X(z) \tag{102}$$

where l is an integer. The region of convergence of $\mathcal{Z}\{x(n+l)\}$ is the same as the region of convergence of $X(z)$, except for the possible inclusion or exclusion of the regions $z=0$ and $|z|=\infty$.

Properties of the z transform

Proof:

By definition

$$\mathcal{Z}\{x(n+l)\} = \sum_{n=-\infty}^{\infty} x(n+l)z^{-n} \tag{103}$$

Making the change of variables $m = n + l$, we have that

$$\mathcal{Z}\{x(n+l)\} = \sum_{m=-\infty}^{\infty} x(m)z^{-(m-l)} = z^{l} \sum_{m=-\infty}^{\infty} x(m)z^{-m} = z^{l}X(z) \tag{104}$$

noting that the multiplication by z^{l} can either introduce or exclude poles at $z = 0$ and $|z| = \infty$. $\square$

Properties of the z transform

4. Multiplication by an exponential

$$\alpha^{-n}x(n) \longleftrightarrow X(\alpha z) \tag{105}$$

where, if the region of convergence of $X(z)$ is $r_1 < |z| < r_2$, then the region of convergence of $\mathcal{Z}\{\alpha^{-n}x(n)\}$ is $\frac{r_1}{|\alpha|} < |z| < \frac{r_2}{|\alpha|}$.

Proof:

$$\mathcal{Z}\{\alpha^{-n}x(n)\} = \sum_{n=-\infty}^{\infty} \alpha^{-n}x(n)z^{-n} = \sum_{n=-\infty}^{\infty} x(n)(\alpha z)^{-n} = X(\alpha z) \tag{106}$$

where the summation converges for $r_1 < |\alpha z| < r_2$, which is equivalent to $\frac{r_1}{|\alpha|} < |z| < \frac{r_2}{|\alpha|}$.

$\square$

Properties of the z transform

5. Complex differentiation

$$nx(n) \longleftrightarrow -z\frac{\mathrm{d}X(z)}{\mathrm{d}z} \tag{107}$$

where the region of convergence of $\mathcal{Z}\{nx(n)\}$ is the same as the one of $X(z)$, that is $r_1 < |z| < r_2$.

Properties of the z transform

Proof:

$$
\begin{aligned}
\mathcal{Z}\{nx(n)\} &= \sum_{n=-\infty}^{\infty} nx(n)z^{-n} \\
&= z \sum_{n=-\infty}^{\infty} nx(n)z^{-n-1} \\
&= -z \sum_{n=-\infty}^{\infty} x(n)\left(-nz^{-n-1}\right) \\
&= -z \sum_{n=-\infty}^{\infty} x(n)\frac{\mathrm{d}}{\mathrm{d}z}\{z^{-n}\} \\
&= -z\frac{\mathrm{d}X(z)}{\mathrm{d}z} \qquad (108)
\end{aligned}
$$

Properties of the z transform

From equations (16) and (17), we have that, if the region of convergence of $X(z)$ is $r_1 < |z| < r_2$, then

$$r_1 = \lim_{n\to\infty} |x(n)|^{\frac{1}{n}} \tag{109}$$

$$r_2 = \lim_{n\to-\infty} |x(n)|^{\frac{1}{n}} \tag{110}$$

Therefore, if the region of convergence of $\mathcal{Z}\{nx(n)\}$ is given by $r_1' < |z| < r_2'$, then

$$r_1' = \lim_{n\to\infty} |nx(n)|^{\frac{1}{n}} = \lim_{n\to\infty} |n|^{\frac{1}{n}} \lim_{n\to\infty} |x(n)|^{\frac{1}{n}} = \lim_{n\to\infty} |x(n)|^{\frac{1}{n}} = r_1 \tag{111}$$

$$r_2' = \lim_{n\to-\infty} |nx(n)|^{\frac{1}{n}} = \lim_{n\to-\infty} |n|^{\frac{1}{n}} \lim_{n\to-\infty} |x(n)|^{\frac{1}{n}} = \lim_{n\to-\infty} |x(n)|^{\frac{1}{n}} = r_2 \tag{112}$$

implying that the region of convergence of $\mathcal{Z}\{nx(n)\}$ is the same as the one of $X(z)$.

$\square$

Properties of the z transform

6. Complex conjugation

$$x^*(n) \longleftrightarrow X^*(z^*) \tag{113}$$

The regions of convergence of $X(z)$ and $\mathcal{Z}\{x^*(n)\}$ being the same.

Properties of the z transform

Proof

$$\begin{aligned}
\mathcal{Z}\{x^*(n)\} &= \sum_{n=-\infty}^{\infty} x^*(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \left[x(n)\left(z^*\right)^{-n}\right]^* \\
&= \left[\sum_{n=-\infty}^{\infty} x(n)\left(z^*\right)^{-n}\right]^* \\
&= X^*(z^*)
\end{aligned} \tag{114}$$

from which it follows trivially that the region of convergence of $\mathcal{Z}\{x^*(n)\}$ is the same as the one of $X(z)$. $\square$

Properties of the z transform

7. Real and imaginary sequences:

$$\text{Re}\{x(n)\} \longleftrightarrow \frac{1}{2}\left(X(z) + X^*\left(z^*\right)\right) \tag{115}$$

$$\text{Im}\{x(n)\} \longleftrightarrow \frac{1}{2\text{j}}\left(X(z) - X^*\left(z^*\right)\right) \tag{116}$$

where Re$\{x(n)\}$ and Im$\{x(n)\}$ are the real and imaginary parts of the sequence $x(n)$, respectively. The regions of convergence of $\mathcal{Z}\{\text{Re}\{x(n)\}\}$ and $\mathcal{Z}\{\text{Im}\{x(n)\}\}$ contain the ones of $X(z)$.

Properties of the z transform

Proof:

$$\mathcal{Z}\{\mathrm{Re}\{x(n)\}\} = \mathcal{Z}\left\{\frac{1}{2}\left(x(n)+x^*(n)\right)\right\} = \frac{1}{2}\left(X(z)+X^*\left(z^*\right)\right) \tag{117}$$

$$\mathcal{Z}\{\mathrm{Im}\{x(n)\}\} = \mathcal{Z}\left\{\frac{1}{2\mathrm{j}}\left(x(n)-x^*(n)\right)\right\} = \frac{1}{2\mathrm{j}}\left(X(z)-X^*\left(z^*\right)\right) \tag{118}$$

with the respective regions of convergence following trivially from the above expressions.

□

Properties of the z transform

8. Initial value theorem

If $x(n) = 0$, for $n < 0$, then

$$x(0) = \lim_{z \to \infty} X(z) \tag{119}$$

Proof:

If $x(n) = 0$, for $n < 0$, then

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \sum_{n=0}^{\infty} x(n) z^{-n} = \sum_{n=0}^{\infty} \lim_{z \to \infty} x(n) z^{-n} = x(0) \tag{120}$$

$\square$

Properties of the z transform

9. Convolution theorem

$$x_1(n) * x_2(n) \longleftrightarrow X_1(z)X_2(z) \tag{121}$$

- The region of convergence of $\mathcal{Z}\{x_1(n) * x_2(n)\}$ is the intersection of the regions of convergence of $X_1(z)$ and $X_2(z)$.
- If a pole of $X_1(z)$ is canceled by a zero of $X_2(z)$, or vice versa, then the region of convergence of $\mathcal{Z}\{x_1(n) * x_2(n)\}$ can be larger than the ones of both $X_1(z)$ and $X_2(z)$.

Properties of the z transform

Proof:

$$
\begin{aligned}
\mathcal{Z}\{x_1(n) * x_2(n)\} &= \mathcal{Z}\left\{\sum_{l=-\infty}^{\infty} x_1(l)x_2(n-l)\right\} \\
&= \sum_{n=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty} x_1(l)x_2(n-l)\right)z^{-n} \\
&= \sum_{l=-\infty}^{\infty} x_1(l) \sum_{n=-\infty}^{\infty} x_2(n-l)z^{-n} \\
&= \left(\sum_{l=-\infty}^{\infty} x_1(l)z^{-l}\right)\left(\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}\right) \\
&= X_1(z)X_2(z) \qquad (122)
\end{aligned}
$$

$\square$

Properties of the z transform

10. Product of two sequences

$$x_1(n)x_2(n) \longleftrightarrow \frac{1}{2\pi j}\oint_{C_1} X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv = \frac{1}{2\pi j}\oint_{C_2} X_1\left(\frac{z}{v}\right)X_2(v)v^{-1}dv \tag{123}$$

where C_1 is a contour contained in the intersection of the regions of convergence of $X_1(v)$ and $X_2\left(\frac{z}{v}\right)$, and C_2 is a contour contained in the intersection of the regions of convergence of $X_1\left(\frac{z}{v}\right)$ and $X_2(v)$. Both C_1 and C_2 are assumed to be counterclockwise oriented.

If the region of convergence of $X_1(z)$ is $r_1 < |z| < r_2$ and the region of convergence of $X_2(z)$ is $r_1' < |z| < r_2'$, then the region of convergence of $\mathcal{Z}\{x_1(n)x_2(n)\}$ is

$$r_1 r_1' < |z| < r_2 r_2' \tag{124}$$

Properties of the z transform

Proof:

By expressing $x_2(n)$ as a function of its z transform, $X_2(z)$ (equation (36)), then changing the order of the integration and summation, and using the definition of the z transform, we have that

$$
\begin{aligned}
\mathcal{Z}\{x_1(n)x_2(n)\} &= \sum_{n=-\infty}^{\infty} x_1(n)x_2(n)z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x_1(n)\left[\frac{1}{2\pi j}\oint_{C_2} X_2(v)v^{(n-1)}dv\right]z^{-n} \\
&= \frac{1}{2\pi j}\oint_{C_2} \sum_{n=-\infty}^{\infty} x_1(n)z^{-n}v^{(n-1)}X_2(v)dv \\
&= \frac{1}{2\pi j}\oint_{C_2} \left[\sum_{n=-\infty}^{\infty} x_1(n)\left(\frac{v}{z}\right)^n\right]X_2(v)v^{-1}dv \\
&= \frac{1}{2\pi j}\oint_{C_2} X_1\left(\frac{z}{v}\right)X_2(v)v^{-1}dv
\end{aligned}
\qquad (125)
$$

Properties of the z transform

If the region of convergence of $X_1(z)$ is $r_1 < |z| < r_2$, then the region of convergence of $X_1\left(\frac{z}{v}\right)$ is

$$r_1 < \frac{|z|}{|v|} < r_2 \tag{126}$$

which is equivalent to

$$\frac{|z|}{r_2} < |v| < \frac{|z|}{r_1} \tag{127}$$

In addition, if the region of convergence of $X_2(v)$ is $r'_1 < |v| < r'_2$, then the contour C_2 must lie in the intersection of the two regions of convergence, that is, C_2 must be contained in the region

$$\max\left\{\frac{|z|}{r_2}, r'_1\right\} < |v| < \min\left\{\frac{|z|}{r_1}, r'_2\right\} \tag{128}$$

Properties of the z transform

Therefore, we must have

$$\min\left\{\frac{|z|}{r_1}, r_2'\right\} > \max\left\{\frac{|z|}{r_2}, r_1'\right\} \tag{129}$$

which holds if $r_1 r_1' < |z| < r_2 r_2'$.

$\square$

- Equation (123) is also known as the complex convolution theorem .
- Although at first it does not have the form of a convolution, if we express $z = \rho_1 e^{j\theta_1}$ and $v = \rho_2 e^{j\theta_2}$ in polar form, then it can be rewritten as

$$\mathcal{Z}\{x_1(n)x_2(n)\}\big|_{z=\rho_1 e^{j\theta_1}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1\left(\frac{\rho_1}{\rho_2}e^{j(\theta_1-\theta_2)}\right) X_2\left(\rho_2 e^{j\theta_2}\right) d\theta_2 \tag{130}$$

which has the form of a convolution in θ_1.

Properties of the z transform

11. Parseval's theorem

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv \tag{131}$$

where x^* denotes the complex conjugate of x and C is a contour contained in the intersection of the convergence regions of $X_1(v)$ and $X_2^*\left(\frac{1}{v^*}\right)$.

Properties of the z transform

Proof:

We begin by noting that

$$\sum_{n=-\infty}^{\infty} x(n) = X(z)\big|_{z=1} \tag{132}$$

Therefore

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \mathcal{Z}\{x_1(n)x_2^*(n)\}\big|_{z=1} \tag{133}$$

By using equation (123) and the complex conjugation property in equation (113), we have that the above equation implies that

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv \tag{134}$$

$\square$

Properties of the z transform

Table of basic z transforms

- The following table contains some commonly used sequences and their corresponding z transforms, along with their regions of convergence.
  - Although it only contains the z transforms for right-handed sequences, the results for left-handed sequences can be readily obtained by making $u(n) = x(-n)$ and applying the time-reversal property.

