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Problems for Chapter 20 of *Advanced Mathematics for Applications*

Theory of Distributions

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1 General

1. Do the following linear operators acting on the infinitely differentiable functions ϕ with compact support define distributions?

$$(a) \quad T_1(\phi) = \sum_{n=0}^{N} \phi^{(n)}(0), \qquad (b) \quad T_2(\phi) = \int_0^1 \phi^{(k)}(t)dt.$$

Here superscripts in parenthesis indicate derivatives and N and k are arbitrary integers.

2. Show that the following equalities hold in the sense of distributions, i.e. as linear continuous functionals over the space of test functions

$$e^x\delta(x) = \delta(x), \qquad \sin(ax)\delta'(x) = -a\delta(x).$$

3. Prove that, in the sense of distributions,

$$x\,\mathrm{Pf}\left(\frac{1}{x}\right) = 1\,.$$

4. Using the result

$$\sum_{m=1}^{N-1} \sin\, mx = \frac{\sin\,(Nx/2)\ \sin[(N-1)x/2]}{\sin(x/2)}$$

prove that, in the distributional sense,

$$\sum_{m=1}^{\infty} m\cos mx = -\frac{1}{4}\mathrm{cosec}^2(\frac{1}{2}x).$$

5. Is it true that $|x|^{-1/2}|x|^{-1/2} = |x|^{-1}$?

6. If, as shown in section 3.6, Poisson's formula

$$u(r,\theta) = \frac{R^2-r^2}{2\pi}\int_{-\pi}^{\pi} \frac{f(\theta-\phi)}{R^2-2rR\cos\phi+r^2}d\phi.$$

furnishes the solution to the boundary-value problem $\nabla^2 u = 0$ for $0 \le r \le R$, $0 \le \phi < 2\pi$, $u = f(\phi)$ for $r = R$, then it must be that

$$\frac{1}{2\pi}\frac{R^2-r^2}{R^2-2rR\cos\phi+r^2} \to \delta(0),$$

as $r \to R$. Prove this fact.

7. For $k = 1, 2, \ldots$ let
$$s_k = \frac{1}{\pi}\frac{k}{1+k^2x^2}.$$
Show that
$$\lim < s_k, \phi >= \phi(0).$$
What is
$$\lim \left\langle \frac{2k^3x}{\pi(1+k^2x^2)^2}, \phi \right\rangle ?$$

8. Show that
$$\lim_{k\to\infty} \frac{1-\cos kx}{x} = \mathrm{Pf}\left(\frac{1}{x}\right).$$

9. Show that $|x|' = \operatorname{sgn} x$.

10. Show that, for $\lambda \neq 0,\ -1,\ -2,\ \ldots,$
$$\frac{\mathrm{d}}{\mathrm{d}x} x_+^\lambda = \lambda x_+^{\lambda-1}.$$

11. Find the second distributional derivatives of $\exp(-|x|)$ and $\sin|x|$.

12. The function $\tanh x^{-1}$ has a jump discontinuity at $x = 0$. Find its distributional derivative.

13. Prove that, in the sense of distributions,
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^\alpha H(x)\right] = \alpha x^{\alpha-1} H(x), \qquad x^\alpha H(x) = \frac{1}{2}|x|^\alpha \left(1 + \operatorname{sgn} x\right).$$

14. Show that $u(x) = H(x)J_0(x)$ is a solution of Bessel's equation of order 0
$$\frac{1}{x}\frac{d}{dx}\left(x\frac{du}{dx}\right) + u(x) = 0$$

15. Show that
$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\, H(x)\right] = \frac{\mathrm{d}f}{\mathrm{d}x} H(x) + f(0)\delta(x).$$
If f is continuous except for jumps of magnitude $f_1,\ f_2,\ \ldots$ at $x = a_1,\ a_2, \ldots,$ what is $f'(x)$?

16. Prove that, in the distributional sense,
$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[f(x+\epsilon) - f(x)\right] = f'(x).$$

17. Calculate
$$I = \int_{-1}^{1} |x| f''(x) dx$$
by expressing $|x|$ in terms of the Heaviside (or step) distribution $H(x)$ and integrating by parts.

