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Appendix 12W.2
Brief History of Leontief Inverses with Errors in the Coefficients of A

A large literature exists on "sensitivity" or "error" input-output analysis. Much of this is concerned with the situation in which all (or many) elements in **A** are perturbed to $a_{ij} + \Delta a_{ij}$. This is where Woodbury's extension (1950) of Sherman and Morrison's results (1949, 1950) becomes relevant. Published works on how errors in an **A** matrix influence elements in the associated Leontief inverse frequently cite results in Dwyer and Waugh (1953) and Evans (1954). An article by Henderson and Searle (1981) provides a comprehensive and unified treatment on inverses of sums of matrices. It is not concerned specifically with input-output models and does not cite either of these earlier studies. Henderson and Searle's results, when applied to the input-output case, show that what may appear to be differing results in the input-output literature are in fact simply the same wine in different bottles.

A12W.2.1 Mathematical Background

Much of the work on alternative expressions for the inverse of the sum of matrices is grounded in results on inverses of partitioned matrices. Recall the two results in Appendix A for the inverse of $\left[\begin{array}{c|c} \mathbf{E} & \mathbf{F} \\ \hline \mathbf{G} & \mathbf{H} \end{array}\right]$, which we labeled $\left[\begin{array}{c|c} \mathbf{S} & \mathbf{T} \\ \hline \mathbf{U} & \mathbf{V} \end{array}\right]$; that is,

$$\left[\begin{array}{c|c} \mathbf{E} & \mathbf{F} \\ \hline \mathbf{G} & \mathbf{H} \end{array}\right]\left[\begin{array}{c|c} \mathbf{S} & \mathbf{T} \\ \hline \mathbf{U} & \mathbf{V} \end{array}\right] = \mathbf{I} = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I} \end{array}\right]$$

Building on the assumption that $\mathbf{E}^{-1}$ is known, we have

$$\begin{array}{ll} \mathbf{S} = \mathbf{E}^{-1}(\mathbf{I} - \mathbf{FU}) & \mathbf{T} = -\mathbf{E}^{-1}\mathbf{FV} \\ \mathbf{U} = -\mathbf{VGE}^{-1} & \mathbf{V} = (\mathbf{H} - \mathbf{GE}^{-1}\mathbf{F})^{-1} \end{array} \qquad \text{(A12W.2.1)}$$

[This is (A.5) in Appendix A.] Alternative expressions result if we begin with the assumption that $\mathbf{H}^{-1}$ is known. This leads to

$$\begin{array}{ll} \mathbf{S} = (\mathbf{E} - \mathbf{FH}^{-1}\mathbf{G})^{-1} & \mathbf{T} = -\mathbf{SFH}^{-1} \\ \mathbf{U} = -\mathbf{H}^{-1}\mathbf{GS} & \mathbf{V} = \mathbf{H}^{-1}(\mathbf{I} - \mathbf{GT}) \end{array} \qquad \text{(A12W.2.2)}$$

[This is (A.6).]

From the observation that corresponding expressions for $\mathbf{S}$, $\mathbf{T}$, $\mathbf{U}$ or $\mathbf{V}$ in (A12W.2.1) and (A12W.2.2) must be equal, we equate the two expressions for **U**, giving

$$(\mathbf{H} - \mathbf{GE}^{-1}\mathbf{F})^{-1}\mathbf{GE}^{-1} = \mathbf{H}^{-1}\mathbf{G}(\mathbf{E} - \mathbf{FH}^{-1}\mathbf{G})^{-1} \qquad \text{(A12W.2.3)}$$

Consider two special cases of this result.