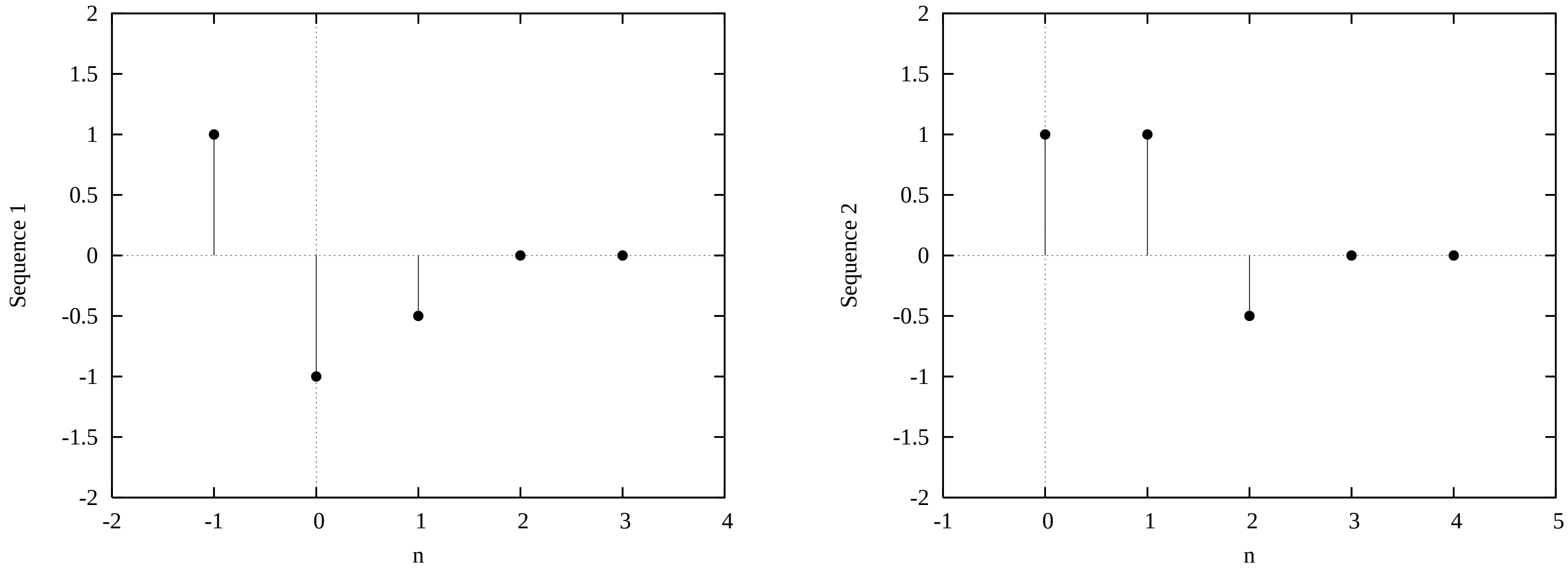
$x(n)$	$X(z)$	Convergence region
$\delta(n)$	1	$z \in \mathbb{C}$
$u(n)$	$\frac{z}{(z-1)}$	$\lvert z\rvert > 1$
$(-a)^n u(n)$	$\frac{z}{(z+a)}$	$\lvert z\rvert > a$
$nu(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$n^2 u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$e^{an} u(n)$	$\frac{z}{(z-e^a)}$	$\lvert z\rvert > \lvert e^a\rvert$
$\binom{n-1}{k-1} e^{a(n-k)} u(n-k)$	$\frac{1}{(z-e^a)^k}$	$\lvert z\rvert > \lvert e^a\rvert$

$\cos(\omega n)u(n)$	$\dfrac{z[z-\cos(\omega)]}{z^2-2z\cos(\omega)+1}$	$\lvert z\rvert > 1$
$\sin(\omega n)u(n)$	$\dfrac{z\sin(\omega)}{z^2-2z\cos(\omega)+1}$	$\lvert z\rvert > 1$
$\dfrac{1}{n}u(n-1)$	$\ln\left(\dfrac{z}{z-1}\right)$	$\lvert z\rvert > 1$
$\sin(\omega n+\theta)u(n)$	$\dfrac{z^2\sin(\theta)+z\sin(\omega-\theta)}{z^2-2z\cos(\omega)+1}$	$\lvert z\rvert > 1$
$e^{an}\cos(\omega n)u(n)$	$\dfrac{z^2-ze^a\cos(\omega)}{z^2-2ze^a\cos(\omega)+e^{2a}}$	$\lvert z\rvert > \lvert e^a\rvert$
$e^{an}\sin(\omega n)u(n)$	$\dfrac{ze^a\sin(\omega)}{z^2-2ze^a\cos(\omega)+e^{2a}}$	$\lvert z\rvert > \lvert e^a\rvert$

Properties of the z transform

Example 2.11

Calculate the linear convolution of the sequences in the figure below using the z transform. Plot the resulting sequence.



(a) (b)

Properties of the z transform

Solution

From the above figure, we can see that the z transforms of the two sequences are

$$X_1(z) = z - 1 - \frac{1}{2}z^{-1}; \quad X_2(z) = 1 + z^{-1} - \frac{1}{2}z^{-2} \tag{135}$$

According to property 2.9, the z transform of the convolution is the product of the z transforms, and then

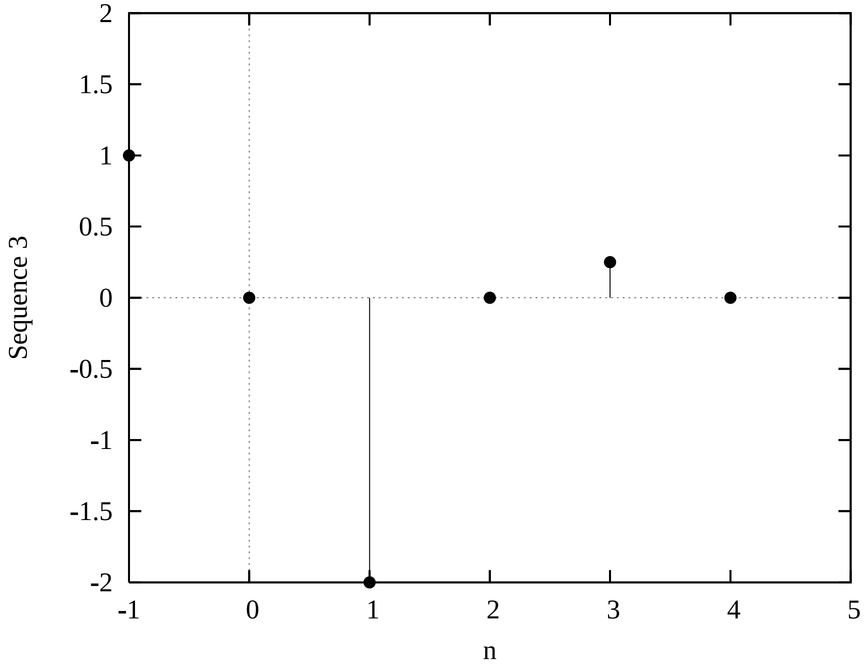
$$\begin{aligned} Y(z) = X_1(z)X_2(z) &= \left(z - 1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1} - \frac{1}{2}z^{-2}\right) \\ &= z + 1 - \frac{1}{2}z^{-1} - 1 - z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2} + \frac{1}{4}z^{-3} \\ &= z - 2z^{-1} + \frac{1}{4}z^{-3} \end{aligned} \tag{136}$$

In the time domain the result is

$$y(-1) = 1,\ y(0) = 0,\ y(1) = -2,\ y(2) = 0,\ y(3) = \frac{1}{4},\ y(4) = 0,\ \cdots \tag{137}$$

Properties of the z transform

which is depicted in the figure below.



$\triangle$

Properties of the z transform

Example 2.12

If $X(z)$ is the z transform of the sequence

$$x(0) = a_0,\ x(1) = a_1,\ x(2) = a_2,\ \ldots,\ x(i) = a_i,\ \ldots, \tag{138}$$

determine the z transform of the sequence

$$y(-2) = a_0,\ y(-3) = -a_1 b,\ y(-4) = -2a_2 b^2,\ \ldots,\ y(-i-2) = -ia_i b^i,\ \ldots \tag{139}$$

as a function of $X(z)$.

Properties of the z transform

Solution

We have that $X(z)$ and $Y(z)$ are

$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_i z^{-i} + \cdots \quad (140)$$

$$Y(z) = a_0 z^2 - a_1 b z^3 - 2a_2 b^2 z^4 - \cdots - i a_i b^i z^{i+2} - \cdots \quad (141)$$

We start solving this problem by using property 2.5, that is, if $x_1(n) = nx(n)$, then

$$\begin{aligned} X_1(z) &= -z\frac{dX(z)}{dz} \\ &= -z\left(-a_1 z^{-2} - 2a_2 z^{-3} - 3a_3 z^{-4} - \cdots - i a_i z^{-i-1} - \cdots\right) \\ &= a_1 z^{-1} + 2a_2 z^{-2} + 3a_3 z^{-3} + \cdots + i a_i z^{-i} + \cdots \end{aligned} \quad (142)$$

The next step is to create $x_2(n) = b^n x_1(n)$. From property 2.4,

$$X_2(z) = X_1\left(\frac{z}{b}\right) = a_1 b z^{-1} + 2a_2 b^2 z^{-2} + 3a_3 b^3 z^{-3} + \cdots + i a_i b^i z^{-i} + \cdots \quad (143)$$

Properties of the z transform

We then generate $X_3(z) = z^{-2}X_2(z)$ as follows

$$X_3(z) = a_1bz^{-3} + 2a_2b^2z^{-4} + 3a_3b^3z^{-5} + \cdots + ia_ib^iz^{-i-2} + \cdots \quad (144)$$

and make $X_4(z) = X_3(z^{-1})$ such that

$$X_4(z) = a_1bz^3 + 2a_2b^2z^4 + 3a_3b^3z^5 + \cdots + ia_ib^iz^{i+2} + \cdots \quad (145)$$

The transform $Y(z)$ of the desired sequence is then

$$\begin{aligned} Y(z) &= a_0z^2 - a_1bz^3 - 2a_2b^2z^4 - 3a_3b^3z^5 - \cdots - ia_ib^iz^{i+2} - \cdots \\ &= a_0z^2 - X_4(z) \end{aligned} \quad (146)$$

Properties of the z transform

Using equations (142) to (146) we can express the desired result as

$$
\begin{aligned}
Y(z) &= a_0 z^2 - X_4(z) \\
&= a_0 z^2 - X_3(z^{-1}) \\
&= a_0 z^2 - z^2 X_2(z^{-1}) \\
&= a_0 z^2 - z^2 X_1\left(\frac{z^{-1}}{b}\right) \\
&= a_0 z^2 - z^2 \left(-z\frac{dX(z)}{dz}\right)\bigg|_{z=\frac{z^{-1}}{b}} \\
&= a_0 z^2 + \frac{z}{b}\,\frac{dX(z)}{dz}\bigg|_{z=\frac{z^{-1}}{b}}
\end{aligned}
\tag{147}
$$

$\triangle$

Transfer functions

- As we have seen in Chapter 1, a discrete-time linear system can be characterized by a difference equation. In this section, we show how the z transform can be used to solve difference equations, and therefore characterize linear systems.
- The general form of a difference equation associated with a linear system is given by

$$\sum_{i=0}^{N} a_i y(n-i) - \sum_{l=0}^{M} b_l x(n-l) = 0 \tag{148}$$

Applying the z transform on both sides and using the linearity property, we find that

$$\sum_{i=0}^{N} a_i \mathcal{Z}\{y(n-i)\} - \sum_{l=0}^{M} b_l \mathcal{Z}\{x(n-l)\} = 0 \tag{149}$$

Applying the time-shift theorem, we obtain

$$\sum_{i=0}^{N} a_i z^{-i} Y(z) - \sum_{l=0}^{M} b_l z^{-l} X(z) = 0 \tag{150}$$

Transfer functions

- Therefore, for a linear system, given $X(z)$, the z-transform representation of the input, and the coefficients of its difference equation, we can use equation (150) to find $Y(z)$, the z transform of the output.
- Applying the inverse z-transform relation in equation (36), the output $y(n)$ can be computed for all n.
  - One should note that, since equation (150) uses z transforms, which consist of summations for $-\infty < n < \infty$, then the system has to be describable by a difference equation for $-\infty < n < \infty$.
  - This is the case only for initially relaxed systems, that is, systems that produce no output if the input is zero for $-\infty < n < \infty$.
  - In our case, this does not restrict the applicability of equation (150), because we are only interested in linear systems, which, as seen in Chapter 1, must be initially relaxed.

Transfer functions

- Making $a_0 = 1$, without loss of generality, we then define

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\displaystyle\sum_{l=0}^{M} b_l z^{-l}}{\displaystyle 1 + \sum_{i=1}^{N} a_i z^{-i}} \tag{151}$$

as the transfer function of the system relating the output $Y(z)$ to the input $X(z)$.

- Applying the convolution theorem to equation (151), we have that

$$Y(z) = H(z)X(z) \quad \longleftrightarrow \quad y(n) = h(n) * x(n) \tag{152}$$

that is, the transfer function of the system is the z transform of its impulse response.

  – Indeed, equations (150) and (151) are the equivalent expressions of the convolution sum in the z-transform domain when the system is described by a difference equation.

Transfer functions

- Equation (151) gives the transfer function for the general case of recursive (IIR) filters.
  - For nonrecursive (FIR) filters, all $a_i = 0$, for $i = 1, 2, \ldots, N$, and the transfer function simplifies to
$$H(z) = \sum_{l=0}^{M} b_l z^{-l} \tag{153}$$
- Transfer functions are widely used to characterize discrete-time linear systems.
- We can describe a transfer function through its poles p_i and zeros z_l, yielding
$$H(z) = H_0 \frac{\prod_{l=1}^{M}(1 - z^{-1} z_l)}{\prod_{i=1}^{N}(1 - z^{-1} p_i)} = H_0 z^{N-M} \frac{\prod_{l=1}^{M}(z - z_l)}{\prod_{i=1}^{N}(z - p_i)} \tag{154}$$

Transfer functions

- As discussed above, for a causal stable system, the convergence region of the z transform of its impulse response must include the unit circle.
  - Indeed, this result is more general as for *any* stable system, the convergence region of its transfer function must necessarily include the unit circle.
  - We can show this by noting that, for z_0 on the unit circle ($|z_0| = 1$), we have

$$|H(z_0)| = \left| \sum_{n=-\infty}^{\infty} z_0^{-n} h(n) \right| \leq \sum_{n=-\infty}^{\infty} |z_0^{-n} h(n)| = \sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad (155)$$

    which implies that $H(z)$ converges on the unit circle.

- Since in the case of a causal system the convergence region of the transfer function is $|z| > r_1$, then all the poles of a causal stable system must be inside the unit circle.

- For a noncausal stable system, since the convergence region is $|z| < r_2$, then all its poles must be outside the unit circle, with the possible exception of a pole at $z = 0$.

Stability in the z domain

- In this section, we present a method for determining whether the roots of a polynomial are inside the unit circle of the complex plane. This method can be used to assess the BIBO stability of a causal discrete-time system.
- Given an Nth-order polynomial in z

$$D(z) = a_N + a_{N-1}z + \cdots + a_0 z^N \tag{156}$$

with $a_0 > 0$, we have that a necessary and sufficient condition for its zeros (the poles of the given transfer function) to be inside the unit circle of the z plane is given by the following algorithm:

(i) Make $D_0(z) = D(z)$.

(ii) For $k = 0, 1, \ldots, (N-2)$:

(a) Form the polynomial $D_k^i(z)$ such that

$$D_k^i(z) = z^{N+k} D_k(z^{-1}) \tag{157}$$

(b) Compute α_k and $D_{k+1}(z)$, such that

$$D_k(z) = \alpha_k D_k^i(z) + D_{k+1}(z) \tag{158}$$

where the terms in z^j, for $j = 0, 1, \ldots, k$, of $D_{k+1}(z)$ are zero.

- In other words, $D_{k+1}(z)$ is the remainder of the division of $D_k(z)$ by $D_k^i(z)$, when it is performed upon the terms of smallest degree.

(iii) All the roots of $D(z)$ are inside the unit circle if the following conditions are satisfied:

- $D(1) > 0$
- $D(-1) > 0$ for N even and $D(-1) < 0$ for N odd
- $|\alpha_k| < 1$, for $k = 0, 1, \ldots, (n-2)$.

Stability in the z domain

Example 2.13

Test the stability of the causal system whose transfer function possesses the denominator polynomial $D(z) = 8z^4 + 4z^3 + 2z^2 - z - 1$.

