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Problems for Chapter 15 of *Advanced Mathematics for Applications*

Green's Functions: Ordinary Differential Equations

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1 Two-point boundary value problems

1.1 Separated boundary conditions

1. Construct the Green's function for the problem

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x)$$

with $a_{11}u(0) + a_{12}[\mathrm{d}u/\mathrm{d}x]_{x=0} = U_0$ and $a_{21}u(b) + a_{22}[\mathrm{d}u/\mathrm{d}x]_{x=b} = U_b$, in which a_{ij}, U_0 and U_b are given constants.

2. Construct the Green's function for the problem

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) = f(x)$$

with $u(0) = 0$ and $[\mathrm{d}u/\mathrm{d}x]_{x=1} = U_b$, with U_b a given constant.

3. In the interval $R_1 < r < R_2$ find the Green's function for the problem

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) + \left(k^2 - \frac{m^2}{r^2}\right)u = f(r)\,,$$

with $u(R_1) = u(R_2) = 0$. Here $k^2 > 0$ and m is an integer.

4. Find the Green's function for the equation

$$u'' + 4c^2x^2u = f \qquad 0 < x < L,$$

corresponding to the boundary conditions $u(0) = a$, $u(L) = b$. The operator appearing in the equation is reducible to a Bessel operator by a substitution of the type $u = x^a v(z)$, where $z = x^b$.

5. Write the left-hand side of the equation

$$x\,\frac{d^2u}{dx^2} - \frac{du}{dx} + 4k^2x^3u = x^2\,f(x),$$

in the standard Sturm-Liouville form and then find the Green's function that expresses the solution to the equation satisfying the boundary conditions

$$u(0) = 0, \qquad \left.\frac{du}{dx}\right|_{x=1} = 0.$$

Are there special values of k such that the procedure breaks down? What can you say about the existence and uniqueness of the solution in these cases? (The transformation $y = x^\alpha$ to reduce the equation to a known form. Strictly speaking this problem is singular, but if f is bounded at 0 the usual theory goes through after the transformation.)

6. Determine, by finding the appropriate Green's function, the solution to the problem

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} - \frac{2}{x}\frac{\mathrm{d}v}{\mathrm{d}x} + \left(k^2 + \frac{2}{x^2}\right) v = f(x),$$

in the interval $0 < x < L$, subject to the conditions $v'(0) = 0$, $v(L) = 0$. For what values of k is the existence of a solution not guaranteed? If solvability conditions for such cases were satisfied, would the solution be unique? (Make the transformation $v = x\,u$ after which the usual theory goes through even though, as posed, the problem is singular.)

7. Find the Green's function for the equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\right] = f \qquad 0 < x < 1,$$

corresponding to the boundary conditions $u(0) = 0$, $\lim_{x\to 1-}(1-x^2)u'(x) = 0$.

8. Find the Green's function for the equation

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}u}{\mathrm{d}x}\right] = f \qquad -1 < x < 1,$$

corresponding to the boundary conditions $\lim_{x\to\pm 1} u(x) = \text{finite}$.

1.2 Mixed boundary conditions

1. In $0 < x < 1$ construct the Green's function for the problem

$$u'' = f(x)\,, \qquad u(0) = u(1)\,, \qquad u'(0) = u'(1)\,.$$

2. In $0 < x < 1$ construct the Green's function for the problem

$$u'' = f(x)\,, \qquad u(0) = u(1)\,, \qquad u'(0) = u'(1)\,.$$

3. In $0 < x < 1$ construct the Green's function for the problem

$$\frac{1}{2}u'' = f(x)\,, \qquad u(0) + u(1) = 0\,, \qquad u'(0) = 0\,.$$

4. In $0 < x < 1$ construct the Green's function for the problem

$$u'' - u = f(x)\,, \qquad u(0) - u'(0) + u(1) = 0\,, \qquad u(0) + u'(0) + 2u'(1) = 0\,.$$

5. In $0 < x < \pi$ construct the Green's function for the problem

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - u = f(x)\,, \qquad u(0) = 0, \quad u'(0) = u(\pi)\,.$$

6. Explain why it is impossible to find a Green's function for the problem

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - k^2 u = f(x)$$

with $u(0) = u(1)$, $[\mathrm{d}u/\mathrm{d}x]_{x=0} = -[\mathrm{d}u/\mathrm{d}x]_{x=1}$.

