Chapter 3: Limits and Continuity

Part B: Continuity

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Continuity at a point

A function f is said to be **continuous at** p if

$$\lim_{x \to p} f(x) = f(p).$$

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The concept of continuity is only to be applied to points which are in the domain of f. In fact they need to be in an open interval which is completely contained in the domain of f, so that the limit can be talked about.

Failure of continuity

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The following functions are not continuous at 0 because their limit does not exist at 0:

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases} \quad \text{and} \quad \operatorname{sgn}(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

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Next, we have functions which are not continuous at 0 because their limit at 0 does not equal their value at 0:

$$E(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases} \quad \text{and} \quad F(x) = \begin{cases} x & \text{if } x = 1/n, \ n \in \mathbb{N}, \\ 1 & \text{if } x = 0, \\ 0 & \text{else.} \end{cases}$$

(In both cases the limit is 0 but the function value is 1.)

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These functions are continuous at every point of $\mathbb{R}$:

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On the other extreme, the Dirichlet function is not continuous at any point!

Continuity of combinations of functions

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Theorem 1

Let $f(x)$ and $g(x)$ be continuous at p. Then the following functions are also continuous at p.

1. $C\,f(x)$.
2. $f(x) \pm g(x)$.
3. $f(x)g(x)$.
4. $\dfrac{f(x)}{g(x)}$, *if* $g(p) \neq 0$.

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So, by the Algebra of Limits, $\displaystyle\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{\lim_{x\to p} f(x)}{\lim_{x\to p} g(x)} = \frac{f(p)}{g(p)}$. □

Polynomials and rational functions

Theorem 2

1. *Any polynomial is continuous at every point of* $\mathbb{R}$.
2. *Any rational function is continuous at every point of its domain.*

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The functions $y = a_0$ and $y = x$ are continuous. By repeated application of (3) of the previous theorem, every function $y = x^i$ ($i \in \mathbb{N}$) is continuous.

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For continuity of rational funcions, combine continuity of polynomials with part 4 of the previous theorem. □

One-sided Continuity

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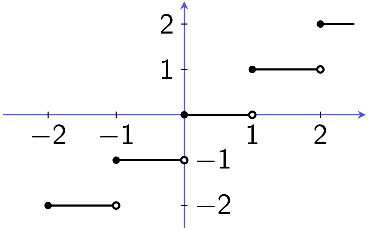
It is **right-continuous** at p if $\lim_{x\to p+} f(x) = f(p)$.

Example: The greatest integer function is right-continuous at every point. It is left-continuous at all points except the integers.



One-sided and two-sided continuity

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Theorem 3

A function f is continuous at p if and only if it is left and right-continuous at p.

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Proof.

$$\lim_{x\to p} f(x) = f(p) \iff \lim_{x\to p+} f(x) = f(p) \text{ and } \lim_{x\to p-} f(x) = f(p).$$

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□

For example, we can argue that the Heaviside step function $H(x)$ is not continuous at $x = 0$ because it is right continuous but not left continuous.

Types of discontinuity

Removable discontinuity: $\lim_{x \to a} f(x)$ exists but does not equal $f(a)$.
We can make f continuous at a by changing its value at a to $\lim_{x \to a} f(x)$.

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Jump discontinuity: $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exist but are not equal. The quantity $\lim_{x \to a+} f(x) - \lim_{x \to a-} f(x)$ is called the **jump** of f at a.

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Essential discontinuity: Either $\lim_{x\to a+} f(x)$ or $\lim_{x\to a-} f(x)$ fails to exist.

Continuity on an interval

A function f is called **continuous on an interval** I if

1. f is continuous at every interior point of I,
2. f is right-continuous at the left endpoint, if the left endpoint is in I,
3. f is left-continuous at the right endpoint, if the right endpoint is in I.

Composition and Limits

Theorem 4

Let f and g be real functions such that their composition $g \circ f$ is defined on an interval (a, b). Let $p \in (a, b)$ with $q = \lim_{x \to p} f(x)$ and suppose g is continuous at q. Then

$$\lim_{x \to p} g(f(x)) = g(q) = g(\lim_{x \to p} f(x)).$$

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Proof. Let $\epsilon > 0$.