Solution

If $D(z) = 8z^4 + 4z^3 + 2z^2 - z - 1$, then we have that

- $D(1) = 12 > 0$
- $N = 4$ is even and $D(-1) = 6 > 0$
- Computation of α_0, α_1, and α_2:

$$
\begin{aligned}
D_0(z) &= D(z) = 8z^4 + 4z^3 + 2z^2 - z - 1 && (159)\\
D_0^i(z) &= z^4(8z^{-4} + 4z^{-3} + 2z^{-2} - z^{-1} - 1)\\
&= 8 + 4z + 2z^2 - z^3 - z^4 && (160)
\end{aligned}
$$

Since $D_0(z) = \alpha_0 D_0^i(z) + D_1(z)$,

$$
\begin{array}{l|l}
8 + 4z + 2z^2 - z^3 - z^4 & -1 - z + 2z^2 + 4z^3 + 8z^4 \\
\hline
-\frac{1}{8} & \underline{+1 + \frac{1}{2}z + \frac{1}{4}z^2 - \frac{1}{8}z^3 - \frac{1}{8}z^4} \\
 & \qquad -\frac{1}{2}z + \frac{9}{4}z^2 + \frac{31}{8}z^3 + \frac{63}{8}z^4
\end{array}
$$

then $\alpha_0 = -\frac{1}{8}$ and

$$D_1(z) = -\frac{1}{2}z + \frac{9}{4}z^2 + \frac{31}{8}z^3 + \frac{63}{8}z^4 \tag{161}$$

$$
\begin{aligned}
D_1^i(z) &= z^{4+1}\left(-\frac{1}{2}z^{-1} + \frac{9}{4}z^{-2} + \frac{31}{8}z^{-3} + \frac{63}{8}z^{-4}\right) \\
&= -\frac{1}{2}z^4 + \frac{9}{4}z^3 + \frac{31}{8}z^2 + \frac{63}{8}z
\end{aligned} \tag{162}
$$

Since $D_1(z) = \alpha_1 D_1^i(z) + D_2(z)$,

$$
\begin{array}{l|l}
\frac{63}{8}z + \frac{31}{8}z^2 + \frac{9}{4}z^3 - \frac{1}{2}z^4 & -\frac{1}{2}z + \frac{9}{4}z^2 + \frac{31}{8}z^3 + \frac{63}{8}z^4 \\
\hline
-\frac{4}{63} & +\frac{1}{2}z + \frac{31}{126}z^2 + \frac{1}{7}z^3 - \frac{2}{63}z^4 \\
\cline{2-2}
 & 2.496z^2 + 4.018z^3 + 7.844z^4
\end{array}
$$

then $\alpha_1 = -\frac{4}{63}$ and

$$D_2(z) = 2.496z^2 + 4.018z^3 + 7.844z^4 \tag{163}$$

$$
\begin{aligned}
D_2^i(z) &= z^{4+2}(2.496z^{-2} + 4.018z^{-3} + 7.844z^{-4}) \\
&= 2.496z^4 + 4.018z^3 + 7.844z^2
\end{aligned} \tag{164}
$$

Since $D_2(z) = \alpha_2 D_2^i(z) + D_3(z)$, we have that $\alpha_2 = \frac{2.496}{7.844} = 0.3182$.

Stability in the z domain

Thus,

$$|\alpha_0| = \frac{1}{8} < 1, \quad |\alpha_1| = \frac{4}{63} < 1, \quad |\alpha_2| = 0.3182 < 1 \tag{165}$$

and, consequently, the system is stable.

$\triangle$

Example 2.14

Given the polynomial $D(z) = z^2 + az + b$, determine the choices for a and b such this polynomial represents the denominator of a stable discrete-time system. Plot $a \times b$ highlighting the stability region.

Stability in the z domain

Solution

Since the order of the polynomial is even,

$$D(1) > 0 \quad \Rightarrow \quad 1 + a + b > 0 \Rightarrow a + b > -1 \tag{166}$$

$$D(-1) > 0 \quad \Rightarrow \quad 1 - a + b > 0 \Rightarrow -a + b > -1 \tag{167}$$

Since $N - 2 = 0$, there exists only α_0. So

$$D_0(z) = z^2 + az + b \tag{168}$$

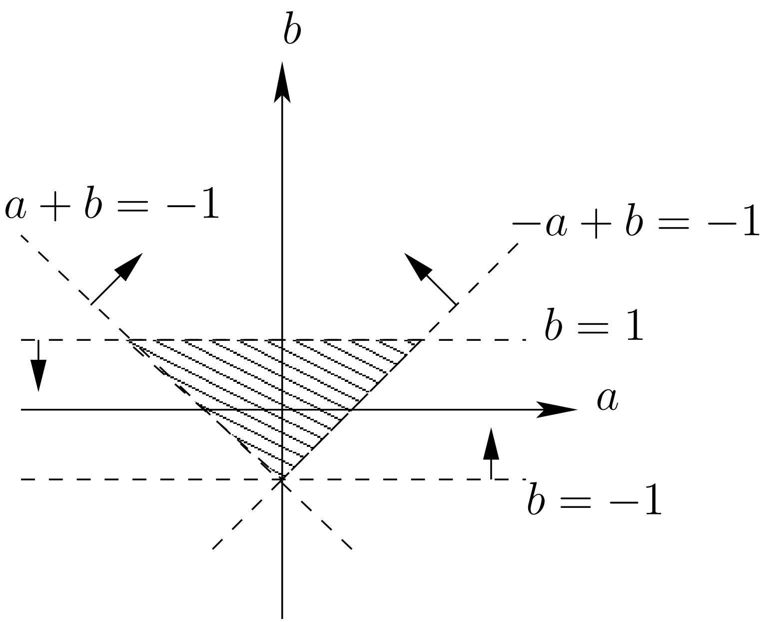
$$D_0^i(z) = z^2(z^{-2} + az^{-1} + b) = 1 + az + bz^2 \tag{169}$$

$$\begin{array}{l|l}
1 + az + bz^2 & b + az + z^2 \\
\cline{1-1}
b & -b - abz - b^2 z^2 \\
\cline{2-2}
 & (1-b)az + (1-b^2)z^2
\end{array}$$

Stability in the z domain

then $|\alpha_0| = |b| < 1$. As such, the conditions are:

$$\left.\begin{array}{l} a + b > -1 \\ -a + b > -1 \\ |b| < 1 \end{array}\right\} \tag{170}$$



△

Frequency response

- As mentioned before, when an exponential z^n is input to a linear system with impulse response $h(n)$, then its output is an exponential $H(z)z^n$.
- Since, as seen above, stable systems are guaranteed to have the z transform on the unit circle, it is natural to try to characterize these systems on the unit circle.
  - Complex numbers on the unit circle are of the form $z = e^{j\omega}$, for $0 \leq \omega < 2\pi$.
  - This implies that the corresponding exponential sequence is a sinusoid $x(n) = e^{j\omega n}$.
  - Therefore, we can state that if we input a sinusoid $x(n) = e^{j\omega n}$ to a linear system, then its output is also a sinusoid of the same frequency, that is

$$y(n) = H\left(e^{j\omega}\right)e^{j\omega n} \tag{171}$$

- If $H(e^{j\omega})$ is a complex number with magnitude $|H(e^{j\omega})|$ and phase $\Theta(\omega)$, then $y(n)$ can be expressed as

$$y(n) = H(e^{j\omega})e^{j\omega n} = |H(e^{j\omega})|e^{j\Theta(\omega)}e^{j\omega n} = |H(e^{j\omega})|e^{j\omega n + j\Theta(\omega)} \tag{172}$$

Frequency response

- This indicates that the output of a linear system for a sinusoidal input is a sinusoid of the same frequency, but with its amplitude multiplied by $|H(e^{j\omega})|$ and phase increased by $\Theta(\omega)$.
  - Thus, when we characterize a linear system in terms of $H(e^{j\omega})$, we are in fact specifying, for every frequency ω, the effect that the linear system has on the input signal's amplitude and phase.
- Therefore, $H(e^{j\omega})$ is commonly known as the **frequency response of the system**.
- It is important to emphasize that $H(e^{j\omega})$ is the value of the z transform $H(z)$ on the unit circle.
  - This implies that we need to specify it only for one turn around the unit circle, that is, for $0 \leq \omega < 2\pi$.
  - Indeed, since, for $k \in \mathbb{Z}$

$$H(e^{j(\omega+2\pi k)}) = H(e^{j2\pi k}e^{j\omega}) = H(e^{j\omega}) \tag{173}$$

then $H(e^{j\omega})$ is periodic with period 2π.

Frequency response

- Another important characteristic of a linear discrete-time system is its group delay. This is defined as the derivative of the phase of its frequency response,

$$\tau(\omega) = -\frac{d\Theta(\omega)}{d\omega} \tag{174}$$

When the group delay $\Theta(\omega)$ is a linear function of ω, that is,

$$\Theta(\omega) = \beta\omega \tag{175}$$

then the output $y(n)$ of a linear system to a sinusoid $x(n) = e^{j\omega n}$ is, according to equation (172),

$$y(n) = |H(e^{j\omega})|e^{j\omega n + j\beta\omega} = |H(e^{j\omega})|e^{j\omega(n+\beta)} \tag{176}$$

Frequency response

- The above equation, together with equation (174), implies that the output sinusoid is delayed by

$$-\beta = -\frac{\mathrm{d}\Theta(\omega)}{\mathrm{d}\omega} = \tau(\omega) \tag{177}$$

samples, irrespective of the frequency ω.

  – Because of this property, the group delay is commonly used as a measure of how a linear time-invariant system delays sinusoids of different frequencies.

Frequency response

Example 2.15

Find the frequency response and the group delay of the FIR filter characterized by the following difference equation

$$y(n) = \frac{x(n) + x(n-1)}{2} \tag{178}$$

Frequency response

Solution

$y(n) = \frac{x(n)+x(n-1)}{2}$

Taking the z transform of $y(n)$, we find

$$Y(z) = \frac{X(z) + z^{-1}X(z)}{2} = \frac{1}{2}(1 + z^{-1})X(z) \tag{179}$$

and then the transfer function of the system is

$$H(z) = \frac{1}{2}(1 + z^{-1}) \tag{180}$$

Making $z = e^{j\omega}$, the frequency response of the system becomes

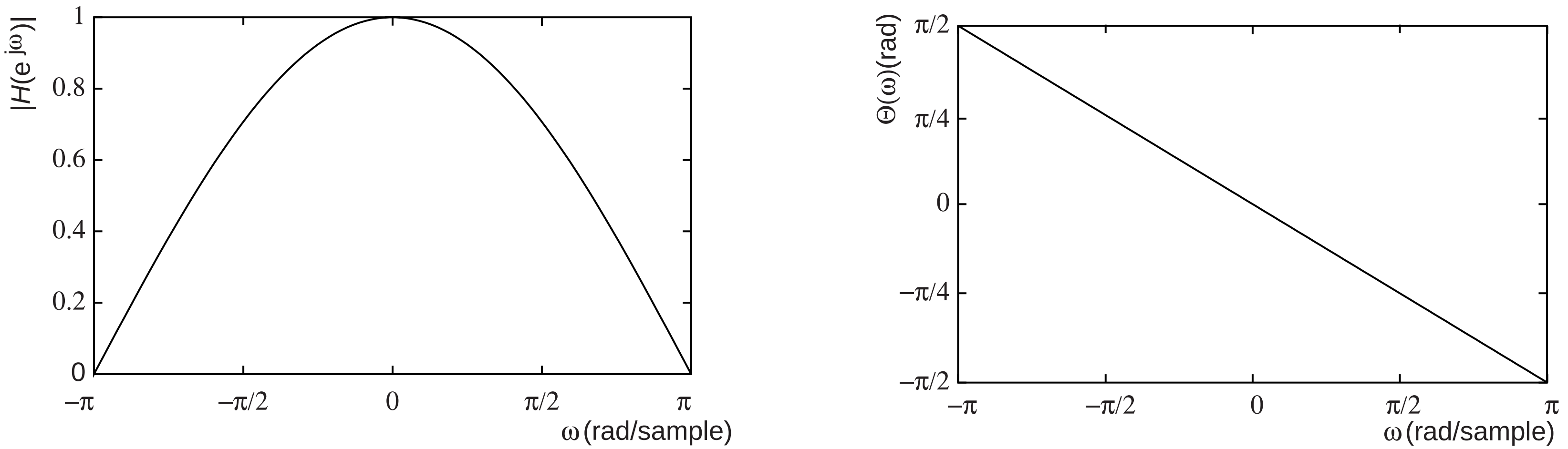
$$H(e^{j\omega}) = \frac{1}{2}(1 + e^{-j\omega}) = \frac{1}{2}e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} + e^{-j\frac{\omega}{2}}\right) = e^{-j\frac{\omega}{2}}\cos\left(\frac{\omega}{2}\right) \tag{181}$$

Since $\Theta(\omega) = -\frac{\omega}{2}$, then the system delays all sinusoids equally by half a sample, that is, the group delay $\tau(\omega) = \frac{1}{2}$ sample.

Frequency response

The magnitude and phase responses of $H(e^{j\omega})$ are below.

- Note that the frequency response is plotted for $-\pi \leq \omega < \pi$, rather than for $0 \leq \omega < 2\pi$.
- In practice, those two ranges are equivalent, since both comprise one period of $H(e^{j\omega})$.



△

Frequency response

Example 2.16

A discrete-time system with impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$ is excited with $x(n) = \sin(\omega_0 n + \theta)$. Find the output $y(n)$ using the frequency response of the system.

