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Chapter 5

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Supplement: Derivation of the basic Boyd-Kleinman expression for SHG with circular Gaussian beams

The derivation of the "Boyd-Kleinman focusing formula" is long and tedious and was mercifully omitted from the text of "Compact Blue–Green Lasers." However, some particularly intrepid readers may wish to see how this formula can be derived, rather than accepting it uncritically. In this chapter of the supplement, the derivation of Equation 2.32 in the text is presented.

Boyd and Kleinman [Boyd and Kleinman (1968)] examined the case of Type I second-harmonic generation using a focused, circular Gaussian beam in a material that exhibits walkoff. The treatment given here closely follows their analysis, although the notation is slightly different and we will use *SI*, rather than *cgs*, units. (Some of the symbols and notation used here differs from that used by Boyd and Kleinman, in order that the usage here be consistent with the rest of "CBGL").

We begin with an input electric field which is a Gaussian beam:

$$E_1(r,z,t) = E_o \frac{w_o}{w(z)} e^{-\alpha_1 z} e^{-\frac{r^2}{w^2(z)}} \cos\left(\omega_1 t - k_1 z + \Psi(r,z) + \Phi(z)\right) \qquad (5.1)$$

where α_1 is the attenuation coefficient, and ω_1 and k_1 are as previously defined. Since we assume that the beam is radially symmetric about the z axis, we specify the distance perpendicular to the z-axis by $r = [x^2 + y^2]^{\frac{1}{2}}$. At a distance $r = w(z)$ from the axis, the amplitude of the electric field has fallen to $1/e$ of its value on the z-axis. This value $w(z)$ is called the $1/e$ radius of the beam. The minimum value of this $1/e$ radius occurs at $z = 0$ and is designated as w_o, the "beam waist". The $1/e$ radius of the beam varies as:

$$w\left(z\right)=w_o\left[1+\left(\frac{\lambda z}{\pi n w_o^2}\right)^2\right]^{\frac{1}{2}} \tag{5.2}$$

The term $\frac{w_o}{w\left(z\right)}$ in Eq. 5.1 indicates that as the beam propagates away from $z=0$ and the beam expands, the peak amplitude falls.

In a plane wave, the phase is constant on any given plane perpendicular to the direction of propagation. However, in a Gaussian beam, the surfaces of constant phase are curved; this curvature is described by the term:

$$\Psi\left(r,z\right)=\frac{kr^2}{2R(z)} \tag{5.3}$$

where $R(z)$ is given by:

$$R\left(z\right)=z\left[1+\left(\frac{\pi n w_o^2}{\lambda z}\right)^2\right] \tag{5.4}$$

At $z=0$, $R(z)=\infty$, i.e., the surface of constant phase is a plane. At $z=\infty$, we also have $R(z)=\infty$, i.e., sufficiently far away from the waist, the beam behaves as a plane wave does. At a distance $z=\frac{\pi n w_o^2}{\lambda}$, the curvature of the wavefronts is maximum. This distance is called the Rayleigh range, z_r. An important related quantity is the *confocal parameter* $b=2z_r$. The far-field divergence angle of the beam is $\theta_o=\lambda/\pi n w_o$.

Finally, in Eq. 5.1, there is a z-dependent phase shift term, given by:

$$\Phi\left(z\right)=\tan^{-1}\left(\frac{z}{z_r}\right) \tag{5.5}$$

The approach underlying the Boyd-Kleinman treatment is illustrated in Figure S-5-1 (which is the same as Figure 2.15 in “CBGL”). We conceptually divide up the crystal into slices of infinitesimal width. In each slice, the interaction of the fundamental beam with the nonlinear crystal generates second-harmonic light. This second-harmonic light propagates through the crystal to arrive at the “observation plane”. To determine the total second-harmonic field produced by the crystal, we must coherently add all the contributions from these infinitesimal slices.

