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Problems for Chapter 1 of *Advanced Mathematics for Applications*

The Classical Field Equations

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1 Vector fields

1. Verify that the vector field

$$\mathbf{V} = (3yz + x^2)\mathbf{i} + (2y^2 + 3xz)\mathbf{j} + (z^2 + 3xy)\mathbf{k}$$

is irrotational and find its (scalar) potential.

2. Verify that the vector field given in cylindrical coordinates by

$$\mathbf{V} = 2rz\sin\phi\,\mathbf{e}_r + r^2 z\cos\phi\,\mathbf{e}_\phi + r^2\sin\phi\,\mathbf{k}$$

is irrotational and find its (scalar) potential (see Figure 6.1 p. 146 for a definition of symbols).

3. Show that, if $\mathbf{V}$ is a constant vector, then the divergence of the vector field $\mathbf{A} = |\mathbf{x}|\,(\mathbf{V}\times\mathbf{x})$ vanishes. Find a vector potential for $\mathbf{A}$.

4. In cylindrical coordinates the infinitesimal displacement $d\mathbf{x}$ of a point $\mathbf{x} = (r, \phi, z)$ when the coordinates are incremented by $(\,\mathrm{d}r\,,\, \mathrm{d}\phi\,,\, \mathrm{d}z\,)$ can be expressed as

$$\mathrm{d}\mathbf{x} = h_r\,dr\,\mathbf{e}_r + h_\phi\;\mathrm{d}\phi\;\mathbf{e}_\phi + h_z\;\mathrm{d}z\;\mathbf{k}$$

where $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{k}$ are unit vectors pointing in the direction in which the coordinates (r, ϕ, z) respectively, increase (see figure 6.1 p.146) and h_r, h_ϕ and h_z are called the metric coefficients (see p. 487). Show explicitly that the metric coefficients in this coordinate system are given by $h_r = h_z = 1$, $h_\phi = r$.

5. By proceeding as in the previous problem in the case of a spherical coordinate system (figure 7.1 p. 170) write

$$\mathrm{d}\mathbf{x} = h_r\,dr\,\mathbf{e}_r + h_\theta\;\mathrm{d}\theta\;\mathbf{e}_\theta + h_\phi\;\mathrm{d}\phi\;\mathbf{e}_\phi$$

and show that $h_r = 1$, $h_\theta = r$, $h_\phi = r\,\sin\theta$.

6. Using the reasoning described at the top of p. 4, verify the expressions of $\boldsymbol{\nabla} u$ in cylindrical and spherical coordinates given in Tables 6.4 p. 148 and 7.3 p. 173.

7. Use the relation in (1.1.3) p. 4

$$\boldsymbol{\nabla}\cdot\mathbf{A} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\oint_S \mathbf{A}\cdot\mathbf{n}\,\mathrm{d}S$$

to verify the expression of $\boldsymbol{\nabla}\cdot\mathbf{A}$ in cylindrical coordinates given in Table 6.4 p. 148. For this purpose build a small volume ΔV with sides parallel to the directions of $\mathbf{e}_r$, $\mathbf{e}_\phi$ and $\mathbf{k}$ at a generic point (r, ϕ, z).

8. Repeat the previous problem for a spherical coordinate system and verify the expression of $\boldsymbol{\nabla}\cdot\mathbf{A}$ given in Table 7.3 p. 173.

9. By using the relation (1.1.5) p. 4

$$\mathbf{n}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) \;=\; \lim_{\Delta S\to 0}\frac{1}{\Delta S}\oint_L \mathbf{A}\cdot\mathbf{t}\,\mathrm{d}\ell$$

verify the expression of $\boldsymbol{\nabla}\times\mathbf{A}$ in cylindrical coordinates given in Table 6.4 p. 148. For this purpose build small planar areas normal to each coordinate direction in turn and having sides parallel to the other two directions.

10. Repeat the previous problem for a spherical coordinate system and verify the expression of $\boldsymbol{\nabla}\times\mathbf{A}$ given in Table 7.3 p. 173.

11. Let $f(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ be a scalar and a vector field respectively. Calculate $\boldsymbol{\nabla}\times(f\mathbf{A})$ and $\boldsymbol{\nabla}\cdot(f\mathbf{A})$.

12. With the aid of the formulae in Table 6.4 p. 148 give an explicit expression for the double curl $\mathbf{B}=\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A}$ and the gradient of the divergence $\mathbf{C}=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})$ of a vector field $\mathbf{A}=(A_r,A_\phi,A_z)$ in cylindrical coordinates. By using the identity (1.1.10), find an expression for $\nabla^2\mathbf{A}$ in this coordinate system.