Frequency response

Solution

Since

$$x(n) = \sin(\omega_0 n + \theta) = \frac{e^{j(\omega_0 n+\theta)} - e^{-j(\omega_0 n+\theta)}}{2j} \tag{182}$$

then, the output $y(n) = \mathcal{H}\{x(n)\}$ is

$$\begin{aligned} y(n) &= \mathcal{H}\left\{\frac{e^{j(\omega_0 n+\theta)} - e^{-j(\omega_0 n+\theta)}}{2j}\right\} \\ &= \frac{1}{2j}\left(\mathcal{H}\{e^{j(\omega_0 n+\theta)}\} - \mathcal{H}\{e^{-j(\omega_0 n+\theta)}\}\right) \\ &= \frac{1}{2j}\left(H(e^{j\omega_0})e^{j(\omega_0 n+\theta)} - H(e^{-j\omega_0})e^{-j(\omega_0 n+\theta)}\right) \\ &= \frac{1}{2j}\left(|H(e^{j\omega_0})|e^{j\Theta(\omega_0)}e^{j(\omega_0 n+\theta)} - |H(e^{-j\omega_0})|e^{j\Theta(-\omega_0)}e^{-j(\omega_0 n+\theta)}\right) \end{aligned} \tag{183}$$

Frequency response

Since $h(n)$ is real, from property 2.7, one has that $H(e^{j\omega}) = H^*(e^{-j\omega})$. This implies that

$$|H(e^{-j\omega})| = |H(e^{j\omega})| \quad \text{and} \quad \Theta(-\omega) = -\Theta(\omega) \tag{184}$$

Using this result, equation (183) becomes

$$\begin{aligned} y(n) &= \frac{1}{2j}\left(|H(e^{j\omega_0})|e^{j\Theta(\omega_0)}e^{j(\omega_0 n+\theta)} - |H(e^{j\omega_0})|e^{-j\Theta(\omega_0)}e^{-j(\omega_0 n+\theta)}\right) \\ &= |H(e^{j\omega_0})|\left[\frac{e^{j(\omega_0 n+\theta+\Theta(\omega_0))} - e^{-j(\omega_0 n+\theta+\Theta(\omega_0))}}{2j}\right] \\ &= |H(e^{j\omega_0})|\sin(\omega_0 n + \theta + \Theta(\omega_0)) \end{aligned} \tag{185}$$

Since the system transfer function is

$$H(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1-\frac{1}{2}z^{-1}} \tag{186}$$

Frequency response

we have that

$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} = \frac{1}{\sqrt{\frac{5}{4} - \cos\omega}} e^{-j\arctan\left(\frac{\sin\,\omega}{2 - \cos\,\omega}\right)} \tag{187}$$

and then

$$|H(e^{j\omega})| = \frac{1}{\sqrt{\frac{5}{4} - \cos\omega}} \tag{188}$$

$$\Theta(\omega) = -\arctan\left(\frac{\sin\omega}{2 - \cos\omega}\right) \tag{189}$$

Substituting these values of $|H(e^{j\omega})|$ and $\Theta(\omega)$ in equation (185), the output $y(n)$ becomes

$$y(n) = \frac{1}{\sqrt{\frac{5}{4} - \cos\omega_0}} \sin\left[\omega_0 n + \theta - \arctan\left(\frac{\sin\omega_0}{2 - \cos\omega_0}\right)\right] \tag{190}$$

$\triangle$

Frequency response

- In general, when we design a discrete-time system, we have to satisfy pre-determined magnitude, $|H(e^{j\omega})|$, and phase, $\Theta(\omega)$, characteristics.
- One should note that, when processing a continuous-time signal using a discrete-time system, we should translate the analog frequency Ω to the discrete-time frequency ω that is restricted to the interval $[-\pi, \pi)$.
  - If $\Omega_s = \frac{2\pi}{T}$ is the sampling frequency, then

$$e^{j\omega n} = x(n) = x_a(nT) = e^{j\Omega nT} \tag{191}$$

  - Therefore, the relation between the digital frequency ω and the analog frequency Ω is

$$\omega = \Omega T = 2\pi \frac{\Omega}{\Omega_s} \tag{192}$$

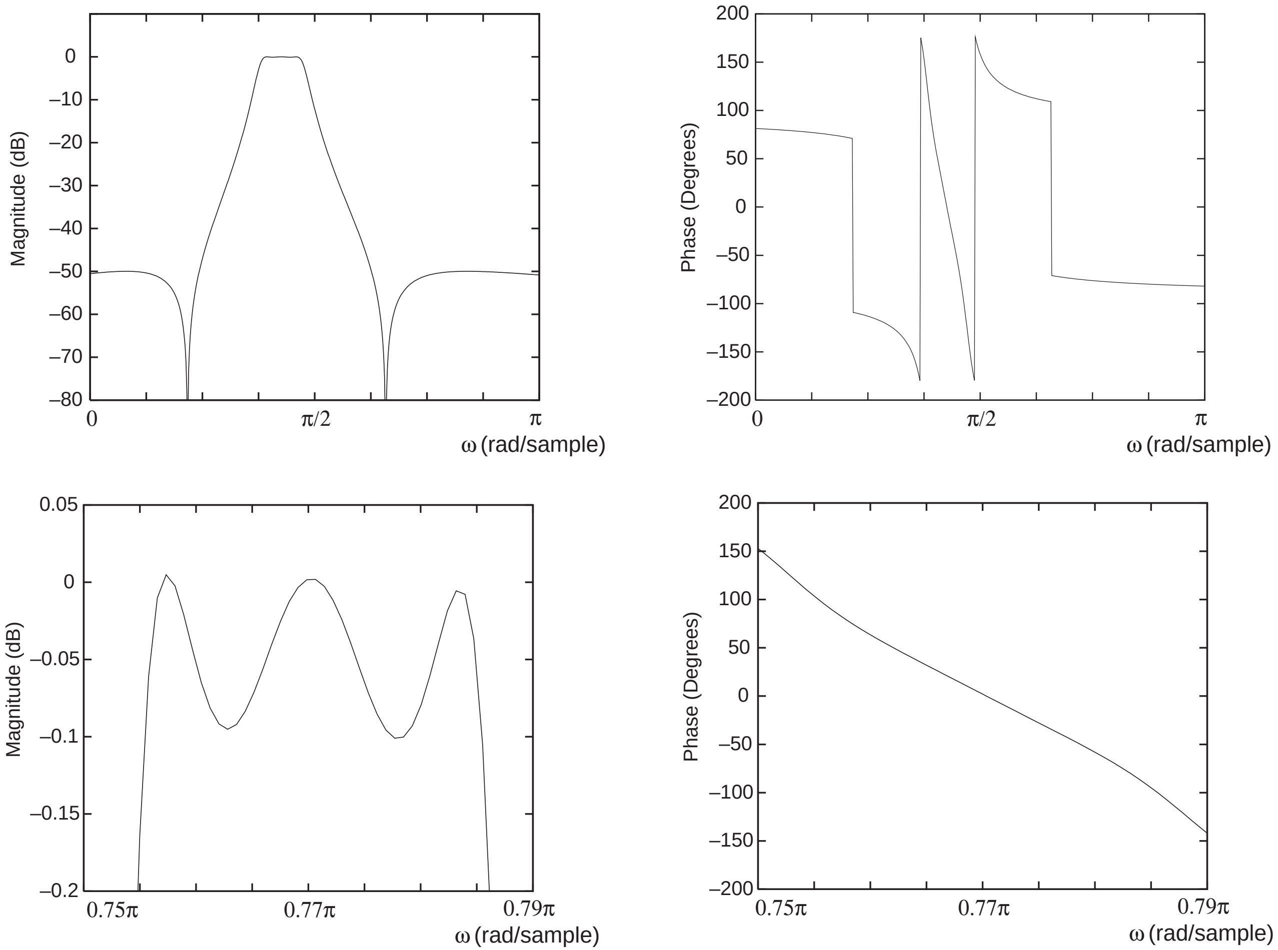
indicating that the frequency interval $[-\pi, \pi)$ for the discrete-time frequency response corresponds to the frequency interval $\left[-\frac{\Omega_s}{2}, \frac{\Omega_s}{2}\right)$ in the analog domain.

Frequency response

Example 2.17

The sixth-order discrete-time lowpass elliptic filter, whose frequency response is shown in the figure below, is used to process an analog signal. If the sampling frequency used in the analog-to-digital conversion is 8000 Hz, determine the passband of the equivalent analog filter. Consider the passband as the frequency range when the magnitude response of the filter is within 0.1 dB of its maximum value.

Frequency response



Frequency response of a sixth-order elliptic filter: (a) magnitude response; (b) phase response; (c) magnitude response in the passband; (d) phase response in the passband.

Frequency response

Solution

From the above figure, we see that the digital bandwidth in which the magnitude response of the system is within 0.1 dB of its maximum value is approximately from $\omega_{p_1} = 0.755\pi$ rad/sample to $\omega_{p_2} = 0.785\pi$ rad/sample.

As the sampling frequency is

$$f_s = \frac{\Omega_s}{2\pi} = 8000 \text{ Hz} \tag{193}$$

then the analog passband is such that

$$\Omega_{p_1} = 0.755\pi\frac{\Omega_s}{2\pi} = 0.755\pi \times 8000 = 6040\pi \text{ rad/s} \Rightarrow f_{p_1} = \frac{\Omega_{p_1}}{2\pi} = 3020 \text{ Hz} \tag{194}$$

$$\Omega_{p_2} = 0.785\pi\frac{\Omega_s}{2\pi} = 0.785\pi \times 8000 = 6280\pi \text{ rad/s} \Rightarrow f_{p_2} = \frac{\Omega_{p_2}}{2\pi} = 3140 \text{ Hz} \tag{195}$$

$\triangle$

Frequency response

- The positions of the poles and zeros of a transfer function are very useful in the determination of its characteristics.
- For example, one can determine the frequency response $H(e^{j\omega})$ using a geometric method.
- Expressing $H(z)$ as a function of its poles and zeros as in equation (154), we have that $H(e^{j\omega})$ becomes

$$H(e^{j\omega}) = H_0 e^{j\omega(N-M)} \frac{\prod_{l=1}^{M}(e^{j\omega} - z_l)}{\prod_{i=1}^{N}(e^{j\omega} - p_i)} \tag{196}$$

Frequency response

- The magnitude and phase responses of $H(e^{j\omega})$ are then

$$|H(e^{j\omega})| = |H_0| \frac{\prod_{l=1}^{M} |e^{j\omega} - z_l|}{\prod_{i=1}^{N} |e^{j\omega} - p_i|} \tag{197}$$

$$\Theta(\omega) = \omega(N - M) + \sum_{l=1}^{M} \angle(e^{j\omega} - z_l) - \sum_{i=1}^{N} \angle(e^{j\omega} - p_i) \tag{198}$$

where $\angle z$ denotes the angle of the complex number z.

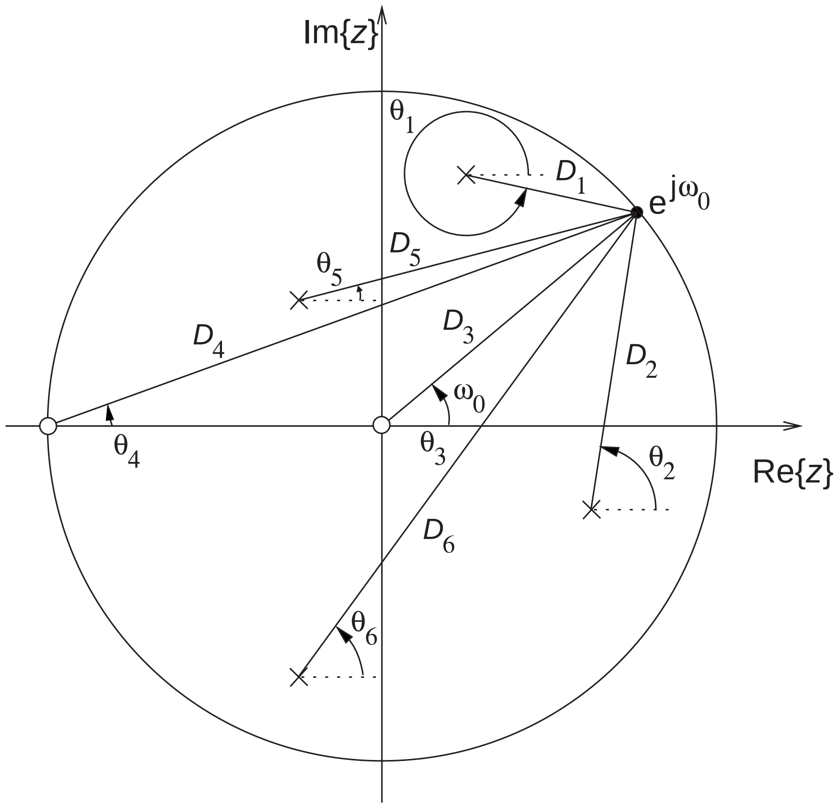
  - The terms of the form $|e^{j\omega} - c|$ represent the distance between the point $e^{j\omega}$ on the unit circle and the complex number c.
  - The terms of the form $\angle(e^{j\omega} - c)$ represent the angle of the line segment joining $e^{j\omega}$ and c and the real axis, measured in the counterclockwise direction.

Frequency response

- For example, for $H(z)$ having its poles and zeros as in the following figure,

$$|H(e^{j\omega_0})| = \frac{D_3 D_4}{D_1 D_2 D_5 D_6} \tag{199}$$

$$\Theta(\omega_0) = \theta_3 + \theta_4 - \theta_1 - \theta_2 - \theta_5 - \theta_6 \tag{200}$$



Fourier transform

- In the previous section, we characterized linear discrete-time systems using the frequency response, which describes the behavior of a system when the input is a complex sinusoid.
- In this section, we present the Fourier transform of discrete-time signals, which is a generalization of the concept of frequency response.
  - It is equivalent to the decomposition of a discrete-time signal into an infinite sum of complex discrete-time sinusoids.
- In Chapter 1, when deducing the sampling theorem, we formed, from the discrete-time signal $x(n)$, a continuous-time signal $x_i(t)$ consisting of a train of impulses with amplitude $x(n)$ at $t = nT$. Its expression is

$$x_i(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT) \tag{201}$$

Fourier transform

- Since the Fourier transform of $\delta(t - Tn)$ is $e^{-j\Omega Tn}$, the Fourier transform of the continuous-time signal $x_i(t)$ becomes

$$X_i(j\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega Tn} \tag{202}$$

- Therefore, one can say that

$$X_i(j\Omega) = X(e^{j\Omega T}) \tag{203}$$

  – This equation means that the Fourier transform at a frequency Ω of the continuous-time signal $x_i(t)$, that is generated by replacing each sample of the discrete-time signal $x(n)$ by an impulse of the same amplitude located at $t = nT$, is equal to the z transform of the signal $x(n)$ at $z = e^{j\Omega T}$.

  – This fact implies that $X(e^{j\omega})$ holds the information about the frequency content of the signal $x_i(t)$, and therefore is a natural candidate to represent the frequency content of the discrete-time signal $x(n)$.

Fourier transform

- We can show that $X(e^{j\omega})$ does indeed represent the frequency content of $x(n)$ by applying the inverse z-transform formula in equation (36) with C being the closed contour $z = e^{j\omega}$, for $-\pi \leq \omega < \pi$, resulting in

$$
\begin{aligned}
x(n) &= \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \\
&= \frac{1}{2\pi j} \oint_{z=e^{j\omega}} X(z) z^{n-1} dz \\
&= \frac{1}{2\pi j} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega(n-1)} j e^{j\omega} d\omega \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad (204)
\end{aligned}
$$

indicating that the discrete-time signal $x(n)$ can be expressed as an infinite summation of sinusoids.

– The sinusoid of frequency ω, $e^{j\omega n}$, has a complex amplitude proportional to $X(e^{j\omega})$.

– Thus computing $X(e^{j\omega})$ is equivalent to decomposing the discrete-time signal $x(n)$ into a sum of complex discrete-time sinusoids.

- In the continuous case, the direct and inverse Fourier transforms are given by

$$\left.\begin{aligned} X_a(j\Omega) &= \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt \\ x_a(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{j\Omega t}d\Omega \end{aligned}\right\} \tag{205}$$

– This pair of equations indicates that a continuous-time signal $x_a(t)$ can be expressed as an infinite summation of continuous-time sinusoids, where the sinusoid of frequency Ω, $e^{j\Omega t}$, has a complex amplitude proportional to $X_a(j\Omega)$.

– Computing $X_a(j\Omega)$ is equivalent to decomposing the continuous-time signal into a sum of complex continuous-time sinusoids.

Fourier transform

- From the above discussion, we see that $X(e^{j\omega})$ defines a Fourier transform of the discrete-time signal $x(n)$, with its inverse given by equation (204).

- The direct and inverse Fourier transforms of a sequence $x(n)$ are formally defined as

$$\left.\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \end{aligned}\right\} \tag{206}$$

- The Fourier transform $X(e^{j\omega})$ of a discrete-time signal $x(n)$ is periodic with period 2π, since

$$X(e^{j\omega}) = X(e^{j(\omega+2\pi k)}), \quad \text{for all } k \in \mathbb{Z} \tag{207}$$

$\Rightarrow$ The Fourier transform of a discrete-time signal requires specification only for a range of 2π, as, for example, $\omega \in [-\pi, \pi]$ or $\omega \in [0, 2\pi]$.