However, a complication arises for anisotropic media. Here we will treat the case of so-called “Type-I” phasematching, in which the fundamental and second-harmonic waves have orthogonal linear polarizations. As discussed in Chapter 2 of “Compact Blue–Green Lasers,” many advantages accrue if the direction of propagation lies along a crystallographic axis, so that non-critical phasematching is achieved. However, it is very often the case that phasematching cannot be achieved for the wavelengths

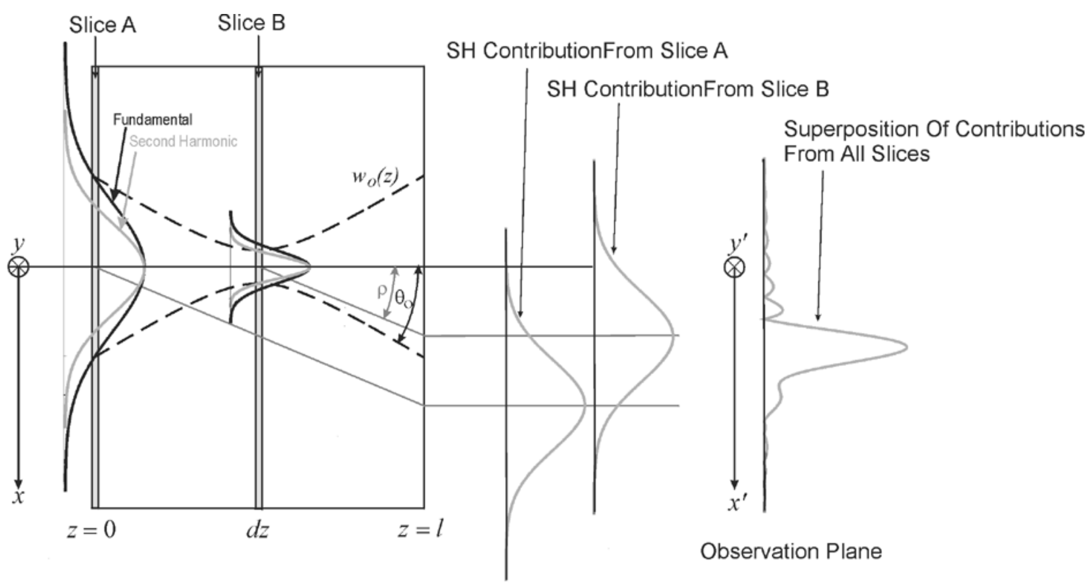

Figure 5.1: Figure S-5-1. Geometry for the Boyd-Kleinman analysis of SHG by a focused Gaussian beam.

of interest unless propagation is at some angle to the crystallographic axes (critical phasematching). When propagation is not along a crystallographic axis, the effect of birefringent walkoff must be taken into account. Thus, as shown in Figure S-5-1, the generated second-harmonic wave propagates through the crystal at an angle ρ relative to the fundamental wave. The result of this walkoff is that the SH contribution from slices near the input end of the crystal are spatially displaced to a greater degree than the SH contributions arising from near the output end of the crystal. This lateral spatial displacement results in an output SH beam which may no longer have a Gaussian distribution.

We now apply the procedure just described in a more quantitative way. We can write the Gaussian beam of Eq. 5.1 as:

$$E_1(r,z,t) = \frac{E_o}{2}\frac{w_o}{w\left(z\right)}e^{-\alpha_1 z}e^{-\frac{r^2}{w^2(z)}}\left\{e^{j\omega_1 t}e^{-jk_1 z}e^{j\Psi(r,z)}e^{j\Phi(z)} + c.c.\right\} \tag{5.6}$$

If we take the Fourier transform of this expression, we obtain:

$$\mathcal{E}_1\left(r,z,\omega\right) = 2\pi e^{-\alpha_1 z}\left[\widetilde{E}_1\left(r,z\right)\delta\left(\omega-\omega_1\right) + \widetilde{E}_1^*\left(r,z\right)\delta\left(\omega+\omega_1\right)\right] \tag{5.7}$$

where

$$\widetilde{E}_1\left(r,z\right) = \frac{E_o}{2}\frac{w_o}{w\left(z\right)}e^{-\frac{\alpha_1}{2}z}e^{-\frac{r^2}{w^2(z)}}e^{-jk_1 z}e^{j\Psi(r,z)}e^{j\Phi(z)} \tag{5.8}$$

