Section 11.5
and Problem 11.51

"Radiative Transfer"

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by

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11.5 Formation of spectral lines (radiative transfer)

Radiation propagating through a gas is transformed by emission and absorption processes. The result is the observed spectrum including spectral lines. Here we set up the differential equation for an elementary case of *radiative transfer* and solve it for several different conditions. This allows us to understand the formation of spectral lines in terms of the frequency dependence of the optical depth.

Radiative transfer equation (RTE)

The differential equation that governs the absorption and emission in a layer of gas follows from the geometry of Fig. 17. A uniform cloud ("source") of temperature T_s, depth Λ, and optical depth τ_Λ lies between the observer and a background source at some other temperature T_0.

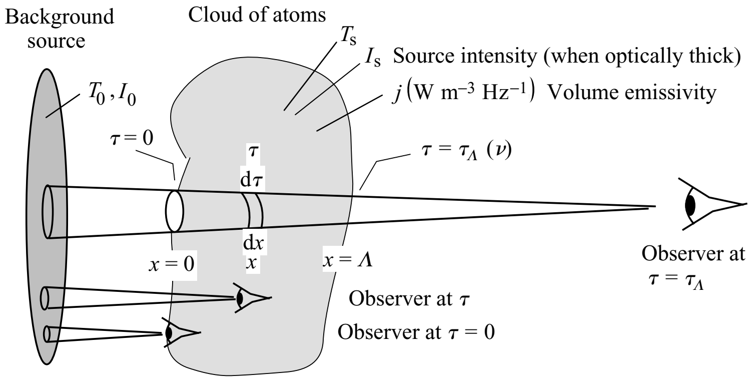

Figure 11.17. Geometry for the radiative transfer equation. The background surface emits with specific intensity I_0 and the intervening gas cloud emits thermal radiation with specific intensity I_s when it is optically thick. An observer in the cloud at position x, or optical depth τ viewing leftward will detect radiation from the cloud atoms at lesser τ and from the background source to the extent it is not absorbed by the cloud.

For the immediate discussion, we refer to radiated intensities at some single frequency (in a differential band) without regard to the entire spectrum. In fact, the overall spectral shape may be inconsistent with a single temperature. Hence we discuss intensities without necessarily defining a temperature. Nevertheless, in stellar atmospheres, temperatures can be defined in regions of *local thermodynamic equilibrium (LTE)*. In this case, a higher intensity at a given frequency does represent a higher temperature.

In the absence of the cloud, the observer in Fig. 17 would detect a specific intensity I_0 (W m^{-2} Hz^{-1} sr^{-1}) from the background source (T_0) at the frequency in question. We refer to I_0 as the *background intensity*. If the intervening cloud is in place, it will absorb some of the radiation from the background source. In addition, the cloud will emit its own thermal radiation in the direction of the observer. If the cloud is optically thick, the emerging radiation would exhibit the specific intensity I_s characteristic of blackbody radiation at T_s.

Intensity differentials

Consider a beam of photons moving in the direction of an observer at some location in the cloud. The differential equation that describes absorption of the photons in a differential path length dx is, from (10.17), $\mathrm{d}N/N = -\sigma n\,\mathrm{d}x$, where $\mathrm{d}N/N$ is the fractional change in the number of photons in the beam, σ (m^2) is the cross section per atom, and n (m^{-3}) is the number density of atoms. The fractional change of the photon number will be equal to the fractional change in the specific intensity, giving,

$$\frac{dI_1}{I} = -\sigma n \, dx \qquad \text{(Absorption in layer d}x\text{)} \qquad (11.34)$$

where dI_1 is one of two contributions to the total change dI.

The cloud also contributes photons to the beam. The thermal emission originating in the layer at x in dx of the cloud can be described with the volume emissivity j (W m^{-3} Hz^{-1}). This gives rise to an element of specific intensity from the layer in question which is, from (8.48),

$$dI_2 = \frac{j \, dx}{4\pi} \qquad \text{(Thermal emission from gas)} \qquad (11.35)$$

The sum of these two effects yields the net change in intensity I of the beam,

$$dI = -I\sigma n \, dx + \frac{j \, dx}{4\pi} \qquad \text{(Net change in } I \text{ in d}x \text{ at } x\text{)} \qquad (11.36)$$

This is the differential equation that allows us to find, by integration, the variation of beam intensity as it traverses the material on its way to the observer.