Fourier transform

Example 2.18

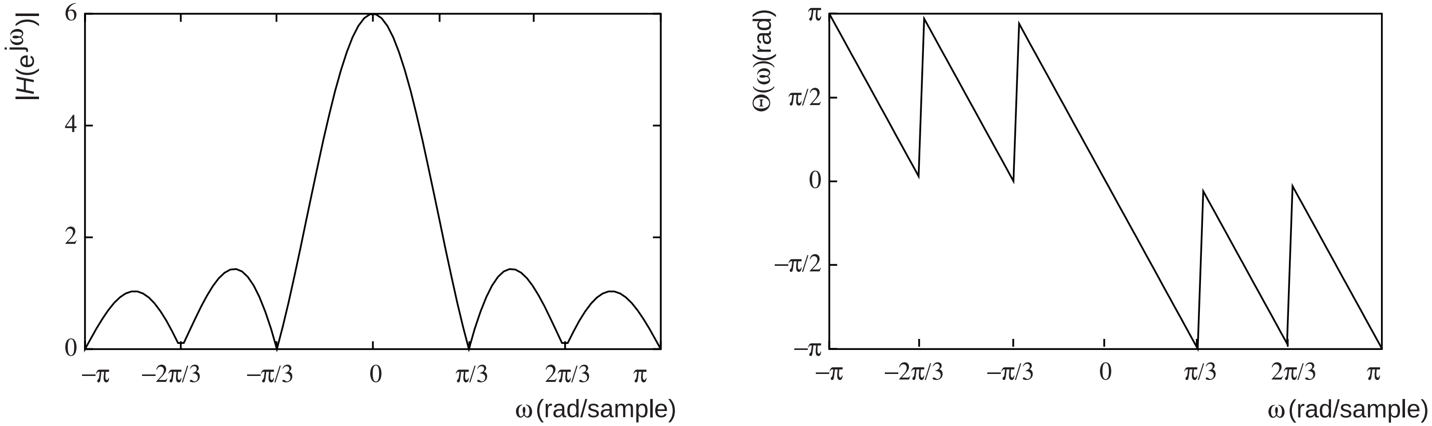
Compute the Fourier transform of the sequence

$$x(n) = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{otherwise} \end{cases} \tag{208}$$

Solution

$$X(e^{j\omega}) = \sum_{k=0}^{5} e^{-j\omega k} = \frac{1 - e^{-6j\omega}}{1 - e^{-j\omega}} = \frac{e^{-3j\omega}(e^{3j\omega} - e^{-3j\omega})}{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}})} = e^{-j\frac{5\omega}{2}} \frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)} \tag{209}$$

- The magnitude and phase responses of $X(e^{j\omega})$ are depicted below.



- Note the phase has been wrapped to fit in the interval $[-\pi, \pi)$
  - Instead of plotting the phase $\Theta(\omega)$, we plot $\overline{\Theta}(\omega) = \Theta(\omega) + 2k(\omega)\pi$, with $k(\omega)$ an integer such that $\overline{\Theta}(\omega) \in [-\pi, \pi)$ for all ω. $\triangle$

Fourier transform

- In order for the Fourier transform of a discrete-time signal to exist for all ω, its z transform must converge for $|z| = 1$.

- We have seen before that whenever

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty \tag{210}$$

then the z transform converges on the unit circle, and therefore the Fourier transform exists for all ω.

Fourier transform

- One example in which equation (210) does not hold and the Fourier transform does not exist for all ω is given by the sequence $x(n) = u(n)$. This case is discussed below.

- In Exercise 2.23 there is an example in which equation (210) does not hold but the Fourier transform does exist for all ω.

$\Rightarrow$ Therefore, the condition in equation (210) is sufficient, but not necessary for the Fourier transform to exist.

- In addition, we have that not all sequences which have a Fourier transform have a z transform.
  - For example, the z transform of any sequence is continuous in its convergence region, and hence the sequences whose Fourier transforms are discontinuous functions of ω do not have a z transform.

Fourier transform

Example 2.19

Compute the Fourier transform of the sequence $x(n) = u(n)$.

Solution

As we have seen before, the z transform of $x(n) = u(n)$ is

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{211}$$

- We know that the z transform of $x(n)$ does not converge neither for $|z| < 1$ nor for $z = 1$ and $z = -1$.
- However, one can say nothing about the other values of z on the unit circle, that is, for $z = \mathrm{e}^{\mathrm{j}\omega}$ for any given ω.

Fourier transform

- Since with equation (211) we can compute $X(z)$ for $|z| > 1$ (that is, for $z = \rho e^{j\omega}$ with $\rho > 1$), we try to compute $X(e^{j\omega})$ as

$$\begin{aligned} X(e^{j\omega}) &= \lim_{\rho \to 1} X(\rho e^{j\omega}) \\ &= \lim_{\rho \to 1} \frac{1}{1 - \rho e^{-j\omega}} \\ &= \frac{1}{1 - e^{-j\omega}} \\ &= \frac{e^{j\frac{\omega}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}} \\ &= \frac{e^{j\frac{\omega}{2}}}{2j \sin\left(\frac{\omega}{2}\right)} \\ &= \frac{e^{j\left(\frac{\omega}{2} - \frac{\pi}{2}\right)}}{2 \sin\left(\frac{\omega}{2}\right)} \end{aligned} \tag{212}$$

Fourier transform

- From the above equation, we can see that the Fourier transform of $x(n)$ does not exist for sin $\left(\frac{\omega}{2}\right) = 0$, that is, for $\omega = k\pi$, which corresponds to $z = \pm 1$.
- However, $X(\mathrm{e}^{\mathrm{j}\omega})$ does exist for all $\omega \neq k\pi$.
  - Although this result indicates that one can use $X(\mathrm{e}^{\mathrm{j}\omega})$ as the frequency content of $x(n)$, its implications should be considered with caution.
  - For example, the inverse Fourier transform in equation (204) is based on the convergence of $X(z)$ on the unit circle.

  $\Rightarrow$ Since $X(z)$ does not converge on the whole unit circle, then equation (204) is not valid for computing $x(n)$.
- See Exercise 2.22 for a way to compute $x(n)$ from $X(\mathrm{e}^{\mathrm{j}\omega})$.

$\triangle$

Fourier transform

Example 2.20

Compute the Fourier transform of the sequence $x(n) = e^{j\omega_0 n}$.

Fourier transform

Solution

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j(\omega_0 - \omega)n} \tag{213}$$

We have seen before that

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j\frac{2\pi}{T}nt} \tag{214}$$

By making $T = 2\pi$ and $t = (\omega_0 - \omega)$ in the above equation, one can express the transform in equation (213) as

$$X(e^{j\omega}) = 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega_0 - \omega - 2\pi n) = 2\pi \sum_{n=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi n) \tag{215}$$

That is, the Fourier transform of a complex sinusoid of infinite duration and frequency ω_0 is an impulse centered at frequency ω_0 and repeated with period 2π. $\triangle$

Fourier transform

- The relation between the Fourier transform of the train of impulses $x_i(t)$ and the one of the original analog signal $x_a(t)$ is

$$X_i(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\Omega - j\frac{2\pi}{T}k\right) \tag{216}$$

- If the discrete-time signal $x(n)$ is such that $x(n) = x_a(nT)$, then we can use this equation to derive the relation between the Fourier transforms of the discrete-time and continuous-time signals, $X(e^{j\omega})$ and $X_a(j\Omega)$. In fact,

$$X_i(j\Omega) = X(e^{j\Omega T}) \tag{217}$$

and making the change of variables $\Omega = \frac{\omega}{T}$ in equation (216), yields

$$X(e^{j\omega}) = X_i\left(j\frac{\omega}{T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(j\frac{\omega - 2\pi k}{T}\right) \tag{218}$$

that is, $X(e^{j\omega})$ is composed of copies of $X_a(j\frac{\omega}{T})$ repeated in intervals of 2π.

Properties of the Fourier transform

- As seen previously, the Fourier transform $X(e^{j\omega})$ of a sequence $x(n)$ is equal to its z transform $X(z)$ at $z = e^{j\omega}$.
- Therefore, most properties of the Fourier transform derive from the ones of the z transform by simple substitution of z by $e^{j\omega}$.
- In what follows, we state them without proof, except in the cases where the properties do not have a straightforward z-transform correspondent.

Properties of the Fourier transform

1. Linearity

$$k_1 x_1(n) + k_2 x_2(n) \longleftrightarrow k_1 X_1(e^{j\omega}) + k_2 X_2(e^{j\omega}) \tag{219}$$

2. Time-reversal

$$x(-n) \longleftrightarrow X(e^{-j\omega}) \tag{220}$$

3. Time-shift theorem

$$x(n+l) \longleftrightarrow e^{j\omega l} X(e^{j\omega}) \tag{221}$$

where l is an integer.

Properties of the Fourier transform

4. Multiplication by a complex exponential (frequency shift, modulation)

$$e^{j\omega_0 n}x(n) \longleftrightarrow X(e^{j(\omega-\omega_0)}) \tag{222}$$

5. Complex differentiation

$$nx(n) \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega} \tag{223}$$

6. Complex conjugation

$$x^*(n) \longleftrightarrow X^*(e^{-j\omega}) \tag{224}$$

Properties of the Fourier transform

7. Real and imaginary sequences

Before presenting the properties of the Fourier transforms of real, imaginary, symmetric, and antisymmetric sequences, it is useful to give the precise definitions below:

- A symmetric (even) function is such that $f(u) = f(-u)$.
- An antisymmetric (odd) function is such that $f(u) = -f(-u)$.
- A conjugate symmetric function is such that $f(u) = f^*(-u)$.
- A conjugate antisymmetric function is such that $f(u) = -f^*(-u)$.

The following properties hold:

$$\mathrm{Re}\{x(n)\} \longleftrightarrow \frac{1}{2}\left(X(e^{j\omega}) + X^*(e^{-j\omega})\right) \tag{225}$$

$$\mathrm{Im}\{x(n)\} \longleftrightarrow \frac{1}{2j}\left(X(e^{j\omega}) - X^*(e^{-j\omega})\right) \tag{226}$$

If $x(n)$ is real, then Im$\{x(n)\} = 0$. Hence, from equation (226),

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \tag{227}$$

that is, the Fourier transform of a real sequence is conjugate symmetric. The following properties for real $x(n)$ follow directly from equation (227):

- The real part of the Fourier transform of a real sequence is even:

$$\text{Re}\{X(e^{j\omega})\} = \text{Re}\{X(e^{-j\omega})\} \tag{228}$$

- The imaginary part of the Fourier transform of a real sequence is odd:

$$\text{Im}\{X(e^{j\omega})\} = -\text{Im}\{X(e^{-j\omega})\} \tag{229}$$

- The magnitude of the Fourier transform of a real sequence is even:

$$|X(e^{j\omega})| = |X(e^{-j\omega})| \tag{230}$$

- The phase of the Fourier transform of a real sequence is odd:

$$\angle[X(e^{j\omega})] = -\angle[X(e^{-j\omega})] \tag{231}$$

Similarly, if $x(n)$ is imaginary, then Re$\{x(n)\} = 0$. Hence, from equation (225),

$$X(e^{j\omega}) = -X^*(e^{-j\omega}) \tag{232}$$

Properties similar to the ones in equations (228) to (231) can be deduced for imaginary sequences, which is left as an exercise to the reader.

Properties of the Fourier transform

8. Symmetric and antisymmetric sequences

- If $x(n)$ is real and symmetric, $X(e^{j\omega})$ is also real and symmetric.

Proof:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(-n)e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{\infty} x(m)e^{j\omega m} = X(e^{-j\omega})$$

$$= \sum_{m=-\infty}^{\infty} x(m)\left(e^{-j\omega m}\right)^* = \sum_{m=-\infty}^{\infty} \left(x(m)e^{-j\omega m}\right)^* = X^*(e^{j\omega})$$

Hence, if $X(e^{j\omega}) = X(e^{-j\omega})$, then $X(e^{j\omega})$ is even, and if $X(e^{j\omega}) = X^*(e^{j\omega})$, then $X(e^{j\omega})$ is real.

$\square$

Properties of the Fourier transform

- If $x(n)$ is imaginary and even, then $X(e^{j\omega})$ is imaginary and even.
- If $x(n)$ is real and odd, then $X(e^{j\omega})$ is imaginary and odd.
- If $x(n)$ is imaginary and odd, then $X(e^{j\omega})$ is real and odd.

  Proof:

$$
\begin{aligned}
X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x^*(-n)e^{-j\omega n} \\
&= \sum_{m=-\infty}^{\infty} x^*(m)e^{j\omega m} = \sum_{m=-\infty}^{\infty} \left(x(m)e^{-j\omega m}\right)^* = X^*(e^{j\omega})
\end{aligned}
$$

  and then $X(e^{j\omega})$ is real. $\square$

- If $x(n)$ is conjugate antisymmetric, then $X(e^{j\omega})$ is imaginary.

Properties of the Fourier transform

10. Convolution theorem

$$x_1(n) * x_2(n) \longleftrightarrow X_1(e^{j\omega})X_2(e^{j\omega}) \tag{233}$$

11. Product of two sequences

$$\begin{aligned} x_1(n)x_2(n) \longleftrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\Omega})X_2(e^{j(\omega-\Omega)})d\Omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j(\omega-\Omega)})X_2(e^{j\Omega})d\Omega \\ &= X_1(e^{j\omega}) \circledast X_2(e^{j\omega}) \end{aligned} \tag{234}$$

Properties of the Fourier transform

12. Parseval's theorem

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})X_2^*(e^{j\omega})d\omega \tag{235}$$

- If we make $x_1(n) = x_2(n) = x(n)$, then Parseval's theorem becomes

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \tag{236}$$

- The left-hand side of this equation corresponds to the energy of the sequence $x(n)$ and its right-hand side corresponds to the energy of $X(e^{j\omega})$ divided by 2π.

- Hence, equation (236) means that the energy of a sequence is the same as the energy of its Fourier transform divided by 2π.

Fourier transform for periodic sequences

- A special case of the Fourier transform that should be considered is the one of the periodic sequences.
- In what follows, we will obtain the expression of the Fourier transform of a periodic sequence. From it, we will define a Fourier series for discrete-time signals.

