Chapter 8: Taylor and Fourier Series

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Power Series

A **power series** is an expression of the form

$$c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \quad \text{or} \quad \sum_{n=0}^{\infty} c_n(x-a)^n$$

in which a and the c_n are constants and x is a variable.

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Main Question: For which values of x will a power series converge?

Example 1

The geometric series $\sum_{n=0}^{\infty} x^n$ is a power series centred at 0 and with all coefficients equal to 1. It converges absolutely for $|x| < 1$ and diverges for $|x| \geq 1$.

Examples

Example 2

$\sum_{n=0}^{\infty} x^n/n!$ is centred at 0 and has coefficients $1/n!$. We saw earlier, using the Ratio Test, that it converges absolutely for every x.

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$\sum_{n=0}^{\infty} n!\, x^n$ converges only at $x = 0$.

Example 4

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(x-1)^n$. is centred at 1 and has coefficients $c_n = (-1)^n/n$, with $c_0 = 0$.

Apply the Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-1)^{n+1} n}{(n+1)(x-1)^n}\right| \to |x-1|$.

So it converges absolutely if $|x-1| < 1$ and diverges if $|x-1| > 1$.

At $x = 0$ it becomes the harmonic series and diverges, while at $x = 2$ it becomes the alternating harmonic series and converges.

So it converges for $0 < x \leq 2$, and diverges elsewhere.

Convergence on an interval

In proving facts about power series, it is enough to consider the $a = 0$ case, since the substitution $y = x - a$ converts a power series with centre a to one with centre 0.

Theorem 5

Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges at $x = r \neq 0$. Then it converges absolutely at every x which satisfies $|x| < |r|$.

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In proving facts about power series, it is enough to consider the $a = 0$ case, since the substitution $y = x - a$ converts a power series with centre a to one with centre 0.

Theorem 5

Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges at $x = r \neq 0$. Then it converges absolutely at every x which satisfies $|x| < |r|$.

Proof. The convergence of $\sum_{n=0}^{\infty} c_n r^n$ implies $c_n r^n \to 0$, hence $|c_n r^n|$ is bounded by some real M. Now consider any x such that $|x| < |r|$. Then

$$|c_n x^n| = |c_n r^n| \left|\frac{x}{r}\right|^n \leq M \left|\frac{x}{r}\right|^n .$$

By the Comparison Theorem, $\sum_{n=0}^{\infty} |c_n x^n|$ converges. □

Radius of Convergence

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Theorem 6

Consider a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. *One of the following will occur:*

1. *The series converges only if* $x = a$.
2. *The series converges absolutely for every* $x \in \mathbb{R}$.
3. *There is* $R \in \mathbb{R}$ *such that the series converges absolutely for* $|x-a| < R$ *and diverges for* $|x-a| > R$.

R is called the **radius of convergence**. In the first case we say $R = 0$. In the second case, we say $R = \infty$.

Radius of Convergence

Proof. We may assume $a = 0$. Let $S = \left\{ h \in \mathbb{R} : \sum_{n=0}^{\infty} c_n h^n \text{ converges} \right\}$. We know $0 \in S$, so $S \neq \emptyset$.

If S is not bounded above then $S = \mathbb{R}$: For any $t \in \mathbb{R}$, there is $h \in S$ such that $h > |t|$. Therefore $t \in S$. This is the $R = \infty$ case.

If S is bounded above, take $R = \sup S$. Suppose $|x| < R$. There is $h \in S$ such that $|x| < h < R$. Since $\sum c_n h^n$ converges, $\sum c_n x^n$ converges absolutely. Hence the series converges for every $x \in (-R, R)$.

Finally, suppose the series converges for some x with $|x| > R$. Take any t such that $R < t < |x|$. Then the series converges at t, violating the definition of R. □

Interval of Convergence

The **interval of convergence** consists of all the x for which the power series converges.

It can be $\{a\}$, $(-\infty, \infty)$, $(a-R, a+R)$, $(a-R, a+R]$, $[a-R, a+R)$ or $[a-R, a+R]$.

The Ratio and Root Tests usually suffice to find R. Some other test would have to be applied to the end-points $a \pm R$.

Power Series as Functions

Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ have radius of convergence $R > 0$.

It defines a function $f\colon (a-R, a+R) \to \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$.

The endpoints $a \pm R$ can also be included in the domain, provided the series converges there.

Is $f(x)$ differentiable? The obvious candidate for its derivative is its **derived power series** $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ obtained by **term-by-term differentiation**.

We will prove this is the right candidate.

Convergence of Desired Series

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Theorem 7

Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ *have radius of convergence* $R > 0$*. Then its derived series* $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ *also has radius of convergence* R*.*

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Proof. Assume $a = 0$. Let $0 < x < R$.

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Proof. Assume $a = 0$. Let $0 < x < R$.
Pick h such that $0 < x < x+h < R$. Then

$$\frac{f(x+h)-f(x)}{h} = \sum_{n=1}^{\infty} c_n \frac{(x+h)^n - x^n}{h} = \sum_{n=1}^{\infty} n\, c_n\, y_n^{n-1}$$

for some $x < y_n < x+h$, by the Mean Value Theorem.

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for some $x < y_n < x + h$, by the Mean Value Theorem.
Since the series for $f(x+h)$ and $f(x)$ converge absolutely, so does $\sum n\, c_n\, y_n^{n-1}$. By Comparison Test, $\sum n\, c_n\, x^{n-1}$ converges absolutely.

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So the derived series converges for $0 < x < R$. Hence its radius of convergence is at least R. On the other hand, for $n > x$ we have $|n\, c_n\, x^{n-1}| \geq |c_n\, x^n|$ and so the derived series diverges for $x > R$.

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So the derived series converges for $0 < x < R$. Hence its radius of convergence is at least R. On the other hand, for $n > x$ we have $|n\, c_n\, x^{n-1}| \geq |c_n\, x^n|$ and so the derived series diverges for $x > R$.
Hence the radius of convergence is exactly R. □

Differentiability of Power Series

Theorem 8

Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ *have radius of convergence* $R > 0$. *Then*

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \text{ for } |x-a| < R.$$

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Proof. Again let $a = 0$. We investigate the gap between the difference quotient and the candidate derivative $g(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$.

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$$\frac{f(y)-f(x)}{y-x} - g(x) = \sum_{n=1}^{\infty} c_n \frac{y^n - x^n}{y-x} - \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$= \sum_{n=1}^{\infty} c_n \left[\frac{y^n - x^n}{y-x} - n x^{n-1} \right]$$

(continued ...)

Differentiability of Power Series

(. . . continued)

$$\frac{f(y) - f(x)}{y - x} - g(x) = \sum_{n=1}^{\infty} c_n \Big[\sum_{k=0}^{n-1} y^k x^{n-k-1} - n x^{n-1} \Big]$$
$$= \sum_{n=1}^{\infty} c_n \Big[\sum_{k=1}^{n-1} x^{n-k-1} (y^k - x^k) \Big].$$

Fix a number ρ such that $|x|, |y| < \rho < R$. Then

$$|y^k - x^k| = |y - x| \Big| \sum_{j=0}^{k-1} y^j x^{k-1-j} \Big| \leq k|y - x|\rho^{k-1}.$$

(continued . . .)