(a) Let $\mathbf{E} = \mathbf{I}$, $\mathbf{F} = -\mathbf{I}$, $\mathbf{G} = \boldsymbol{\alpha}$ and $\mathbf{H} = \mathbf{I}$, then (A12W.2.3) is

$$(\mathbf{I}+\boldsymbol{\alpha})^{-1}\boldsymbol{\alpha}=\boldsymbol{\alpha}(\mathbf{I}+\boldsymbol{\alpha})^{-1} \quad \text{(A12W.2.4)}$$

(b) Let $\mathbf{E}=\mathbf{I}$, $\mathbf{F}=-\boldsymbol{\beta}$, $\mathbf{G}=\boldsymbol{\alpha}$ and $\mathbf{H}=\mathbf{I}$, then (A12W.2.3) is

$$(\mathbf{I}+\boldsymbol{\alpha}\boldsymbol{\beta})^{-1}\boldsymbol{\alpha}=\boldsymbol{\alpha}(\mathbf{I}+\boldsymbol{\beta}\boldsymbol{\alpha})^{-1} \quad \text{(A12W.2.5)}$$

Further,

$$(\mathbf{I}+\boldsymbol{\alpha})^{-1}(\mathbf{I}+\boldsymbol{\alpha})=\mathbf{I}\Rightarrow(\mathbf{I}+\boldsymbol{\alpha})^{-1}\mathbf{I}-(\mathbf{I}+\boldsymbol{\alpha})^{-1}\boldsymbol{\alpha}=\mathbf{I}-\boldsymbol{\alpha}(\mathbf{I}+\boldsymbol{\alpha})^{-1} \quad \text{(A12W.2.6)}$$

[from (A12W.2.4)].

We now address the general problem of $(\mathbf{M}+\mathbf{PNQ})^{-1}$. Straightforward algebra leads to

$$(\mathbf{M}+\mathbf{PNQ})^{-1}=(\mathbf{M}+\mathbf{MM}^{-1}\mathbf{PNQ})^{-1}=[\mathbf{M}(\mathbf{I}+\mathbf{M}^{-1}\mathbf{PNQ})]^{-1}=(\mathbf{I}+\mathbf{M}^{-1}\mathbf{PNQ})^{-1}\mathbf{M}^{-1}$$

From (A12W.2.4) above, with $\boldsymbol{\alpha}=\mathbf{M}^{-1}\mathbf{PNQ}$, the first term is

$$(\mathbf{I}+\mathbf{M}^{-1}\mathbf{PNQ})^{-1}=\mathbf{I}-(\mathbf{I}+\mathbf{M}^{-1}\mathbf{PNQ})^{-1}\mathbf{M}^{-1}\mathbf{PNQ}$$

and so

$$\begin{aligned}(\mathbf{M}+\mathbf{PNQ})^{-1}&=[\mathbf{I}-(\mathbf{I}+\mathbf{M}^{-1}\mathbf{PNQ})^{-1}\mathbf{M}^{-1}\mathbf{PNQ}]\mathbf{M}^{-1}\\&=[\mathbf{M}^{-1}-(\mathbf{I}+\mathbf{M}^{-1}\mathbf{PNQ})^{-1}\mathbf{M}^{-1}\mathbf{PNQM}^{-1}]\end{aligned} \quad \text{(A12W.2.7)}$$

This is the foundation for Henderson and Searle's alternative expressions.

A12W.2.2 Application to Leontief Inverses

Let $\Delta\mathbf{A}$ represent a matrix of (additive) errors (or perturbations) in the elements of a direct input coefficients matrix, so $\mathbf{A}^*=\mathbf{A}+\Delta\mathbf{A}$. Also, $\mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}$ and

$$(\mathbf{I}-\mathbf{A}^*)=(\mathbf{I}-\mathbf{A})-\Delta\mathbf{A},\ \mathbf{L}^*=(\mathbf{I}-\mathbf{A}^*)^{-1}=(\mathbf{I}-\mathbf{A}-\Delta\mathbf{A})^{-1}$$