13. With the aid of the formulae in Table 7.3 p. 173 give an explicit expression for the double curl $\mathbf{B}=\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A}$ and the gradient of the divergence $\mathbf{C}=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})$ of a vector field $\mathbf{A}=(A_r,A_\theta,A_\phi)$ in spherical polar coordinates. By using the identity (1.1.10), find an expression for $\nabla^2\mathbf{A}$ in this coordinate system.

14. Given a function $u(\mathbf{x})$ define its spherical mean over a sphere centered at $\mathbf{x}$ and having radius r by

$$M_u(\mathbf{x},r) \;=\; \frac{1}{4\pi}\int_{|e|=1} u(\mathbf{x}+r\mathbf{e})\,\mathrm{d}S$$

Note that this expression provides a definition of M_u also for $r<0$ and shows that $M_u(\mathbf{x},r)=M_u(\mathbf{x},-r)$ so that M_u is an even function of r. With the aid of the divergence theorem show that M_u satisfies the Darboux equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)M_u \;=\; \nabla_x^2 M_u$$

where ∇_x^2 is the Laplacian with respect to the coordinate $\mathbf{x}$.

2 Elliptic equations

1. Let u be harmonic, i.e., let it satisfy the Laplace equation $\nabla^2 u=0$. Is u^2 also harmonic? What is the most general class of functions f such that $f(u)$ is harmonic?

2. Show that the expression

$$u(x,y,z) \;=\; \int_{-\pi}^{\pi} f\left(z+ix\cos u+iy\sin u,\; u\right)\;\mathrm{d}u$$

is a solution of the Laplace equation $\nabla^2 u=0$. After converting this expression from Cartesian to spherical polar coordinates, find the function f which gives rise to the solid harmonic $r^\ell Y_\ell^m(\theta,\phi)$ (cf. section 13.3.1).

3. Prove that, if $u(\mathbf{x})$ satisfies Laplace's equation $\nabla^2 u = 0$ in any number of dimensions, then so does the function

$$v(\mathbf{x}) = |\mathbf{x}|^{2-n} u\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)$$

where $|\mathbf{x}| = \sqrt{\mathbf{x}\cdot\mathbf{x}}$. This is a generalization of the Kelvin inversion of section 7.9.

4. Show that the function given by

$$u(x,y) = \mathrm{Re} \int_0^z J_0\left(k\sqrt{(z-\zeta)\overline{\zeta}}\right) f(\zeta)\,\mathrm{d}\zeta$$

satisfies the two-dimensional Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0\,,$$

in which k is a constant. Here $z = x + iy$, $\overline{z} = x - iy$, $\zeta = \xi + i\eta$ and J_0 is the Bessel function of order 0 defined by (see p. 305)

$$J_0(t) = \sum_{n=0}^{\infty} \frac{(-t^2/4)^n}{(n!)^2}\,.$$

5. Show that the general solution of the biharmonic equation (p. 7)

$$\nabla^4 u = 0$$

in a simply-connected two-dimensional domain can be represented as

$$u(x,y) = \mathrm{Re}\,[\overline{z}\,v(z) + w(z)]$$

where $\overline{z} = x - iy$ and v and w are arbitrary analytic functions of $z = x + iy$.

3 Hyperbolic equations

1. By effecting the change of variables

$$\xi = x\,, \qquad \eta = iy$$

transform the two dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

into the formally hyperbolic equation

$$\frac{\partial^2 u}{\partial \xi^2} - \frac{\partial^2 u}{\partial \eta^2} = 0$$

In this way deduce the formula

$$u(x,y) = \frac{1}{2}\Phi(x+iy) + \frac{1}{2}\overline{\Phi(x+iy)}$$

expressing any solution of the two-dimensional Laplace equation as the real part of some analytic function of the variable $x + iy$.

2. Starting from the general d'Alembert solution (1.2.9) p. 8 find, for $x > 0$ and $t > 0$, the solution to the homogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

subject to the initial conditions $u(x, 0) = f(x)$, $\partial u/\partial t|_{t=0} = g(x)$ and to the boundary condition $\partial u/\partial t = au$ at $x = 0$. Is there a special value of a such that the problem, in general, admits no solution? Give a physical interpretation of your answer.

3. Show that, in three-dimensional space, the function

$$u(r, t) = \frac{1}{r} \left[\Phi_+(r - ct) + \Phi_-(r + ct)\right]$$

with r the distance from the origin, is a spherically symmetric solution of the wave equation. Using this fact, find the solution of the wave equation corresponding to the initial data $u(r, 0) = 0$, $\partial u/\partial t|_{t=0} = g(r)$ with g an even function of its argument. For the special case

$$g(r) = \begin{cases} 1 & \text{for } 0 < r < a \\ 0 & \text{for } a < r \end{cases}$$

find u explicitly in the different regions bounded by the cones $r = a \pm ct$. Locate the discontinuities of the solution arising from the discontinuity of the initial data.

