A First Course in Digital Communications

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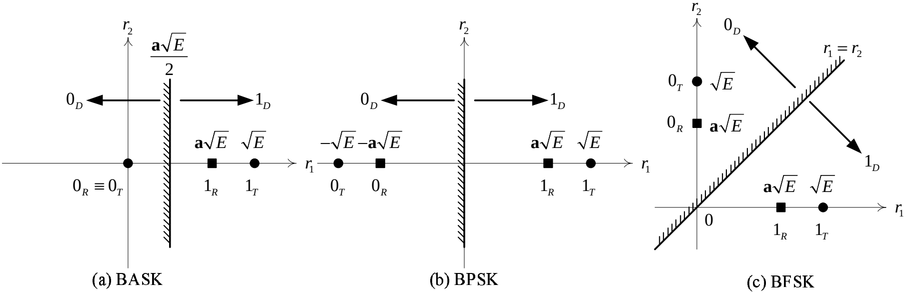
Introduction

- We have considered that the transmitted signal is only degraded by AWGN and it might be subjected to filtering.
- There are communication channels (e.g., mobile wireless channels) where the received signal is not subjected to a known transformation or filtering.
- Typically the gain and/or phase of a digitally modulated transmitted signal is not known precisely at the receiver.
- It is common to model these parameters as *random*.
- Shall consider channel models where the amplitude and/or phase of the received signal is random.

Demodulation with Random Amplitude

$$\mathbf{r}(t) = \mathbf{a}s(t) + \mathbf{w}(t),$$

where $\mathbf{a}$ is a random variable with known pdf $f_{\mathbf{a}}(a)$.



Performance of BPSK and BFSK

$$\begin{aligned}\mathbf{P}[\text{error}] &= Q\left(\mathbf{a}\sqrt{\frac{2E_b}{N_0}}\right) \quad \text{(antipodal)},\\ \mathbf{P}[\text{error}] &= Q\left(\mathbf{a}\sqrt{\frac{E_b}{N_0}}\right) \quad \text{(orthogonal)}.\end{aligned}$$

$$\begin{aligned}E\{\mathbf{P}[\text{error}]\} &= \int_0^\infty Q\left(a\sqrt{\frac{2E_b}{N_0}}\right) f_{\mathbf{a}}(a)\mathrm{d}a \quad \text{(antipodal)},\\ E\{\mathbf{P}[\text{error}]\} &= \int_0^\infty Q\left(a\sqrt{\frac{E_b}{N_0}}\right) f_{\mathbf{a}}(a)\mathrm{d}a \quad \text{(orthogonal)}.\end{aligned}$$

Optimum Demodulation of BASK

- Optimum receiver is determined by the maximum likelihood ratio:

$$\frac{f_{\mathbf{r}_1}(r_1|1_T)}{f_{\mathbf{r}_1}(r_1|0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} 1.$$

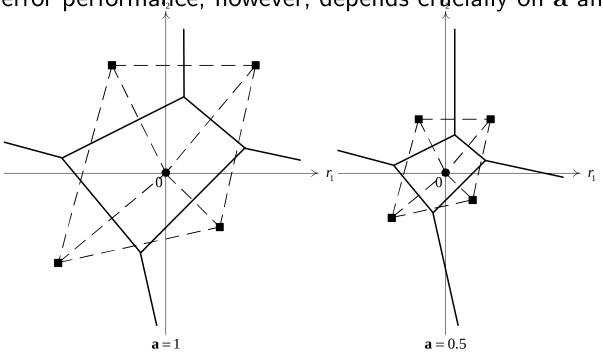
- $f_{\mathbf{r}_1}(r_1|0_T)$ is $\mathcal{N}(0, N_0/2)$, while

$$\begin{aligned} f_{\mathbf{r}_1}(r_1|1_T) &= \int_0^{\infty} f_{\mathbf{r}_1}(r_1|1_T, \mathbf{a}=a) f_{\mathbf{a}}(a)\mathrm{d}a \\ &= E\left\{f_{\mathbf{r}_1}(r_1|1_T, \mathbf{a}=a)\right\}. \end{aligned}$$

- Need to know $f_{\mathbf{a}}(a)$ to proceed further.
- In general the threshold (and hence the decision regions) is a balance between the different regions given by the values that $\mathbf{a}$ takes on weighted by the probability that $\mathbf{a}$ takes on these values.

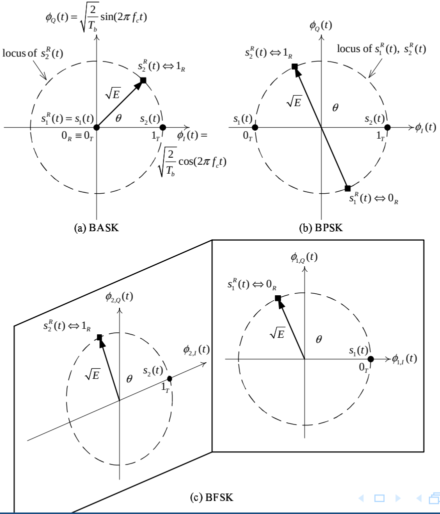
M-ary Demodulation with Random Amplitude

- If all the signal points lie at distance of $\sqrt{E_s}$ from the origin (i.e., equal energy), then the optimum decision regions are *invariant* to any scaling by $\mathbf{a}$, provided that $\mathbf{a} \geq 0$.
- The matched-filter or correlation receiver structure is still optimum, one does not even need to know $f_{\mathbf{a}}(a)$.
- The error performance, however, depends crucially on $\mathbf{a}$ and $f_{\mathbf{a}}(a)$.



Demodulation with Random Phase

Phase uncertainty can be modeled as a uniform *random variable* (over $[0, 2\pi]$ or $[-\pi, \pi]$). It does not change the energy of the received signal.



(a) BASK

(b) BPSK

(c) BFSK

Optimum Receiver for Noncoherent BASK

$$\mathbf{r}(t) = \begin{cases} \mathbf{w}(t), & \text{if "}0_T\text{"} \\ \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t - \boldsymbol{\theta}) + \mathbf{w}(t), & \text{if "}1_T\text{"} \end{cases}.$$

$$\mathbf{r}_I = \begin{cases} \mathbf{w}_I, & \text{if "}0_T\text{"} \\ \sqrt{E}\cos\boldsymbol{\theta} + \mathbf{w}_I, & \text{if "}1_T\text{"} \end{cases}, \quad \mathbf{r}_Q = \begin{cases} \mathbf{w}_Q, & \text{if "}0_T\text{"} \\ \sqrt{E}\sin\boldsymbol{\theta} + \mathbf{w}_Q, & \text{if "}1_T\text{"} \end{cases},$$

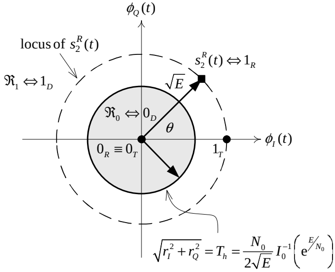
where $\mathbf{w}_I$ and $\mathbf{w}_Q$ are statistically independent $\mathcal{N}(0, N_0/2)$.