Our interest is in expressing $\mathbf{L}^*$ as a function of $\mathbf{L}$ and $\Delta\mathbf{A}$ [1]. Henderson and Searle (1981) present a number of such expressions. We particularize the results in (A12W.2.7) to the Leontief inverse case by letting

$$\mathbf{M}=(\mathbf{I}-\mathbf{A}),\ \mathbf{N}=\Delta\mathbf{A},\ \mathbf{P}=-\mathbf{I} \text{ and } \mathbf{Q}=\mathbf{I} \text{ [and recalling that } \mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}]$$

so that $(\mathbf{M}+\mathbf{PNQ})^{-1}=(\mathbf{I}-\mathbf{A}-\Delta\mathbf{A})^{-1}$. Of interest are the following from Henderson and Searle; as the authors note, they display an interesting pattern.[2]

$$\mathbf{L}^*=\mathbf{L}+[\mathbf{I}-\mathbf{L}(\Delta\mathbf{A})]^{-1}\mathbf{L}(\Delta\mathbf{A})\mathbf{L} \quad \text{(A12W.2.8)}$$

This is exactly (A12W.2.7), with the appropriate substitutions.

Repeatedly using (A12W.2.5) generates the remaining three results of interest:

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[1] Many publications use **D** (for "difference" matrix) or **E** (for "error" matrix) in place of $\Delta\mathbf{A}$.

[2] These are Equations (21), (22), (24) and (26) in the article, with notation adjusted for the present input-output case.

$$\mathbf{L}^* = \mathbf{L} + \mathbf{L}[\mathbf{I} - (\Delta\mathbf{A})\mathbf{L}]^{-1}(\Delta\mathbf{A})\mathbf{L} \quad \text{(A12W.2.9)}$$

$$\mathbf{L}^* = \mathbf{L} + \mathbf{L}(\Delta\mathbf{A})[\mathbf{I} - \mathbf{L}(\Delta\mathbf{A})]^{-1}\mathbf{L} \quad \text{(A12W.2.10)}$$

$$\mathbf{L}^* = \mathbf{L} + \mathbf{L}(\Delta\mathbf{A})\mathbf{L}[\mathbf{I} - (\Delta\mathbf{A})\mathbf{L}]^{-1} \quad \text{(A12W.2.11)}$$

The various (seemingly different) relationships presented in published input-output work are simply variations on these results.

Dwyer and Waugh (1953)

In Dwyer and Waugh (1953)[3] we find

$$\mathbf{L}^* = \mathbf{L}[\mathbf{I} - (\Delta\mathbf{A})\mathbf{L}]^{-1} \quad \text{(A12W.2.12)}$$

$$\mathbf{L}^* = [\mathbf{I} - \mathbf{L}(\Delta\mathbf{A})]^{-1}\mathbf{L} \quad \text{(A12W.2.13)}$$

$$\mathbf{L}^* - \mathbf{L} = \mathbf{L}(\Delta\mathbf{A})\mathbf{L}[\mathbf{I} - (\Delta\mathbf{A})\mathbf{L}]^{-1} \quad \text{(A12W.2.14)}$$

$$\mathbf{L}^* - \mathbf{L} = (\mathbf{I} - \mathbf{L}(\Delta\mathbf{A})]^{-1}\mathbf{L}(\Delta\mathbf{A})\mathbf{L} \quad \text{(A12W.2.15)}$$

The latter two results are just (A12W.2.11) and (A12W.2.8), respectively. (Clearly Dwyer and Waugh's results predate the Henderson and Searle article but are not mentioned in it.) It is straightforward to show that (A12W.2.9) and (A12W.2.12) are equivalent, from (A12W.2.4) with $\boldsymbol{\alpha} = -(\Delta\mathbf{A})\mathbf{L}$. Exactly the same sort of reasoning, with $\boldsymbol{\alpha} = -\mathbf{L}(\Delta\mathbf{A})$, shows the equivalence of (A12W.2.10) and (A12W.2.13).