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Proof. Let $\epsilon > 0$.
Since g is continuous at q there is a $\delta' > 0$ such that $|y - q| < \delta'$ implies $|g(y) - g(q)| < \epsilon$.

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Since g is continuous at q there is a $\delta' > 0$ such that $|y - q| < \delta'$ implies $|g(y) - g(q)| < \epsilon$.
There is a $\delta > 0$ such that $0 < |x - p| < \delta$ implies $|f(x) - q| < \delta'$.

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Proof. Let $\epsilon > 0$.
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There is a $\delta > 0$ such that $0 < |x - p| < \delta$ implies $|f(x) - q| < \delta'$.
Hence,

$$0 < |x - p| < \delta \implies |f(x) - q| < \delta' \implies |g(f(x)) - g(q)| < \epsilon.$$

Example

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Calculate $\lim_{x \to 1} \sqrt{\dfrac{x^2 - 1}{x - 1}}$.

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We first note that $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = 2$.

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Since the square root function is continuous at 2 (we proved that for $a > 0$, $\lim_{x\to a} \sqrt{x} = \sqrt{a}$), we have

$$\lim_{x\to 1} \sqrt{\frac{x^2-1}{x-1}} = \sqrt{\lim_{x\to 1} \frac{x^2-1}{x-1}} = \sqrt{2}.$$

Composition and continuity

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Theorem 5

Let f and g be real functions such that their composition $g \circ f$ is defined on an interval (a, b). Let $p \in (a, b)$ such that f is continuous at p and g is continuous at $f(p)$. Then $g \circ f$ is continuous at p.

Composition and continuity

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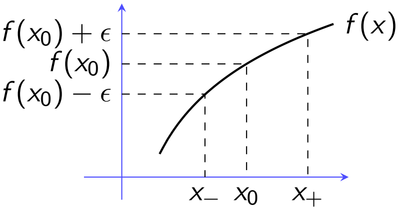
Proof. $\lim_{x \to p} g(f(x)) = g(\lim_{x \to p} f(x)) = g(f(p))$. □

Continuity of monotone functions

Theorem 6

If I, J are intervals and $f : I \to J$ is a surjective monotone function, then f is continuous on I.

Proof. We'll do the case when J is an open interval.
Let $x_0 \in I$ and let $\epsilon > 0$. We may assume that $f(x_0) \pm \epsilon \in J$.
Since f is surjective, there are $x_\pm \in I$ such that $f(x_-) = f(x_0) - \epsilon$ and $f(x_+) = f(x_0) + \epsilon$.



Take $\delta = \min\{x_0 - x_-, x_+ - x_0\}$. □

Continuity of logarithms and exponentials

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Theorem 7

All logarithms and exponential functions are continuous.

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Task: Let $r \geq 0$. Show that the function x^r is continuous on $[0, \infty)$.

Indefinite integrals

Suppose I is an interval and $f: I \to \mathbb{R}$ is integrable on every $[\alpha, \beta] \subseteq I$. Fix a point $a \in I$. Then the function

$$F(x) = \int_a^x f(t)\,dt \qquad (x \in I)$$

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Hence $F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$

Continuity of indefinite integrals

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Theorem 8

Suppose I is an interval, $a \in I$, and $f : I \to \mathbb{R}$ is integrable on every $[\alpha, \beta] \subseteq I$. Then $F(x) = \int_a^x f(t)\,dt$ is continuous on I.

Continuity of indefinite integrals

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Theorem 8

Suppose I is an interval, $a \in I$, and $f : I \to \mathbb{R}$ is integrable on every $[\alpha, \beta] \subseteq I$. Then $F(x) = \int_a^x f(t)\,dt$ is continuous on I.

Proof. For any $x, p \in I$ we have

$$F(x) - F(p) = \int_a^x f(t)\,dt - \int_a^p f(t)\,dt = \int_p^x f(t)\,dt.$$

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Suppose p is not the right endpoint of I.
Then, there is a $\delta > 0$ such that $[p, p + \delta] \subseteq I$.

Continuity of indefinite integrals

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Suppose p is not the right endpoint of I.
Then, there is a $\delta > 0$ such that $[p, p + \delta] \subseteq I$.
Since f is integrable on $[p, p + \delta]$ it is bounded there.