18. Verify formally that, if $a(x) < t < b(x)$,
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\{H[t-a(x)]H[b(x)-t]f(x,t)\}dt.$$

19. Calculate in closed form
$$J = \int_0^\infty H(\sin\,\pi x)\ \exp(-ax)\,dx.$$

20. Calculate the integral
$$I \ = \int_0^b H(x-a)f'(x)dx$$
first directly and then by parts and show that the two results are equal. Consider both $a > b$ and $a < b$.

21. Calculate the first derivative of the distribution defined by
$$\langle \widetilde{x^{-1}}, \phi\rangle \ = \lim_{\epsilon\to 0}\left[\int_\epsilon^\infty \frac{\phi(x)}{x}\,dx + \phi(0)\,\log\epsilon\right]$$
For test functions which can be expanded in Taylor series near the origin, show explicitly that the limit $\epsilon \to 0$ exists and is finite.

2 The δ distribution

1. Show that the following equalities hold in the sense of distributions, i.e. as linear continuous functionals over the space of test functions
$$e^x\delta(x) = \delta(x), \qquad \sin(ax)\delta'(x) = -a\delta(x).$$

2. Prove that
$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}\delta(ax+b) \ = \ \frac{1}{a^m|a|}\,\frac{\mathrm{d}^m}{\mathrm{d}x^m}\delta(x+b/a)$$

3. Show that
$$x^n\delta^{(m)}(x) \ = \ \left\{\begin{array}{cc} 0 & m < n \\ (-1)^n\frac{m!}{(m-n)!}\delta^{(m-n)}(x) & m \geq n \end{array}\right. .$$

4. Show that
$$\delta'(x^3+3x) \ = \ \frac{1}{9}\delta'(x)\,, \qquad \delta''(x^3+3x) \ = \ \delta''(x) - 2\delta(x)\,.$$

5. Show that, if x_0 is the only simple zero of the function $f(x)$,
$$\delta'\left(f(x)\right) \ = \ \frac{1}{[f'(x_0)]^2}\left[\delta'(x-x_0) + \frac{f''(x_0)}{f'(x_0)}\delta(x-x_0)\right].$$

6. Show that
$$\delta\left(a - \frac{1}{x}\right) \ = \ \left\{\begin{array}{cc} a^{-2}\delta(x-1/a) & a > 0 \\ 0 & a \leq 0 \end{array}\right. .$$

7. Prove that, in the sense of distributions,
$$\lim_{\alpha\to\infty}\frac{\alpha}{\sqrt{\pi}}e^{-\alpha^2x^2} \ = \ \delta(x)\,.$$

8. Prove that, in the sense of distributions,
$$\lim_{\alpha\to\infty}\alpha e^{-\alpha|x|} \ = \ 2\delta(x)\,.$$

9. Show that
$$\lim_{\epsilon\to 0} \epsilon |x|^{\epsilon-1} e^{ix} \;=\; 2\delta(x)\,.$$

10. Show that
$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\left(|x|^{\epsilon}-1\right)\,\mathrm{sgn}\,x \;=\; (\mathrm{sgn}\,x)\,\log|x|\,.$$

11. Prove that, in the sense of distributions,
$$\frac{\partial}{\partial a}\delta(x-a) \;=\; -\delta'(x-a)\,.$$

12. Show that
$$x^{-1}\delta^{(n)}(x) \;=\; -\frac{1}{n+1}\delta^{(n+1)}(x) + C\delta(x)$$
where C is an arbitrary constant.

13. Show that the set of distributions $(-1)^n\delta^{(n)}(x)$ and the set of monomials $x^n/n!$, in both cases with $0 \le n < \infty$, satisfy the relation
$$\left\langle (-1)^m\delta^{(m)}, \frac{x^n}{n!} \right\rangle \;=\; \delta_{nm}\,.$$

14. Reduce $f(x)\delta''(x)$ to an expression involving the values of f and its derivatives at 0. Give a distributional verification of your result.

15. Calculate
$$I_1 = \int_{-\infty}^{\infty} \delta(ax^2-b)f(x)dx, \qquad I_2 = \int_{0}^{\infty} \delta(ax^2-b)f(x)dx.$$
Consider all possible sign combinations of the constants a and b.