Fourier transform for periodic sequences

We start by considering a signal $x_f(n)$ with N non-zero samples. Without loss of generality, we make

$$x_f(n) = 0, \quad \text{for } n < 0 \text{ and } n \geq N \tag{237}$$

According to equation (206), its Fourier transform is given by

$$X_f(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_f(n) e^{-j\omega n} = \sum_{n=0}^{N-1} x_f(n) e^{-j\omega n} \tag{238}$$

Fourier transform for periodic sequences

Next, we build from $x_f(n)$ a periodic signal $x(n)$ with period N composed of versions of $x_f(n)$ shifted to the positions kN, for all $k \in \mathbb{Z}$,

$$x(n) = \sum_{k=-\infty}^{\infty} x_f(n + kN) \tag{239}$$

Using the time-shift property (3), its Fourier transform is

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} e^{j\omega kN} X_f(e^{j\omega}) = X_f(e^{j\omega}) \sum_{k=-\infty}^{\infty} e^{j\omega kN} \tag{240}$$

Fourier transform for periodic sequences

Making $T = \frac{2\pi}{N}$ and $t = \omega$, we have that the above equation becomes

$$\begin{aligned} X(e^{j\omega}) &= X_f(e^{j\omega})\frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi}{N}k\right) \\ &= \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}X_f\left(e^{j\frac{2\pi}{N}k}\right)\delta\left(\omega - \frac{2\pi}{N}k\right) \\ &= \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}X(k)\delta\left(\omega - \frac{2\pi}{N}k\right) \end{aligned} \tag{241}$$

where

$$X(k) = X_f\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=0}^{N-1}x_f(n)e^{-j\frac{2\pi}{N}k} = \sum_{n=0}^{N-1}x(n)e^{-j\frac{2\pi}{N}k} \tag{242}$$

Fourier transform for periodic sequences

By computing the inverse Fourier transform of equation (241), we have that

$$\begin{aligned}
x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2\pi}{N}\sum_{k=-\infty}^{\infty} X(k)\delta\left(\omega-\frac{2\pi}{N}k\right)e^{j\omega n}d\omega \\
&= \frac{1}{N}\sum_{k=-\infty}^{\infty} X(k)\int_{-\pi}^{\pi}\delta\left(\omega-\frac{2\pi}{N}k\right)e^{j\omega n}d\omega \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}kn} \qquad (243)
\end{aligned}$$

Fourier transform for periodic sequences

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}kn} \tag{244}$$

- This represents the expansion of a discrete-time signal $x(n)$ of period N as a sum of complex discrete-time sinusoids with frequencies that are multiples of $\frac{2\pi}{N}$, the fundamental frequency of $x(n)$.
  - Therefore, the above equation can be regarded as a Fourier series expansion of $x(n)$.
  - Note that equations (242) and (243) define a Fourier series pair for the periodic signal, that can be used whenever one wants to avoid the impulses in frequency that come from the Fourier transform in equation (241).

Random signals in transform domain

- The representation of a random process in the transform domain is not straightforward.
  - The direct application of the z or Fourier transforms on a random process does not make much of a sense, since in such a cases the waveform $x(n)$ is by definition unknown.
  - In addition, several types of random processes present infinite energy, requiring some specific mathematical treatment.
- This, however, does not indicate that there are no transform-domain tools for random signal analysis.
- Random signal analysis can indeed benefit a great deal from transform-domain tools.
- Now we introduce a frequency representation for the autocorrelation function, and use it to characterize the input-output relationship of a linear system when processing random signals.

Power spectral density

- The so-called power spectral density (PSD) function, $\Gamma_X(e^{j\omega})$, is defined as the Fourier transform of the autocorrelation function of a given random process $\{X\}$.
  - For a WSS process, we have that

$$\Gamma_X(e^{j\omega}) = \sum_{\nu=-\infty}^{\infty} R_X(\nu)e^{-j\omega\nu} \tag{245}$$

in such a way that

$$R_X(\nu) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_X(e^{j\omega})e^{j\omega\nu}\, d\omega \tag{246}$$

- Equations (245) and (246) are jointly referred to as the Wiener-Khinchin theorem
- In particular, by setting $\nu = 0$ in equation (246), we get

$$R_X(0) = E\{X^2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_X(e^{j\omega})\, d\omega \tag{247}$$

Power spectral density

- If a random signal $x(n)$ from a WSS process $\{X\}$ is filtered by a linear time-invariant system, with impulse response $h(n)$, the PSD function for the output signal $y(n)$, using equation (245), is given by

$$\begin{aligned}\Gamma_Y(e^{j\omega}) &= \sum_{\nu=-\infty}^{\infty} R_Y(\nu)e^{-j\omega\nu} \\ &= \sum_{\nu=-\infty}^{\infty}\left(\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} R_X(\nu-k_1+k_2)h(k_1)h(k_2)\right)e^{-j\omega\nu}\end{aligned} \tag{248}$$

By defining the auxiliary time variable $\nu' = (\nu - k_1 + k_2)$, such that $\nu = (\nu' + k_1 - k_2)$, then

Power spectral density

$$\begin{aligned}
\Gamma_Y(e^{j\omega}) &= \sum_{\nu'=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} R_X(\nu')h(k_1)h(k_2)e^{-j\omega(\nu'-k_1+k_2)} \\
&= \sum_{\nu'=-\infty}^{\infty} R_X(\nu')e^{-j\omega\nu'} \left[\sum_{k_1=-\infty}^{\infty} h(k_1)e^{j\omega k_1} \left(\sum_{k_2=-\infty}^{\infty} h(k_2)e^{-j\omega k_2} \right) \right] \\
&= \Gamma_X(e^{j\omega})H^*(e^{j\omega})H(e^{j\omega}) \qquad (249)
\end{aligned}$$

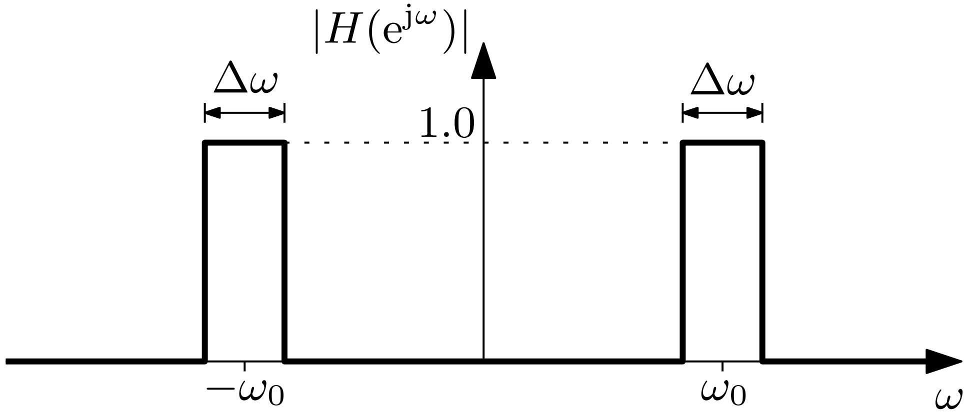
or, equivalently,

$$\Gamma_Y(e^{j\omega}) = \Gamma_X(e^{j\omega}) \left| H(e^{j\omega}) \right|^2 \qquad (250)$$

- Therefore, the output PSD function is the input PSD function multiplied by the squared magnitude response of the linear system.
  - This is the equivalent frequency-domain description for random signals of the input-output relationship of a linear time-invariant system whose input is a deterministic signal.

Power spectral density

- Consider the processing of a random signal $x(t)$ by a filter with unit gain and narrow passband, $(\omega_0 - \frac{\Delta\omega}{2}) \leq \omega \leq (\omega_0 + \frac{\Delta\omega}{2})$, as depicted below.



- If $\Delta\omega$ is small enough, we may consider the input PSD constant around ω_0, in such a way that, using equation (247), the output mean squared value may be written as

Power spectral density

$$
\begin{aligned}
E\{Y^2(n)\} &= R_Y(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_Y(e^{j\omega})\, d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_X(e^{j\omega}) \left|H(e^{j\omega})\right|^2 d\omega \\
&= \frac{2}{2\pi}\int_{\omega_0-\frac{\Delta\omega}{2}}^{\omega_0+\frac{\Delta\omega}{2}} \Gamma_X(e^{j\omega})\, d\omega \\
&\approx \frac{\Delta\omega}{\pi}\Gamma_X(e^{j\omega_0}) \qquad (251)
\end{aligned}
$$

and then

$$
\Gamma_X(e^{j\omega_0}) \approx \frac{E\{Y^2(n)\}}{\frac{\Delta\omega}{\pi}} \qquad (252)
$$

- This result indicates that the value of the PSD at ω_0 is a measure of the density of signal power around that frequency, what justifies the nomenclature used for that function.

Power spectral density

Example 2.21

Determine the PSD of the random process $\{X\}$ described by

$$x_m(n) = \cos(\omega_0 n + \Theta_m) \tag{253}$$

where Θ is a continuous random variable with uniform PDF within the interval $[0, 2\pi]$.

Power spectral density

Solution

$x_m(n) = \cos(\omega_0 n + \Theta_m)$

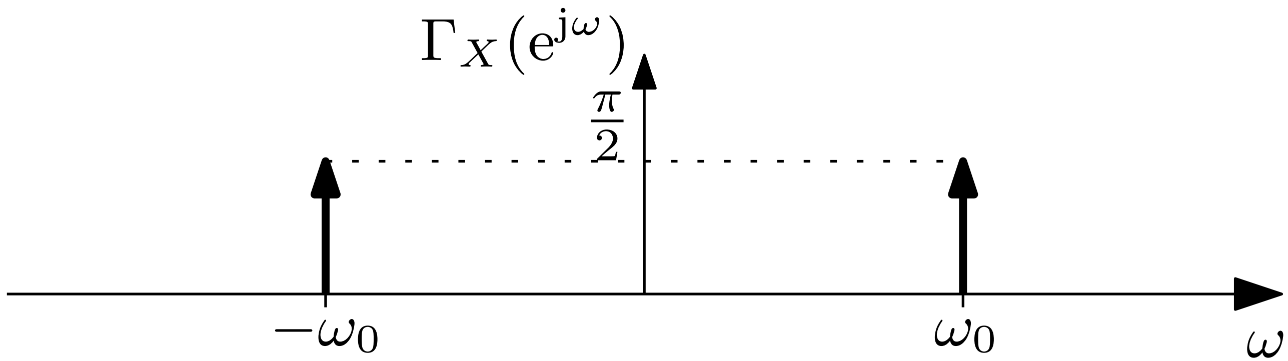
As determined before, the random process $\{X\}$ is WSS with autocorrelation function given by

$$R_X(\nu) = \frac{1}{2}\cos(\omega_0\nu) = \frac{e^{j\omega_0\nu} + e^{-j\omega_0\nu}}{4} \tag{254}$$

Therefore, the PSD function of $\{X\}$ is given by

$$\Gamma_X(e^{j\omega}) = \frac{\pi}{2}\sum_{n=-\infty}^{\infty}[\delta(\omega - \omega_0 + 2\pi n) + \delta(\omega + \omega_0 + 2\pi n)] \tag{255}$$

This result indicates that the random process $\{X\}$ only presents signal power around the frequencies $(\pm\omega_0 + 2k\pi)$, for $k \in \mathbb{Z}$, despite its random phase component Θ. This is depicted in the following figure for frequencies in the interval $[-\pi, \pi]$.

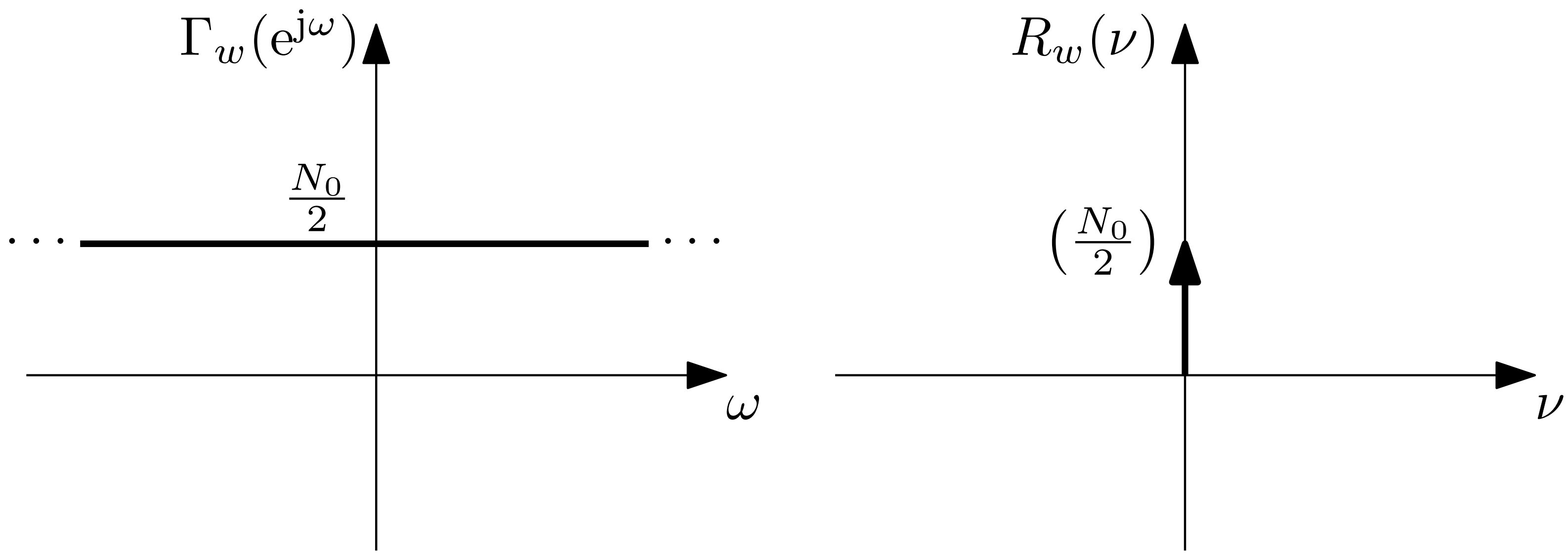


$\triangle$

White noise

- A very important class of random processes includes the so-called white noise, characterized by a constant PSD function for all values of ω.
- Using Wiener-Khinchin theorem, we easily infer that the autocorrelation function for the white noise is an impulse at the lag origin $\nu = 0$.
  - This indicates that for any non-zero lag ν, the white noise samples are statistically uncorrelated.
- The white noise nomenclature comes from the analogy to white light which includes all frequency components with similar power.
- White noise is a powerful tool in signal analysis, due to its simple descriptions in both frequency and time (lag) domains, as visualized in the next figure, making it quite suitable to model any unpredictable portion of a process.

White noise



(a) Frequency domain (b) Lag domain

- Due to its infinite average power, a perfect white noise can not be found in nature.
  - It, however, can be approximated well by pseudo-random generating algorithms, such as the MATLAB commands `rand` and `randn`, as will be explored in the Do-it-yourself section.