Differentiability of Power Series

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(...continued)

Hence,
$$\left|\frac{f(y)-f(x)}{y-x}-g(x)\right| \leq |y-x|\sum_{n=1}^{\infty}|c_n|\left[\sum_{k=1}^{n-1}k\rho^{n-2}\right]$$
$$= |y-x|\sum_{n=1}^{\infty}|c_n|\frac{n(n-1)}{2}\rho^{n-2}$$
$$= M|y-x|.$$

Therefore, $\lim_{y\to x}\left(\frac{f(y)-f(x)}{y-x}-g(x)\right)=0$ and $f'(x)=g(x)$. □

Example

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Example 9

We have seen that the power series $\sum_{n=0}^{\infty} x^n/n!$ converges for every $x \in \mathbb{R}$. Thus, it defines a differentiable function $f : \mathbb{R} \to \mathbb{R}$. Note that

$$f(0) = 1 \quad \text{and} \quad f'(x) = \sum_{n=1}^{\infty} n\frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = f(x).$$

Therefore $f(x)$ is the exponential function, and we get

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

In particular, $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Integration of Power Series

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Theorem 10

Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ *with radius of convergence* $R > 0$. *Then*

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C.$$

Further, for any $b \in (a-R, a+R)$,

$$\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} \int_a^b c_n(x-a)^n\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(b-a)^{n+1}.$$

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Proof. Consider the power series $h(x) = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$.
As $f(x)$ is the derived series of $h(x)$, $h(x)$ also has radius of convergence R, and it is the anti-derivative of $f(x)$.

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This proves the first part.

Integration of Power Series

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Further, for any $b \in (a-R, a+R)$*,*

$$\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} \int_a^b c_n(x-a)^n\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(b-a)^{n+1}.$$

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As $f(x)$ is the derived series of $h(x)$, $h(x)$ also has radius of convergence R, and it is the anti-derivative of $f(x)$.
This proves the first part.
The second part follows from the Second Fundamental Theorem. □

Applications of Geometric Series

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Consider the function defined by the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots, \qquad \text{for } |x| < 1.$$

We can use various substitutions to get power series expansions of other functions.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - + \cdots, \qquad \text{for } |x| < 1.$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - + \cdots, \qquad \text{for } |x| < 1.$$

$$\frac{1}{2-x} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{x^n}{2^n} = \frac{1}{2} + \frac{x}{4} + \frac{x^2}{8} + \cdots, \qquad \text{for } |x| < 2.$$

Applications of Geometric Series

Differentiating the geometric series repeatedly gives more power series expansions, each valid for $|x| < 1$.

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \cdots .$$

$$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2} = 1 + 3x + 6x^2 + \cdots .$$

$$\frac{1}{(1-x)^k} = \sum_{n=k-1}^{\infty} \binom{n}{k-1} x^{n-k+1}.$$

Applications of Geometric Series

Integration gives other interesting series.

$$\begin{aligned}\log(1+x) &= \int_0^x \frac{1}{1+t}\,dt = \int_0^x \left(\sum_{n=0}^{\infty}(-1)^n t^n\right) dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots \quad \text{for } |x| < 1.\end{aligned}$$

Task 1

Prove that $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots$ *for* $|x| < 1$.

Abel's Theorem

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Theorem 11

Suppose a power series with centre at $x = a$ has radius of convergence $R > 0$. If the power series converges at $a + R$ then the function defined by it is left continuous at that point.

Abel's Theorem

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Suppose a power series with centre at $x = a$ has radius of convergence $R > 0$. If the power series converges at $a + R$ then the function defined by it is left continuous at that point.

Proof. We may assume the power series has the form $\sum_{n=0}^{\infty} c_n x^n$, $R = 1$, and $\sum_{n=0}^{\infty} c_n$ converges. Now take any x such that $0 < x < 1$.

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Denote $S_n = c_0 + \cdots + c_n$ and $S = \lim S_n$.

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$$\sum_{n=0}^{m} c_n x^n = x^m S_m + \sum_{n=0}^{m-1} (x^n - x^{n+1}) S_n = x^m S_m + (1 - x) \sum_{n=0}^{m-1} x^n S_n,$$

obtained by substituting $c_n = S_n - S_{n-1}$ and regrouping.

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$$\sum_{n=0}^{m} c_n x^n = x^m S_m + \sum_{n=0}^{m-1} (x^n - x^{n+1}) S_n = x^m S_m + (1-x) \sum_{n=0}^{m-1} x^n S_n,$$

obtained by substituting $c_n = S_n - S_{n-1}$ and regrouping.
By letting $m \to \infty$ we obtain

$$\sum_{n=0}^{\infty} c_n x^n = (1-x) \sum_{n=0}^{\infty} x^n S_n.$$

(continued ...)

Abel's Theorem

(…continued)

Hence, $\sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n = (1-x) \sum_{n=0}^{\infty} x^n (S_n - S)$.

Abel's Theorem

(. . . continued)

Hence, $\sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n = (1-x)\sum_{n=0}^{\infty} x^n (S_n - S)$.

Given $\epsilon > 0$ we first choose N such that $n \geq N$ implies $|S_n - S| < \epsilon/2$.

Abel's Theorem

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Given $\epsilon > 0$ we first choose N such that $n \geq N$ implies $|S_n - S| < \epsilon/2$.

Then,

$$\left|\sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n\right| \leq |1-x| \sum_{n=0}^{N-1} |x|^n |S_n - S| + |1-x| \sum_{n=N}^{\infty} |x|^n |S_n - S|$$

$$\leq |1-x| \sum_{n=0}^{N-1} |S_n - S| + \epsilon/2.$$

Abel's Theorem

(…continued)

Hence, $\sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n = (1-x)\sum_{n=0}^{\infty} x^n (S_n - S)$.
Given $\epsilon > 0$ we first choose N such that $n \geq N$ implies $|S_n - S| < \epsilon/2$.
Then,

$$\left|\sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n\right| \leq |1-x| \sum_{n=0}^{N-1} |x|^n |S_n - S| + |1-x| \sum_{n=N}^{\infty} |x|^n |S_n - S|$$
$$\leq |1-x| \sum_{n=0}^{N-1} |S_n - S| + \epsilon/2.$$

The choice of N was independent of x, hence $\sum_{n=0}^{N-1} |S_n - S|$ is also independent of x. Therefore, for x close enough to 1, we'll have $|1-x| \sum_{n=0}^{N-1} |S_n - S| < \epsilon/2$. This completes the proof. □

Applications of Abel's Theorem

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It follows from Abel's Theorem that $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}$ is left continuous at 1. Hence,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = f(1) = \lim_{x \to 1-} f(x) = \lim_{x \to 1-} \log(1+x) = \log 2.$$

Task 2

Prove that

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4\Big(1 - \frac{1}{3} + \frac{1}{5} - + \cdots\Big).$$

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Taylor Series

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Suppose a smooth function $f(x)$ is expressible as a power series with centre a and radius of convergence $R > 0$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad \text{for } |x-a| < R.$$

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By differentiating repeatedly at $x = a$, we get the following.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \implies f(a) = c_0,$$

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \implies f'(a) = c_1,$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2} \implies f''(a) = 2c_2.$$

(continued ...)

Taylor Series

(...continued)

Differentiating k times gives

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n(x-a)^{n-k} \quad \Longrightarrow f^{(k)}(a) = k!\, c_k$$

$$\Longrightarrow \; c_k = \frac{f^{(k)}(a)}{k!}.$$

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(. . . continued)

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$$\implies c_k = \frac{f^{(k)}(a)}{k!}.$$

The power series $T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the **Taylor series** of $f(x)$ with centre at a.

Taylor Series

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Every smooth function $f(x)$ has a unique Taylor series $T_f(x)$ centered at a. But they may not be equal to each other.

