Elastic strain energy

The objective of this section is to introduce the elastic strain energy and coenergy and compute it for a few simple examples.

Elastic strain energy density (strain energy per unit volume) is given by:

$$w_{\mathrm{es}} = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right). \tag{D1}$$

The total strain energy within a linear elastic body is the integral of the energy density over the volume of the body V

$$W_{\mathrm{es}} = \int_V w_{\mathrm{es}} dv = \frac{1}{2}\int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right) dv\,. \tag{D2}$$

Simple examples

Let us evaluate the strain energy for a few simple examples: axially loaded bar, rod in torsion, and a beam in pure bending.

Axially loaded beams

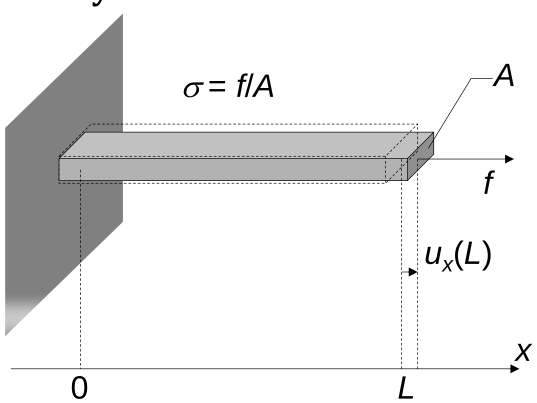

Figure 1: A bar in tension.

The stress is given by

$$\sigma_x = \frac{f_x}{A}\,. \tag{D3}$$

Assuming small displacement, within the linear range, the Hooke's law holds and

$$\varepsilon_x = \frac{\sigma_x}{Y} = \frac{f_x}{AY}\,. \tag{D4}$$

Inserting Eqs. (D3) and (D4) into Eq. (D2) yields the total elastic strain energy

$$W_{\mathrm{es}} = \frac{1}{2}\int_V \frac{f_x^2}{A^2Y} dv = \frac{1}{2}\int_0^L \frac{f_x^2}{A^2Y} A dx = \frac{1}{2}\int_0^L \frac{f_x^2}{AY} dx\,. \tag{D5}$$

Assuming constant area A and constant Young's modulus Y, elastic energy simplifies to

$$W_{\mathrm{es}} = \frac{1}{2}\frac{L}{AY} f_x^2 = \frac{1}{2} a_{\mathrm{eq}} f_x^2 \,. \tag{D6}$$

where a_{eq} is the equivalent flexibility coefficient.

In the case A = const and Y = const, integration of strain ε_x over the length of the beam yields the maximum displacement at the tip of the beam

$$u_x(L) = \int_0^L \varepsilon_x dx = \frac{L}{AY} f_x \,. \tag{D7}$$

The same result can be obtain by differentiation:

$$u_x(L) = \frac{\partial W_{\mathrm{st}}}{\partial f_x} = \frac{L}{AY} f_x \,. \tag{D8}$$

Expressing f_x in terms of the maximum displacement and inserting the result into Eq. (D5) yields

$$W_{\mathrm{es}} = \frac{1}{2}\frac{AY}{L} u_x^2(L) = \frac{1}{2} k_{\mathrm{eq}} u_x^2 \,, \tag{D9}$$

where k_{eq} is the equivalent spring constant. Compare Eq. (D6) with Eq. (D9) – they appear dual, just like energy and coenergy of a capacitor given by Eqs. (D10) and (D11), respectively.

$$W_{\mathrm{e}} = \frac{1}{2}\frac{q^2}{C} \quad \text{(electrostatic energy)} \tag{D10}$$

$$W'_{\mathrm{e}} = \frac{1}{2} C v^2 \quad \text{(electrostatic coenergy)} \tag{D11}$$

However, we have not employed the Legendre transformation here. Just as in the case of a linear capacitor, where one can express the energy in terms of voltage via insertion of the constitutive relation $q = Cv$, here we expressed elastic energy in terms of two different state variable using constitutive relations (displacement and force). Energy and coenergy are equal only in case of a linear capacitor. The same holds true for elastic energy and elastic coenergy (also known as complementary energy in mechanics of solids). The elastic coenergy will be introduced in more details in the next section. Before that, we will evaluate elastic energy for a rod in torsion and a beam in pure bending.

Torsion of a circular rod

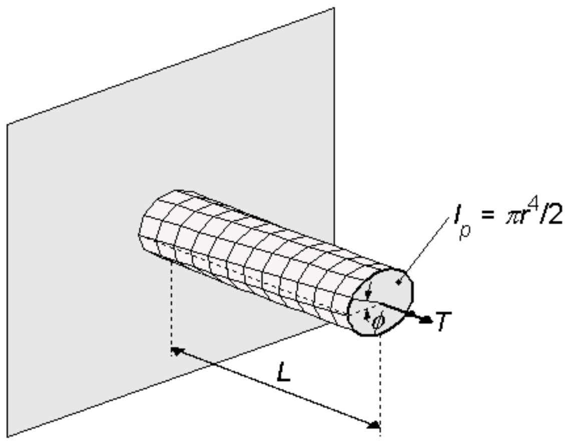

Figure 2: Rods under torsional stress. Torsion applied to a uniform, circular rod. Each infinitesimal, elemental disk undergoes rotation with no deformation in the axial direction.