Intensity variation with optical depth

Rewrite (36) to be a function of optical depth τ. Recall the definition of the opacity, $\kappa \equiv \sigma n/\rho$ (10.24), where ρ (kg/m^3) is the mass density. Opacity is the cross section per kilogram of material (m^2/kg). Substitute $\kappa\rho$ for σn into (36) and rearrange,

$$\frac{1}{\kappa\rho}\frac{dI}{dx} = -I + \frac{j}{4\pi\kappa\rho} \qquad (11.37)$$

The product $\kappa\rho$ or σn is simply the inverse of the mean free path x_m with units of (m^{-1}); see Table 10.1. Thus the product $\kappa\rho x$ is the *number of mean free paths* in the distance x for fixed κ and ρ. In other words it is the *optical depth* $\tau = \kappa\rho x$, a dimensionless quantity previously defined (10.29). The denominator $\kappa\rho \, dx$ of the left side of (37) is thus equal to $d\tau$ since κ and ρ do not change appreciably in an incremental distance dx.

The equality (37) demands that the rightmost term have units of specific intensity. Since j is the volume emissivity of our cloud, we define this term to be the cloud intensity, or the *source intensity* I_s,

$$I_s \equiv \frac{j \, (\text{W m}^{-3}\ \text{Hz}^{-1})}{4\pi(\text{sr}) \, \kappa\rho \, (\text{m}^{-1})} \qquad \text{(Source intensity defined: W m}^{-2}\ \text{Hz}^{-1}\ \text{sr}^{-1}\text{)} \qquad (11.38)$$

This expression has the form of (8.53), the relation between j and I for an optically thin plasma of thickness Λ, namely $I = j_{av}\Lambda/4\pi$. Here, the mean free path $(\kappa\rho)^{-1}$ plays the role of the cloud thickness Λ. In our optically thick case, an observer can "see" only a depth of about one mean free path into the cloud. The source intensity (38) is thus the intensity an observer embedded in the cloud would measure if her view were limited by optically thick conditions ($\tau \gg 1$).

It follows from the above considerations that the differential equation (37) may now be written as

$$\Rightarrow \quad \frac{\mathrm{d}I(\tau)}{\mathrm{d}\tau} = -I(\tau) + I_\mathrm{s} \qquad \text{(Equation of radiative transfer)} \qquad (11.39)$$

where we express I as a function of τ, the optical depth of the cloud in the observer's line of sight (Fig. 17). This is the differential *radiative transfer equation* (RTE) which may be solved for the unknown quantity $I(\tau)$, the specific intensity at optical depth τ for our chosen frequency.

In (39), $I(\tau = 1)$ is the specific intensity measured by an observer within the cloud at the depth of one mean free path into the gas. Note that depth is measured from the left edge of the cloud. At $\tau = 0.1$ or $\tau = 3$, the function $I(\tau)$ is the specific intensity at depths of 0.1 and 3 mean free paths respectively. If the entire cloud has optical depth τ_Λ (corresponding to thickness Λ), the function $I(\tau_\Lambda)$ is the specific intensity measured by the observer outside the cloud.

The quantity $I(\tau)$ is distinct from I_s. It includes the radiation from the background source I_0 as modified by absorption and emission in the cloud. The background radiation is the "initial condition" we apply to the differential equation (39). The source function reflects the volume emissivity of the cloud itself.

The quantities τ, $I(\tau)$ and I_s in (39) are all functions of frequency; namely $\tau(\nu)$, $I(\nu)$, and $I_\mathrm{s}(\nu)$. We continue to consider one frequency only and suppress the argument ν. The function j, and hence I_s, can vary with position in the cloud, i.e., both can be functions of τ. This is the case in stellar atmospheres where the temperature varies continuously with altitude. In the following, we consider I_s to be a constant throughout the cloud; the important consequences of (39) are well illustrated in this case.

Local thermodynamic equilibrium

If the gas of the cloud were in *complete thermodynamic equilibrium*, the radiation and matter would all be in thermal equilibrium at some temperature T; the specific intensity $I(\tau)$ would not vary throughout the cloud. In this case, the derivative in (39) equals zero, $\mathrm{d}I/\mathrm{d}\tau = 0$, and the observed intensity $I(\tau)$ is given by

$$I(\tau) = I_\mathrm{s} \qquad \text{(Perfect thermal equilibrium)} \qquad (11.40)$$

which is independent of τ. Since $I(\tau)$ is the specific intensity for complete thermodynamic equilibrium, its spectrum must be the Planck (blackbody) function (23). In turn, the source intensity I_s must also have a blackbody spectrum.