Do-it-yourself: The z and Fourier transforms

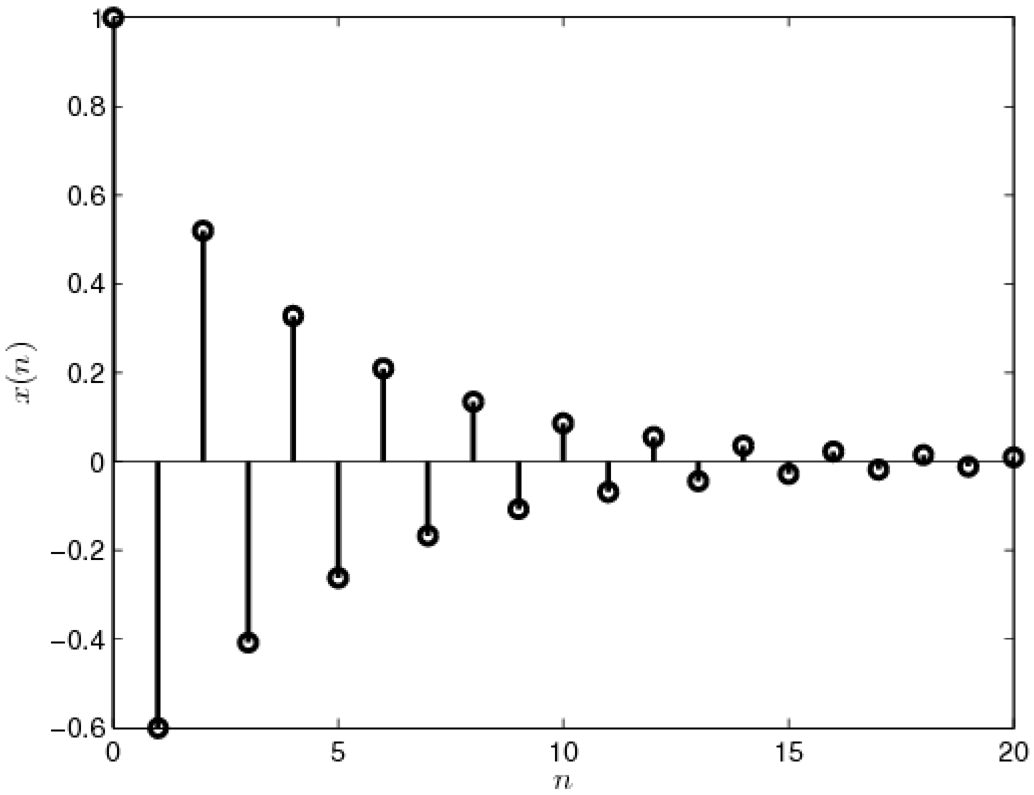
Experiment 2.1:

- We have that the residue of a pole p_k of $X(z)z^{n-1}$ with multiplicity m_k is equal to the coefficient c_{k1} of its partial-fraction expansion.
  - Therefore, the MATLAB command `residue` may be used to perform the inverse z transform of a given transfer function.
  - In this context, the `residue` command shall receive only two input parameters containing the numerator and denominator polynomials of $X(z)z^{n-1}$ in descending powers of z.
- Revisiting Example 2.3, the associated impulse response $x(n)$ can be determined for $0 \leq n \leq P$, with $P = 20$, as:

Do-it-yourself: The z and Fourier transforms

```
x = zeros(1,P+1);
num = [1 zeros(1,P+1)];
den = [1 0.6 -0.16];
for n = 0:P,
   [r,p,k] = residue(num(1:n+2),den);
   x(n+1) = sum(r);
end;
stem(0:P,x);
```

- In this short script, the `r` variable receives the desired residue values which are summed up to determine the $x(n)$ sequence depicted in the following figure.



- For a pole of multiplicity $m > 1$, the `residue` command evaluates equation (34) for $m_k = 1, 2, \ldots, m$.
  - In such cases, we must consider only the residue for $m_k = m$ in the summation that determines $x(n)$.

Do-it-yourself: The z and Fourier transforms

Experiment 2.2:

- In Experiment 1.3, we analyzed the behavior of a system whose output-input relationship is described by

$$y(n) = \frac{x(n) + x(n-1) + \cdots + x(n-N+1)}{N} \tag{256}$$

Taking this relation into the z domain and using the time-shift property associated to the z transform, we get the causal-form transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1} + \cdots + z^{-N+1}}{N} \tag{257}$$

- As explained before, by letting $z = e^{j\omega}$, with $0 \leq \omega \leq 2\pi$, we determine the frequency response of the system described in equation (257).

Do-it-yourself: The z and Fourier transforms

- In MATLAB, however, this response is easily obtained using the command `freqz`.
- In order to do so, we must first rewrite $H(z)$ in its rational-polynomial form with non-negative exponents.
- For large values of N, the numerator and denominator polynomials of $H(z)$ are efficiently defined with the matrix commands `zeros` and `ones` as exemplified here for $N = 10$:

```
num10 = ones(1,N);
den10 = [N, zeros(1,N-1)];
[H10,W] = freqz(num10,den10);
figure(1); plot(W,abs(H10));
```

- The last line generates the resulting magnitude response, which for $N = 10$ corresponds to the dashed curve in the next Figure.
- The phase response could have been obtained in a similar fashion replacing `abs` by the command `angle`.

Do-it-yourself: The z and Fourier transforms

- The group delay can be determined from the phase response or directly with the `grpdelay` command whose input and output arguments are the same as of the `freqz` command.

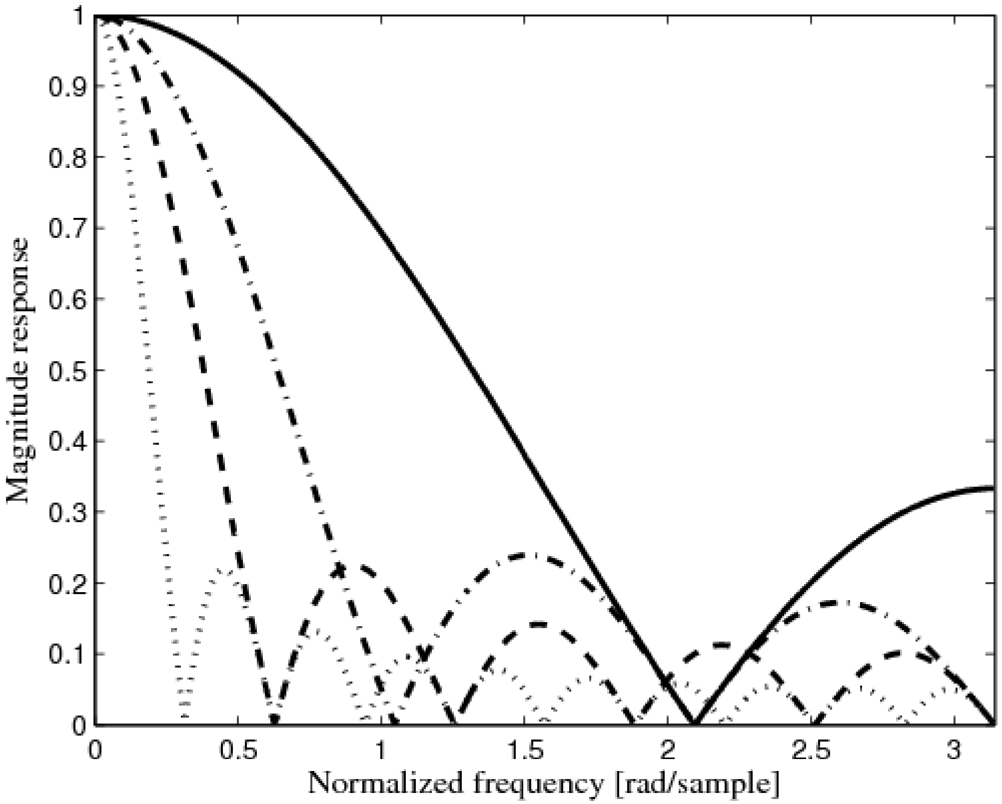
- Repeating the script above for $N = 3, 6, 20$, and changing the variable names accordingly, we generate next Figure, which indicates that equation (257) corresponds to a lowpass system whose bandwidth decreases with N.

- In Experiment 1.3, two sinusoidal components of frequencies $f_1 = 1$ Hz and $f_2 = 50$ Hz were digitally processed by equation (256) with $F_s = 1000$ samples/second.
  - In the normalized-frequency scale employed in the next Figure, these components correspond to $\omega_1 = \frac{2\pi f_1}{F_s} \approx 0.006$ and $\omega_2 = \frac{2\pi f_2}{F_s} \approx 0.314$ rad/sample.

Do-it-yourself: The z and Fourier transforms

- The next figure explains how the time-averaging process of equation (256) is able to reduce the amount of noise as N increases.
  - However, even the sinusoidal components are affected significantly if N becomes too large.
- This motivates us to search for better ways to process $x(n)$ in order to reduce noise without affecting the original signal components.

Do-it-yourself: The z and Fourier transforms



Magnitude responses of linear system defined in equation (256) for $N = 3$ (solid-line), $N = 6$ (dash-dotted line), $N = 10$ (dashed line), and $N = 20$ (dotted line).

Do-it-yourself: The z and Fourier transforms

- Determining the causal form of equation (257), we get that

$$H(z) = \frac{z^{N-1} + z^{N-2} + \cdots + 1}{Nz^{N-1}} \tag{258}$$

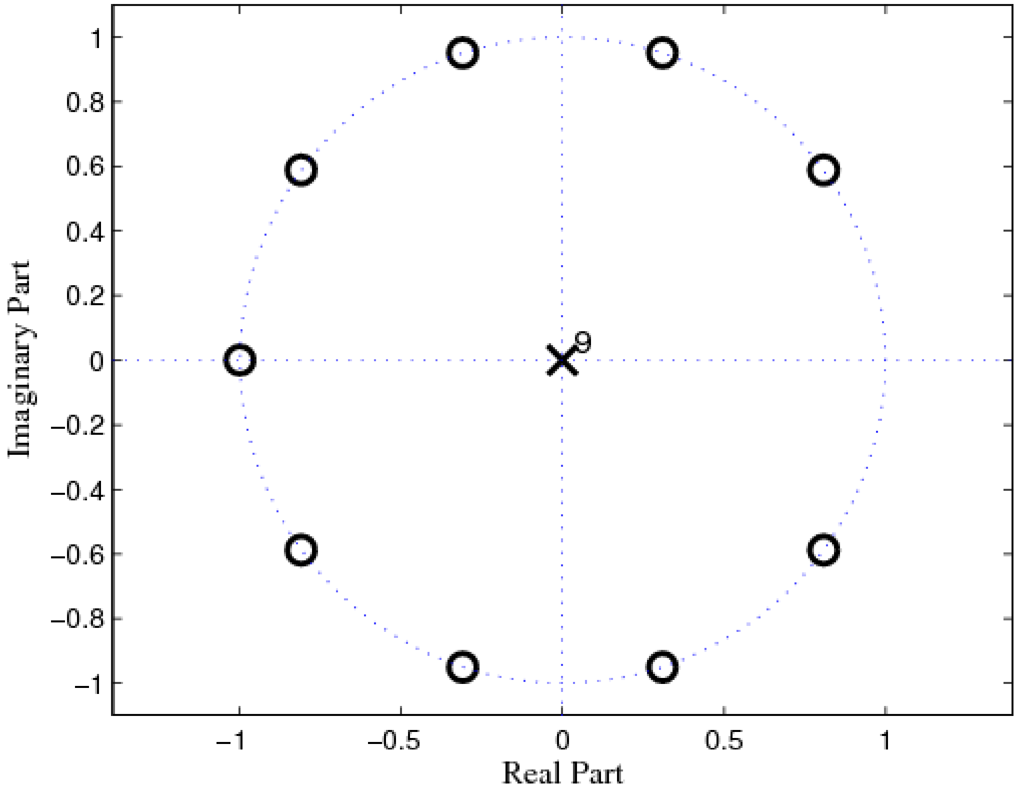
- Clearly, the associated system presents $(N-1)$ poles at the origin of the complex z plane and $(N-1)$ zeros equally spread (except at $z=1$) around the unit circle, as indicated in the next for $N=10$, obtained with the single command line:

```
zplane(num10,den10);
```

  where `num10` and `den10` are line vectors as specified above.

  - The numerical values of these zeros and poles can be determined by the `roots` command, which, as the name indicates, calculates the roots of a given polynomial, or by using the auxiliary commands `tf2zp` and `zp2tf`, which decomposes the numerator and denominator polynomials of a given transfer function into first-order factors and vice versa.

Do-it-yourself: The z and Fourier transforms



Zero (circles) and pole (cross) constellation of transfer function given in equation (258) with $N = 10$.

Do-it-yourself: The z and Fourier transforms

Experiment 2.3:

- The geometric computation of the magnitude and phase of a transfer function can be used to perform intuitive designs of digital filters.
  - The functions `zp2tf`, which generates a transfer function given the positions of its poles and zeros, and `freqz`, which generates the magnitude and phase responses of a given transfer function are important tools for such designs.

Do-it-yourself: The z and Fourier transforms

- Suppose we want to design a filter that provides a significant magnitude response only for frequencies around $\frac{\pi}{4}$.
  - One way of achieving this is to generate a transfer function that has a pole near the unit circle with phase $\frac{\pi}{4}$ (one should remember that for transfer functions with real coefficients, a complex pole or zero must be accompanied by its complex conjugate).
  - This is so because, since the denominator tends to be small around this pole, the magnitude response tends to be large.
  - In addition, we can decrease it at the other frequencies by placing zeros at $z = 1$ and $z = -1$, forcing a zero response at the frequencies $\omega = 0$ and $\omega = \pi$ rad/sample.
  - This pole-zero placement is depicted in Figure 1a, where $p_1 = 0.9e^{j\frac{\pi}{4}}$.

Do-it-yourself: The z and Fourier transforms

- The corresponding magnitude response is shown in Figure 1b.
- In can be seen that the designed filter has indeed the desired magnitude response, with a pronounced peak around $\frac{\pi}{4}$.
  - Note that, the closer the magnitude of the pole is to the unit circle, the more pronounced is the peak of the magnitude response. A MATLAB code that generates this example is given below.

```
p1 = 0.9*exp(j*pi/4);
Z = [1 -1 ].'; P = [p1 p1'].';
[num,den] = zp2tf(Z,P,1);
[h,w] = freqz(num,den);
plot(w,abs(h)/max(abs(h)));
```

- The reader is encouraged to explore the effect of the pole's magnitude on the frequency response.