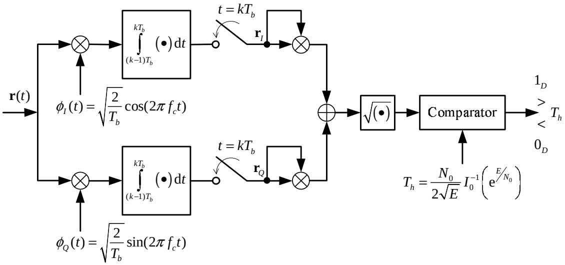
$$f(r_I, r_Q|0_T) = f(r_I|0_T)f(r_Q|0_T) = \frac{1}{\pi N_0}\exp\left(-\frac{r_I^2+r_Q^2}{N_0}\right).$$

$$\begin{aligned} f(r_I, r_Q|1_T) &= \int_0^{2\pi} \frac{1}{\pi N_0}\exp\left[-\frac{\left(r_I - \sqrt{E}\cos\theta\right)^2 + \left(r_Q - \sqrt{E}\sin\theta\right)^2}{N_0}\right]\frac{1}{2\pi}\mathrm{d}\theta \\ &= \frac{1}{\pi N_0}\mathrm{e}^{-\frac{r_I^2+r_Q^2}{N_0}}\mathrm{e}^{-\frac{E}{N_0}}I_0\left(\frac{2\sqrt{E}}{N_0}\sqrt{r_I^2+r_Q^2}\right). \end{aligned}$$

$$\mathrm{e}^{-\frac{E}{N_0}}I_0\left(\frac{2\sqrt{E}}{N_0}\sqrt{r_I^2+r_Q^2}\right) \underset{0_D}{\overset{1_D}{\gtrless}} 1 \Rightarrow \sqrt{r_I^2+r_Q^2} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{N_0}{2\sqrt{E}}I_0^{-1}\left(\mathrm{e}^{\frac{E}{N_0}}\right).$$

Decision Regions of BASK with Random Phase

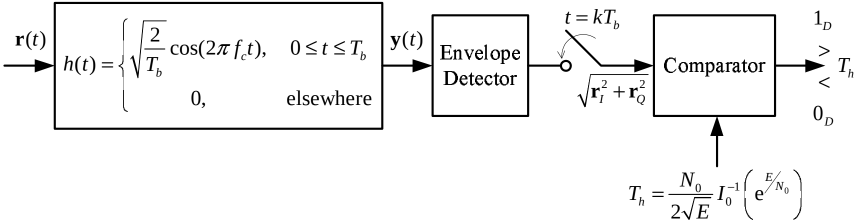




Different Implementation of BASK Demodulation

$$y(t) = \int_{-\infty}^{\infty} r(\lambda)\sqrt{\frac{2}{T_b}}\cos(2\pi f_c(t-\lambda))[u(t-\lambda)-u(t-\lambda-T_b]\mathrm{d}\lambda$$
$$= \sqrt{y_I^2(t)+y_Q^2(t)}\cos\left[2\pi f_c\left(t-\tan^{-1}\frac{y_Q(t)}{y_I(t)}\right)\right],$$

where $\sqrt{y_I^2(t)+y_Q^2(t)}$ is the envelope. At the sampling instant, $t = kT_b$, then $y_I(kT_b) = r_I$ and $y_Q(kT_b) = r_Q$.



Error Performance

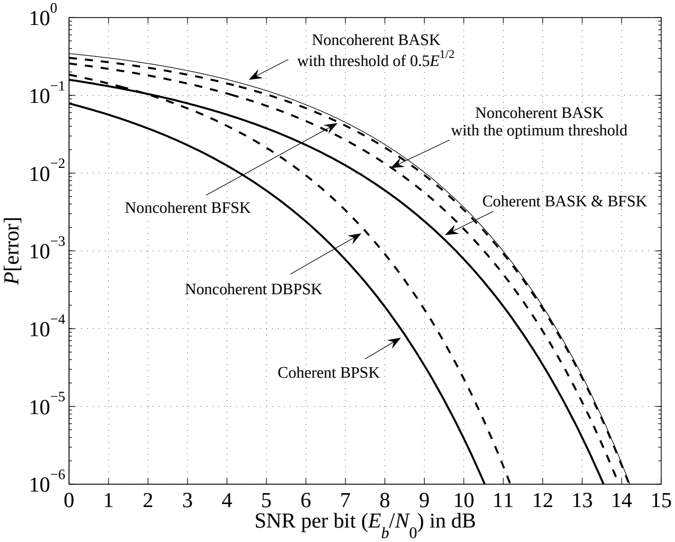
$$
\begin{aligned}
P[\text{error}|0_T] &= \iint_{\Re_1} f(r_I, r_Q|0_T)\mathrm{d}r_I\mathrm{d}r_Q = \iint_{\Re_1} \frac{1}{\pi N_0}\mathrm{e}^{-\frac{r_I^2+r_Q^2}{N_0}}\mathrm{d}r_I\mathrm{d}r_Q, \\
&= \frac{1}{\pi N_0}\int_{\alpha=0}^{2\pi}\int_{\rho=T_h}^{\infty} \rho \mathrm{e}^{-\frac{\rho^2}{N_0}}\mathrm{d}\rho\mathrm{d}\alpha = \mathrm{e}^{-T_h^2/N_0}.
\end{aligned}
$$

$$
\begin{aligned}
P[\text{error}|1_T] &= 1 - P[\text{correct}|1_T] = 1 - \iint_{\Re_1} f(r_I, r_Q|1_T)\mathrm{d}r_I\mathrm{d}r_Q \\
&= 1 - Q\left(\sqrt{\frac{2E}{N_0}}, \sqrt{\frac{2}{N_0}}T_h\right).
\end{aligned}
$$

where $Q(\alpha,\beta) = \int_\beta^\infty x\mathrm{e}^{-\frac{x^2+\alpha^2}{2}} I_0(\alpha x)\mathrm{d}x$ is Marcum's Q-function.

$$
P[\text{error}] = \frac{1}{2}\mathrm{e}^{-T_h^2/N_0} + \frac{1}{2}\left[1 - Q\left(\sqrt{\frac{2E}{N_0}}, \sqrt{\frac{2}{N_0}}T_h\right)\right].
$$

There is about 0.3 dB penalty in power when using such a simpler suboptimum threshold, $\frac{\sqrt{E}}{2} = \sqrt{\frac{E_b}{2}}$.



Optimum Receiver for Noncoherent BFSK

$$\mathbf{r}(t) = \begin{cases} \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t - \boldsymbol{\theta}) + \mathbf{w}(t), & \text{if "}0_T\text{"} \\ \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t - \boldsymbol{\theta}) + \mathbf{w}(t), & \text{if "}1_T\text{"} \end{cases}.$$

0_T

$$\begin{aligned} \mathbf{r}_{1,I} &= \sqrt{E}\cos\boldsymbol{\theta} + \mathbf{w}_{1,I} \\ \mathbf{r}_{1,Q} &= \sqrt{E}\sin\boldsymbol{\theta} + \mathbf{w}_{1,Q} \\ \mathbf{r}_{2,I} &= \mathbf{w}_{2,I} \\ \mathbf{r}_{2,Q} &= \mathbf{w}_{2,Q} \end{aligned}$$

1_T

$$\begin{aligned} \mathbf{r}_{1,I} &= \mathbf{w}_{1,I} \\ \mathbf{r}_{1,Q} &= \mathbf{w}_{1,Q} \\ \mathbf{r}_{2,I} &= \sqrt{E}\cos\boldsymbol{\theta} + \mathbf{w}_{2,I} \\ \mathbf{r}_{2,Q} &= \sqrt{E}\sin\boldsymbol{\theta} + \mathbf{w}_{2,Q} \end{aligned}$$

$$f(r_{1,I}, r_{1,Q}, r_{2,I}, r_{2,Q}|0_T) = \frac{1}{\pi N_0}\mathrm{e}^{-\frac{r_{1,I}^2+r_{1,Q}^2}{N_0}} \times$$