Evans (1954)

The results of interest presented in Evans [1954, (1.5) and (1.6), respectively] are

$$\mathbf{L}^* = \mathbf{L}[\mathbf{I} - (\Delta\mathbf{A})\mathbf{L}]^{-1} \quad \text{(A12W.2.16)}$$

and

$$\mathbf{L}^* = [\mathbf{I} - \mathbf{L}(\Delta\mathbf{A})]^{-1}\mathbf{L} \quad \text{(A12W.2.17)}$$

These are (A12W.2.12) and (A12W.2.13) from Dwyer and Waugh and hence equivalent to (A12W.2.9) and (A12W.2.10). (Evans is also not cited by Henderson and Searle).

West (1982)

Here we find [West's (4)]

$$\mathbf{L}^* = \mathbf{L} + [\mathbf{I} - \mathbf{L}(\Delta\mathbf{A})]^{-1}\mathbf{L}(\Delta\mathbf{A})\mathbf{L} \quad \text{(A12W.2.18)}$$

which is Dwyer and Waugh's (A12W.2.15) and hence (A12W.2.8) from Henderson and Searle. [West cites Evans but not Dwyer and Waugh and then provides his own derivation of the result in (A12W.2.18).][4]

[3] These are Equations (3.2), (3.2)', (3.6) and (3.6)' in the article.

Sherman-Morrison and Sonis-Hewings

The matrix statement in (A12W.2.8),

$$\mathbf{L}^* = \mathbf{L} + [\mathbf{I} - \mathbf{L}(\Delta\mathbf{A})]^{-1}\mathbf{L}(\Delta\mathbf{A})\mathbf{L} = \mathbf{L} + \Delta\mathbf{L}$$

is one possible generalization of the Sherman-Morrison result in Appendix 12.1 [in (A12.1.4)],

$$l^*_{rs(ij)} = l_{rs} + \frac{l_{ri}l_{js}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} = l_{rs} + \Delta l_{rs(ij)}$$

To demonstrate the equivalence of the general case, namely of $\Delta l_{rs(ij)}$ in $\Delta\mathbf{L}$ [in (A12W.2.8)] and its counterpart in (A12.1.4), requires fairly complicated notation, including representing elements in the inverse in (A12W.2.8).

Instead, we illustrate the point for a three-sector model with a change only in a_{23}; that is, $\Delta\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta a_{23} \\ 0 & 0 & 0 \end{bmatrix}$. This will generate (the interested reader should check this)

$$\mathbf{L}(\Delta\mathbf{A}) = \begin{bmatrix} 0 & 0 & l_{12}\Delta a_{23} \\ 0 & 0 & l_{22}\Delta a_{23} \\ 0 & 0 & l_{32}\Delta a_{23} \end{bmatrix} \text{ and } \mathbf{L}(\Delta\mathbf{A})\mathbf{L} = \begin{bmatrix} l_{12}l_{31} & l_{12}l_{32} & l_{12}l_{33} \\ l_{22}l_{31} & l_{22}l_{32} & l_{22}l_{33} \\ l_{32}l_{31} & l_{32}l_{32} & l_{32}l_{33} \end{bmatrix}(\Delta a_{23})$$

Notice that $\mathbf{L}(\Delta\mathbf{A})\mathbf{L} = \mathbf{F}[2,3](\Delta a_{23})$, using the "field of influence" notation of Sonis and Hewings. In this small example,

$$[\mathbf{I} - \mathbf{L}(\Delta\mathbf{A})]^{-1} = \begin{bmatrix} 1 & 0 & l_{12}\Delta a_{23} \\ 0 & 1 & l_{22}\Delta a_{23} \\ 0 & 0 & 1 \end{bmatrix}\left(\frac{1}{[1 - l_{32}\Delta a_{23}]}\right)$$

This leads, finally, to

$$\Delta\mathbf{L}_{(23)} = [\mathbf{I} - \mathbf{L}(\Delta\mathbf{A})]^{-1}\mathbf{L}(\Delta\mathbf{A})\mathbf{L} = \begin{bmatrix} l_{12}l_{31} & l_{12}l_{32} & l_{12}l_{33} \\ l_{22}l_{31} & l_{22}l_{32} & l_{22}l_{33} \\ l_{32}l_{31} & l_{32}l_{32} & l_{32}l_{33} \end{bmatrix}\left(\frac{\Delta a_{23}}{[1 - l_{32}\Delta a_{23}]}\right)$$

$$= \mathbf{F}[2,3]\left(\frac{\Delta a_{23}}{[1 - l_{32}\Delta a_{23}]}\right)$$

Clearly, any element $\Delta l_{rs(23)}$ in this expression has exactly the structure of the Sherman-Morrison result in (A12.1.4).