16. Calculate in closed form
$$I = \int_0^{\infty} \delta(\sin\pi x)\exp(-qx)dx,$$
where q is a positive constant.

17. If a and b are non-negative real numbers with $a \neq b$, show that, in the sense of distributions,
$$\int_0^{\infty} J_0(ax)\,\cos bx\,dx \;=\; \frac{H(a-b)}{\sqrt{a^2-b^2}}. \tag{1}$$
Hint: Recall that
$$J_0(z) \;=\; \frac{1}{\pi}\int_0^{\pi} \cos\left(z\sin\,\theta\right)d\theta. \tag{2}$$
Substitute, exchange, think of the relation between $\delta(x)$ and its Fourier transform ...

18. By calculating
$$\left\langle \frac{d^n}{dx^n}\,\delta\left(ax-b\right), \phi(x) \right\rangle$$
or otherwise, derive an equivalent expression for $(d^n/dx^n)\,\delta\left(ax-b\right)$; a and b are real constants.

19. Show that
$$\int_{-\infty}^{\infty} \delta(x-a)\delta(b-x)\,\mathrm{d}x \;=\; \delta(b-a)\,, \qquad \int_{-\infty}^{\infty} \delta^{(m)}(x-a)\delta^{(n)}(b-x)\,\mathrm{d}x \;=\; \delta^{(m+n)}(b-a)\,.$$

20. Show that, if $x = \ell \cosh u \cos\phi$, $y = \ell \sinh u \sin\phi$, with $0 < u < \infty$, $0 < \phi < 2\pi$, then

$$\delta(x - x_0)\delta(y - y_0) \;=\; \frac{\delta(u - u_0)\delta(\phi - \phi_0)}{\ell^2(\cosh^2 u - \cos^2 v)}\,,$$

where u_0 and ϕ_0 are the values of u and ϕ corresponding to (x_0, y_0).

21. Define the generalized function $\delta(xy)$ by

$$\delta(xy) \;=\; |x|^{-1}\delta(y) + |y|^{-1}\delta(x)$$

and show that

$$xy\,\delta'(xy) + \delta(xy) \;=\; 0\,.$$

22. Show that

$$\delta(xy) \;\equiv\; |x|^{-1}\delta(y) + |y|^{-1}\delta(x) \;=\; \frac{\delta(x) + \delta(y)}{\sqrt{x^2 + y^2}}\,.$$

3 Convolution

1. Prove that (section 20.9)

$$[xH(x)] * [e^x H(x)] \;=\; (e^x - x - 1)H(x)\,.$$

2. Prove that (section 20.9)

$$[\sin x\, H(x)] * [\cos x\, H(x)] \;=\; \frac{1}{2}x\, \sin x\, H(x)\,.$$

3. Prove that (section 20.9)

$$\left[\frac{(-1)^n}{n!}\delta^{(n)}(x - \xi)\right] * \mathrm{Pf}\left(\frac{1}{x}\right) \;=\; \mathrm{Pf}\left(\frac{1}{(x - \xi)^n}\right)\,.$$

4. Calculate (section 20.9)

$$e^{-|x|} * e^{-|x|}\,.$$

5. Calculate (section 20.9)

$$e^{-ax^2} * \left[xe^{-ax^2}\right]\,.$$

4 Fourier and Laplace transforms

1. Show that $\mathcal{F}\{e^{ax}\} \;=\; (2\pi)^{-1/2}\delta(k - ia)$.

2. Show that $\mathcal{F}\{x^{-1}\} \;=\; (\pi/2)^{1/2} i\,\mathrm{sgn}\, k$.

3. Show that

$$\mathcal{F}\{x^{-m}\} \;=\; (\pi/2)^{1/2}\frac{i^m}{(m-1)!}k^{m-1}\mathrm{sgn}\, k\,.$$

4. Calculate the (distributional) Fourier transform of $\mathrm{Pf}(x^{-1})$. On the basis of this result and of the known properties of the Fourier transform, find the transform of $\mathrm{Pf}(x^{-2})$.