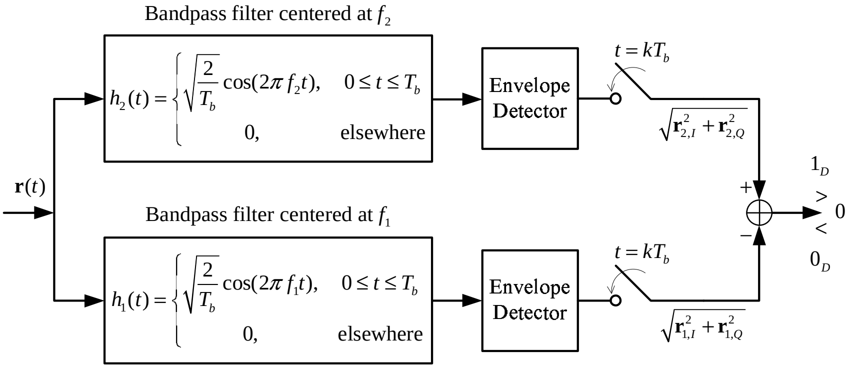
$$\mathrm{e}^{-\frac{\sqrt{E}}{N_0}} I_0\left(\frac{2\sqrt{E}}{N_0}\sqrt{r_{1,I}^2 + r_{1,Q}^2}\right)\frac{1}{\pi N_0}\mathrm{e}^{-\frac{r_{2,I}^2+r_{2,Q}^2}{N_0}},$$

$$f(r_{1,I}, r_{1,Q}, r_{2,I}, r_{2,Q}|1_T) = \frac{1}{\pi N_0}\mathrm{e}^{-\frac{r_{1,I}^2+r_{1,Q}^2}{N_0}} \times$$

$$\frac{1}{\pi N_0}\mathrm{e}^{-\frac{r_{2,I}^2+r_{2,Q}^2}{N_0}}\mathrm{e}^{-\frac{\sqrt{E}}{N_0}} I_0\left(\frac{2\sqrt{E}}{N_0}\sqrt{r_{2,I}^2 + r_{2,Q}^2}\right).$$

Optimum Demodulator for BFSK with Random Phase

$$\frac{I_0\left(\frac{2\sqrt{E}}{N_0}\sqrt{r_{2,I}^2+r_{2,Q}^2}\right)}{I_0\left(\frac{2\sqrt{E}}{N_0}\sqrt{r_{1,I}^2+r_{1,Q}^2}\right)} \underset{0_D}{\overset{1_D}{\gtrless}} 1 \Rightarrow \sqrt{r_{2,I}^2+r_{2,Q}^2} \underset{0_D}{\overset{1_D}{\gtrless}} \sqrt{r_{1,I}^2+r_{1,Q}^2}.$$



The demodulator finds the envelope at the two frequencies and chooses the larger one at the sampling instant.

Error Performance of BFSK with Random Phase

By symmetry $P[\text{error}] = P[\text{error}|0_T] = P\left(\mathbf{r}_{2,I}^2 + \mathbf{r}_{2,Q}^2 \geq \mathbf{r}_{1,I}^2 + \mathbf{r}_{1,Q}^2\right)$.

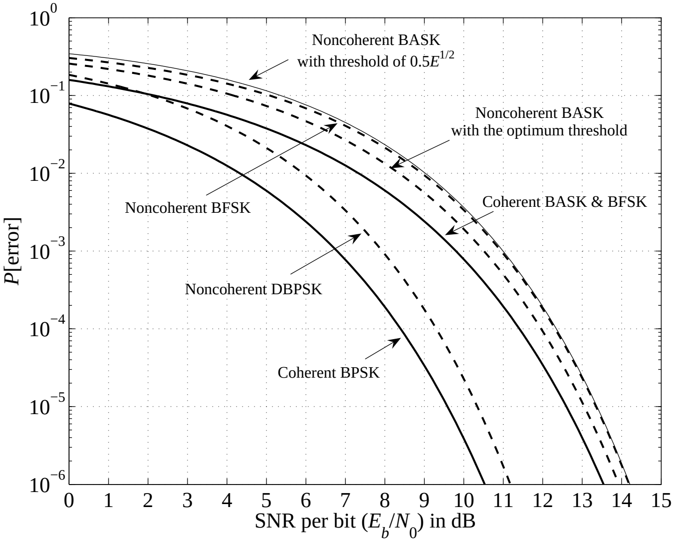
$$P\left[\text{error}\middle|0_T, \mathbf{r}_{1,I}^2 + \mathbf{r}_{1,Q}^2 = R^2\right] =$$

$$P\left[(\mathbf{r}_{2,I}, \mathbf{r}_{2,Q}) \text{ falls outside the circle of radius } R\middle|0_T\right]$$

$$= \frac{1}{\pi N_0}\int_{\alpha=0}^{2\pi}\int_{\rho=R}^{\infty}\rho \mathrm{e}^{-\frac{\rho^2}{N_0}}\mathrm{d}\rho \mathrm{d}\alpha = \mathrm{e}^{-\frac{R^2}{N_0}} = \mathrm{e}^{-\frac{\left(r_{1,I}^2+r_{1,Q}^2\right)}{N_0}}.$$

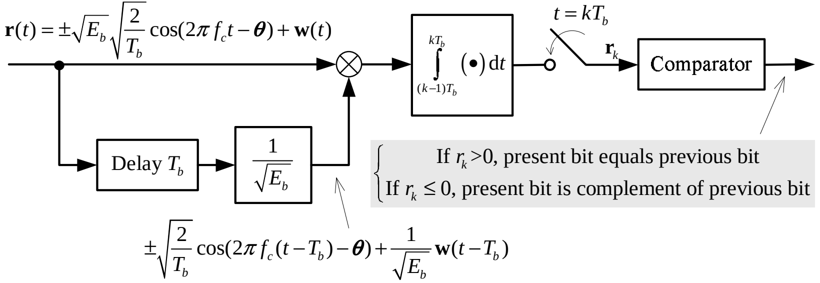
$$\begin{aligned}
P[\text{error}] &= E\left\{\int_{r_{1,I}=-\infty}^{\infty}\int_{r_{1,Q}=-\infty}^{\infty}\mathrm{e}^{-\frac{\left(r_{1,I}^2+r_{1,Q}^2\right)}{N_0}} f(r_{1,I}, r_{1,Q}|0_T)\mathrm{d}r_{1,I}\mathrm{d}r_{1,Q}\right\} \\
&= \int_{\alpha=0}^{2\pi}\left[\int_{r_{1,I}=-\infty}^{\infty}\mathrm{e}^{-\frac{r_{1,I}^2}{N_0}}\mathcal{N}\left(\sqrt{E}\cos\alpha, \frac{N_0}{2}\right)\mathrm{d}r_{1,I}\right] \\
&\quad \left[\int_{r_{1,Q}=-\infty}^{\infty}\mathrm{e}^{-\frac{r_{1,Q}^2}{N_0}}\mathcal{N}\left(\sqrt{E}\sin\alpha, \frac{N_0}{2}\right)\mathrm{d}r_{1,Q}\right] f_{\boldsymbol{\theta}}(\alpha)\mathrm{d}\alpha. \\
&= \int_{\alpha=0}^{2\pi}\frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{E\cos^2\alpha}{2N_0}}\frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{E\sin^2\alpha}{2N_0}}\frac{1}{2\pi}\mathrm{d}\alpha = \frac{1}{2}\mathrm{e}^{-\frac{E}{2N_0}} = \frac{1}{2}\mathrm{e}^{-\frac{E_b}{2N_0}}.
\end{aligned}$$

Noncoherent BASK is about 0.3 dB more power efficient than noncoherent BFSK.



Differential BPSK

- Coherent BPSK is 3 dB better than coherent BASK or BFSK: Is it possible to use BPSK on a channel with phase uncertainty?
- Possible if a *phase reference* can be established at the receiver that is matched to the received signal.
- If the phase uncertainty changes relatively slowly with time, the received signal in one bit interval can act as a phase reference for the succeeding bit interval.



But the above method can lead to error propagation!