The solutions we seek are, in general, not for complete thermodynamic equilibrium because they involve a gas at one temperature and incoming photons representative of a slightly different temperature. Also, the limited extent of the cloud implies that radiation is leaving the volume of the cloud, so that complete equilibrium can not exist near the surface. Nevertheless, in solving the RTE one can make the assumption of *local thermodynamic equilibrium* (LTE).

Under LTE, the matter (e.g., protons and electrons) in a local region is in equilibrium with itself, but not necessarily with the radiation. That is, the matter obeys strictly the Boltzmann–Saha–Maxwell statistics, i.e., (9.14) and (9.15), for the local temperature, but the photon distribution is allowed to deviate slightly from it. Nevertheless, the radiation emitted from the local region follows the frequency dependence of the blackbody function for the local temperature, according to (40). The source function I_s for radiation emitted in a local region is therefore the blackbody function (23) for the (matter) temperature of the local region.

One can show that in the solar photosphere, the number density of particles is $\sim 10^5$ times that of the photons. Since every such photon or particle has about the same energy, $\sim kT$, in thermal equilibrium, the energy content is overwhelmingly contained in the particles. They can thus maintain their own temperature and radiate at that temperature in their local region even if photons from a lower and slightly hotter region diffuse up into their region and minimally distort the overall photon spectrum.

Solution of the RTE

Insight into the behavior of $I(\tau)$ according to the radiative transfer equation (39) can be gained simply from knowledge of the relative magnitudes of $I(\tau)$ and I_s. If $I(\tau) < I_s$ at some depth τ, the derivative in (39) is positive which tells us that $I(\tau)$ increases with optical depth. This is shown as the heavy line in Fig. 18a; note that it lies below the horizontal dashed line for I_s. Recall that in our case we hold I_s constant throughout the cloud. If, on the other hand, $I(\tau) > I_s$, then $I(\tau)$ decreases with depth (heavy line in Fig. 18b). In each case, $I(\tau)$ moves toward I_s and asymptotically approaches it at large optical depth.

At zero optical depth, $I(0)$ is equal to the background intensity I_0 because only the background source, and no part of the cloud, is in the observer's line of sight as is clear from Fig. 17. We also see this in both panels of Fig. 18. This obvious result also follows from the formal solution of the RTE to which we now proceed. We will find that the solution naturally provides for absorption and emission lines.

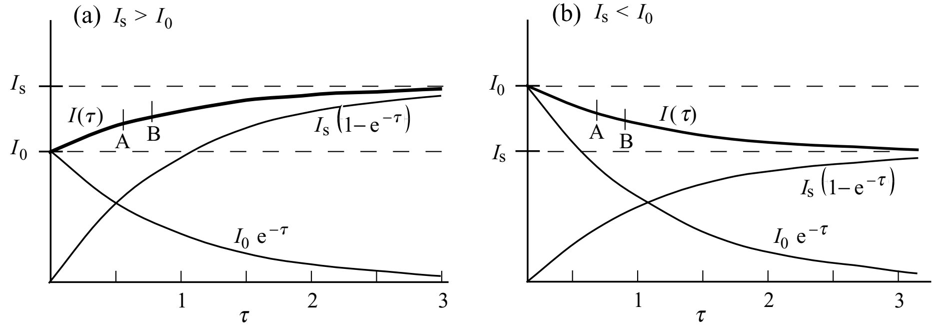

Figure 11.18. Plots of intensity $I(\tau)$ vs. optical depth τ from (44) for two cases: (a) source (cloud) intensity greater than the background intensity, $I_s > I_0$, and (b) $I_s < I_0$. As frequency is varied, the optical depth becomes higher at a resonance. If depth "A" is off resonance and depth "B" is centered at the resonance, case (a) yields an emission line and case (b) an absorption line.