7. Compare the two Green's functions for the operator

$$\mathsf{L}u = \frac{\mathrm{d}^2u}{\mathrm{d}x^2} + x\frac{\mathrm{d}u}{\mathrm{d}x}$$

and its adjoint. The boundary conditions are $u(0) = u(1)$ and $[\mathrm{d}u/\mathrm{d}x]_{x=0} + [\mathrm{d}u/\mathrm{d}x]_{x=1} = 0$.

8. Calculate the Green's function for the problem

$$x^2u'' + 2xu' + \lambda^2x^2u = f(x) \qquad 0 < x < 1 \qquad\qquad (*)$$

subject to $u(0) = u_0$, $u(1) = u_1$. Find the eigenvalues and eigenfunctions of the associated problem

$$x^2v_n'' + 2xv_n' = -\lambda^2x^2v_n \qquad v_n(0) = v_n(1) = 0$$

by determining the values of λ for which a solution of (*) does not exist for arbitrary f.

9. Construct the Green's function for the problem

$$x^2u'' + xu' + (\lambda x^2 - 1)u = f(x) \qquad 0 < x < 1$$

subject to $u(0) = u_0$, $u(1) = u_1$.

2 Regular Sturm-Liouville problems

2.1 Separated boundary conditions

1. Find eigenvalues and eigenfunctions of the operator

$$\mathsf{L}u \equiv u'' , \qquad u(0) = u'(1) = 0$$

in $0 < x < 1$.

2. Find eigenvalues and eigenfunctions of the operator

$$\mathsf{L}u \equiv u'' , \qquad u(-1) = u(1) = 0$$

in $-1 < x < 1$.

3. Find eigenvalues and eigenfunctions of the operator

$$\mathsf{L}u \equiv u'' + 4c^2x^2u , \qquad u(0) = 0 , \quad u(L) = 0 .$$

(The differential equation is reducible to the Bessel form by a substitution of the type $u = x^a v(z)$, where $z = x^b$.)

4. In $1 < x < e$ consider the Sturm-Liouville problem

$$\frac{d}{dx}\left(x^2\frac{du}{dx}\right) + \frac{1}{4}u = f(x)$$

where $f(x)$ is given, subject to the boundary conditions

$$u(1) = a , \qquad u(e) = b .$$

   (i) Construct the Green's function $G(x,\xi)$ for the problem and write down the solution for general f, a, b

(ii) Consider next the associated eigenvalue problem

$$\frac{d}{dx}\left(x^2\frac{du_n}{dx}\right) + \frac{1}{4}u_n \;=\; \lambda_n^2 u_n$$

$$u_n(1) \;=\; u_n(e) \;=\; 0\,.$$

Find eigenvalues and normalized eigenfunctions.

(iii) Solve the original problem with $a = b = 0$ by expanding u in a series of u_n's; comparing with the previous solution write down the expression of G as a series of eigenfunctions

(iv) Verify that this expression is correct by taking the scalar product of both sides with u_n.

(v) In $1 < x < e$ consider the diffusion problem

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right) + \frac{1}{4}u \;=\; \frac{\partial u}{\partial t}$$

subject to

$$u(1,t) \;=\; u(e,t) \;=\; 0\,, \qquad u(x,0) \;=\; F(x)\,,$$

and write down the general solution.

[Hints: One solution of the differential equation is easy; for the other solution you can use (2.2.33). The change of variable $y = \log x$ is useful to deal with the integrals that arise.]

2.2 Mixed boundary conditions

1. Find eigenvalues and eigenfunctions of the operator

$$\mathsf{L}u \equiv u''\,, \qquad u(0) \;=\; u(1)\,, \qquad u'(0) = u'(1)$$

in $0 < x < 1$.

2. Find eigenvalues and eigenfunctions of the operator

$$\mathsf{L}u \equiv u''\,, \qquad u(0) \;=\; u(1)\,, \qquad u'(0) \;=\; u'(1)$$

in $0 < x < 1$.

3. Find eigenvalues and eigenfunctions of the operator

$$\mathsf{L}u \equiv \frac{1}{2}u''\,, \qquad u(0) + u(1) \;=\; 0\,, \qquad u'(0) \;=\; 0$$

in $0 < x < 1$.

4. Find eigenvalues and eigenfunctions of the operator

$$\mathsf{L}u \equiv u'' - u\,, \qquad u(0) - u'(0) + u(1) \;=\; 0\,, \qquad u(0) + u'(0) + 2u'(1) \;=\; 0$$

in $0 < x < 1$.