Do-it-yourself: The z and Fourier transforms

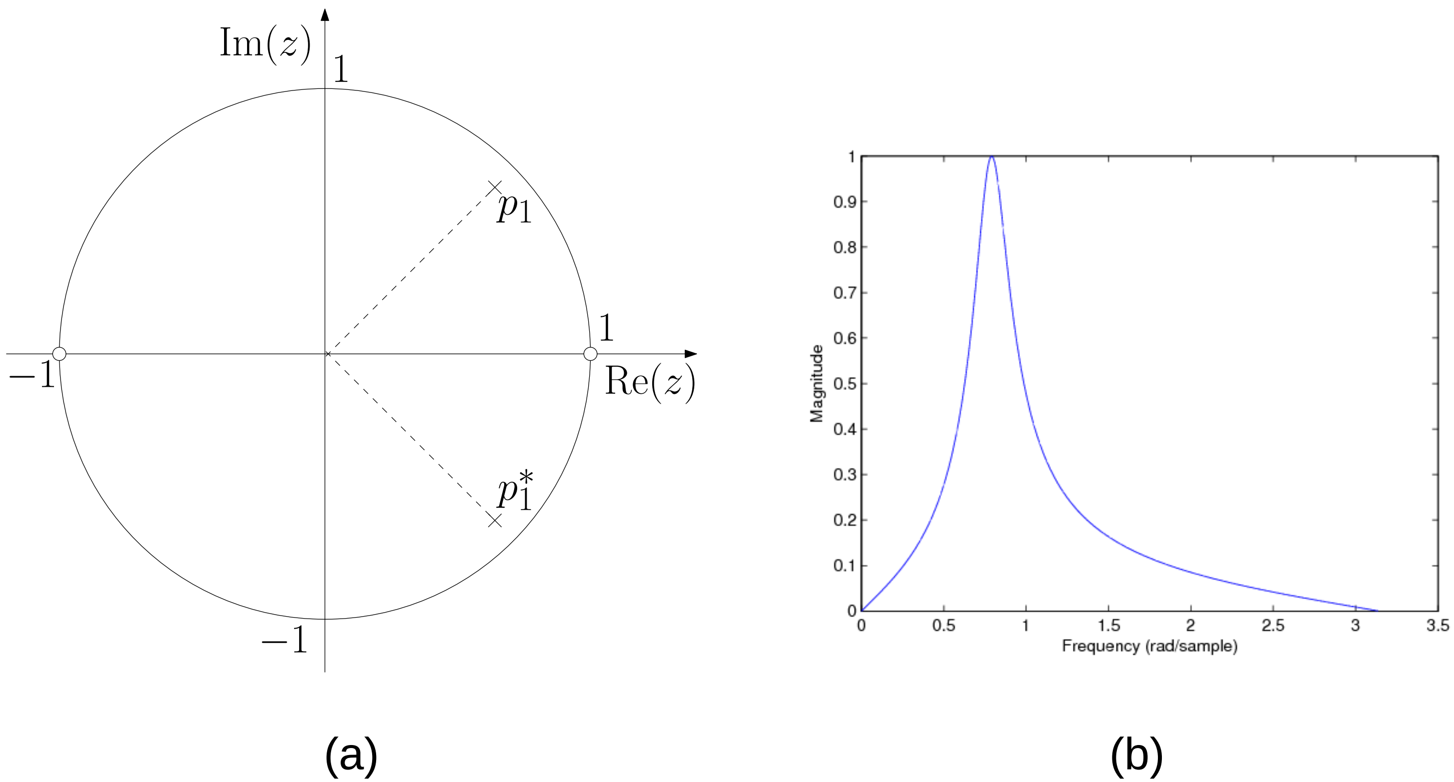
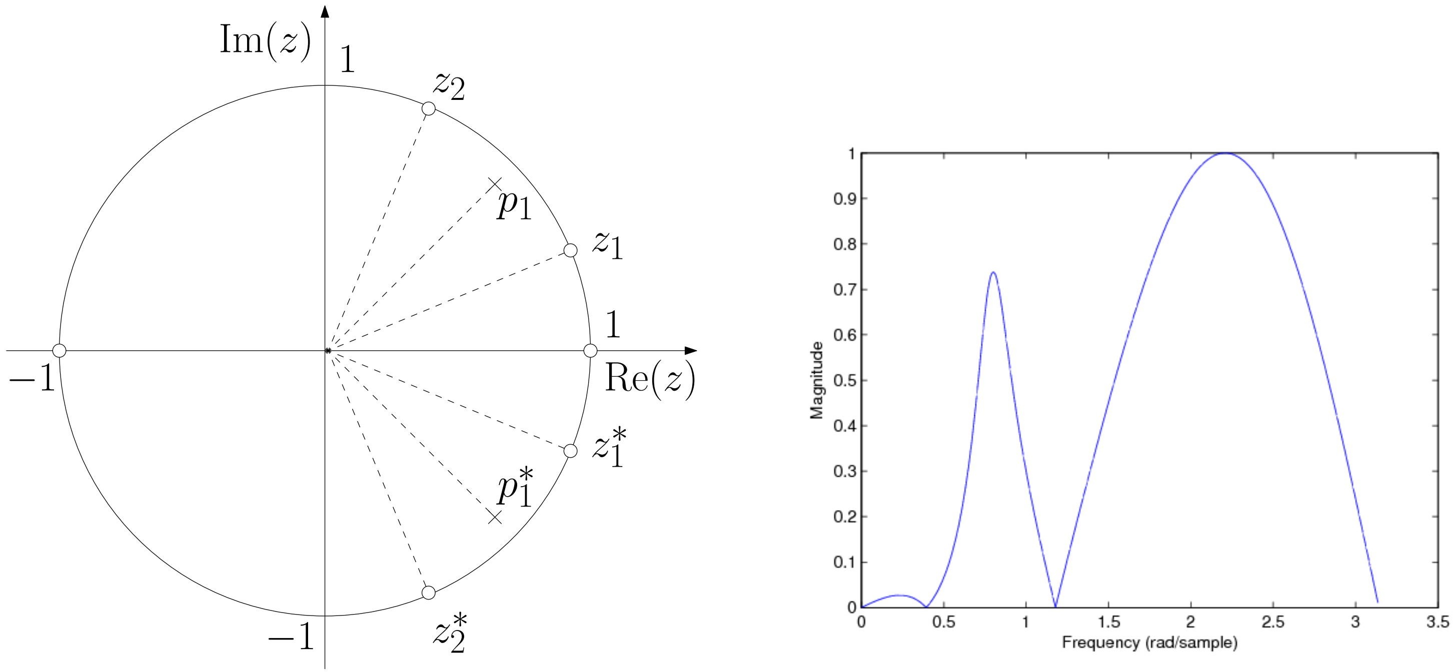


Figure 1: Pole-zero placement for Experiment 2.3 and its corresponding frequency response. $p_1 = 0.9e^{j\frac{\pi}{4}}$.

Do-it-yourself: The z and Fourier transforms

- If we want the filter to be even more selective, one option is to place zeros around the center frequency.
  - In order to achieve this, we have to insert four extra zeros on the unit circle, one conjugate pair with phases $\pm\frac{\pi}{8}$ and another with phases $\pm\frac{3\pi}{8}$.
  - The effect obtained is depicted in Figure 2.

Do-it-yourself: The z and Fourier transforms



(a) (b)

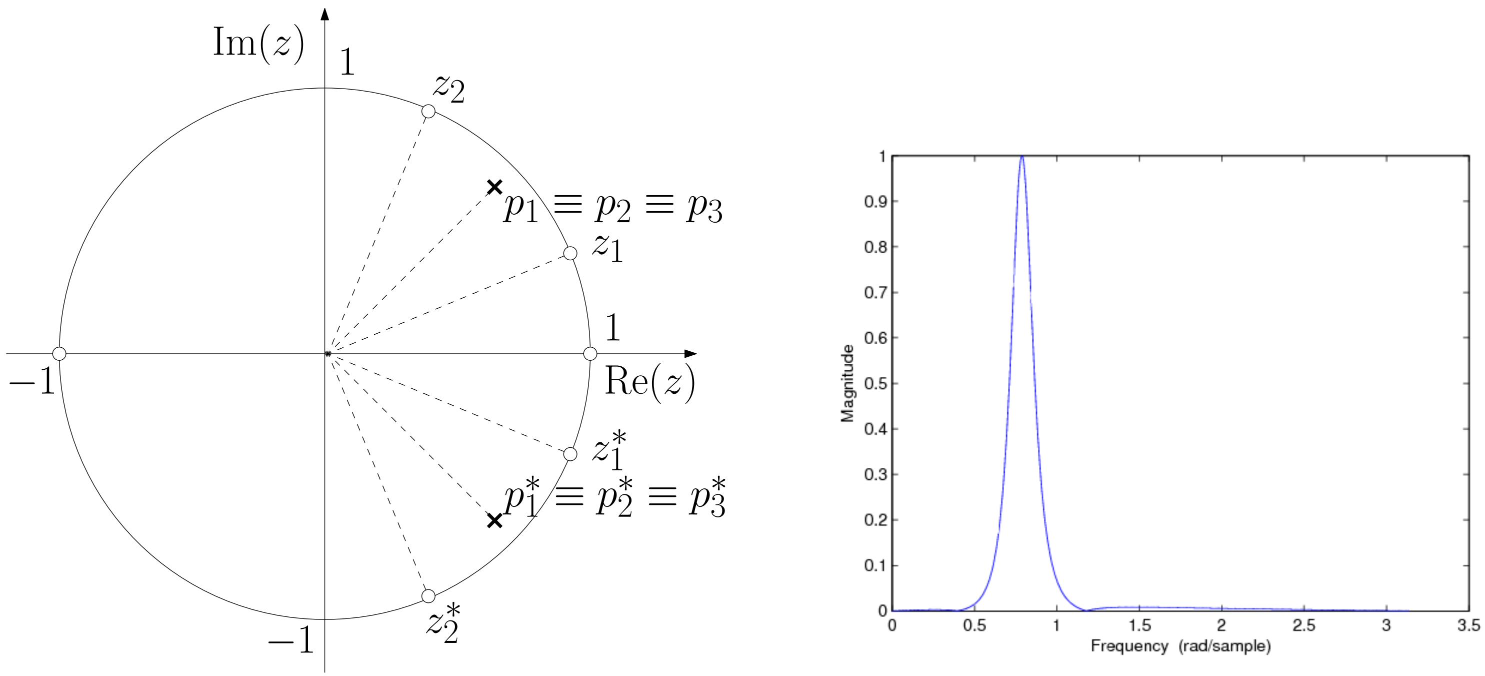
Figure 2: Pole-zero placement for Experiment 2.3 and its corresponding frequency response. $p_1 = 0.9e^{j\frac{\pi}{4}}$, $z_1 = e^{j\frac{\pi}{8}}$ and $z_2 = e^{j\frac{3\pi}{8}}$.

Do-it-yourself: The z and Fourier transforms

- We can see that the inserted zeros have produced, as an undesirable side effect, a large magnitude response around $\omega = \frac{3\pi}{2}$.
- This draws our attention to a care that must be taken when designing transfer functions through the placement of poles and zeros.
  - At frequencies far from the locations of the poles and zeros, all the product terms in both the numerator and the denominator of equation (197) tend to be large.
  - Then, when the number of factors in the numerator (zeros) is larger than the number of factors in the denominator (poles), the magnitude of the transfer function tends to be also large.
  - One way to solve this problem is to have as much poles as zeros in the transfer function.
  - In the present case, since we have 6 zeros (at ± 1, $e^{\pm \frac{\pi}{8}}$ and $e^{\pm \frac{3\pi}{8}}$), we can achieve this by making the poles at $z = 0.9e^{\pm \frac{\pi}{4}}$ triple.
- The magnitude response obtained is depicted in Figure 3.

Do-it-yourself: The z and Fourier transforms

- One can see that the counterbalancing of the six zeros with six poles has the desired effect.



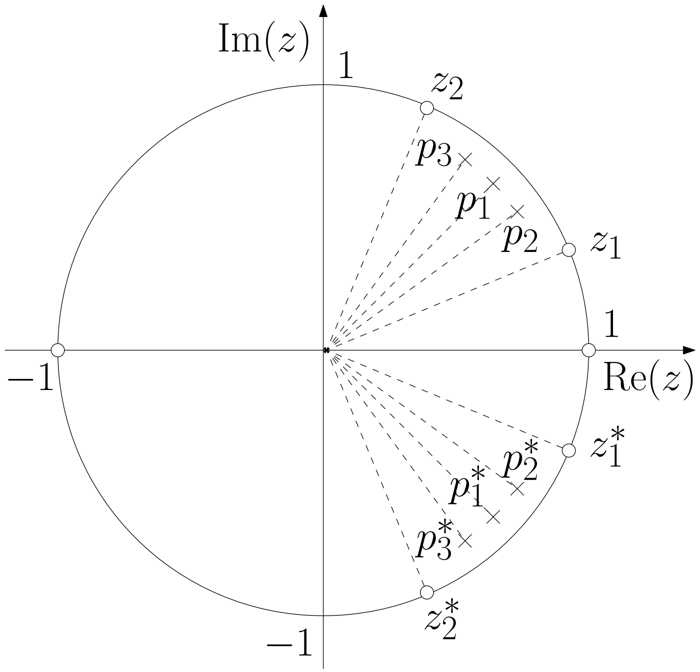
(a) (b)

Figure 3: Pole-zero placement for Experiment 2.3 and its corresponding frequency response. $p_1 = p_2 = p_3 = 0.9e^{j\frac{\pi}{4}}$, $z_1 = e^{j\frac{\pi}{8}}$ and $z_2 = e^{j\frac{3\pi}{8}}$.

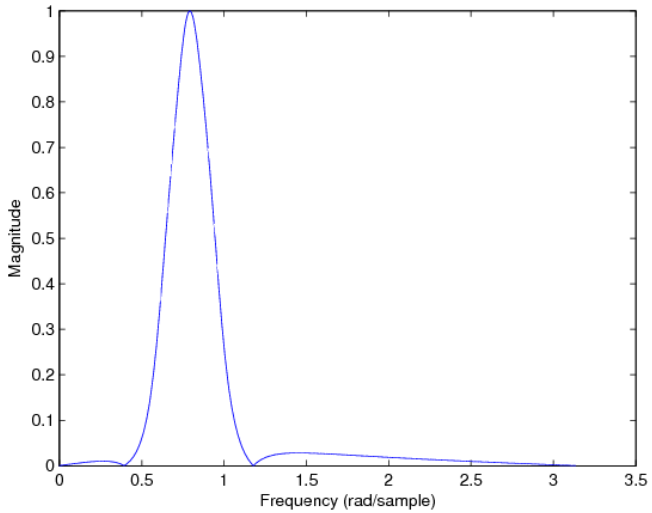
Do-it-yourself: The z and Fourier transforms

- As a further illustration, we now attempt to make the magnitude response in Figure 3b less spiky without significantly affecting its bandwidth.
  - In order to do so, we slightly move the two poles p_2 and p_3 away from p_1.
  - Figure 3 shows the effect obtained when p_2 and p_3 are rotated by $-\frac{\pi}{20}$ and $\frac{\pi}{20}$, respectively.
  - One can see that the magnitude response is indeed less spiky.

Do-it-yourself: The z and Fourier transforms



(a)



(b)

Figure 4: Pole-zero placement for Experiment 2.3 and its corresponding frequency response. $p_1 = 0.9e^{j\frac{\pi}{4}}$, $p_2 = p_1 e^{-j\frac{\pi}{20}}$, $p_3 = p_1 e^{j\frac{\pi}{20}}$, $z_1 = e^{j\frac{\pi}{8}}$ and $z_2 = e^{j\frac{3\pi}{8}}$.

Do-it-yourself: The z and Fourier transforms

- The MATLAB code to generate Figure 4 is shown below.

```
z1 = exp(j*pi/8);
z2 = exp(j*3*pi/8);
p1 = 0.9*exp(j*pi/4);
p2 = 0.9*exp(j*pi/4 - j*pi/20);
p3 = 0.9*exp(j*pi/4 + j*pi/20);
Z = [1 -1 z1 z1' z2 z2'].';
P = [p1 p1' p2 p2' p3 p3'].';
[num,den] = zp2tf(Z,P,1);
[h,w] = freqz(num,den);
plot(w,abs(h)/max(abs(h)));
```