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[4] Lahr (2001; Appendix 1) contains a very brief discussion of Dwyer and Waugh (1953), Evans (1954) and West (1982).

We finish with a variant of the numerical example in section 12.3.4 in the text, where we had $\mathbf{A}=\begin{bmatrix} .15 & .25 & .05 \\ .20 & .05 & .40 \\ .30 & .25 & .05 \end{bmatrix}$ and $\mathbf{L}=\begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix}$. To see how the algebra plays out, let a_{23} increase by ten percent— $\Delta a_{23}=0.04$ —so $\Delta\mathbf{A}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.04 \\ 0 & 0 & 0 \end{bmatrix}$.

Applying (A12W.2.8), we need

$$\mathbf{L}(\Delta\mathbf{A})=\begin{bmatrix} 0 & 0 & 0.0170 \\ 0 & 0 & 0.0539 \\ 0 & 0 & 0.0196 \end{bmatrix},\ \mathbf{L}(\Delta\mathbf{A})\mathbf{L}=\begin{bmatrix} 0.0097 & 0.0083 & 0.0219 \\ 0.0307 & 0.0264 & 0.0695 \\ 0.0111 & 0.0096 & 0.0252 \end{bmatrix}$$

$$[\mathbf{I}-\mathbf{L}(\Delta\mathbf{A})]^{-1}=\begin{bmatrix} 1 & 0 & 0.0173 \\ 0 & 1 & 0.0550 \\ 0 & 0 & 1.0200 \end{bmatrix}$$

Then

$$\Delta\mathbf{L}_{23}=[\mathbf{I}-\mathbf{L}(\Delta\mathbf{A})]^{-1}\mathbf{L}(\Delta\mathbf{A})\mathbf{L}=\begin{bmatrix} 0.0099 & 0.0085 & 0.0224 \\ 0.0313 & 0.0269 & 0.0709 \\ 0.0114 & 0.0098 & 0.0257 \end{bmatrix}$$

Alternatively, with

$$\mathbf{F}[2,3]=\begin{bmatrix} 0.2423 & 0.2080 & 0.5480 \\ 0.7682 & 0.6593 & 1.7370 \\ 0.2787 & 0.2392 & 0.6302 \end{bmatrix} \text{ and } \Delta a_{23}/[1-l_{32}\Delta a_{23}]=0.0408$$

we have

$$\Delta\mathbf{L}_{23}=\mathbf{F}[2,3]\Delta a_{23}/[1-l_{32}\Delta a_{23}]=\begin{bmatrix} 0.0099 & 0.0085 & 0.0224 \\ 0.0313 & 0.0269 & 0.0709 \\ 0.0114 & 0.0098 & 0.0257 \end{bmatrix}$$

Then

$$\mathbf{L}+\Delta\mathbf{L}_{23}=\begin{bmatrix} 1.3750 & 0.4337 & 0.2733 \\ 0.5587 & 1.3750 & 0.6662 \\ 0.5812 & 0.4988 & 1.3142 \end{bmatrix}$$

and, as expected, for $\mathbf{A}^* = \begin{bmatrix} 0.1500 & 0.2500 & 0.0500 \\ 0.2000 & 0.0500 & 0.4400 \\ 0.3000 & 0.2500 & 0.0500 \end{bmatrix}$,

$$\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 1.3750 & 0.4337 & 0.2733 \\ 0.5587 & 1.3750 & 0.6662 \\ 0.5812 & 0.4988 & 1.3142 \end{bmatrix}$$

References for Appendix 12W.2

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