Examples

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We already know the following Maclaurin series expansions.

$$\frac{1}{(1-x)^k} = 1 + \binom{k}{k-1}x + \binom{k+1}{k-1}x^2 + \cdots \quad \text{for } |x| < 1,\ k \in \mathbb{N},$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots \quad \text{for } |x| < 1,$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots \quad \text{for } |x| < 1,$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for } x \in \mathbb{R}.$$

Binomial Series

Theorem 12

Fix $r \in \mathbb{R}$. Then the Maclaurin series of $f(x) = (1+x)^r$ is given by

$$T(x) = \sum_{n=0}^{\infty} \binom{r}{n} x^n$$

and equals $f(x)$ for $|x| < 1$. We have used the notation

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}, \text{ with the convention } \binom{r}{0} = 1.$$

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Proof. We compute the Maclaurin series by repeated differentiation:

$$\begin{aligned} f(x) &= (1+x)^r &\implies f(0) &= 1 &\implies c_0 &= 1, \\ f'(x) &= r(1+x)^{r-1} &\implies f'(0) &= r &\implies c_1 &= r, \\ f''(x) &= r(r-1)(1+x)^{r-2} &\implies f''(0) &= r(r-1) &\implies c_2 &= \frac{r(r-1)}{2!}. \end{aligned}$$

(continued . . .)

Binomial Series

(…continued)

Continuing in this fashion gives $c_n = \dfrac{r(r-1)\cdots(r-n+1)}{n!} = \dbinom{r}{n}$ and we obtain the desired form of the Maclaurin series.

Binomial Series

(...continued)

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Now we have to address its convergence.

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Continuing in this fashion gives $c_n = \dfrac{r(r-1)\cdots(r-n+1)}{n!} = \dbinom{r}{n}$ and we obtain the desired form of the Maclaurin series.
Now we have to address its convergence.
First apply the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{c_{n+1}x^{n+1}}{c_nx^n}\right| = |x|\lim_{n\to\infty}\left|\frac{r(r-1)\cdots(r-n)}{r(r-1)\cdots(r-n+1)}\cdot\frac{n!}{(n+1)!}\right|$$
$$= |x|\lim_{n\to\infty}\left|\frac{r-n}{n+1}\right| = |x|.$$

Binomial Series

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Thus the Maclaurin series has radius of convergence 1 and defines a function $T(x)$ for $|x| < 1$.

(continued ...)

Binomial Series

(...continued)
To obtain the equality of f and T we consider their derivatives:

$$f'(x) = r(1+x)^{r-1} \implies (1+x)f'(x) = rf(x), \text{ and}$$

$$\begin{aligned}(1+x)T'(x) &= \sum_{n=1}^{\infty} n\binom{r}{n}x^{n-1} + \sum_{n=1}^{\infty} n\binom{r}{n}x^n \\ &= r + \sum_{n=1}^{\infty}\left[(n+1)\binom{r}{n+1} + n\binom{r}{n}\right]x^n \\ &= r + \sum_{n=1}^{\infty} r\binom{r}{n}x^n = rT(x).\end{aligned}$$

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It follows that $T(x) = cf(x)$.
Checking the values at $x = 0$ gives $T(x) = f(x)$, for $|x| < 1$. □

Taylor Polynomials and Series

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Recall that the Taylor polynomials of $f(x)$ are defined by

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The Taylor polynomials of a smooth function are the partial sums for its Taylor series. Hence

$$T_f(x) = \lim_{n\to\infty} T_n(x).$$

So the question of whether $f(x)$ equals $T_f(x)$ can be addressed by checking whether $\lim_{N\to\infty} R_N(x) = 0$, where $R_N(x)$ is the remainder after N terms.

Example: Sine function

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Example 13

From the Taylor polynomial calculations in Chapter 6 we know that the Maclaurin series of the sine function is

$$T_{\sin}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots .$$

The remainder terms are given by

$$R_{2n+1}(x) = \sin x - \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sin^{(2k+2)}(c) \frac{x^{2k+2}}{(2k+2)!}.$$

Hence, $|R_{2n+1}(x)| \leq \dfrac{|x|^{2k+2}}{(2k+2)!} \to 0 \implies R_{2n+1}(x) \to 0.$

Therefore, $\sin x = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots .$

Exercises

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Task 3

Prove that the cosine function equals its Maclaurin series:

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots .$$

Task 4

Let $f : (c - R, c + R) \to \mathbb{R}$ *be a* C^∞ *function such that there is a constant* M *with* $|f^{(n)}(x)| \le M^n$ *for all* x *and* n. *Prove that*
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n$ *for every* x.

Taylor Series and Limits

Substituting a Taylor series for the corresponding function can often be useful in computing limits. For example,

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{(x - x^3/3! + x^5/5!\cdots) - x}{x^3} = \lim_{x\to 0}\frac{-x^3/3! + x^5/5!\cdots}{x^3}$$
$$= \lim_{x\to 0}\left(-\frac{1}{3!} + \frac{x^2}{5!}\cdots\right) = -\frac{1}{3!}.$$

To compute a limit at a, we use Taylor series expansions centred at a, provided the series equal the corresponding functions. Here is another example:

$$\lim_{x\to 1}\frac{\log x}{\sqrt{x}-1} = \lim_{x\to 1}\frac{(x-1)-(x-1)^2/2\cdots}{(x-1)/2-(x-1)^2/8\cdots}$$
$$= \lim_{x\to 1}\frac{1-(x-1)/2\cdots}{1/2-(x-1)/8\cdots} = 2.$$

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Our goal is to express periodic functions in terms of the sine and cosine functions. To begin, we consider functions whose period is 2π. It is enough to analyse such functions on the interval $[-\pi, \pi]$.

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It is enough to consider the following sub-collection of these functions: $\{1\} \cup \{\sin mx \mid m = 1, 2, 3, \dots\} \cup \{\cos mx \mid m = 1, 2, 3, \dots\}$.
A **trigonometric polynomial** is a linear combination of such functions,

$$T(x) = \frac{a_0}{2} + \sum_{n=1}^{M} a_n \cos nx + \sum_{n=1}^{M} b_n \sin nx.$$

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An immediate question is: If we are able to process T in terms of finding information like its integrals, can we recover the defining numbers a_n and b_n? This is part of a general approach of trying to recover the original data from knowledge of averages.

Orthogonality Relations

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Task 5

Suppose $m, n \in \mathbb{N}$. *Show the following.*

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n, \\ \pi & \text{if } m = n, \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0,$$

$$\int_{-\pi}^{\pi} 1 \cdot \cos mx \, dx = \int_{-\pi}^{\pi} 1 \cdot \sin mx \, dx = 0,$$

$$\int_{-\pi}^{\pi} 1 \cdot 1 \, dx = 2\pi.$$

Orthogonality Relations

From the orthogonality relations, we can conclude that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} T(x)\cos nx\,dx = a_n,$$
$$\frac{1}{\pi}\int_{-\pi}^{\pi} T(x)\sin nx\,dx = b_n,$$
$$\frac{1}{\pi}\int_{-\pi}^{\pi} T(x)\cdot 1\,dx = a_0.$$

Setting the constant part of T to $a_0/2$ has enabled a uniform formula for all a_n.

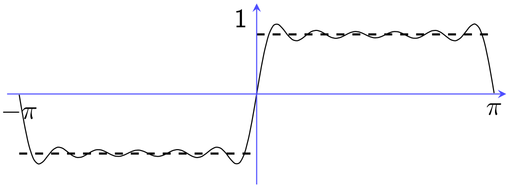
Example

Example 14

An example of a trigonometric polynomial is

$$T(x) = \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin 11x}{11}.$$

We plot its graph and observe it appears to be approaching a square wave shape:



This example shows that trigonometric polynomials have the potential to approximate discontinuous periodic functions.

Trigonometric Polynomials

A **trigonometric series** is an expression of the form,

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Let f be a function with period 2π and which is integrable on $[-\pi, \pi]$. Then the functions $f(x)\cos nx$ and $f(x)\sin nx$ are also integrable. Hence, we can define,

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The corresponding trigonometric series is called the **Fourier series** of f and we write:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

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$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

The symbol $\sim$ indicates that the right hand side is the Fourier series of f but we have not established its convergence to f.

