H Appendix H The adiabatic atmosphere

Let us consider an adiabatic atmosphere in which movements are fast enough for heat transfer between different parts to be negligible.

$$dU = dQ - PdV = -PdV \tag{H.53}$$

We also assume that it is made of perfect gases for which the internal energy is only a function of the temperature and use C_V as the heat capacity at constant volume:

$$dU = C_V dT \tag{H.54}$$

Combining the two equations and using the law of perfect gases to eliminate P, we get

$$C_V \frac{dT}{T} + R\frac{dV}{V} = 0 \tag{H.55}$$

The gas density ρ relates to the volume through $\rho V = M$ (molar mass), and therefore:

$$\frac{d\rho}{\rho} = \frac{R}{C_V}\frac{dT}{T} \tag{H.56}$$

which we can be integrated as:

$$\frac{\rho}{\rho_0} = \left(\frac{T}{T_0}\right)^{\frac{C_V}{R}} \tag{H.57}$$

where the subscript 0 refers to surface conditions (elevation $z = 0$). The next task is to find a relationship between temperature and elevation in the atmosphere. To this end, we will use the law of perfect gases one more time to eliminate V from (H.55):

$$R\frac{dP}{P} = (R + C_V)\frac{dT}{T} \tag{H.58}$$

and combine it with the hydrostatic condition:

$$dP = -\rho g dz = -\frac{M}{V} g dz \tag{H.59}$$

to obtain

$$-M\frac{RT}{PV}dz = (R + C_V)\, dT \tag{H.60}$$

or, using $R = C_P - C_V$:

$$\frac{dT}{dz} = -\frac{Mg}{C_P} \tag{H.61}$$

an expression known as the adiabatic gradient. This equation can be integrated to give the variation of temperature in the atmosphere:

$$T = T_0 \left(1 - \frac{Mg}{C_P T_0} z\right) \tag{H.62}$$

and for the density:

$$\rho = \rho_0 \left(1 - \frac{Mg}{C_P T_0} z\right)^{\frac{C_V}{R}} \tag{H.63}$$

For a diatomic gas such as N_2 or O_2, $C_V = \frac{5}{2} R$, so the density of the atmosphere decreases with the elevation much faster than its temperature.