Looking at Eqs. 5.2 and 5.4, we see that the quantity $\frac{\lambda z}{\pi n w_o^2} = \frac{z}{z_r} = \frac{2z}{b}$ or its reciprocal appears frequently. In order to compact our notation we introduce $\zeta = \frac{2z}{b}$. Then we can write:

$$w\left(z\right) = w_o \left[1+\zeta^2\right]^{\frac{1}{2}} \tag{5.9}$$

$$e^{j\Psi(r,z)} = e^{j\frac{kr^2}{2R(z)}} = e^{j\frac{\zeta r^2}{w_o^2(1+\zeta^2)}} \tag{5.10}$$

$$e^{j\Phi(z)} = \frac{1-j\zeta}{\sqrt{1+\zeta^2}} \tag{5.11}$$

and Equation 5.8 becomes:

$$\widetilde{E}_1\left(r,z\right) = \frac{E_o}{2} e^{-\frac{\alpha_1}{2}z} \frac{1}{\sqrt{1+\zeta^2}} e^{-\frac{r^2}{w_o^2(1+\zeta^2)}} e^{-jk_1 z} e^{j\frac{\zeta r^2}{w_o^2(1+\zeta^2)}} \frac{1-j\zeta}{\sqrt{1+\zeta^2}} \tag{5.12}$$

$$= \frac{E_o}{2} e^{-\frac{\alpha_1}{2}z} \frac{1}{1+j\zeta} e^{-\frac{r^2}{w_o^2(1+j\zeta)}} e^{-jk_1 z} \tag{5.13}$$

The Gaussian beam as we have written it here has its waist at $z = 0$, which is normally where we consider the input face of the crystal to be. We can make our expression more general by placing the waist at some position $z = f$, and we can achieve this change in Equation 5.12 simply by now making $\zeta = 2\left(z-f\right)/b$. The induced polarization is given by:

$$\widetilde{P}^{(\omega_3)} = 2\epsilon_o d_{eff}\left[\widetilde{E}^{(\omega_1)}\right]^2 = \epsilon_o d_{eff} \frac{E_o^2}{2} e^{-\alpha_1 z}\left(\frac{1}{1+j\zeta}\right)^2 e^{-\frac{2r^2}{w_o^2(1+j\zeta)}} e^{-2jk_1 z} \tag{5.14}$$

and we can write Equation **??** as

$$d\widetilde{E}_3\left(x\right) = \frac{-j}{2\epsilon_o n_3^2} k_3 e^{jk_3 z} \tilde{P}^{(\omega_3)} dz \tag{5.15}$$

$$= \frac{-jk_3}{2n_3^2} e^{j\Delta k z} d_{eff} \frac{E_o^2}{2} e^{-\alpha_1 z} \frac{1}{1+j\zeta}\left\{\frac{1}{1+j\zeta} e^{-\frac{2r^2}{w_o^2(1+j\zeta)}}\right\} dz \tag{5.16}$$

This expression gives the contribution to the second-harmonic field from a slab of the nonlinear crystal with width dz, located at a position z inside the crystal ($0 < z < l$).

Note by comparing Eq.5.16 to Eq. 5.13 that the quantity in the braces has the form appropriate to a Gaussian beam with spotsize $w_o/\sqrt{2}$, confocal parameter b, and focus at $z = f$.

We would like to determine the contribution of this increment on the total second-harmonic field detected by a remote observer, stationed outside the crystal at location $x^{'}, y^{'}, z^{'}$. In order to avoid having to deal with reflection and refraction at the ends of the crystal, we will consider the crystal to be immersed in a medium having the same refractive index. Because of walkoff (which we assume to occur in the $x - z$ plane), the contribution generated at the point x, y, z contributes to the detected field at the point $x^{'} = x + \rho\,(l - z)$ and $y^{'} = y$.The Gaussian portion in the braces above can be "propagated" to the observer's location by substituting for ζ the quantity $\zeta^{'} = 2\left(z^{'} - f\right)/b$ and by using the relationship between x and $x^{'}$ and y and $y^{'}$ given above:

$$d\widetilde{E}_3\left(r^{'}, z^{'}\right) = \frac{-jk_3}{2n_3^2} e^{j\Delta kz} d_{eff} \frac{E_o^2}{2} e^{-\alpha_1 z} e^{-\frac{\alpha_3}{2}(l-z)} \frac{1}{1+j\zeta} \left\{ \frac{1}{1+j\zeta^{'}} e^{-\frac{2\left\{\left[x^{'}-\rho(l-z)\right]^2+\left[y^{'}\right]^2\right\}}{w_o^2\left(1+j\zeta^{'}\right)}} \right\} dz \tag{5.17}$$