Differential BPSK Modulation and Demodulation

$$0_T\text{: no phase change,}$$
$$1_T\text{: } \pi \text{ phase change.}$$

- The decision rule is:

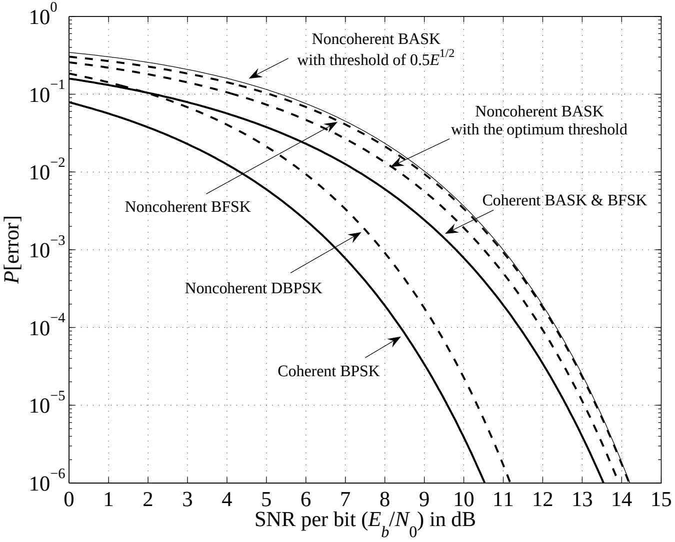
$$r_k \underset{0_D}{\overset{1_D}{\gtrless}} 0.$$

  which is independent of the previous decision.
- Since DBPSK is orthogonal signaling, the error analysis for noncoherent BFSK therefore applies to DBPSK:

$$P[\text{error}]_{\text{DBPSK}} = \frac{1}{2}\mathrm{e}^{-E_b/N_0}.$$

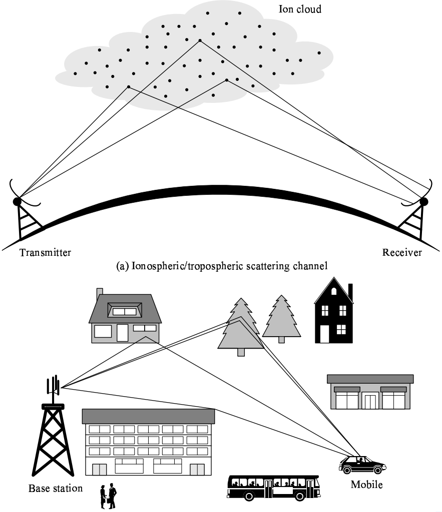
- The only difference is rather than E_b joules/bit, the energy in DBPSK becomes $2E_b$. This is because the received signal over two bit intervals is used to make a decision.

DBPSK shows about 1 dB degradation over coherent BPSK.



Detection over Fading Channels

Fading channel model arises when there are multiple transmission paths from the transmitter to the receiver.



(a) Ionospheric/tropospheric scattering channel

(b) Mobile wireless channel

Rayleigh Fading Channel Model

- Consider the transmitted signal $s_T(t) = s(t)\cos(2\pi f_c t)$, where $s(t) = \pm\sqrt{E_b}\sqrt{\frac{2}{T_b}}$ over the bit interval with bit rate $r_b \ll f_c$ (*lowpass* signal).
- The received signal is:

$$\begin{aligned}\mathbf{r}(t) &= \sum_j \mathbf{r}_j(t) = \sum_j s(t-\mathbf{t}_j)\boldsymbol{\alpha}_j \cos(2\pi f_c(t-\mathbf{t}_j)) \\ &\approx s(t)\sum_j \boldsymbol{\alpha}_j \cos\left(2\pi f_c t - 2\pi f_c \mathbf{t}_j\right) = s(t)\sum_j \boldsymbol{\alpha}_j \cos\left(2\pi f_c t - \boldsymbol{\theta}_j\right)\end{aligned}$$

  where $\boldsymbol{\alpha}_j$ represents the attenuation and $\mathbf{t}_j$ the delay along the jth path, which are *random* variables. Also because $s(t)$ is lowpass, we approximate $s(t) \approx s(t-t_j)$.
- Since $\mathbf{t}_j \sim 1/f_c$, the random phase $\boldsymbol{\theta}_j$ lies in the range $[0, 2\pi)$. Now

$$\mathbf{r}(t) = s(t)\left[\left(\sum_j \boldsymbol{\alpha}_j \cos\boldsymbol{\theta}_j\right)\cos(2\pi f_c t) + \left(\sum_j \boldsymbol{\alpha}_j \sin\boldsymbol{\theta}_j\right)\sin(2\pi f_c t)\right].$$

Rayleigh Fading Channel Model

$\mathbf{n}_{F,I} = \left(\sum_j \boldsymbol{\alpha}_j \cos \boldsymbol{\theta}_j\right)$ and $\mathbf{n}_{F,Q} = \left(\sum_j \boldsymbol{\alpha}_j \sin \boldsymbol{\theta}_j\right)$ have the following moments:

$$E\left\{\mathbf{n}_{F,I}\right\} = \sum_j E\{\boldsymbol{\alpha}_j\}E\{\cos \boldsymbol{\theta}_j\} = 0,\ E\left\{\mathbf{n}_{F,Q}\right\} = \sum_j E\{\boldsymbol{\alpha}_j\}E\{\sin \boldsymbol{\theta}_j\} = 0,$$

$$E\left\{\mathbf{n}_{F,I}^2\right\} = \sum_j E\left\{\boldsymbol{\alpha}_j^2\right\} E\left\{\cos^2 \boldsymbol{\theta}_j\right\} = \frac{\sigma_F^2}{2},$$

$$E\left\{\mathbf{n}_{F,Q}^2\right\} = \sum_j E\left\{\boldsymbol{\alpha}_j^2\right\} E\left\{\sin^2 \boldsymbol{\theta}_j\right\} = \frac{\sigma_F^2}{2},$$

$$E\left\{\mathbf{n}_{F,I}\mathbf{n}_{F,Q}\right\} = E\left\{\sum_j \boldsymbol{\alpha}_j \cos \boldsymbol{\theta}_j \sum_k \boldsymbol{\alpha}_k \sin \boldsymbol{\theta}_k\right\}$$

$$= \sum_j \sum_k E\left\{\boldsymbol{\alpha}_j \boldsymbol{\alpha}_k\right\} \underbrace{E\left\{\cos \boldsymbol{\theta}_j \sin \boldsymbol{\theta}_k\right\}}_{=0} = 0,$$

Since the number of multipaths is large, the *central limit theorem* says that $\mathbf{n}_{F,I}$, $\mathbf{n}_{F,Q}$ are Gaussian random variables.

- $\mathbf{n}_{F,I}$ and $\mathbf{n}_{F,Q}$ are statistically independent Gaussian random variables, zero-mean, variance $\sigma_F^2/2$:

$$f_{\mathbf{n}_I,\mathbf{n}_Q}(n_I, n_Q) = f_{\mathbf{n}_I}(n_I) f_{\mathbf{n}_Q}(n_Q) = \mathcal{N}\left(0, \frac{\sigma_F^2}{2}\right)\mathcal{N}\left(0, \frac{\sigma_F^2}{2}\right).$$

- The received signal is therefore:

$$\begin{aligned}\mathbf{r}(t) &= s(t)\left[\mathbf{n}_{F,I}\cos(2\pi f_c t) + \mathbf{n}_{F,Q}\sin(2\pi f_c t)\right] \\ &= s(t)\left[\boldsymbol{\alpha}\cos(2\pi f_c t - \boldsymbol{\theta})\right],\end{aligned}$$

where $\boldsymbol{\alpha} = \sqrt{\mathbf{n}_{F,I}^2 + \mathbf{n}_{F,Q}^2}$, $\boldsymbol{\theta} = \tan^{-1}\left(\frac{\mathbf{n}_{F,Q}}{\mathbf{n}_{F,I}}\right)$ and

$$\begin{aligned} f_{\boldsymbol{\theta}}(\theta) &= \frac{1}{2\pi} \quad \text{(uniform)}, \\ f_{\boldsymbol{\alpha}}(\alpha) &= \frac{2\alpha}{\sigma_F^2}\exp\left\{-\frac{\alpha^2}{\sigma_F^2}\right\} u(\alpha) \quad \text{(Rayleigh)}.\end{aligned}$$

- The term "Rayleigh fading" comes from the envelope distribution.
- The phase of the received signal severely degraded but that the amplitude is affected as well: The incoming signals add not only constructively but also destructively.