The total elastic strain energy of a rod is in pure shear

$$W_{\mathrm{es}} = \frac{1}{2}\int_V \tau\gamma dv = \frac{1}{2}\int_V \frac{\tau^2}{G} dv = \frac{1}{2}\int_0^L \frac{T^2}{GI_p} dx , \qquad \text{(D12)}$$

where we employed the constitutive relation for shear $\tau = G\gamma$ (see Eq. D21 and D22 of the text).

If G and I_p do not vary along the rod, Eq. (D12) simplifies to

$$W_{\mathrm{es}} = \frac{1}{2}\frac{L}{GI_p}T^2 . \qquad \text{(D13)}$$

Torque T can be viewed as the generalized load. Thus, generalized displacement δ is obtained as

$$\delta = \frac{\partial W_{\mathrm{es}}}{\partial T} = \frac{L}{GI_p}T . \qquad \text{(D14)}$$

This generalized displacement is the angle ϕ at the free end of the rod (see Figure 2).

Beam in pure bending

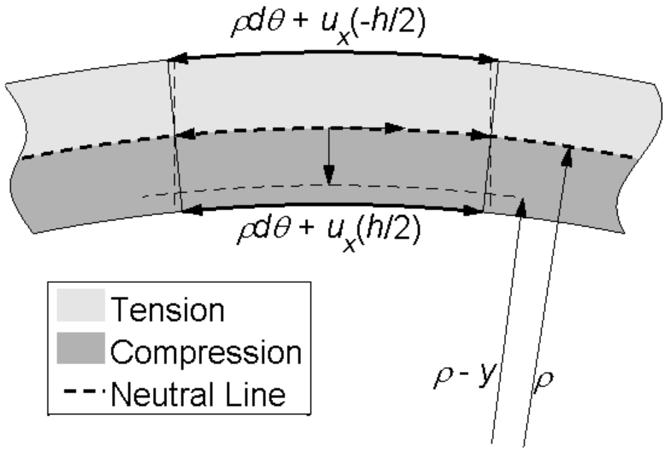

Figure 3: Beam element in pure bending, with bending exaggerated for clarity. Each plane surface of the unbent beam undergoes rotation about the neutral axis, but no other form of distortion affects these planes.

Elastic strain in pure bending is given by (see Section D.7.2)

$$\sigma_x = \frac{My}{I_x} \tag{D15}$$

Inserting Eq. (D15) into Eq. (D2) yields

$$W_{\text{es}} = \frac{1}{2}\int_V \frac{\sigma_x^2}{Y} dv = \frac{1}{2}\int_V \frac{M^2 y^2}{I_x^2} dv = \frac{1}{2}\int_0^L \frac{M^2}{YI_x} dx \tag{D16}$$

In case of a constant moment, Eq. (D16) simplifies to

$$W_{\text{es}} = \frac{1}{2}\frac{L}{YI_x}M^2 \tag{D17}$$

Here the moment M can be viewed as the generalized load and L / (YI_x) as the generalized compliance. The generalized displacement δ is obtain by differentiating the elastic strain energy with respect to the generalized load

$$\delta = \frac{\partial W_{\text{es}}}{\partial M} = \frac{L}{YI_x} M \tag{D18}$$

The generalized displacement δ is angle at the tip of the beam.

Energy coenergy and hybrid energies in mechanics of solids

We have seen above that in case of linear materials with a single generalized load and a single generalized displacement, one does not have to worry too much about energy and coenergy because they are algebraically equal.

Let us first examine the case with one force and displacement in more detail using the axial bar example. In general, an increment of the mechanical energy is defined as the product of force and the incremental displacement.

$$dW_{es} = fdu \tag{D19}$$

Integration of the force over the length of the bar yields the area under the curve of Figure 4 (the area between the curve and the *u* axis)

$$W_{es} = \int_0^L fdu\,. \tag{D20}$$

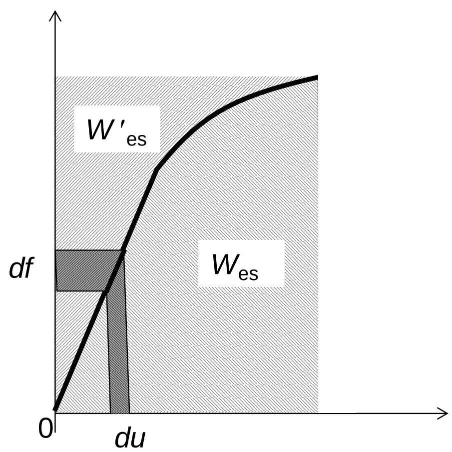

Figure 4: Energy and coenergy of an axially loaded bar.