The RTE (39) can be solved for $I(\tau)$ by integration as follows. Multiply (39) by e^{τ},

$$\frac{dI}{d\tau}e^{\tau} + I e^{\tau} = I_s\, e^{\tau}, \tag{11.41}$$

rewrite the left side as $d(I e^{\tau})/d\tau$, and integrate from 0 to τ,

$$\int_0^{\tau} d(I e^{\tau}) = \int_0^{\tau} I_s\, e^{\tau}\, d\tau \tag{11.42}$$

For our cloud with I_s independent of optical depth τ,

$$I(\tau)\, e^{\tau}|_0^{\tau} = I_s\, e^{\tau}|_0^{\tau} \tag{11.43}$$

Insert the limits and divide through by e^{τ},

➡ $$I(\tau) = I_0\, e^{-\tau} + I_s(1 - e^{-\tau}) \qquad \text{(Solution of radiative transfer equation)} \tag{11.44}$$

This is the solution of the RTE. The first term on the right shows the decreasing effect of the background radiation I_0 as the optical depth increases, while the second term shows the increasing effect of the source (cloud) emission. These two terms and their sum are plotted in Fig. 18. These plots illustrate the variation of intensity with τ for a single chosen frequency.

Limiting cases

The solution (44) readily illustrates the formation of spectral lines if we consider the variation of τ (and also I_0 and I_s) with frequency. There are four cases to consider, one of which has two possibilities:

$I_0 = 0$: there is no background radiation illuminating the cloud
(*i*) $\tau \ll 1$: the gas is optically thin
(*ii*) $\tau \gg 1$: the gas is optically thick
$I_0 > 0$: background radiation illuminates the back of the cloud
(*iii*) $\tau \ll 1$: the gas is optically thin (for $I_s > I_0$ and $I_s < I_0$)
(*iv*) $\tau \gg 1$: the gas is optically thick

Case 1: $I_0 = 0, \tau \ll 1$

The condition $I_0 = 0$ means that $I(\tau)$ will be affected only by radiation from the cloud. The $\tau \ll 1$ condition allows us to expand the exponential, $e^{-\tau} \approx 1-\tau$. The solution (44) then reduces to

$$I(\tau) = \tau I_s \qquad (I_0 = 0, \tau \ll 1) \qquad (11.45)$$

This tells us that the emission is proportional to the optical depth, for $\tau \ll 1$. This is reasonable because, for an observer located at $\tau \approx 0$ with leftward viewing detectors (Fig. 17) , there are no atoms in view. The optical depth is zero and so is the detected intensity. As the observer moves to the right, toward increasing τ, the number of atoms in the line of sight increases linearly with τ. The cloud is optically thin so every layer $d\tau$ of the cloud that is in view contributes equally to the intensity (Fig. 8.8); hence $I \propto \tau$. Note that changes in mass density ρ and opacity κ along the line of sight are automatically incorporated into τ.

Now we address the frequency variation of the quantities in (45). Let the atoms in the cloud have an atomic transition or *resonance* at some frequency. At that frequency the cross section σ for absorption of incoming photons is high, and hence, so is the optical depth τ. In general, τ is a function of frequency and therefore so is the intensity I. We therefore rewrite (45) as

$$I(\nu) = \tau(\nu)\, I_s(\nu) \qquad (I_0 = 0, \tau \ll 1) \qquad (11.46)$$

Resonances at two distinct frequencies are hypothesized and illustrated in Fig. 19a (left panel) which is a plot of τ vs. ν for an observer at fixed position x. From (46), we see that high optical depths at these frequencies lead to high emerging fluxes $I(\nu)$ at these same frequencies, provided that I_s is a smooth function of frequency. A plot of I vs. ν for an arbitrarily chosen spectrum $I_s(\nu)$ (Fig. 19a, right panel) shows emission lines at the two resonance frequencies. Note that the spectrum lies well below the source spectrum I_s because $\tau \ll 1$, in accord with (46).