Continuity of indefinite integrals

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Suppose p is not the right endpoint of I.
Then, there is a $\delta > 0$ such that $[p, p + \delta] \subseteq I$.
Since f is integrable on $[p, p + \delta]$ it is bounded there.
Hence there is a positive number M such that $-M \leq f(x) \leq M$ for every $x \in [p, p + \delta]$.

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Then, there is a $\delta > 0$ such that $[p, p+\delta] \subseteq I$.
Since f is integrable on $[p, p+\delta]$ it is bounded there.
Hence there is a positive number M such that $-M \leq f(x) \leq M$ for every $x \in [p, p+\delta]$. Now for $p < x < p+\delta$,

$$-M(x-p) = \int_p^x (-M)\,dt \leq \int_p^x f(t)\,dt \leq \int_p^x M\,dt = M(x-p).$$

Continuity of indefinite integrals

(*proof continued*)

By the Sandwich Theorem, we have $\lim_{x \to p+} |F(x) - F(p)| = 0$.

Therefore $\lim_{x \to p+} F(x) = F(p)$.

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Similarly, we check that if p is not the left endpoint of I then f is left-continuous at p.

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This establishes the continuity of f on I. □

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2 Intermediate Value Theorem

3 Trigonometric Functions

Intermediate Value Theorem, ver. 1

Theorem 9

Suppose f is continuous on $[a, b]$ and $f(a)f(b) < 0$. Then there is a number $c \in (a, b)$ such that $f(c) = 0$.

Intermediate Value Theorem, ver. 1

Theorem 9

Suppose f is continuous on $[a, b]$ and $f(a)f(b) < 0$. Then there is a number $c \in (a, b)$ such that $f(c) = 0$.

Proof. Assume that $f(x)$ is never zero.

Intermediate Value Theorem, ver. 1

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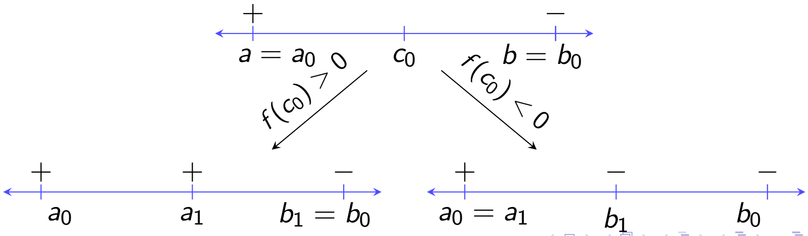

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From the Completeness Axiom we obtain a number c such that $a_n \leq c \leq b_n$ for every n.

Intermediate Value Theorem, ver. 1

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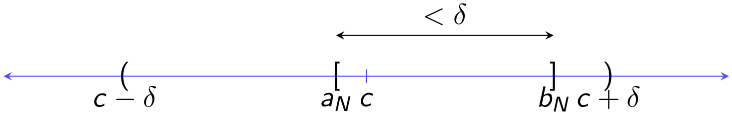

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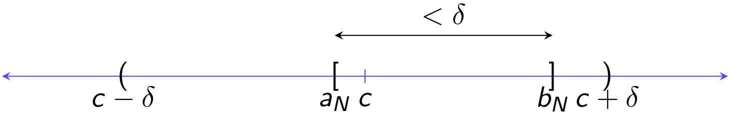

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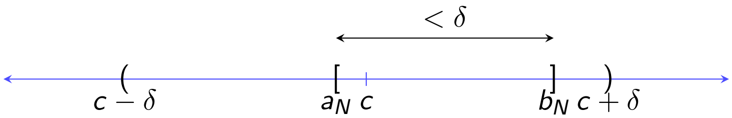

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The $f(c) < 0$ case similarly leads to a contradiction. □

Solving equations

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The value of $f(x)$ at its midpoint is $f(1.5) = 10.8 > 0$.
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$f(1.25) = 3.3 \implies$ solution is in $(1, 1.25)$.
$f(1.125) = 0.81 \implies$ solution is in $(1, 1.125)$.