Example: Square Wave

Example 15

Consider the **square wave function** defined by $S(x) = -1$ if $x \in (-\pi, 0]$ and $S(x) = 1$ if $x \in \{-\pi\} \cup (0, \pi]$. Then the Fourier coefficients of S are:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\cos nx\,dx = 0 + 0 = 0,$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\sin nx\,dx \\ &= \frac{2}{\pi}\int_{0}^{\pi}(1)\sin nx\,dx = -\frac{2}{\pi}\cdot\frac{\cos nx}{n}\Big|_0^{\pi} = \frac{2}{\pi}\cdot\frac{1-(-1)^n}{n} \\ &= \begin{cases} 0 & \text{if } n \text{ even} \\ 4/\pi n & \text{if } n \text{ odd.} \end{cases} \end{aligned}$$

Hence the Fourier series is $S(x) \sim \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$.

Example: Square Wave

Let us see Fourier's own method for establishing the convergence of the Fourier series for the square wave function. Let $S_m(x)$ be the partial sum of the first m terms of the series,

$$\frac{\pi}{4} S_m(x) = \sin x + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2m-1)x}{2m-1}.$$

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$$\frac{\pi}{4} S_m(x) = \sin x + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2m-1)x}{2m-1}.$$

Differentiate, and multiply both sides by $2\sin 2x$:

$$\begin{aligned}\frac{\pi}{2} S_m'(x)\sin 2x &= 2\cos x \sin 2x + 2\cos 3x \sin 2x + \cdots + 2\cos(2m-1)x\sin 2x \\ &= (\sin 3x - \sin(-x)) + (\sin 5x - \sin x) + \cdots \\ &\quad + (\sin(2m+1)x - \sin(2m-3)x) \\ &= \sin(2m+1)x + \sin(2m-1)x = 2\sin(2mx)\cos x.\end{aligned}$$

Example: Square Wave

Let us see Fourier's own method for establishing the convergence of the Fourier series for the square wave function. Let $S_m(x)$ be the partial sum of the first m terms of the series,

$$\frac{\pi}{4} S_m(x) = \sin x + \frac{\sin 3x}{3} + \cdots + \frac{\sin(2m-1)x}{2m-1}.$$

Differentiate, and multiply both sides by $2\sin 2x$:

$$\begin{aligned}
\frac{\pi}{2} S_m'(x)\sin 2x &= 2\cos x\sin 2x + 2\cos 3x\sin 2x + \cdots + 2\cos(2m-1)x\sin 2x \\
&= (\sin 3x - \sin(-x)) + (\sin 5x - \sin x) + \cdots \\
&\quad + (\sin(2m+1)x - \sin(2m-3)x) \\
&= \sin(2m+1)x + \sin(2m-1)x = 2\sin(2mx)\cos x.
\end{aligned}$$

Hence, $S_m'(x) = \dfrac{2}{\pi}\dfrac{\sin 2mx}{\sin x}$.

Example: Square Wave

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Hence, $S_m'(x) = \dfrac{2}{\pi}\dfrac{\sin 2mx}{\sin x}$. Note that $S_m(\pi/2) = (4/\pi)(1 - 1/3 + 1/5 + \cdots + (-1)^{m-1}/(2m-1)) \to 1$, by the Gregory-Leibniz formula which we observed earlier.
(continued ...)

Example: Square Wave

(...continued)
Now, for any $a \in [\pi/2, \pi)$,

$$\begin{aligned}
S_m(a) &= S_m(\pi/2) + \int_{\pi/2}^{a} S_m'(x)\,dx = S_m(\pi/2) + \frac{2}{\pi}\int_{\pi/2}^{a} \sin 2mx \csc x\,dx \\
&= S_m(\pi/2) - \frac{2}{\pi}\frac{\cos 2mx \csc x}{2m}\bigg|_{\pi/2}^{a} + \frac{1}{m\pi}\int_{\pi/2}^{a} \cos 2mx \csc' x\,dx \\
&= S_m(\pi/2) - \frac{\cos 2ma \csc a - (-1)^m}{m\pi} + \frac{\cos 2m\xi_m}{m\pi}\int_{\pi/2}^{a} \csc' x\,dx \\
&= S_m(\pi/2) - \frac{1}{m\pi}\Big(\cos 2ma \csc a - \cos 2m\xi_m(\csc a - 1) - (-1)^m\Big) \\
&\to 1 \text{ as } m \to \infty.
\end{aligned}$$

The argument also works for $a \in (0, \pi/2]$.

Example: Square Wave

(. . . continued)
Now, for any $a \in [\pi/2, \pi)$,

$$\begin{aligned} S_m(a) &= S_m(\pi/2) + \int_{\pi/2}^{a} S_m'(x)\,dx = S_m(\pi/2) + \frac{2}{\pi}\int_{\pi/2}^{a} \sin 2mx \csc x\,dx \\ &= S_m(\pi/2) - \frac{2}{\pi}\frac{\cos 2mx \csc x}{2m}\bigg|_{\pi/2}^{a} + \frac{1}{m\pi}\int_{\pi/2}^{a} \cos 2mx \csc' x\,dx \\ &= S_m(\pi/2) - \frac{\cos 2ma \csc a - (-1)^m}{m\pi} + \frac{\cos 2m\xi_m}{m\pi}\int_{\pi/2}^{a} \csc' x\,dx \\ &= S_m(\pi/2) - \frac{1}{m\pi}\Big(\cos 2ma \csc a - \cos 2m\xi_m(\csc a - 1) - (-1)^m\Big) \\ &\to 1 \text{ as } m \to \infty. \end{aligned}$$

The argument also works for $a \in (0, \pi/2]$.
We have to separate these cases because in the fourth equality we have used the Mean Value Theorem for Weighted Integration, which requires the weight function to not change sign.
(continued . . .)

Example: Square Wave

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(. . . continued)
Further, for $x \in (-\pi, 0)$, we have $S_m(x) \to -1$ because these are odd functions.

Thus the Fourier series converges to $S(x)$ at the points where it is continuous.

At the points of discontinuity $(-\pi,\ 0,\ \pi)$ the series gives zero, which is the mean of the left and right hand limits.

Task 6

Show that $\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}\left(1 + \dfrac{1}{3} - \dfrac{1}{5} - \dfrac{1}{7} + \cdots\right).$

Task 7

Let S be the square wave function of the last example. Show that the Fourier series of $g(x) = S(x - a)$ converges to 0 at $x = a$.

Dirichlet Kernel

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Let f_m represent a partial sum of the Fourier series expansion of $f(x)$,

$$f_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} a_n \cos nx + \sum_{n=1}^{m} b_n \sin nx.$$

Substitute the formulas for the Fourier coefficients:

$$\begin{aligned} f_m(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,ds + \sum_{n=1}^{m} \Big(\frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\cos ns\,ds\Big)\cos nx \\ &\quad + \sum_{n=1}^{m} \Big(\frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sin ns\,ds\Big)\sin nx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\Big(\frac{1}{2} + \sum_{n=1}^{m}(\cos ns\cos nx + \sin ns\sin nx)\Big)\,ds \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\Big(\frac{1}{2} + \sum_{n=1}^{m}\cos n(s-x)\Big)\,ds. \end{aligned}$$

Dirichlet Kernel

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Task 8

Show that $\frac{1}{2} + \sum_{n=1}^{m} \cos n\theta = \frac{\sin(m+1/2)\theta}{2\sin\theta/2}$. *Give a justification for claiming equality even when* $\theta = 2n\pi$.

The collection of functions $D_m(x) = \frac{\sin(m+1/2)x}{2\pi\sin x/2}$ is called the **Dirichlet kernel**.