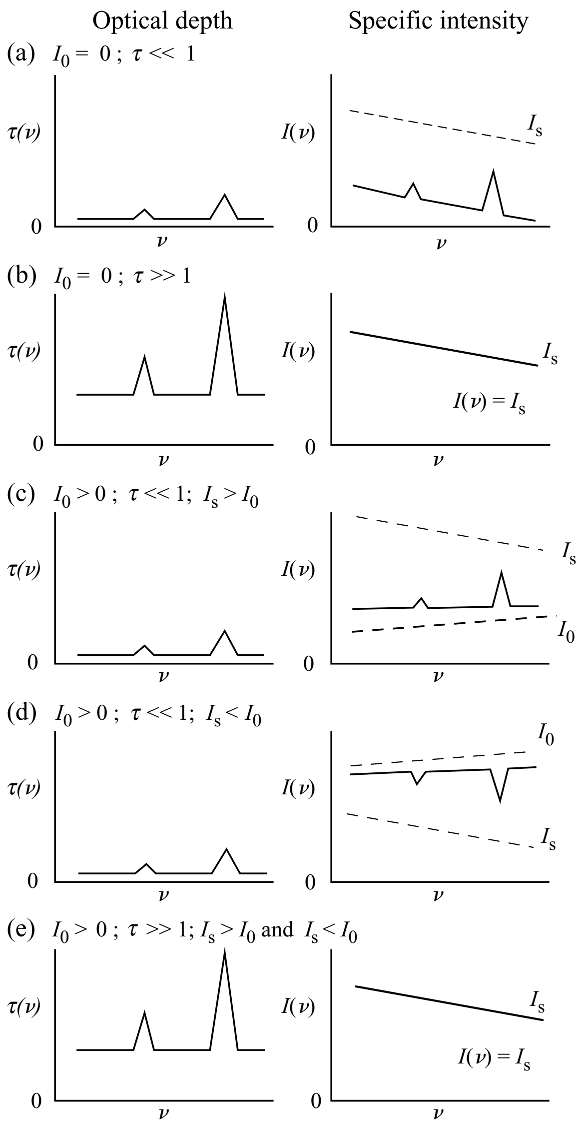

Figure 11.19. Frequency dependence of optical depth $\tau(\nu)$ (left) and specific intensity $I(\nu,\tau)$ (right) for an observer at some fixed position x in the cloud. I follows from $\tau(\nu)$ and the solution (44) of the radiative transfer differential equation. Arbitrary spectral shapes for the source (cloud) spectrum I_s and the background spectrum I_0 are shown as dashed lines. Emission lines are expected if the foreground gas is optically thin, $\tau \ll 1$, and hotter than the background source (c) or, in the limiting case, there is no background source (a). Absorption lines are expected if the background source is hotter than the foreground source, again if the cloud is optically thin (d). If the cloud is optically thick, the continuum spectrum of the cloud is observed (b,e).

At resonance frequencies, with their higher cross sections, the gas behaves as if it contains more atoms, or as if it were thicker. In this case, the observer would seem to "see" more emitting atoms, and hence greater intensity. At frequencies adjacent to a line, τ will be lower by definition and fewer atoms are seen. If τ is constant at these adjacent frequencies, as in the left panel of Fig. 19a, the output spectrum $I(\nu)$ will, according to (46), mimic the spectrum I_s away from the line as shown in the right panel.

Case 2: $I_0 = 0, \tau \gg 1$

Here the gas is very thick ($\tau \gg 1$), and the expression (44) reduces to,

$$I(\nu) = I_s(\nu) \qquad (I_0 = 0, \tau \gg 1) \qquad (11.47)$$

which we have previously deduced (40); here we use the variable ν rather than $\tau(\nu)$.

The output radiation given by (47) equals that of the continuum source specific intensity at all frequencies. If the source spectrum is the blackbody spectrum, the output spectrum at any depth $\tau \gg 1$ is also blackbody. Even though the resonances may exist (Fig. 19b), the intensity $I(\nu)$ has no dependence on τ, and hence no spectral lines will form.

In this case, the local observer sees the maximum possible number of emitting atoms at any frequency so the resonances are not apparent. It is like being immersed in a thick fog that is denser (more opaque) in some directions than others. Nevertheless, the appearance in all directions (frequencies) is uniform as long as the fog is totally impenetrable ($\tau \gg 1$) in all directions.

The observer in the fog sees only to a depth that yields enough water droplets to completely block the view. The same number of water droplets are thus seen in all directions, and the view appears uniform even though in some directions it penetrates less deeply than others. In our case, the increase of opacity at some frequency reduces the depth of view, but the observed intensity does not change.

The optically thick character of the gas allows the photons and particles to interact sufficiently to come into equilibrium thus giving rise to the continuum (blackbody) spectrum characteristic of the cloud.