Intermediate Value Theorem, ver. 2

Theorem 10

Suppose f is continuous on $[a, b]$ and L is a value between $f(a)$ and $f(b)$, i.e., $f(a) < L < f(b)$ or $f(b) < L < f(a)$. Then there is a number $c \in (a, b)$ such that $f(c) = L$.

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The case $f(b) < L < f(a)$ is handled in a similar manner. □

Table of Contents

1. Continuity
2. Intermediate Value Theorem
3. Trigonometric Functions

Angles

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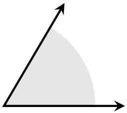
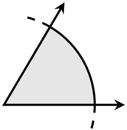
We define an angle to be a region bounded by two rays with a common starting point, as shown below.



To measure the angle we draw a unit circle whose centre is the meeting point of the rays. We take twice the area enclosed by this circle within the angle, and call that the radian measure of the angle. Thus the full circle corresponds to 2π radians while a right angle corresponds to $\pi/2$ radians.



Radians

We have allotted a real number in $[0, 2\pi]$ to each angle. We show that every such number really is the radian measure of some angle.

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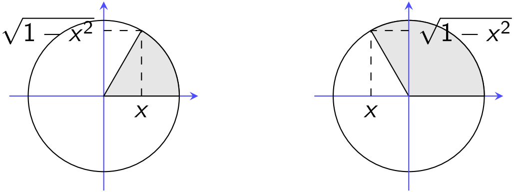

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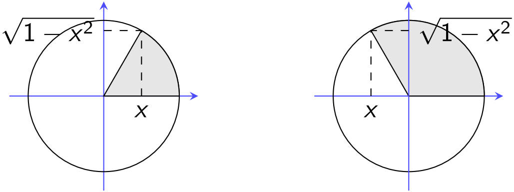

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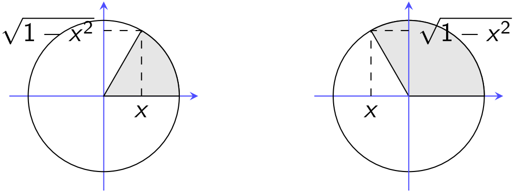

Let $R(x)$ be the radian measure of this angle.
The function $R\colon [-1, 1] \to [0, \pi]$ is defined by

$$R(x) = x\sqrt{1-x^2} + 2\int_x^1 \sqrt{1-t^2}\,dt.$$

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Task: Show that every number between π and 2π is also the radian measure of an angle.

Sine and Cosine

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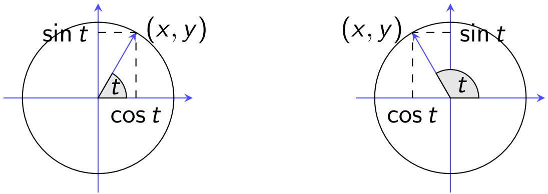
The figures below illustrate the definitions for an acute and an obtuse angle respectively.



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- Similarly, $\cos\colon [0, 2\pi] \to [-1, 1]$ is also onto.

Properties of sine and cosine

- Since $(\cos t, \sin t)$ is a point on the unit circle, we have $\cos^2 t + \sin^2 t = 1$.
- $\cos t, \sin t \in [-1, 1]$.
- Let $y \in [-1, 1]$.
  Then $x = \sqrt{1 - y^2}$ is defined and (x, y) is on the unit circle.
  Let t be the angle between the positive x-axis and the ray emanating from origin and passing through (x, y).
  By the definition of the sine function, $\sin t = y$.
  Therefore, $\sin\colon [0, 2\pi] \to [-1, 1]$ is onto.
- Similarly, $\cos\colon [0, 2\pi] \to [-1, 1]$ is also onto.
- By symmetry, $\sin(\pi/2 - t) = \cos t$ for every $t \in [0, \pi/2]$.