Assuming that f extends beyond $[-\pi, \pi]$ with a period of 2π, the partial sums of the Fourier series can be expressed as

$$f_m(x) = \int_{-\pi}^{\pi} f(s) D_m(s-x)\, ds = \int_{-\pi}^{\pi} f(s+x) D_m(s)\, ds.$$

If we take $f(x) = 1$ then all the partial sums are also 1, and we get

$$1 = \int_{-\pi}^{\pi} D_m(s)\, ds \qquad \text{for every } n.$$

Dirichlet Kernel

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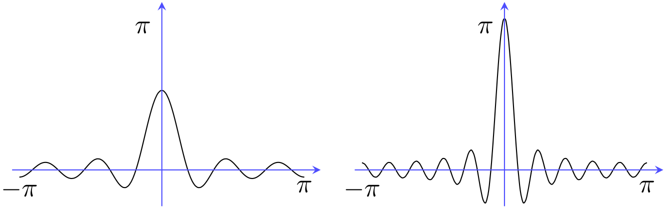
The graphs below depict D_5 and D_{10}.



As n increases, the Dirichlet kernel gets more concentrated near origin. Consequently, the values of $\int_{-\pi}^{\pi} f(s+x) D_n(s)\, ds$ depend more and more on the values of f near x (corresponding to $s = 0$).

If f is continuous at x, this should make the integral approach $f(x)$.

If f has a jump discontinuity, then the left and right limits contribute equally (due to the symmetry of D_n) and so we expect the integral to converge to their mean.

Convergence of Fourier Series

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Theorem 16

Let $f : [-\pi, \pi] \to \mathbb{R}$ *be 'piecewise differentiable' in the following sense.*

1. *The interval* $[-\pi, \pi]$ *has a partition* $-\pi = x_0 < \cdots < x_n = \pi$ *such that* f *is differentiable on each subinterval* (x_i, x_{i+1})*.*
2. *The left and right limits of* f *and* f' *exist at every point.*

Then we have the following convergence results.

1. *If* f *is continuous at* x*, the Fourier series at* x *converges to* $f(x)$*.*
2. *If* f *has a jump discontinuity at* x*, the Fourier series at* x *converges to the mean of the left and right-hand limits at* x*.*

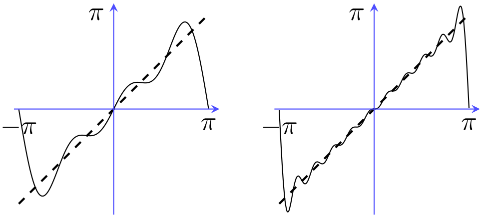
Example: Sawtooth Function

Example 17

Consider the **sawtooth function** f, which has period 2π and satisfies $f(x) = x$ on $(-\pi, \pi)$. Since f is odd, all $a_n = 0$. And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx = (-1)^{n+1}2/n$. Hence, the Fourier series is

$$f(x) \sim 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

We plot two of the partial sums below, with 3 and 10 terms respectively.



Example: Triangular Wave

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Example 18

$C(x) = \pi - |x|$, for $x \in [-\pi, \pi]$, is continuous but fails to be differentiable at 0. Using integration by parts, we can calculate

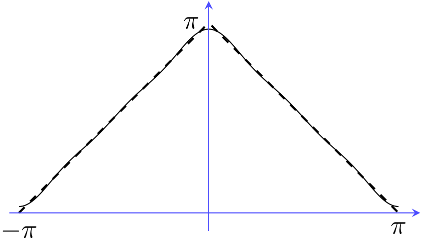
$$a_n = \begin{cases} \pi & \text{if } n = 0 \\ 4/\pi n^2 & \text{if } n \text{ odd} \\ 0 & \text{else} \end{cases}, \quad b_n = 0.$$

Therefore, $C(x) \sim \frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right)$.

Convergence of the Fourier series to $C(x)$ is guaranteed by the last theorem.

Example: Triangular Wave

The partial sum $\dfrac{\pi}{2} + \dfrac{4}{\pi}\left(\cos x + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2}\right)$ is plotted below.



Note how quickly the series converges when the function has no discontinuity.

Task 9

Show that $\dfrac{\pi^2}{8} = 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots.$

Inner Product

Let us use the name $\mathcal{I}$ for the class of integrable functions on $[-\pi, \pi]$. For $f, g \in \mathcal{I}$ we define their **inner product** by

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

Note that the Fourier coefficients of f are given by $a_n = \langle f(x), \cos nx\rangle$ and $b_n = \langle f(x), \sin nx\rangle$.

Task 10

Let $f, g, h \in \mathcal{I}$. Show the following.

1. $\langle f, f\rangle \geq 0$. *(Hence we can define the* **norm** $||f|| = \langle f, f\rangle^{1/2}$.*)*
2. $\langle f, g\rangle = \langle g, f\rangle$.
3. *If* $c \in \mathbb{R}$ *then* $\langle cf, g\rangle = \langle f, cg\rangle = c\langle f, g\rangle$.
4. $\langle f + g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$ *and* $\langle f, g + h\rangle = \langle f, g\rangle + \langle f, h\rangle$.
5. $\langle f, g\rangle = 0 \implies ||f + g||^2 = ||f||^2 + ||g||^2$.

Bessel's Inequality

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Theorem 19

Let $f \in \mathcal{I}$ *and* $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$. *Then*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\, dx.$$

Bessel's Inequality

Theorem 19

Let $f \in \mathcal{I}$ *and* $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$. *Then*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx.$$

Proof. Let f_m be a partial sum of the Fourier series:

$$f_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m} a_k \cos kx + \sum_{k=1}^{m} b_k \sin kx.$$

Bessel's Inequality

Theorem 19

Let $f \in \mathcal{I}$ *and* $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$*. Then*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx.$$

Proof. Let f_m be a partial sum of the Fourier series:

$$f_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m} a_k \cos kx + \sum_{k=1}^{m} b_k \sin kx.$$

Due to the orthogonality relations, we have for $1 \leq n \leq m$,

$$\langle f_m(x), 1 \rangle = a_0, \quad \langle f_m(x), \cos nx \rangle = a_n, \quad \langle f_m(x), \sin nx \rangle = b_n.$$

Bessel's Inequality

Theorem 19

Let $f \in \mathcal{I}$ *and* $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$. *Then*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx.$$

Proof. Let f_m be a partial sum of the Fourier series:

$$f_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m} a_k \cos kx + \sum_{k=1}^{m} b_k \sin kx.$$

Due to the orthogonality relations, we have for $1 \leq n \leq m$,

$$\langle f_m(x), 1 \rangle = a_0, \quad \langle f_m(x), \cos nx \rangle = a_n, \quad \langle f_m(x), \sin nx \rangle = b_n.$$

Hence $||f_m||^2 = \langle f_m, f_m \rangle = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2$.

(continued . . .)