where we have included a term to account for loss of the second-harmonic as it propagates from its point of origin to the observation plane. Note that the exponent no longer contains a simple r^2, circularly-symmetric dependence. The spatial variation of the field in the transverse plane is now more complicated, due to the effects of walkoff. In order to determine the total amplitude at point $x^{'}, y^{'}, z^{'}$, we integrate all the contributions from the different z locations:

$$\widetilde{E}_3\left(r^{'}, z^{'}\right) = \frac{-jk_3}{2n_3^2} d_{eff} \frac{E_o^2}{2} e^{-\frac{\alpha_3}{2}l} \int_0^l \frac{e^{-\alpha z} e^{j\Delta kz}}{1+j\zeta} \left\{ \frac{1}{1+j\zeta^{'}} e^{-\frac{2\left\{\left[x^{'}-\rho(l-z)\right]^2+\left[y^{'}\right]^2\right\}}{w_o^2\left(1+j\zeta^{'}\right)}} \right\} dz \tag{5.18}$$

where $\alpha = \alpha_1 - \frac{1}{2}\alpha_3$.We now let the observation point approach an infinite distance from the crystal, i.e., $\zeta^{'} \to \infty$. Then:

$$\frac{1}{w_o^2\left(1+j\zeta^{\prime}\right)} = \frac{1-j\zeta^{'}}{w_o^2\left(1+\zeta^{\prime 2}\right)} = \frac{1-j\zeta^{'}}{w_o^2 \zeta^{\prime 2}\left(\frac{1}{\zeta^{\prime 2}}+1\right)} \to \frac{1-j\zeta^{'}}{w_o^2\zeta^{\prime 2}} \tag{5.19}$$

We define some additional normalized spatial coordinates:

$$u = \frac{x' - \rho(l-f)}{w_o\zeta'} \tag{5.20}$$

$$v = \frac{y'}{w_o\zeta'} \tag{5.21}$$

$$\beta = \frac{\rho}{\theta_o} \tag{5.22}$$

We can then show that

$$\frac{\left[x' - \rho(l-z)\right]^2}{w_o^2(1+j\zeta')} \approx \frac{\left(1-j\zeta'\right)\left[x' - \rho(l-z)\right]^2}{w_o^2\zeta'^2} = \left(1-j\zeta'\right)\left[u+\beta\frac{\zeta}{\zeta'}\right]^2 \approx u^2\left(1-j\zeta'\right) - 2ju\beta\zeta \tag{5.23}$$

where, in dropping some terms from the last portion of the expression, we have explicitly let $\zeta' \to \infty$.Note that ζ' depends on z' rather than z, so that terms involving it can be taken outside the integral. Then we have:

$$\widetilde{E}_3\left(r', z'\right) = \frac{-k_3}{2n_3^2} d_{eff} \frac{E_o^2}{2\zeta'} e^{-\frac{\alpha_3}{2}l} e^{-2\left(1-j\zeta'\right)\left(u^2+v^2\right)} \int_0^l \frac{e^{-\alpha z} e^{j\Delta kz} e^{4ju\beta\zeta}}{1+j\zeta} dz \tag{5.24}$$

If we make a change of variables in the integral, using $\zeta = 2(z-f)/b$, and therefore $d\zeta = 2\ dz/b$, we obtain for the integral:

$$\int_0^l \frac{e^{-\alpha z} e^{j\Delta kz} e^{4ju\beta\zeta}}{1+j\zeta} dz = \int_{-\frac{2f}{b}}^{\frac{2(l-f)}{b}} \frac{b}{2} \frac{e^{-\frac{\alpha b\zeta}{2}} e^{-\alpha f} e^{j\frac{\Delta kb\zeta}{2}} e^{j\Delta kf} e^{4ju\beta\zeta}}{1+j\zeta} d\zeta \tag{5.25}$$

If we define $\kappa = \dfrac{\alpha b}{2}$, $\sigma = \dfrac{\Delta kb}{2}$, $\sigma' = \sigma + 4\beta u$, $\mu = \dfrac{l-2f}{l}$, and $\xi = \dfrac{l}{b}$, we can write the integral as:

$$\frac{b}{2} e^{-\alpha f} e^{j\Delta kf} \int_{-\xi(1-\mu)}^{\xi(1+\mu)} \frac{e^{-\kappa\zeta} e^{j\sigma'\zeta}}{1+j\zeta} d\zeta \tag{5.26}$$