Noncoherent Demodulation of BFSK in Rayleigh Fading

$$s(t) = \begin{cases} \sqrt{E_b}\sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t), & \text{if "}0_T\text{"} \\ \sqrt{E_b}\sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t), & \text{if "}1_T\text{"} \end{cases},$$

$$\begin{aligned} \mathbf{r}(t) &= \begin{cases} \sqrt{E_b}\sqrt{\frac{2}{T_b}}\boldsymbol{\alpha}\cos(2\pi f_1 t - \boldsymbol{\theta}) + \mathbf{w}(t), & \text{if "}0_T\text{"} \\ \sqrt{E_b}\sqrt{\frac{2}{T_b}}\boldsymbol{\alpha}\cos(2\pi f_2 t - \boldsymbol{\theta}) + \mathbf{w}(t), & \text{if "}1_T\text{"} \end{cases}. \\ &= \begin{cases} \sqrt{E_b}\mathbf{n}_{F,I}\underbrace{\sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t)}_{\phi_{1,I}(t)} + \sqrt{E_b}\mathbf{n}_{F,Q}\underbrace{\sqrt{\frac{2}{T_b}}\sin(2\pi f_1 t)}_{\phi_{1,Q}(t)} + \mathbf{w}(t), & \text{"}0_T\text{"}, \\ \sqrt{E_b}\mathbf{n}_{F,I}\underbrace{\sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t)}_{\phi_{2,I}(t)} + \sqrt{E_b}\mathbf{n}_{F,Q}\underbrace{\sqrt{\frac{2}{T_b}}\sin(2\pi f_2 t)}_{\phi_{2,Q}(t)} + \mathbf{w}(t), & \text{"}1_T\text{"}, \end{cases} \end{aligned}$$

The transmitted signal lies entirely within the signal space spanned by $\phi_{1,I}(t)$, $\phi_{1,Q}(t)$, $\phi_{2,I}(t)$ and $\phi_{2,Q}(t)$.

$\underline{0_T}$

$$\begin{aligned} \mathbf{r}_{1,I} &= \sqrt{E_b}\mathbf{n}_{F,I} + \mathbf{w}_{1,I} \\ \mathbf{r}_{1,Q} &= \sqrt{E_b}\mathbf{n}_{F,Q} + \mathbf{w}_{2,Q} \\ \mathbf{r}_{2,I} &= \mathbf{w}_{2,I} \\ \mathbf{r}_{2,Q} &= \mathbf{w}_{2,Q} \end{aligned}$$

$\underline{1_T}$

$$\begin{aligned} \mathbf{r}_{1,I} &= \mathbf{w}_{1,I} \\ \mathbf{r}_{1,Q} &= \mathbf{w}_{2,Q} \\ \mathbf{r}_{2,I} &= \sqrt{E_b}\mathbf{n}_{F,I} + \mathbf{w}_{2,I} \\ \mathbf{r}_{2,Q} &= \sqrt{E_b}\mathbf{n}_{F,Q} + \mathbf{w}_{2,Q} \end{aligned}$$

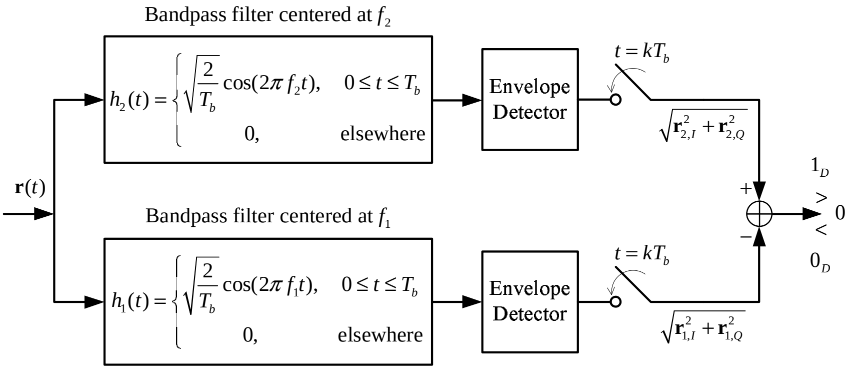
- $\mathbf{w}_{1,I}$, $\mathbf{w}_{1,Q}$, $\mathbf{w}_{2,I}$, $\mathbf{w}_{2,Q}$ are due to thermal noise, are Gaussian, statistically independent, zero-mean, and variance $N_0/2$.
- $\mathbf{n}_{F,I}$ and $\mathbf{n}_{F,Q}$, are also Gaussian, statistically independent, zero-mean and variance $\sigma_F^2/2$.
- The sufficient statistics are Gaussian, statistically independent, zero-mean, with a variance of either $N_0/2$ or $E_b\sigma_F^2/2 + N_0/2$, depending on whether a "0_T" or "1_T".
- Computing the likelihood ratio gives the following decision rule:

$$r_{2,I}^2 + r_{2,Q}^2 \underset{0_D}{\overset{1_D}{\gtrless}} r_{1,I}^2 + r_{1,Q}^2,$$

Equivalently the decision rule can be expressed as:

$$\sqrt{r_{2,I}^2 + r_{2,Q}^2} \underset{0_D}{\overset{1_D}{\gtrless}} \sqrt{r_{1,I}^2 + r_{1,Q}^2}.$$

which is identical to that for noncoherent BFSK!



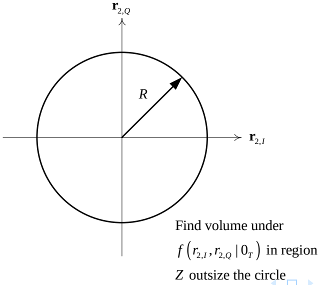
Error Probability of Noncoherent BFSK

$$P[\text{error}] = P[\text{error}|0_T] = P\left[\sqrt{\mathbf{r}_{2,I}^2+\mathbf{r}_{2,Q}^2} \geq \sqrt{\mathbf{r}_{1,I}^2+\mathbf{r}_{1,Q}^2}\,\middle|\, 0_T\right].$$

Fix the value of $\mathbf{r}_{1,I}^2+\mathbf{r}_{1,Q}^2$ at a specific value, say R^2 and compute

$$P\left[\sqrt{\mathbf{r}_{2,I}^2+\mathbf{r}_{2,Q}^2} \geq R \,\middle|\, 0_T, \sqrt{\mathbf{r}_{1,I}^2+\mathbf{r}_{1,Q}^2} = R\right] = \iint\limits_{Z} \frac{1}{\pi N_0} \mathrm{e}^{-\frac{r_{2,I}^2+r_{2,Q}^2}{N_0}} \mathrm{d}r_{2,I}\mathrm{d}r_{2,Q}$$

$$= \int_{\lambda=0}^{2\pi}\int_{\rho=R}^{\infty} \frac{1}{\pi N_0}\rho \mathrm{e}^{-\frac{\rho^2}{N_0}} \mathrm{d}\rho\mathrm{d}\lambda = \mathrm{e}^{-\frac{\left(r_{1,I}^2+r_{1,Q}^2\right)}{N_0}}.$$