Bessel's Inequality

(. . . continued)

Further, for $1 \leq n \leq m$,

$$\langle f(x) - f_m(x), 1\rangle = \langle f(x) - f_m(x), \cos nx\rangle = \langle f(x) - f_m(x), \sin nx\rangle = 0.$$

Bessel's Inequality

(...continued)

Further, for $1 \leq n \leq m$,

$$\langle f(x) - f_m(x), 1\rangle = \langle f(x) - f_m(x), \cos nx\rangle = \langle f(x) - f_m(x), \sin nx\rangle = 0.$$

Hence, $\langle f - f_m, f_m\rangle = 0$. Therefore,

$$||f||^2 = ||(f - f_m) + f_m||^2 = ||f - f_m||^2 + ||f_m||^2 \geq ||f_m||^2.$$

Bessel's Inequality

(... continued)

Further, for $1 \leq n \leq m$,

$$\langle f(x) - f_m(x), 1 \rangle = \langle f(x) - f_m(x), \cos nx \rangle = \langle f(x) - f_m(x), \sin nx \rangle = 0.$$

Hence, $\langle f - f_m, f_m \rangle = 0$. Therefore,

$$||f||^2 = ||(f - f_m) + f_m||^2 = ||f - f_m||^2 + ||f_m||^2 \geq ||f_m||^2.$$

Since this is true for all m, the result follows. □

Bessel's Inequality

(...continued)

Further, for $1 \leq n \leq m$,

$$\langle f(x) - f_m(x), 1\rangle = \langle f(x) - f_m(x), \cos nx\rangle = \langle f(x) - f_m(x), \sin nx\rangle = 0.$$

Hence, $\langle f - f_m, f_m\rangle = 0$. Therefore,

$$||f||^2 = ||(f - f_m) + f_m||^2 = ||f - f_m||^2 + ||f_m||^2 \geq ||f_m||^2.$$

Since this is true for all m, the result follows. □

Task 11 (Riemann Lemma)

Let $f \in \mathcal{I}$ with Fourier coefficients a_n, b_n. Prove that $a_n, b_n \to 0$.

Table of Contents

Complex Numbers

Consider $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with the binary operations given below.

$$\text{Addition: } (x, y) + (u, v) = (x + u, y + v).$$
$$\text{Multiplication: } (x, y) * (u, v) = (xu - yv, xv + yu).$$

We denote the cartesian plane with these operations by $\mathbb{C}$ and call it the **complex plane**. First we observe that $\mathbb{C}$ is a field.

1. The operations $+$ and $*$ are commutative and associative.
2. $(0, 0)$ is the additive identity, and $(-x, -y)$ is the additive inverse of (x, y).
3. $(1, 0)$ is the multiplicative identity, and a nonzero (x, y) has multiplicative inverse $\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)$.
4. Multiplication distributes over addition.

Real and Imaginary Parts

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Now we identify $(1,0)$ with the real number 1 and denote $(0,1)$ by i. Then

$$(x,y) = x(1,0) + y(0,1) = x + iy.$$

In this notation, the rules for $+$ and $*$ become

$$(x+iy) + (u+iv) = (x+u) + i(y+v).$$
$$(x+iy) * (u+iv) = (xu - yv) + i(xv + yu).$$

If $z = x + iy$ with $x, y \in \mathbb{R}$, we say x is the **real part** of z and y is the **imaginary part** of z. We shall use 'Let $z = x + iy$' as an abbreviation for 'Let $z = x + iy$ with $x, y \in \mathbb{R}$'.

Conjugation and Absolute Value

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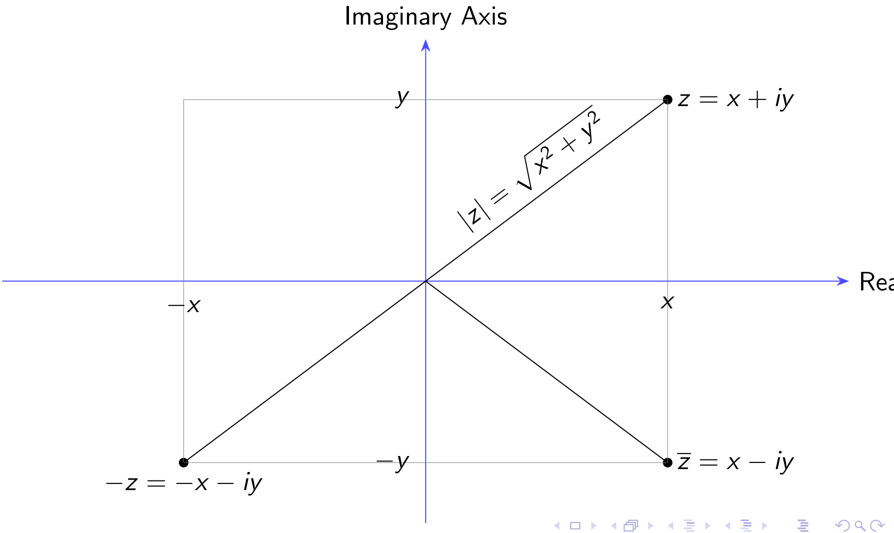
The **absolute value** or **modulus** of z is $|z| = \sqrt{x^2 + y^2}$ and the **conjugate** of z is $\overline{z} = x - iy = x + i(-y)$. We have the following properties:

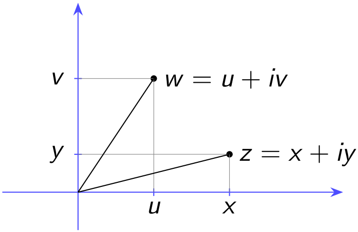
1. $z = 0 \iff |z| = 0$,
2. $z\overline{z} = |z|^2$,
3. $z \neq 0 \implies z^{-1} = \overline{z}/|z|^2$,
4. $\overline{\overline{z}} = z$,
5. $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$,
6. $|z + w| \leq |z| + |w|$,
7. $|z - w| \geq ||z| - |w||$,
8. $|zw| = |z||w|$.

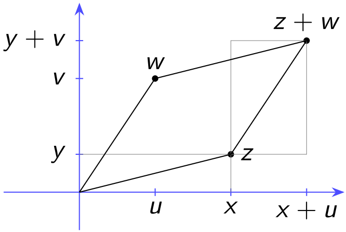
The Argand Plane



Addition

The process of addition is depicted by the 'Parallelogram Rule' in the Argand plane:





Multiplication

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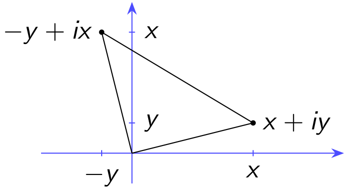
To visualize multiplication, let us first view what happens when a complex number is multiplied by i. We have $i(x + iy) = -y + ix$.



We see that the points $x + iy$, $-y + ix$ and 0 form a right angled triangle. Thus, multiplication by i causes a counter-clockwise rotation by a right angle around the origin. Multiplying by i *twice* gives a total rotation of 180 degrees and hence sends z to $-z$. This is the geometric description of i being the square root of -1.

Polar Coordinates

Any complex number $z = x + iy$ can be expressed as $|z|(\cos\theta + i\sin\theta)$ where θ is the angle between the rays starting at origin and passing through the points $(1,0)$ and (x,y).

Let $w = |w|(\cos\phi + i\sin\phi)$ be another complex number. Let us multiply them:

$$\begin{aligned} zw &= |z||w|(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= |z||w|((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)) \\ &= |z||w|(\cos(\theta+\phi) + i\sin(\theta+\phi)) \end{aligned}$$

Thus the effect of multiplying by z is to stretch the other number by a factor of $|z|$ and also rotate it by θ.

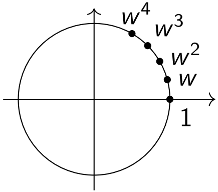
Task 12 (De Moivre's formula)

If $z = |z|(\cos\theta + i\sin\theta) \neq 0$ *and* $n \in \mathbb{Z}$ *then*

$$z^n = |z|^n(\cos n\theta + i\sin n\theta).$$

Roots of Unity

Suppose $z^n = 1$ for $n \in \mathbb{N}$. Write $z = |z|(\cos\theta + i\sin\theta)$.
Then $1 = |z|^n(\cos n\theta + i\sin n\theta)$.
Hence $|z| = 1$ and $n\theta = 2\pi k$ with $k \in \mathbb{Z}$.
Taking $k = 1$, we get the root $w = \cos(2\pi/n) + i\sin(2\pi/n)$.
If we let k vary over $0, \dots, n-1$ we get n distinct **roots of unity**:
$1, w, w^2, \dots, w^{n-1}$. They are equally spaced out on the unit circle:



Task 13

Prove that $1 + w + w^2 + \cdots + w^{n-1} = 0$.