Graphs

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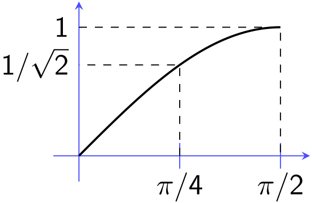
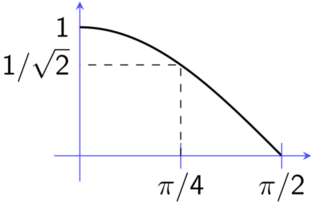
The following values of sine and cosine are obvious:

x	0	$\pi/2$	π	$3\pi/2$	2π
$\sin x$	0	1	0	-1	0
$\cos x$	1	0	-1	0	1

Task: Show that $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$.



sine



cosine

Graphs

Task: $\cos(\pi - t) = \cos(\pi + t) = -\cos t$, for every $t \in [0, \pi]$.
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Graphs

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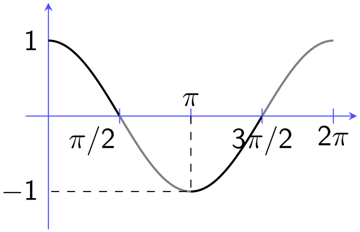
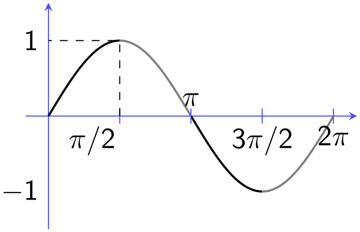
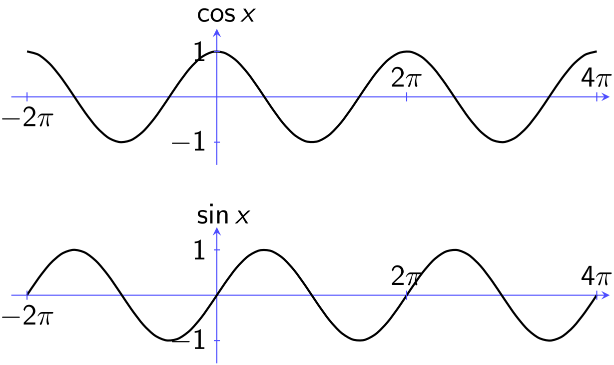
With the help of these identities, we can visualize the graphs over $[0, 2\pi]$, using the pieces for $[0, \pi/2]$ as building blocks:





Graphs

The domains can be extended on each side of $[0, 2\pi]$ by setting $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$:



An angle sum identity

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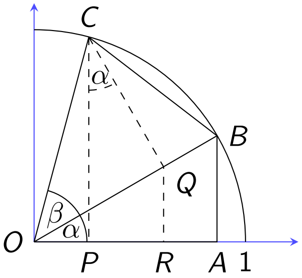

In this figure, CP and QR are perpendicular to OA, while CQ is perpendicular to OB.

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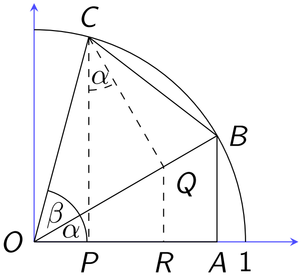

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We have the following calculations:

$$OQ = \cos\beta \implies OR = \cos\alpha\cos\beta,$$
$$CQ = \sin\beta \implies PR = \sin\alpha\sin\beta.$$

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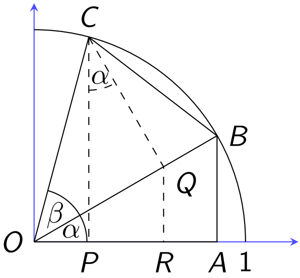

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Hence, $\cos(\alpha+\beta) = OP = OR - PR = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

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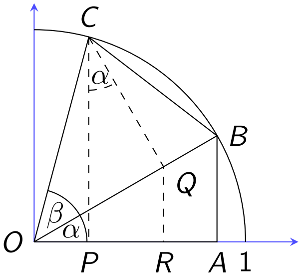

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Our figure is only valid for $0 \le \alpha, \beta$ and with $\alpha + \beta \le \pi/2$. The identity can be extended to arbitrary α, β by other appropriate figures.

Other angle sum identities

The other sum of angle identities can be obtained from this one.

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Replacing α by $\pi/2 - \alpha$ and β by $-\beta$ gives

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Task: Show that $\sin\pi/6 = \cos\pi/3 = 1/2$,

$$\cos\pi/6 = \sin\pi/3 = \sqrt{3}/2.$$

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Task: Prove the **half-angle formulas**:

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1,$$

$$\sin 2x = 2\cos x\sin x.$$

Law of Sines

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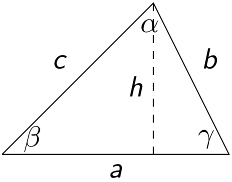
Theorem 11

Consider a triangle whose sides have lengths a, b, c, and the corresponding opposite angles are α, β, γ. Then

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

Proof. Take a as the base of the triangle and let h be the height.