3 Singular Sturm-Liouville problems

1. Find eigenfunctions and eigenvalues of the operator

$$\mathsf{L}u \equiv x\frac{d^2u}{dx^2} - \frac{du}{dx} + 4k^2x^3u$$

with u subject to the boundary conditions

$$u(0) = 0, \qquad \left.\frac{du}{dx}\right|_{x=1} = 0.$$

(The transformation $y = x^\alpha$ to reduce the equation to a known form.)

2. Solve the eigenvalue problem

$$\mathsf{L}u \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}u}{\mathrm{d}x}\right) + \lambda x^2 u = 0$$

with u subject to the boundary conditions

$$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=0} = 0, \qquad u(1) = 0\,.$$

3. Find eigenvalues and eigenfunctions of the operator

$$\mathsf{L}u \equiv xu'' + u'\,, \qquad u(1) = 0$$

in $0 < x < 1$, with $u(0)$ finite.

4. Discuss the eigenfunctions and spectrum of the singular operator

$$\mathsf{L}u \equiv x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (1-x)\frac{\mathrm{d}u}{\mathrm{d}x}$$

in the range $0 < x < \infty$; the scalar product is defined by

$$(v,u) = \int_0^\infty e^{-x}\overline{v}\,u\,\mathrm{d}x\ .$$

This is the operator which gives rise to the Laguerre polynomials (section 13.9, p. 334).

5. Discuss the eigenfunctions and spectrum of the singular operator

$$\mathsf{L}u \equiv (1-x^2)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - x\frac{\mathrm{d}u}{\mathrm{d}x}$$

in the range $-1 < x < 1$; the scalar product is defined by

$$(v,u) = \int_{-1}^1 (1-x^2)^{-1/2}\overline{v}\,u\,\mathrm{d}x\ .$$

This is the operator which gives rise to the Chebyshev polynomials $T_n(x)$ (section 13.9, p. 334).

4 Initial-value problems

Note: For some of these problems use of the Laplace transform is a useful method to construct the Green's function.

4.1 First-order equations

1. Build a Green's function theory suitable for the general ordinary differential equation of the first order

$$\frac{\mathrm{d}u}{\mathrm{d}t} + a(t)u = f(t), \qquad u(0) = u_0$$

and verify that the final expression satisfies the equation; $a(t)$ and $f(t)$ are given functions and u_0 a given constant. Compare with the solution given in (2.2.31) p. 34.

2. Solve by constructing a suitable Green's function the equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + ku = f, \qquad u(0) = u_0$$

where u_0 and k are given constants.

4.2 Second-order equations

1. By constructing the appropriate Green's function, solve for $t > 0$ the initial-value problem

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \frac{\mathrm{d}u}{\mathrm{d}t} = f(t), \qquad u(0) = 0, \quad u'(0) = b.$$

2. By constructing the appropriate Green's function, solve for $t > 0$ the initial-value problem

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + u = f(t), \qquad u(0) = a, \quad u'(0) = 0.$$

3. By constructing the appropriate Green's function, solve for $t > 0$ the initial-value problem

$$t\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - \frac{\mathrm{d}u}{\mathrm{d}t} + 4t^3u = f(t), \qquad u(0) = 0, \quad u'(0) = 0.$$

4. By constructing the appropriate Green's function, solve for $t > 0$ the initial-value problem

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + 2c\frac{\mathrm{d}u}{\mathrm{d}t} + (1+c^2)u = f(t), \qquad u(0) = a, \quad \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{t=0} = b;$$

c is a given real constant.

5. By constructing the appropriate Green's function, solve for $t > 0$ the initial-value problem

$$t\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + 2\frac{\mathrm{d}u}{\mathrm{d}t} - c^2tu = f(t), \qquad u(0) = 1, \quad \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{t=0} = 0,$$

where c is a given real constant,

6. By constructing the appropriate Green's function, solve for $t > 0$ the initial-value problem

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + (a+bt)u = f(t) \qquad u(0) = u_0, \quad \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{t=0} = v_0,$$

in which a and b are given constants.

7. By constructing the appropriate Green's function, solve for $t > 0$ the initial-value problem

$$t\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \frac{\mathrm{d}u}{\mathrm{d}t} - tu = f(t), \qquad u(0) = 0, \quad \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{t=0} = 0.$$

8. Develop a Green's function theory for the solution of the initial-value governed by a third-order ordinary differential equation and apply it to the solution, for $t > 0$, of the initial-value problem

$$\frac{\mathrm{d}^3 u}{\mathrm{d}t^3} + \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} - 2u = f(t), \qquad u(0) = 0, \quad \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{t=0} = 1, \quad \left.\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}\right|_{t=0} = 2.$$