Coenergy is defined analogously to the definition for electrostatic energy (refer to Section 3.2.1)

$$W_{es} + W'_{es} = fu\,. \tag{D21}$$

Differentiating Eq. (D21) and using Eq. (D19) gives the differential of the coenergy

$$dW'_{es} = udf\,. \tag{D22}$$

Integrating the coenergy (see Figure 4) gives the area to the left of the curve (the area between the curve and the *f*-axis)

$$W'_{es} = \int_0^L udf\,. \tag{D23}$$

It is clear from the figure that the two areas (represented by W_{es} and W'_{es}) are in general different. They are the same only when the relationship between the force and displacement is linear.

Now consider the case where two forces are applied simultaneously[1]. The differential of the elastic energy is given by

$$dW_{\mathrm{es}} = f_1 du_1 + f_2 du_2 . \tag{D24}$$

The coenergy is defined in the usual manner

$$W_{\mathrm{es}} + W'_{\mathrm{es}} = f_1 u_1 + f_2 u_2 . \tag{D25}$$

and its differential is

$$dW'_{\mathrm{es}} = u_1 df_1 + u_2 df_2 . \tag{D26}$$

With two degrees of freedom, we can have a mixed representation.

As in Chapter 3, to integrate the energy with two independent displacement we select the integration path. For example we integrate along u'_1 while keeping u'_2 at zero and then integrate along u'_2, with $u'_1 = u_1$.

$$dW_{\mathrm{es}} = \int_0^{u_1} f_1\left(u'_1, u' = 0\right) du'_1 + \int f_2\left(u'_1 = u_1, u'\right) du'_2 . \tag{D27}$$

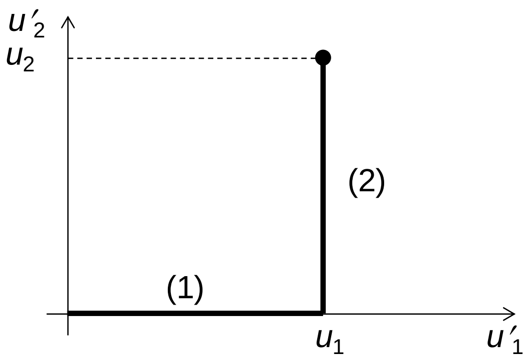

Figure 5: Convenient integration path for the energy function $W_{\mathrm{es}}(u_1, u_2)$.

Consider a simple 2 DOF system of Figure 6. On path (1) only u'_1 is nonzero and the integral is

$$W_{\mathrm{es}}^{(1)} = \int_0^{u_1} (k_1 + k_2) u'_1 du'_1 = \frac{1}{2}\left(k_1 + k_{12}\right) u_1^2 . \tag{D28}$$

On path (2) we have

$$W_{\mathrm{es}}^{(2)} = \int_0^{u_1} \left(k_2 u'_2 + k_{12}\left(u'_2 - u_1\right)\right) du'_2 = \frac{1}{2}\left(k_2 + k_{12}\right) u_2^2 - k_{12} u_1 u_2 . \tag{D29}$$

Finally, adding the two contributions yield the total elastic strain energy

[1] More generally we could consider generalized forces (forces and moments) and generalized displacements (displacements and angles), but there is not a significant gain of employing different symbols to emphasize this generality.

$$W_{\mathrm{es}} = W_{\mathrm{es}}^{(1)} + W_{\mathrm{es}}^{(2)} = \frac{1}{2}(k_1 + k_{12})u_1^2 - k_{12}u_1u_2 + \frac{1}{2}(k_2 + k_{12})u_2^2. \tag{D30}$$

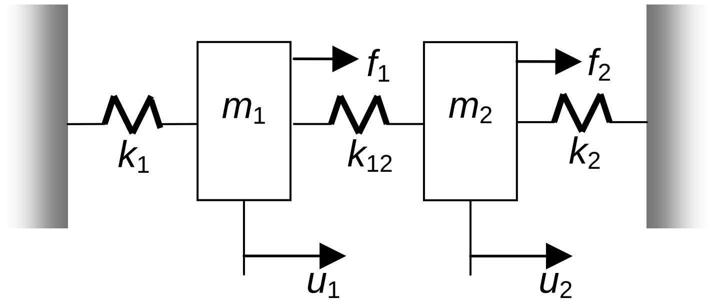

Figure 6: Simple 2 DOF system. Note that the displacements are imparted slowly so that masses do not play a role here.

Taking partial derivatives of W_{es} with respect to the displacements, yield the forces

$$f_1 = \frac{\partial W_{\mathrm{es}}}{\partial u_1} = (k_1 + k_{12})u_1 - k_{12}u_2 \tag{D31}$$

$$f_2 = \frac{\partial W_{\mathrm{es}}}{\partial u_2} = (k_2 + k_{12})u_2 - k_{12}u_1 \tag{D32}$$

or, in matrix form,

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \underbrace{\begin{bmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{bmatrix}}_{=\text{stiffness matrix } \overline{\overline{K}}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{D33}$$