5. Calculate the Fourier transform of $x^n\delta^{(m)}(x)$.

6. Calculate the Fourier transform of $(1-x)^{-3/2}H(1-x)$.

7. Calculate the Fourier transform of $(x^2-4)^{-1}$.

8. Prove that

$$\mathcal{L}\{H(x)\log x\} \;=\; -\frac{\gamma+\log s}{s}$$

where the logarithm has its principal value and γ is Euler's constant. Deduce that

$$\mathcal{L}\left\{x^{-m}H(x)\right\} \;=\; \frac{(-s)^{m-1}}{(m-1)!}\,(C-\gamma-\log s)$$

where C is arbitrary.

9. Find the Laplace transform of $\sum_{n=1}^{N}\delta^{(n)}(x-n)$ with N finite.

10. Show that

$$\mathcal{L}\{\delta(t+\beta)\} \;=\; \frac{e^{\beta s/\alpha}}{|\alpha|}\,.$$

5 Fourier series

1. Show that $\sum_{-\infty}^{\infty} a_n e^{nx}$ is a generalized function if and only if, for $n\to\infty$, $a_n = O(|n|^N)$ for some integer N.

2. For $m>0$ a positive integer, evaluate

$$\sum_{n=-\infty}^{\infty} n^m e^{inx}\,.$$

3. Proceeding formally find the Fourier series of $\delta(x)$ over the interval $-L\le x\le L$. Can you check your result by reducing it to the one given in class for the interval $-\pi\le x\le\pi$? What is the distributional limit, over $-L\le x\le L$, of

$$S_N=\sum_{k=-N}^{N} k^2\exp(ik\pi x/L)?$$

4. On the interval $(-L,L)$ find the Fourier series of the distributions $\delta(x-a)$ and $H(x-a)$, where a is a constant in the same interval, and prove that the first one is the distributional derivative of the second one.

5. Show that

$$\delta(\sin x) \;=\; \frac{1}{\pi}\sum_{n=-\infty}^{\infty} e^{2nix}\,.$$

6. Show that

$$\sum_{n=1}^{\infty} e^{inx} \;=\; \pi\sum_{m=-\infty}^{\infty}\delta(x-2m\pi)+\frac{1}{2}\left(i\cot\frac{x}{2}-1\right)\,.$$

7. If $a_n = O(|n|^N)$, is $\sum_{n=-\infty}^{\infty}\delta(x-n)$ periodic?

6 Asymptotic evaluation of integrals

1. Examine the differences between the behavior of the Fourier transforms of e^{-x^2} and $e^{-x^2}\operatorname{sgn} x$ as $k \to \infty$.

2. Derive the first few terms of the asymptotic expansion of

$$\int_{-\infty}^{\infty} e^{-ikx} x^{-1} \log|x-1| \, \mathrm{d}x$$

with an error of order $|k|^{-2}$.

3. If $f(x)$ is continuously differentiable in $x \geq 0$ and, together with its derivatives, is well-behaved at infinity, show that

$$\int_{-\infty}^{\infty} f(|x|) \operatorname{sgn} x \, e^{-ikx} \, \mathrm{d}x \; = \; \sum_{n=0}^{N-1} \frac{2f^{(2n)}(0)}{(ik)^{2n+1}}$$

for any N, with an error decreasing faster than $|k|^{-2N}$).

7 Algebraic and differential equations

1. Show that the general solution of the equation

$$x^n \, u(x) \; = \; 1$$

is given by $u(x) \; = \; x^{-n} + \sum_{k=1}^{n} C_k \delta^{(k-1)}(x)$.

2. Show that the general solution of the equation

$$x \, u(x) \; = \; \delta^{(m)}(x)$$

is given by $u(x) \; = \; C\delta(x) - \delta^{(m+1)}/(m+1)$.

3. Find the distributional solution of the equation

$$x^m u(x) = \delta(x),$$

for $m = 1$ and calculate explicitly $< u, \phi >$.

4. In the range $-\frac{1}{2} \leq x \leq \frac{1}{2}$ find the distributional solution of the equation

$$(\sin \pi x) u(x) = 1.$$

Hint: The distributional solution must coincide with the ordinary solution at all points such that the coefficients multiplying the unknown do not vanish.