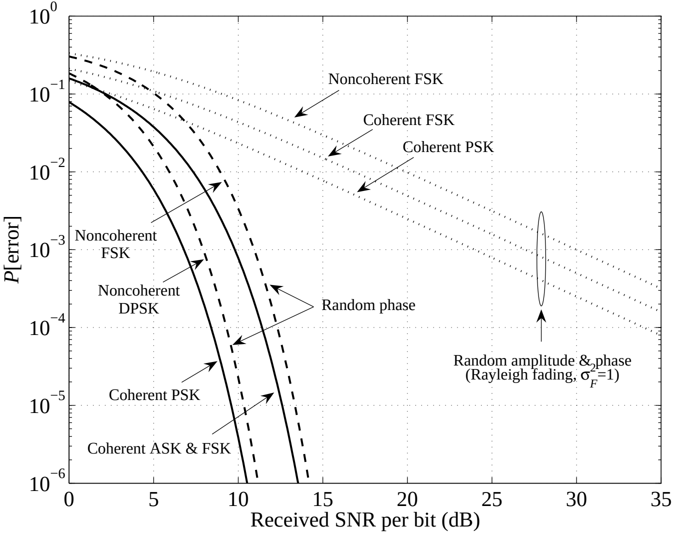


- Average over all possible values of $\mathbf{r}_{1,I},\ \mathbf{r}_{1,Q}$:

$$
\begin{aligned}
& E\left\{ \mathrm{e}^{-\frac{\left(r_{1,I}^2+r_{1,Q}\right)^2}{N_0}} \middle| 0_T \right\} \\
= & \int_{r_{1,I}=-\infty}^{\infty}\int_{r_{1,Q}=-\infty}^{\infty} \mathrm{e}^{-\frac{\left(r_{1,I}^2+r_{1,Q}\right)^2}{N_0}} f(r_{1,I}, r_{1,Q}|0_T)\mathrm{d}r_{1,I}\mathrm{d}r_{1,Q} \\
= & \frac{1}{2+\sigma_F^2\frac{E_b}{N_0}}.
\end{aligned}
$$

- $E_b\sigma_F^2$ can be interpreted as the received energy per bit.
- The behavior is $P[\text{error}] \propto \frac{1}{\text{SNR}}$, a much much slower rate of decay as compared to $P[\text{error}] \propto \mathrm{e}^{-\text{SNR}}$.
- In the log-log plot of the $P[\text{error}]$ versus SNR in dB, the error performance curve appears to be a straight line of slope -1 in the high SNR region.

Compared to noncoherent demodulation of BFSK in random phase only, at an error probability of 10^{-3} about 19 dB more power is needed for noncoherent demodulation of BFSK in Rayleigh fading!



BFSK and BPSK with Coherent Demodulation

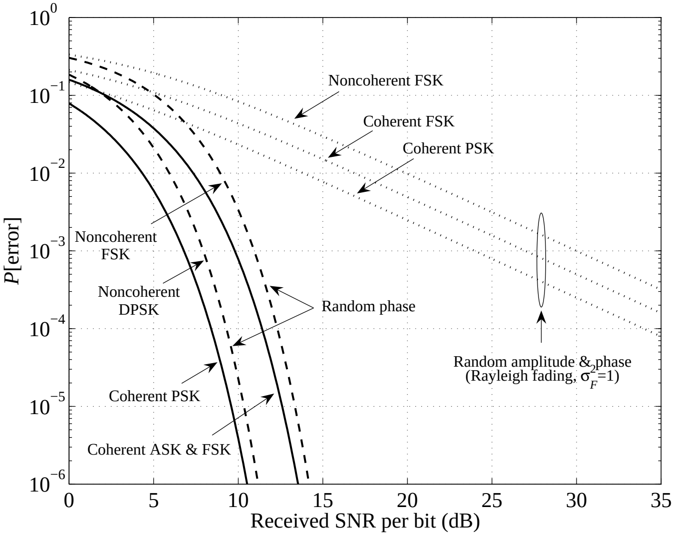
- If the random phase introduced by fading can be perfectly estimated, then coherent demodulation can be achieved $\Rightarrow$ The situation is the same as detection in random amplitude.
- With a Rayleigh fading channel, $\boldsymbol{\alpha}$ is a Rayleigh random variable.
- For BFSK, the optimum decision rule is $r_1 \underset{1_D}{\overset{0_D}{\gtrless}} r_2$ and

$$P[\text{error}] = E\left\{Q\left(\boldsymbol{\alpha}\sqrt{\frac{E_b}{N_0}}\right)\right\} = \frac{1}{2}\left[1 - \sqrt{\frac{\sigma_F^2 \frac{E_b}{N_0}}{2 + \sigma_F^2 \frac{E_b}{N_0}}}\right].$$

- For BPSK, the optimum decision rule is $r_1 \underset{0_D}{\overset{1_D}{\gtrless}} 0$ and

$$P[\text{error}] = \frac{1}{2}\left[1 - \sqrt{\frac{\sigma_F^2 \frac{E_b}{N_0}}{1 + \sigma_F^2 \frac{E_b}{N_0}}}\right].$$

Coherent BPSK is 3 dB more efficient that coherent BFSK, which in turn is 3 dB more efficient than the noncoherent BFSK.



Diversity

- All communication schemes over a Rayleigh fading channel have the same discouraging performance behavior of $P[\text{error}] \propto \frac{1}{\text{SNR}}$.
- The reason is that it is very probable for the channel to exhibit what is called a *deep fade*, i.e, the received signal amplitude becomes very small.
- *Diversity* technique: multiple copies of the same message are transmitted over independent fading channels in the hope that at least one of them will not experience a deep fade.
    - Time diversity: Achieved by transmitting the same message in different time slots.
    - Frequency diversity: Accomplished by sending the message copies in different frequency slots.
    - Antenna diversity: Achieved with the use of antenna arrays

Optimum Demodulation of Binary FSK with Diversity

Consider N transmissions of BFSK over a fading channel:

$$s(t) = \begin{cases} \sqrt{E_b'}\sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t), & \text{if "}0_T\text{"} \\ \sqrt{E_b'}\sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t), & \text{if "}1_T\text{"} \end{cases},$$

$$\begin{aligned}
\mathbf{r}_j(t) &= \begin{cases} \sqrt{E_b'}\sqrt{\frac{2}{T_b}}\boldsymbol{\alpha}_j\cos(2\pi f_1 t - \boldsymbol{\theta}_j) + \mathbf{w}(t), & \text{"}0_T\text{"} \\ \sqrt{E_b'}\sqrt{\frac{2}{T_b}}\boldsymbol{\alpha}_j\cos(2\pi f_2 t - \boldsymbol{\theta}_j) + \mathbf{w}(t), & \text{"}1_T\text{"} \end{cases} \\
&= \begin{cases} \sqrt{E_b'}\mathbf{n}_{j,I}\underbrace{\sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t)}_{\phi_{j,I}^{(1)}(t)} + \sqrt{E_b'}\mathbf{n}_{j,Q}\underbrace{\sqrt{\frac{2}{T_b}}\sin(2\pi f_1 t)}_{\phi_{j,Q}^{(1)}(t)} + \mathbf{w}(t), & \text{"}0_T\text{"} \\ \sqrt{E_b'}\mathbf{n}_{j,I}\underbrace{\sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t)}_{\phi_{j,I}^{(2)}(t)} + \sqrt{E_b'}\mathbf{n}_{j,Q}\underbrace{\sqrt{\frac{2}{T_b}}\sin(2\pi f_2 t)}_{\phi_{j,Q}^{(2)}(t)} + \mathbf{w}(t), & \text{"}1_T\text{"} \end{cases}
\end{aligned}$$

for $(j-1)T_b \le t \le jT_b$ and $j = 1, \ldots, N$.

$$\begin{cases} \phi_{j,I}^{(1)}(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t), \ \phi_{j,Q}^{(1)}(t) = \sqrt{\frac{2}{T_b}}\sin(2\pi f_1 t), \\ \phi_{j,I}^{(2)}(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_2 t), \ \phi_{j,Q}^{(2)}(t) = \sqrt{\frac{2}{T_b}}\sin(2\pi f_2 t), \end{cases} \quad (j-1)T_b \le t \le jT_b, \ j = 1, \ldots, N.$$