We define

$$H\left(\sigma', \kappa, \xi, \mu\right) = \frac{1}{2\pi} \int_{-\xi(1-\mu)}^{\xi(1+\mu)} \frac{e^{-\kappa\zeta} e^{j\sigma'\zeta}}{1+j\zeta} d\zeta \tag{5.27}$$

Substituting into Eq. 5.24 gives:

$$\widetilde{E}_3\left(r^{'}, z^{'}\right) = \frac{-k_3}{2n_3^2} d_{eff} \frac{E_o^2}{2\zeta^{'}} e^{-\frac{\alpha_3}{2}l} e^{-2\left(1-j\zeta^{'}\right)\left(u^2+v^2\right)} \frac{b}{2} e^{-\alpha f} e^{j\Delta k f} \left[2\pi H\left(\sigma^{'}, \kappa, \xi, \mu\right)\right] \quad (5.28)$$

The second-harmonic intensity is:

$$I_3\left(r^{'}, z^{'}\right) = \frac{1}{8\eta_3} \frac{k_3^2}{n_3^4} d_{eff}^2 \frac{E_o^4}{\zeta^{'2}} e^{-\alpha_3 l} e^{-4\left(u^2+v^2\right)} b^2 \pi^2 e^{-2\alpha f} \left|H\left(\sigma^{'}, \kappa, \xi, \mu\right)\right|^2 \quad (5.29)$$

We can determine the total power in the SH beam by integrating the intensity over area. The result is:

$$P_3 = \frac{1}{8\eta_3} \frac{k_3^2}{n_3^4} d_{eff}^2 \frac{E_o^4}{\zeta^{'2}} e^{-\alpha_3 l} b^2 \pi^2 e^{-2\alpha f} \int dx\, dy e^{-4\left(u^2+v^2\right)} \left|H\left(\sigma^{'}, \kappa, \xi, \mu\right)\right|^2 \quad (5.30)$$

$$= \frac{1}{8\eta_3} \frac{k_3^2}{n_3^4} d_{eff}^2 \frac{E_o^4}{\zeta^{'2}} e^{-\alpha_3 l} b^2 \pi^2 e^{-2\alpha f} \left[w_o \zeta^{'} \int_{-\infty}^{\infty} du\, e^{-4u^2} \left|H\left(\sigma^{'}, \kappa, \xi, \mu\right)\right|^2\right] \left[w_o \zeta^{'} \int_{-\infty}^{\infty} dv\, e^{-4v^2}\right] \quad (5.31)$$

We define the quantities:

$$F\left(\sigma, \beta, \kappa, \xi, \mu\right) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} du\, e^{-4u^2} \left|H\left(\sigma + 4\beta u, \kappa, \xi, \mu\right)\right|^2 \quad (5.32)$$

and

$$h(\sigma, \beta, \kappa, \xi, \mu) = \frac{\pi^2}{\xi} e^{\mu \alpha l} F\left(\sigma, \beta, \kappa, \xi, \mu\right) \quad (5.33)$$

We can then write the SH power, after some simplification, as:

$$P_3 = \frac{1}{32\eta_3} \frac{k_3^2}{n_3^4} d_{eff}^2 E_o^4 b^2 \pi w_o^2 e^{-\alpha^{'} l} \xi h(\sigma, \beta, \kappa, \xi, \mu) \quad (5.34)$$

where $\alpha^{'} = \alpha_1 + \frac{1}{2}\alpha_3$. We can calculate the power in the fundamental beam to be:

$$P_1 = \frac{E_o^2}{2\eta_1} \frac{\pi w_o^2}{2} \quad (5.35)$$

After further simplification, we arrive at the result:

$$P_3 = \frac{16\pi^2 d_{eff}^2}{\epsilon_o c \lambda_1^3 n_3 n_1} P_1^2 e^{-\alpha^{'} l} l h(\sigma, \beta, \kappa, \xi, \mu) \quad (5.36)$$

which is Equation 2.32 of “CBGL”.

CT # Bibliography

[Boyd and Kleinman (1968)] Boyd, G. D. and Kleinman, D. A. (1968). Parametric Interaction of Focused Gaussian Light Beams. *J. Appl. Phys.*, **39**, 3597-3639.