4. Prove that, if $v(\mathbf{x}, t; \tau)$ is a solution of

$$\frac{\partial v}{\partial t} - \sum_{n=1}^{d} A_j \frac{\partial v}{\partial x_j} - Bv = 0\,,$$

where d is the number of the space dimensions and $\mathbf{x} = (x_1,\, x_2, \ldots,\, x_d)$, satisfying the inhomogeneous initial data $v(\mathbf{x}, \tau; \tau) = f(\mathbf{x}, \tau)$ then the Duhamel integral

$$u(\mathbf{x}, t) = \int_0^t v(\mathbf{x}, t; \tau)\, \mathrm{d}\tau$$

satisfies the inhomogeneous equation

$$\frac{\partial u}{\partial t} - \sum_{n=1}^{d} A_j \frac{\partial u}{\partial x_j} - Bu = f(\mathbf{x}, t)$$

together with the homogeneous initial condition $u(\mathbf{x}, 0) = 0$ (cf section 5.5.4 p. 141).

5. Show that the retarded potential

$$u(x, y, z, t) = \frac{1}{4\pi} \int\int\int \frac{F(\xi, \eta, \zeta, t - \rho)}{\rho}\, \mathrm{d}\xi\, \mathrm{d}\eta\, \mathrm{d}\zeta$$

in which $\rho = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$, satisfies the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = F(x, y, z, t)$$

together with the homogeneous initial conditions $u(x, y, z, 0) = 0,\ \partial u(x, y, z, t)/\partial t|_{t=0} = 0$.

6. Show that one solution of the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

can be written in the form $u(x, y, t) = \operatorname{Re} f(\theta)$, where f is an analytic function of the argument $\theta = \theta(x, y, t)$ given by the relation

$$t - \frac{x}{c}\,\theta + \frac{y}{c}\,\sqrt{1-\theta^2} = 0\,.$$

7. By effecting a suitable change of variables, find the general solution of the equation

$$a^2\frac{\partial^2 u}{\partial x^2} + 2a\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

where a is a given constant.

8. Show that the function

$$u(x, y, t) = \sum_{n=0}^{\infty} \frac{t^n}{\lambda^n n!}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^n T(x, y)$$

where $T(x, y)$ is an arbitrary polynomial in the variables x and y and λ is a constant, satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \lambda\frac{\partial u}{\partial t}\,.$$

9. Show that, for any $R > 0$ there is a $T > 0$ such that, for all points (x, y) with $x^2 + y^2 < R^2$ and all times $t \geq T$ the solution of the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

subject to $u(x, y, 0) = u_0(x, y),\ \partial u/\partial t|_{t=0} = v(x, y)$ is expressible by means of the converging series

$$u(x, y, t) = \sum_{n=0}^{\infty} \frac{U_m(x, y)}{t^{n+1}}\,.$$

Calculate U_0 and U_1.

4 Diffusion equation

1. Prove that the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

possesses solutions of the form

$$u = t^{\alpha}\, v(\eta), \qquad \eta = \frac{x^2}{4t}.$$

Give explicitly the (ordinary) differential equation satisfied by v and solve it.

2. Consider the equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}$$

Let

$$u = t^{-1/2}\, v(\eta), \qquad \eta = \frac{x^2}{4t}$$

and derive the ordinary differential equation satisfied by v. Show that its solution is unbounded for $\eta \to 0$, i.e., $t \to \infty$.

3. Find the general solution of the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

having the form $u(x,t) = f(x - vt)$, with v a constant. Solutions of this type may represent, for example, the heating of a substance ahead of a detonation wave propagating with velocity v when the chemical reaction maintains the temperature of the wave front constant.

4. Find the general solution of the non-linear one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial}{\partial x}\left(u^n\,\frac{\partial u}{\partial x}\right)$$

having the form $u(x,t) = f(x - vt)$, with v a constant.

5. Consider spherically symmetric solutions of the diffusion equation

$$\frac{\partial u}{\partial t} = D\nabla^2 u$$

in the entire d-dimensional space, i.e., solutions of the form $u(r,t)$ with $r = (x_1^2 + x_2^2 + \ldots + x_d^2)^{1/2}$. Find a transformation of the form $r' = \alpha r$, $t' = \beta t$ which leaves the equation unchanged and deduce the structure of the solution $u(r,t)$.