0_T	
$\mathbf{r}_{1,I}^{(1)} = \sqrt{E_b'}\mathbf{n}_{1,I}^{(1)} + \mathbf{w}_{1,I}^{(1)}$	$\mathbf{r}_{1,Q}^{(1)} = \sqrt{E_b'}\mathbf{n}_{1,Q}^{(1)} + \mathbf{w}_{1,Q}^{(1)}$
$\vdots$	$\vdots$
$\mathbf{r}_{N,I}^{(1)} = \sqrt{E_b'}\mathbf{n}_{N,I}^{(1)} + \mathbf{w}_{N,I}^{(1)}$	$\mathbf{r}_{N,Q}^{(1)} = \sqrt{E_b'}\mathbf{n}_{N,Q}^{(1)} + \mathbf{w}_{N,Q}^{(1)}$
$\mathbf{r}_{1,I}^{(2)} = \mathbf{w}_{1,I}^{(2)}$	$\mathbf{r}_{1,Q}^{(2)} = \mathbf{w}_{1,Q}^{(2)}$
$\vdots$	$\vdots$
$\mathbf{r}_{N,I}^{(2)} = \mathbf{w}_{N,I}^{(2)}$	$\mathbf{r}_{N,Q}^{(2)} = \mathbf{w}_{N,Q}^{(2)}$

1_T	
$\mathbf{r}_{1,I}^{(1)} = \mathbf{w}_{1,I}^{(1)}$	$\mathbf{r}_{1,Q}^{(1)} = \mathbf{w}_{1,Q}^{(1)}$
$\vdots$	$\vdots$
$\mathbf{r}_{N,I}^{(1)} = \mathbf{w}_{N,I}^{(1)}$	$\mathbf{r}_{N,Q}^{(1)} = \mathbf{w}_{N,Q}^{(1)}$
$\mathbf{r}_{1,I}^{(2)} = \sqrt{E_b'}\mathbf{n}_{1,I}^{(2)} + \mathbf{w}_{1,I}^{(2)}$	$\mathbf{r}_{1,Q}^{(2)} = \sqrt{E_b'}\mathbf{n}_{1,Q}^{(2)} + \mathbf{w}_{1,Q}^{(2)}$
$\vdots$	$\vdots$
$\mathbf{r}_{N,I}^{(2)} = \sqrt{E_b'}\mathbf{n}_{N,I}^{(2)} + \mathbf{w}_{N,I}^{(2)}$	$\mathbf{r}_{N,Q}^{(2)} = \sqrt{E_b'}\mathbf{n}_{N,Q}^{(2)} + \mathbf{w}_{N,Q}^{(2)}$

Number the sufficient statistics corresponding to f_1 from 1 to $2N$; sufficient statistics associated with f_2 from $2N+1$ to $4N$. The likelihood ratio test is

$$\frac{f(r_1,\ldots,r_N;r_{2N+1},\ldots,r_{4N}|1_T)}{f(r_1,\ldots,r_N;r_{2N+1},\ldots,r_{4N}|0_T)} =$$

$$\frac{\prod_{j=1}^{2N}\frac{1}{\sqrt{2\pi}\sigma_w}\mathrm{e}^{-r_j^2/(2\sigma_w^2)}\prod_{j=2N+1}^{4N}\frac{1}{\sqrt{2\pi}\sigma_t}\mathrm{e}^{-r_j^2/(2\sigma_t^2)}}{\prod_{j=1}^{2N}\frac{1}{\sqrt{2\pi}\sigma_t}\mathrm{e}^{-r_j^2/(2\sigma_t^2)}\prod_{j=2N+1}^{4N}\frac{1}{\sqrt{2\pi}\sigma_w}\mathrm{e}^{-r_j^2/(2\sigma_w^2)}} \underset{0_D}{\overset{1_D}{\gtrless}} 1,$$

which can be reduced to

$$\sum_{j=2N+1}^{4N} r_j^2 \underset{0_D}{\overset{1_D}{\gtrless}} \sum_{j=1}^{2N} r_j^2.$$

Chi-Square Probability Density Function

Consider $\mathbf{y} = \mathbf{x}_1^2 + \mathbf{x}_2^2 + \cdots + \mathbf{x}_N^2$ where the $\mathbf{x}_i$'s are zero-mean, statistically independent Gaussian random variables with identical variances, σ^2. To find $f_{\mathbf{y}}(y)$ determine the characteristic function $\Phi_{\mathbf{y}}(f)$ and then inverse transform it.

$$\Phi_{\mathbf{y}}(f) = E\left\{\mathrm{e}^{j2\pi f \mathbf{y}}\right\} = E\left\{\mathrm{e}^{j2\pi \sum_{k=1}^{N} \mathbf{x}_k^2}\right\} = E\left\{\prod_{k=1}^{N} \mathrm{e}^{j2\pi f \mathbf{x}_k^2}\right\} = \prod_{k=1}^{N} E\left\{\mathrm{e}^{j2\pi f \mathbf{x}_k^2}\right\}.$$

$$E\left\{\mathrm{e}^{j2\pi f \mathbf{x}_k^2}\right\} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \mathrm{e}^{j2\pi f x_k^2}\mathrm{e}^{-x_k^2/(2\sigma^2)}\mathrm{d}x_k = \frac{1}{\sqrt{1-j4\pi\sigma^2 f}}.$$

Therefore $\Phi_{\mathbf{y}}(f) = \frac{1}{(1-j4\pi\sigma^2 f)^{N/2}}$ and $f_{\mathbf{y}}(y) = \int_{-\infty}^{\infty} \frac{1}{(1-j4\pi\sigma^2 f)^{N/2}}\mathrm{e}^{-j2\pi y f}\mathrm{d}f$, where $y \geq 0$. From the identity $\int_{-\infty}^{\infty}(\beta - ix)^{-\nu}\mathrm{e}^{-ipx}\mathrm{d}x = \frac{2\pi p^{\nu-1}\mathrm{e}^{-\beta p}}{\Gamma(\nu)}u(p)$, where $\mathcal{R}(\nu) > 0$ and $\mathcal{R}(\beta) > 0$, the pdf is

$$f_{\mathbf{y}}(y) = \frac{y^{\frac{N}{2}-1}\mathrm{e}^{-y/(2\sigma^2)}}{2^{\frac{N}{2}}\sigma^N\Gamma\left(\frac{N}{2}\right)}u(y),$$

where $\Gamma(x) = \int_0^{\infty} t^{x-1}\mathrm{e}^{-t}\mathrm{d}t = (x-1)!$ for x integer.

Error Performance of BFSK with Diversity

Define $\boldsymbol{\ell}_1 = \sum_{j=2N+1}^{4N} \mathbf{r}_j^2$ and $\boldsymbol{\ell}_0 = \sum_{j=1}^{2N} \mathbf{r}_j^2$. The decision rule is:

$$\boldsymbol{\ell}_1 \underset{0_D}{\overset{1_D}{\gtrless}} \boldsymbol{\ell}_0.$$

$$P[\text{error}] = P[\text{error}|0_T] = \int_0^{\infty} f(\ell_0|0_T) \left[\int_{\ell_0}^{\infty} f(\ell_1|0_T) \mathrm{d}\ell_1 \right] \mathrm{d}\ell_0.$$

$f(\ell_1|0_T)$ and $f(\ell_0|0_T)$ are chi-square distributions:

$$f(\ell_1|0_T) = \frac{\ell_1^{N-1} \mathrm{e}^{-\ell_1/(2\sigma_w^2)}}{2^N \sigma_w^{2N} \Gamma(N)} u(\ell_1), \; f(\ell_0|0_T) = \frac{\ell_0^{N-1} \mathrm{e}^{-\ell_0/(2\sigma_t^2)}}{2^N \sigma_t^{2N} \Gamma(N)} u(\ell_0).$$

It can be shown that

$$P[\text{error}] = \sum_{j=1}^{N} \left(\frac{\sigma_t^2}{\sigma_w^2} \right)^{N-j} \frac{1}{\left(1 + \frac{\sigma_t^2}{\sigma_w^2}\right)^{2N-j}} \frac{\Gamma(2N-j)}{\Gamma(N)\Gamma(N-j+1)}.$$

- Define $\gamma_T = E_b'\sigma_F^2/N_0$ as the averaged SNR *per transmission*. Recognize that $\frac{\sigma_t^2}{\sigma_w^2} = 1 + \gamma_T$ and $\Gamma(x) = (x-1)!$ for integer x. Then