Roots of Complex Numbers

If $z = |z|(\cos\theta + i\sin\theta)$ is a non-zero complex number then

$$\alpha = |z|^{1/n}(\cos(\theta/n) + i\sin(\theta/n))$$

is an n^{th} root: $\alpha^n = z$.

Let $1, w, \ldots, w^{n-1}$ be the roots of unity.

Then $\alpha, \alpha w, \ldots, \alpha w^{n-1}$ are the n^{th} roots of z.

Thus every non-zero complex number has n distinct n^{th} roots, and they are equally spaced on a circle centred at origin.

Notice how uniform and pleasant this situation is compared to taking roots of real numbers.

Example

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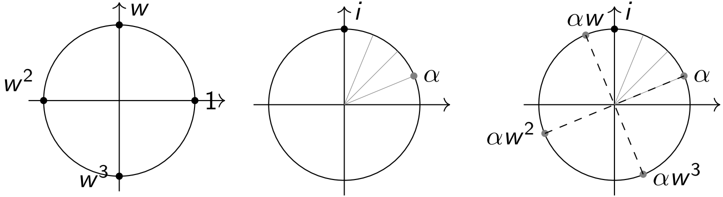
Let us compute the 4th roots of i.

We'll need the 4th roots of unity: 1, $w = i$, $w^2 = -1$, $w^3 = -i$.

We need the root $\alpha = \cos\pi/8 + i\sin\pi/8$ of i.

Then all the roots of i are obtained by multiplying each root of unity by α, that is, by rotating each of them through an angle of $\pi/8$.



Complex Sequences

A **complex sequence** is a function from $\mathbb{N}$ to $\mathbb{C}$, that is, an unending list of complex numbers.

Let (z_n) be a complex sequence, and L a complex number. We say that (z_n) **converges** to L if for every real number $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|z_n - L| < \epsilon$. The number L is called the **limit** of (z_n), and we write $\lim_{n\to\infty} z_n = L$ or $\lim z_n = L$ or $z_n \to L$.

Task 14

Prove the following.

1. $|z_n| \to 0$ *if and only if* $z_n \to 0$.
2. $z_n \to L$ *if and only if* $z_n - L \to 0$.
3. $z_n \to L$ *implies* $|z_n| \to |L|$.

Complex sequences follow the same Algebra of Limits as real sequences. They lack the Sandwich Theorem as the complex numbers are not an ordered field.

A Criterion for Convergence

Theorem 20

Consider a sequence (z_n) and let $z_n = x_n + iy_n$. Further, let $L = M + iN$. Then $z_n \to L$ if and only if $x_n \to M$ and $y_n \to N$.

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Proof. We begin by noting that $|z_n - L|^2 = |x_n - M|^2 + |y_n - N|^2$. Hence,

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$$\begin{aligned} x_n \to M \text{ and } y_n \to N &\implies |x_n - M| \to 0 \text{ and } |y_n - N| \to 0 \\ &\implies |z_n - L|^2 = |x_n - M|^2 + |y_n - N|^2 \to 0 \\ &\implies z_n \to L. \end{aligned}$$

A Criterion for Convergence

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□

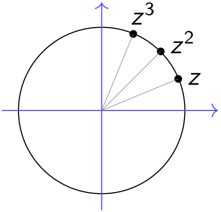
Example

Consider a fixed complex number z and consider the sequence (z^n) of powers of z.

If $|z| > 1$, we already know $|z|^n$ diverges, and hence z^n diverges.

If $|z_n| < 1$ then $|z|^n \to 0$ and hence $z_n \to 0$.

If $|z| = 1$ the sequence (z^n) rotates around the unit circle and so has no limit. The only exception is when $z = 1$.



Complex Series

Given a complex sequence (z_n), we create the corresponding series $\sum_{n=1}^{\infty} z_n$. It has the partial sums $S_n = \sum_{k=1}^{n} z_k$, and we define the **sum** of the series as $\lim S_n$. The series **converges** if this limit exists and **diverges** if it does not.

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Consider a geometric series $\sum_{n=1}^{\infty} z^{n-1} = 1 + z + z^2 + \cdots$ with $z \neq 1$. The partial sums are $S_n = 1 + z + \cdots + z^{n-1} = \dfrac{1 - z^n}{1 - z}$.

By our earlier calculations we see that $\displaystyle\sum_{n=1}^{\infty} z^{n-1} = \frac{1}{1 - z}$ if $|z| < 1$, and diverges otherwise.

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Task 15 (Divergence Test)

If $\sum z_n$ converges then $z_n \to 0$.

Algebra of Series

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Task 16

Consider convergent complex series $\sum a_n = L$ *and* $\sum b_n = M$. *Show that:*

1. *For any* $c \in \mathbb{C}$, $\sum (c\, a_n) = cL$,
2. $\sum (a_n + b_n) = L + M$.

Algebra of Series

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Let $z_n = x_n + iy_n$. Show that $\sum z_n$ converges if and only if both $\sum x_n$ and $\sum y_n$ converge. If these series do converge, prove $\sum z_n = \sum x_n + i \sum y_n$.

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Analysing convergence by separating into real and imaginary parts is not always fruitful, as a simple complex expression may look complicated when viewed in terms of real and imaginary parts.

Absolute Convergence

A complex series $\sum z_n$ is called **absolutely convergent** if $\sum |z_n|$ converges. For example, if $|z| < 1$, the geometric series $\sum z^n$ is absolutely convergent.

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Task 18

If a series is absolutely convergent then it is convergent.

Complex Power Series

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A **complex power series** is an expression of the form

$$\sum_{n=0}^{\infty} c_n(z-a)^n = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

in which a and the c_n are complex numbers and z is a complex variable.

Our main question is: For which values of z will a power series converge?

Example 23

The geometric series $\sum_{n=0}^{\infty} z^n$ is a power series centred at $a = 0$. It converges absolutely on the open disc $|z| < 1$ and diverges for $|z| \geq 1$.

Disc of Convergence

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We can replicate the basic results for real power series, with the same proofs.

Theorem 24

Suppose $\sum_{n=0}^{\infty} c_n z^n$ converges at $z_0 \neq 0$. Then it converges absolutely at every z which satisfies $|z| < |z_0|$.

Theorem 25

Consider a complex power series $\sum c_n(z-a)^n$. One of the following cases will occur.

1. *The series converges only if $z = a$.*
2. *The series converges absolutely for every $z \in \mathbb{C}$.*
3. *There is $R \in \mathbb{R}$ such that the series converges absolutely for $|z-a| < R$ and diverges for $|z-a| > R$.*

Disc of Convergence

The **radius of convergence** R determines a **disc of convergence** inside which the series converges absolutely at every point. The bounding circle of this disc becomes an object of special interest, as the series may converge on some points of this circle and diverge on others.

Complex power series give a convenient procedure for extending real functions to the complex plane. We simply take the power series of a real function and convert the real variable to a complex one. For example, we define

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots ,$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots ,$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots .$$

Identities

Task 19

Use the Ratio or Root Test to prove that the power series that define the complex exponential, sine and cosine functions converge absolutely for every complex z*.*

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$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \ \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

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$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \ \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Proof. We have

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} \cdots,$$

$$e^{-iz} = 1 - iz - \frac{z^2}{2!} + i\frac{z^3}{3!} + \frac{z^4}{4!} - i\frac{z^5}{5!} \cdots.$$

Adding and subtracting these expressions gives the result. □

Properties of Exponential

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Show that $e^{iz} = \cos z + i \sin z$*. In particular, we have* **Euler's identity***:* $e^{i\pi} = -1$*.*

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Theorem 27

For any $z, w \in \mathbb{C}$, $e^{z+w} = e^z e^w$.