5. Solve the distributional differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} = H(x),$$

where H is the Heaviside distribution. After having found the general solution, find the particular one such that $< u, \phi >= 0$ for all test functions with support in $-\infty < x < 0$.

6. Find the distributional solution of
$$xu'(x) = 1$$
by a "fast and dirty" method. [Note that the regular solution must differ from the distributional solution by a distribution with support concentrated at the point where the coefficient vanishes].

7. Find the general distributional solution of the equation
$$\frac{d^2u}{dx^2} = \delta(x-\xi).$$
First, proceed formally as if you were dealing with functions.

8. Find the distributional solution of the equation
$$\frac{d^2u}{dt^2} + a^2u = \delta(t)$$
satisfying the conditions
$$u = 0 \quad \text{for} \quad t < 0, \qquad u \to 0 \quad \text{for} \quad t \to 0+\,.$$

9. Find the general distributional solution of the equation
$$x\,\frac{du}{dx} = \delta(x-a)$$
where $a \neq 0$ is a given constant.

10. Show that the general solution of the differential equation
$$x\frac{\mathrm{d}u}{\mathrm{d}x} - \alpha u = 0\,,$$
with α not a negative integer, is
$$u(x) = C_1 x^\alpha H(x) + C_2(-x)^\alpha H(-x)$$
with C_1 and C_2 arbitrary. If, on the other hand, $\alpha = -n$, a negative integer, then
$$u(x) = C_1 x^{-m} + C_2\delta^{(n-1)}(x)\,.$$

11. Find the distribution which satisfies the differential equation
$$\frac{\mathrm{d}u}{\mathrm{d}x} + 3u = e^{iax}$$
with a real.

12. In the range $0 \leq r < \infty$ find the solution of the equation
$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} - \frac{n(n+1)}{r^2}u = \alpha\delta(r-R),$$
where n is an integer and $0 < R < \infty$, which is regular at $r = 0$ and at infinity and vanishes for $\alpha = 0$.

13. In the range $-1 \leq x \leq 1$ find the general distributional solution of the equation
$$(1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} = \alpha\delta(x-\xi).$$
which is regular at $x = \pm 1$ and vanishes for $\alpha = 0$.

14. Show that the infinite series

$$u = \sum_{n=0}^{\infty} \frac{2^{n+1}\delta^{(n)}(x)}{n!(n+1)!},$$

formally satisfies the first-order ordinary differential equation

$$x^2\frac{du}{dx} - 2u = 0.$$

15. Find the solution of the equation

$$x^2\, u'' - 2u = -\delta(x-\xi)$$

u bounded at $x = 0$ and $x \to \infty$.

16. A point-like heat source of strength Q constant in time is placed at the center of a rod of length $2L$. The initial temperature of the rod is $u = 0$, its two ends are kept at zero temperature for all times, and the sides of the rod are thermally insulated. The system is therefore governed by the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + Q\delta(x), \quad -L \le x \le L,$$

with

$$u(x,0) = 0, \qquad u(\pm L, t) = 0.$$

Find $u(x,t)$ using the usual method of eigenfunction expansion.

17. Solve, in unbounded three-dimensional space $0 \le r < \infty$, the problem

$$\nabla^2 u = \frac{A}{4\pi R^2}\,\delta(r-R)$$

where R and A are constants and u is regular at $r = 0$ and $u \to 0$ for $r \to \infty$.

18. Solve the following differential equation

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) \;=\; -\delta(x-a)$$

in the range $0 \le x \le X$ subject to $u(0)$ bounded, $u = U$ at $x = X$; a is a positive constant and $a < X$.

19. Solve the following equation

$$\frac{\partial G}{\partial t} \;=\; \frac{\partial^2 G}{\partial x^2} + \delta(x-\xi)\,\delta(t-\tau)$$

in the interval $0 \le x \le L$, subject to the boundary conditions $G = 0$ for $x = 0$, L, $G = 0$ for $t = 0$.

20. In the interval $0 < x < \infty$ solve

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} - \left(1 + \frac{m^2}{x^2}\right)u \;=\; \delta(x-a) + \delta(x-b) \qquad\qquad 0 < a < b$$

with $m > 0$, subject to $u(0) = 0$, $u \to 0$ at infinity. ‘