We have $h = c\sin\beta = b\sin\gamma$, hence $\dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$.

Law of Sines

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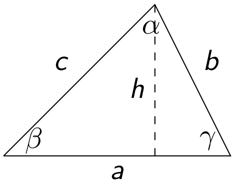


We have $h = c\sin\beta = b\sin\gamma$, hence $\dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$.

Similarly, taking b as the base, we get $\dfrac{\sin\gamma}{c} = \dfrac{\sin\alpha}{a}$.

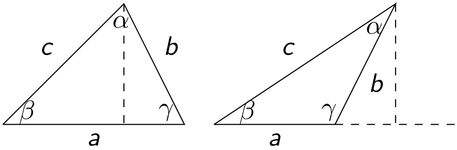
Law of Cosines

Theorem 12

Consider a triangle whose sides have lengths a, b, c, and the corresponding opposite angles are α, β, γ. Then

$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Proof. Take the side a as the base of the triangle. We show two cases below, depending on whether any base angle is obtuse.

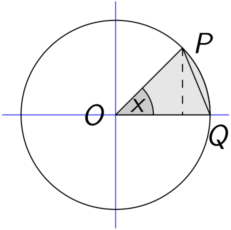


We have $a = c\cos\beta + b\cos\gamma$ in both cases. Hence, $a^2 = ac\cos\beta + ab\cos\gamma$. Similarly, $b^2 = ab\cos\gamma + bc\cos\alpha$, So,

$$c^2 = ac\cos\beta + bc\cos\alpha.$$

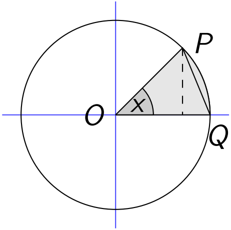
$$a^2 + b^2 = 2ab\cos\gamma + ac\cos\beta + bc\cos\alpha = c^2 + 2ab\cos\gamma.$$ □

Limits at zero



The circle has radius 1.

Limits at zero



The circle has radius 1.
For $0 < x < \pi/2$ we have

$$0 < \text{Area}(\triangle OPQ) < \text{Area}(\triangledown OPQ)$$
$$\implies 0 < \frac{1}{2}\sin x < \frac{x}{2}$$
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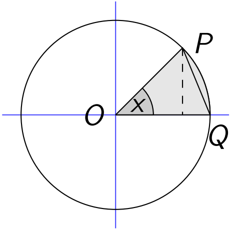

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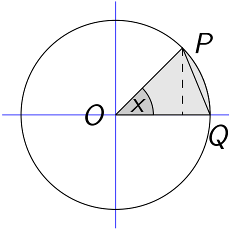

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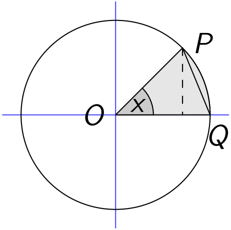

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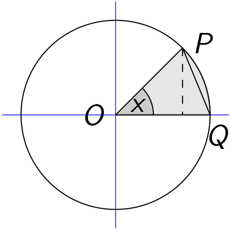

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Apply the half-angle formula: $\lim_{x\to 0} \cos x = \lim_{x\to 0} \left(1 - 2\sin^2 \frac{x}{2}\right) = 1$.

Continuity

$$\lim_{x\to a} \sin x = \lim_{h\to 0} \sin(a+h) = \lim_{h\to 0} (\sin a \cos h + \cos a \sin h) = \sin a,$$

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Task: Calculate $\lim_{x\to 1} \sin\left(\frac{x^2 - 2x + 1}{x^2 - 1}\right)$.

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$$\tan x = \frac{\sin x}{\cos x},\quad \cot x = \frac{\cos x}{\sin x},\quad \sec x = \frac{1}{\cos x},\quad \csc x = \frac{1}{\sin x}.$$

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By the properties of continuity, these functions are continuous at every point of their domains.