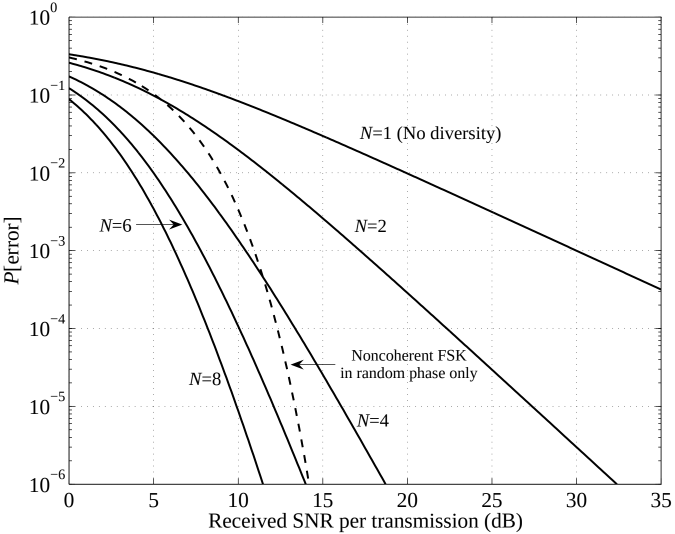
$$P[\text{error}] = \frac{1}{(2+\gamma_T)^N} \sum_{k=0}^{N-1} \binom{N-1+k}{k} \left(\frac{1+\gamma_T}{2+\gamma_T}\right)^k .$$

- For large values of SNR, $1+\gamma_T \approx 2+\gamma_T \approx \gamma_T$ and

$$P[\text{error}] \approx \frac{1}{(\gamma_T)^N} \sum_{k=0}^{N-1} \binom{N-1+k}{k} = \frac{1}{(\gamma_T)^N} \binom{2N-1}{N}.$$

- The error performance now decays inversely with the Nth power of the received SNR.
- The exponent N of the SNR is generally referred to as the *diversity order* of the modulation scheme.

Compared to no diversity, there is a significant improvement in performance with diversity.



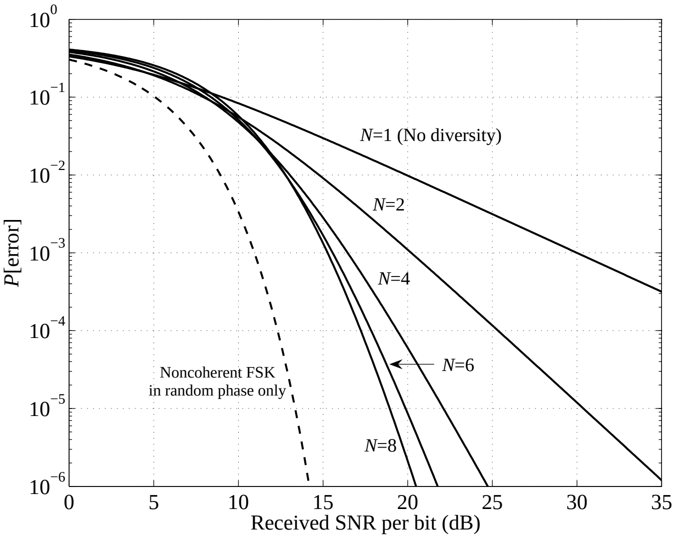
Optimum Diversity

- As the diversity order N increases the error performance improves.
- This improvement comes at the expense of a reduced data rate in the case of time diversity.
- If the transmitter's power or equivalently the energy expended per information bit is constrained to E_b joules then increasing N does not necessarily lead to a better error performance.
- With increased N we increase the probability of avoiding a deep fade, at the same time the energy, $E_b^{'}$, of each transmission is reduced. Therefore the SNR of each transmission is reduced which in turn increases the error probability.
- There is an optimum value for the diversity order N at each level of error probability. An empirical relationship is:

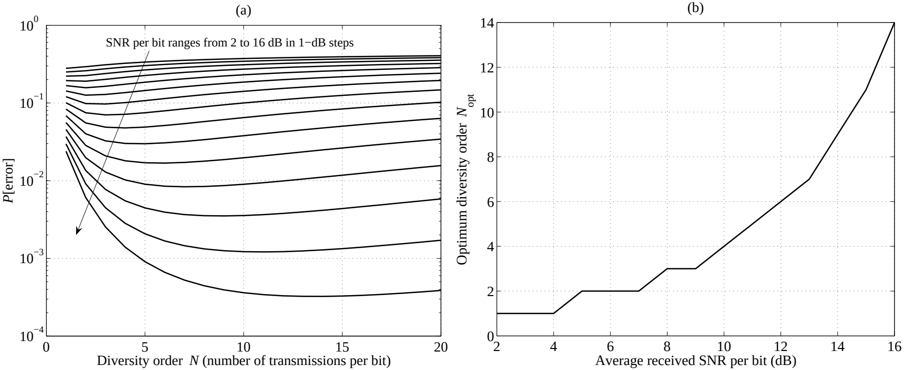
$$N_{\text{opt}} = K\mathrm{e}^{10\log_{10}\gamma_T}.$$

where K is some constant.

P[error] versus the averaged received SNR *per bit*, $10\log_{10}\left(\frac{E_b\sigma_F^2}{N_0}\right)$.



Determining The Optimum Diversity Order



Central Limit Theorem

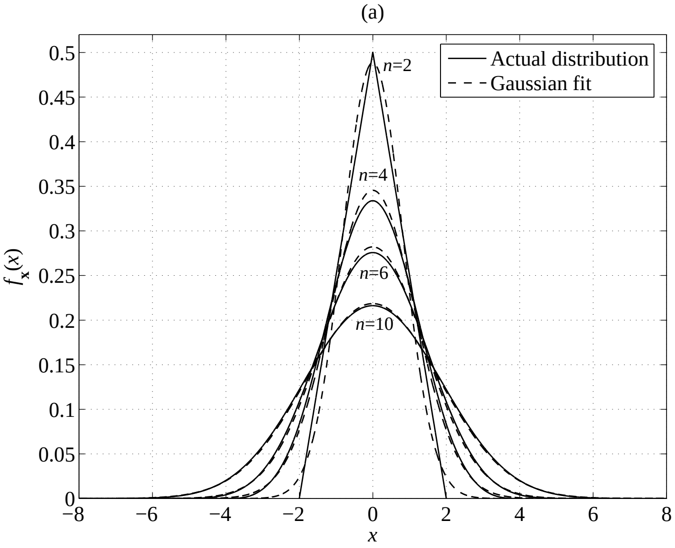
- Central limit theorem states that under certain general conditions the sum of n statistically independent continuous random variables has a pdf that approaches a Gaussian pdf as n increases.
- Let $\mathbf{x} = \sum_{i=1}^{n} \mathbf{x}_i$. where the $\mathbf{x}_i$'s are statistically independent random variables with mean, $E\{\mathbf{x}_i\} = m_i$, and variance, $E\left\{(\mathbf{x}_i - m_i)^2\right\} = \sigma_i^2$. Then $\mathbf{x}$ is a random variable with mean $m_{\mathbf{x}} = \sum_{i=1}^{n} m_i$, variance $\sigma_{\mathbf{x}}^2 = \sum_{i=1}^{n} \sigma_i^2$ and a pdf of

$$f_{\mathbf{x}}(x) = f_{\mathbf{x}_1}(x) * f_{\mathbf{x}_2}(x) * \cdots * f_{\mathbf{x}_n}(x).$$

- By the central limit theorem $f_{\mathbf{x}}(x)$ approaches a Gaussian pdf as n increases, i.e.,

$$f_{\mathbf{x}}(x) \sim \frac{1}{\sqrt{2\pi}\sigma_{\mathbf{x}}} \mathrm{e}^{-\frac{(x-m_{\mathbf{x}})^2}{2\sigma_{\mathbf{x}}^2}}.$$

Example 1: $f_{\mathbf{x}_i}(x_i)$ are Zero-Mean Uniform



Example 2: $f_{\mathbf{x}_i}(x_i)$ are Laplacian