Proof.

$$e^z e^w = \lim_{m\to\infty} \sum_{n=0}^{m} \frac{z^n}{n!} \sum_{n=0}^{m} \frac{w^n}{n!} = \lim_{m\to\infty} \sum_{n=0}^{2m} \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}$$

$$= \lim_{m\to\infty} \sum_{n=0}^{2m} \sum_{k=0}^{n} \binom{n}{k} \frac{z^k w^{n-k}}{n!} = \lim_{m\to\infty} \sum_{n=0}^{2m} \frac{(z+w)^n}{n!} = e^{z+w}.$$

Properties of Exponential

We can now express the exponential function in terms of the real and imaginary parts of z:

$$z = x + iy \implies e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

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Task 22

Prove the following properties of the exponential function.

1. *For every $z \in \mathbb{C}$, $e^z \neq 0$ and $(e^z)^{-1} = e^{-z}$.*
2. *Every non-zero $z \in \mathbb{C}$ can be expressed as $z = e^w$ for some $w \in \mathbb{C}$.*
3. *We have $e^z = 1$ if and only if $z = 2n\pi i$, with $n \in \mathbb{Z}$. Hence $e^z = e^w$ if and only if $w - z \in 2\pi i\mathbb{Z}$.*

Example

Consider the real power series expansion $\frac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots$.

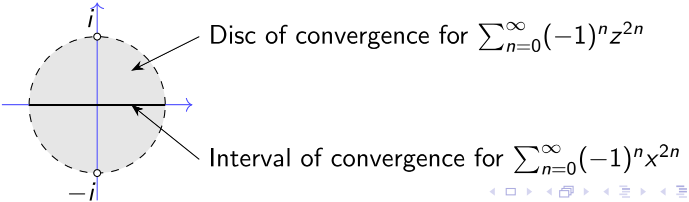
It has $R = 1$. The function itself is smooth on all of $\mathbb{R}$ so it seems odd that the power series representation breaks down at ± 1.

Now consider the complex version $\frac{1}{1+z^2} = 1 - z^2 + z^4 - \cdots$.

The function $1/(1+z^2)$ is not defined at $\pm i$. In fact, as z varies along the imaginary axis, the function takes the following appearance:

$$\frac{1}{1+(it)^2} = \frac{1}{1-t^2}.$$

It takes values which are arbitrarily large in magnitude as $t \to \pm 1$. Hence its power series expansion *must* break down beyond a radius of 1.



Complex Fourier Series

Consider $f\colon \mathbb{R} \to \mathbb{C}$. We express it as $f(x) = u(x) + iv(x)$ with $u, v\colon \mathbb{R} \to \mathbb{C}$. In future, we will just say 'Let $f = u + iv$'.

We say f has period T if $f(x + T) = f(x)$ for every x. This is equivalent to both u, v having period T.

Further, we call f **integrable** on $[a, b]$ if u, v are integrable there, and we define

$$\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx.$$

We shall work with functions that have period 2π. We only need to understand these over the interval $[-\pi, \pi]$. The role of the integral $\int_a^b f(x)g(x)\,dx$ in the real case will be assumed by $\int_a^b f(x)\overline{g(x)}\,dx$ in the complex case.

Trigonometric Polynomials

A trigonometric polynomial $T(x)$ can be expressed in terms of exponentials:

$$\begin{aligned} T(x) &= \frac{a_0}{2} + \sum_{n=1}^{M} a_n \cos nx + \sum_{n=1}^{M} b_n \sin nx \\ &= \frac{a_0}{2} + \sum_{n=1}^{M} a_n \frac{e^{inx} + e^{-inx}}{2} - i \sum_{n=1}^{M} b_n \frac{e^{inx} - e^{-inx}}{2} \\ &= \frac{a_0}{2} + \sum_{n=1}^{M} \frac{a_n - ib_n}{2} e^{inx} + \sum_{n=1}^{M} \frac{a_n + ib_n}{2} e^{-inx} \\ &= \sum_{n=-M}^{M} c_n e_n(x). \end{aligned}$$

Similarly, a trigonometric series can be expressed as

$$\sum_{n=-\infty}^{\infty} c_n e_n(x) = \lim_{M\to\infty} \sum_{n=-M}^{M} c_n e_n(x).$$

Orthogonality Relations

For $n \in \mathbb{Z}$, define $e_n(x) = e^{inx}$. Then

$$\int_{-\pi}^{\pi} e_n(x)\,dx = \int_{-\pi}^{\pi} \cos nx\,dx + i\int_{-\pi}^{\pi} \sin nx\,dx = \begin{cases} 0 & \text{if } n \neq 0, \\ 2\pi & \text{if } n = 0. \end{cases}$$

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This leads to the following calculation:

$$\int_{-\pi}^{\pi} e_n(x)\overline{e_m(x)}\,dx = \int_{-\pi}^{\pi} e_{n-m}(x)\,dx = \begin{cases} 0 & \text{if } n \neq m, \\ 2\pi & \text{if } n = m. \end{cases}$$

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We can use integration to extract the coefficients of a trigonometric polynomial:

$$\begin{aligned} \int_{-\pi}^{\pi} T(x)\overline{e_m(x)}\,dx &= \int_{-\pi}^{\pi} \sum_{n=-M}^{M} c_n e_n(x)\overline{e_m(x)}\,dx \\ &= \sum_{n=-M}^{M} c_n \int_{-\pi}^{\pi} e_n(x)\overline{e_m(x)}\,dx \\ &= 2\pi c_m. \end{aligned}$$

Fourier Series

Given an integrable function f with period 2π we try to express it as a trigonometric series by first defining its **Fourier coefficients**:

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{e_n(x)}\, dx, \qquad (n \in \mathbb{Z})$$

and then its **Fourier series**

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e_n(x).$$

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For $f, g \in \mathcal{I}$ we define $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$ and $||f||^2 = \langle f, f\rangle$. Note that $\hat{f}(n) = \langle f, e_n\rangle$.

Fourier Series

Task 23

Show that $\langle f, g \rangle = 0 \implies ||f+g||^2 = ||f||^2 + ||g||^2.$

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Complex numbers reveal a connection between trigonometric series and power series. Consider $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and let us describe z as $re^{i\theta}$. If we fix θ and vary r we get a power series in the real variable r,

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$$f^{\theta}(r) = \sum_{n=0}^{\infty} (c_n e^{in\theta}) r^n.$$

On the other hand, if we fix r and vary θ we get a trigonometric series,

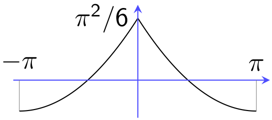
$$f_r(\theta) = \sum_{n=0}^{\infty} (c_n r^n) e^{in\theta}.$$

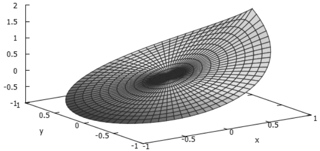
Example

Consider $f(z) = \sum_{n=1}^{\infty} z^n/n^2$. It has $R = 1$ and converges absolutely at each point of the unit circle. We look at the values on the unit circle:

$$f_1(\theta) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n^2} = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} + i \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}.$$

The real part of this series is the Fourier series of the even function given by $g(\theta) = \frac{1}{4}(\theta - \pi)^2 - \pi^2/12$ for $0 \le \theta \le \pi$. This function is not differentiable at $\theta = 0$. Below, we see the graph of g as well as that of the real part of $f(z)$, showing the kink corresponding to $r = 1$ and $\theta = 0$.





This lack of differentiability prevents the series from converging beyond the unit circle.