Case 3: $I_0 > 0, \tau \ll 1$

In this case, there is a source behind the cloud. Since $\tau \ll 1$, we again use the Taylor expansion, $e^{-\tau} \approx 1 - \tau$, so that (44) becomes

$$I = I_0 + \tau(I_s - I_0) \qquad (I_0 > 0, \tau \ll 1) \qquad (11.48)$$

Consider two cases here, $I_s > I_0$ and $I_s < I_0$. In the former case, the output intensity is the background intensity I_0 plus another positive term. If the optical depth τ is higher at some frequency (i.e., greater opacity κ) than at surrounding frequencies, the emerging flux will be greater at that frequency. This yields an emission line (Fig. 19c).

In the case of $I_s < I_0$, the rightmost term is negative, and the emerging intensity is less than the background intensity. If again, the optical depth τ is especially large at some frequency, the emerging intensity is depressed even more at that frequency. This yields an absorption line (Fig. 19d).

These same conclusions extend to somewhat larger optical depths, $\tau \lesssim 2$, as illustrated in the plots of the function $I(\tau)$ vs. τ (Fig. 18). In the case of $I_s > I_0$ (Fig. 18a), the observed intensity I increases with optical depth τ. At a given frequency not at a resonance, the intensity I might be given by the value at point A in the plot. At the frequency of a resonance, the optical depth is higher (by definition), and the observed intensity is therefore higher (point B). Thus an emission line is observed. For $I_s < I_0$ (Fig. 18b), the increase in opacity again moves the observer from A to B, but in this case it yields a decrease in intensity, or an absorption line.

If the functions I_0 and I_s are each blackbody spectra, the one with the higher temperature will have the greater intensity at any frequency (Fig. 8). Thus we have $T_s > T_0$ for the $I_s > I_0$ case, and $T_s < T_0$ for the $I_s < I_0$ case. We conclude therefore that if the temperature of the foreground cloud T_s is *greater* than the background temperature T_0, a spectrum with *emission lines* will emerge, and that if the cloud is *cooler* than the background, a spectrum with *absorption lines* will emerge.

In most stellar atmospheres at the depth seen in visible light (the photosphere), the temperature decreases with altitude, i.e., toward the observer. The deeper hot layers are then the background radiation for the higher, cooler regions. Absorption lines are thus prevalent in stellar spectra at visible wavelengths.

In contrast, radiation from the sun at ultraviolet frequencies yields emission lines. The observed ultraviolet radiation comes from high in the solar atmosphere because the higher opacities in the ultraviolet limit the depth into which the observer can "see". In these higher chromospheric regions, the temperature is increasing with height (moving toward the 10^6-K corona). Thus the higher temperatures are in the foreground, and the spectra characteristically exhibit emission lines.

Case 4: $I_0 > 0, \tau \gg 1$

In this case, the gas is optically thick and (44) again yields

$$I(\nu) = I_s(\nu) \tag{11.49}$$

This is the same expression obtained when there was no background intensity (47). Since the gas is optically thick, the presence of the background source is immaterial (Fig. 19e). The radiation at any τ is simply the continuum source (blackbody) spectrum of the optically thick cloud. It does not matter whether $I_0 > I_s$ or $I_0 < I_s$.

Summary

This concludes our discussion of the limiting cases of the solution to the radiative transfer equation. In each case the result is a continuum spectrum that reflects one or both of the spectra I_0 and I_s with, in some cases, superimposed lines created by increases in the optical depth τ at certain frequencies. If the foreground cloud intensity (temperature) is greater than the background intensity (temperature), emission lines are formed. If the opposite is true, absorption lines are formed.

Problem 11.51. Consider a stellar atmosphere where I_s varies with depth in the cloud as $I_s = a + b\tau$ where a is a positive constant and b is a constant that can be positive or negative. (In the text, we took I_s to be constant throughout the cloud.) Assume that conditions of local thermodynamic equilibrium are satisfied, and that the observer views the atmosphere head on, as in Fig. 17. (The variation in I_s arises through a variation the volume emissivity with position (38) which in turn is a consequence of temperature variation within the atmosphere. (a) Find the solution $I(\tau)$ of the equation of radiation transfer (39) for this situation. (b) Evaluate the solution for the case of no background source, $I_0 = 0$, with $\tau \ll 1$ and with $\tau \gg 1$. (c) Explain why spectral lines will or will not be formed in each of these two cases. If they are, what are the condition(s) on b that result in emission or absorption lines? In the $\tau \ll 1$ case, how would you constrain b so that only emission lines occur in the region $\tau < 0.1$, in the context of your approximations? [Ans. (b) $I(\tau \ll 1) \approx a\tau + (b - a)(\tau^2/2)$]