Solutions to exercises in chapter 2

1. **Error propagation**

   a) The surface area of a sphere is: $A = 4\pi r^2 = \pi d^2$. Therefore the error in the surface is: $\sigma_A = \frac{\partial A}{\partial d}\sigma_d = 2\pi d\sigma_d$, or in terms of relative errors: $r_A = 2r_d$. Numerically: $A = 100\pi\mu m^2 = \pi \cdot 10^{-10}m^2$ and $\sigma_A = \pi 10\mu m^2$, or $r_A = 0.1$.

   b) The volume of a sphere is: $V = 4\pi r^3/3 = \pi d^3/6$. Therefore the volumes error is: $\sigma_V = \frac{\partial V}{\partial d}\sigma_d = \pi d^2\sigma_d/2$, or in terms of relative errors: $r_V = 3r_d$. Numerically: $V = 1000\pi/6\mu m^3 = \pi/6 \cdot 10^{-15}m^3$ and $\sigma_V = 250 \cdot \pi\mu m^3$, or $r_V = 0.15$.

2. **Absolute and relative errors**

   a) Relative error of pressure: $r_p = \frac{\sigma_p}{p} = \frac{1Pa}{1000Pa} = 10^{-3}$

   b) Error in pressure difference: $\sigma_{\Delta p} = \sqrt{\sigma_{p1}^2 + \sigma_{p2}^2} = \sqrt{1Pa^2 + 1Pa^2} = \sqrt{2}Pa \simeq 1.4Pa$

   c) Relative error of pressure difference: $r_{\Delta p} = \frac{\sigma_{\Delta p}}{\Delta p} = \frac{1.4Pa}{10Pa} = 0.14$

3. **Data analysis**

   (a) On the one hand, there is the uncertainty in the distance travelled. This is certainly bigger than the ticks on the meter scale (which are typically cm), such that our initial estimate might be 0.01 m. However, if the plane does not fly perfectly straight, there is an additional uncertainty and will always increase the actually travelled distance. For a few degrees uncertainty in the direction, we obtain an uncertainty of a few %. Finally, there is an uncertainty that always decreases the distance travelled, namely the parallax that is obtained when observing the entire distance from its middle. This is of the same order as the directional uncertainty, such that we have uncertainties in both directions of a few %. Assuming 2%, we obtain $\sigma_{\Delta L} = 0.02 * L = 0.02 * 4.8m \simeq 0.1m$.

   For the timing uncertainty, we again could use the precision of the stopwatch, i.e. 0.01 s, however the uncertainty in the timing is given by the uncertainty of pressing the stopwatch. Note that this is not the reaction time of the person measuring the time, since that will be the same for the start and the stop, such that in the time difference this systematic uncertainty drops out. The uncertainty in pressing however (which corresponds to the fluctuation in reaction times) is roughly given by 0.1 s, i.e. $\sigma_{\Delta t} = 0.1s$.

   (b) The average is given by $\langle\Delta t\rangle = \frac{1}{N}\sum_i \Delta t_i$. We have a total of 15 measurements, i.e. N = 15 and inserting the different values Δt_i we obtain: $\langle\Delta t\rangle = (1.20 + 1.16 + 1.23 + 1.06 + 1.12 + 1.14 + 1.05 + 1.28 + 1.15 + 1.18 + 1.07 + 1.11 + 1.14 + 1.17 + 1.10)s/12 = 1.144s$.

   (c) The varaince of the average is given by $var = \frac{1}{N-1}\sum_i(\Delta t_i - \langle\Delta t\rangle)^2$. For the differences from the average, we have (in seconds): 0.056; 0.016; 0.086; -0.084; -0.024; -0.004; -0.094; 0.136; 0.006; 0.036; -0.074; -0.034; -0.004; 0.026; -0.044. Therfore the variance is: $var = (0.056^2 + 0.016^2 + 0.086^2 + 0.084^2 + 0.024^2 + 0.004^2 + 0.094^2 + 0.136^2 + 0.006^2 + 0.036^2 + 0.074^2 + 0.034^2 + 0.004^2 + 0.026^2 + 0.044^2)/14s^2 = 0.0564/14s^2 = 0.004s^2$. The standard deviation (and hence the estimate for the error of a single time measurement) is then $std = 0.07$ s, which is not too far away from the estimate we have given above. The actual error of the average is finally given by $\sigma_{\Delta t} = std/\sqrt{N} = 0.02s$.

   (d) So our result is $\Delta x = 4.8 \pm 0.1m \simeq 4.8(1)m$ and $\Delta t = 1.14 \pm 0.02s = 1.14(2)s$. For the speed we need to add the relative errors in squares, i.e. $\frac{\sigma_v^2}{v^2} = \frac{\sigma_{\Delta t}^2}{\Delta t^2} + \frac{\sigma_{\Delta x}^2}{\Delta x^2}$, which numerically gives: $\frac{\sigma_v^2}{v^2} = \frac{0.02^2}{1.14^2} + \frac{0.1^2}{4.8^2} = \frac{0.0004}{1.3} + \frac{0.01}{23} = 0.00074$. The relative error in the speed thus is $\sigma_v = 0.03 * v$, where $v = \Delta x/\Delta t = 4.8/1.14m/s = 4.2m/s$, hence $\sigma_v = 0.1m/s$. The final result therefore is: $v = 4.2(1)m/s$.

4. **Absolute and relative errors 2**

   (a) The error of the 250 μl is given by $0.01 \cdot 250\mu$l = 2.5 μl. For the minimum pipette amount of 25 μl we have a relative error of 5%, or in other words, the error of these 25 μl = $0.05 \cdot 25\mu$l =

1.25 μl. Therefore, the error of the total amount is given by: $\sigma_{tot}^2 = (2.5^2 + 1.25^2)\mu l^2 = 7.81\mu l^2$ or $\sigma_{tot} = 2.9\mu$l

(b) The error of the 150 μl is $0.01 \cdot 150\mu$l $= 1.5$ μl. The same calculation can be done for the 125 μl, i.e. $0.01 \cdot 125\mu$l $= 1.25$ μl. Therefore, the error of the total amount is given by: $\sigma_{tot}^2 = (1.5^2 + 1.25^2)\mu l^2 = 3.81\mu l^2$ or $\sigma_{tot} = 1.9\mu$l

(c) The error of the 5 μl is 0.01 5 μl $= 0.05$ μl. This is the same every time of the 5 times the amount is pipetted. Therefore, the error of the total amount is given by: $\sigma_{tot}^2 = 55 \cdot 0.05^2\mu l^2 = 0.14\mu l^2$ or $\sigma_{tot} = 0.37\mu$l

5. **Error propagation 2**

a) We know from Gaussian error propagation that $\sigma_c^2 = \left(\frac{\partial c}{\partial \lambda}\right)^2 \sigma_\lambda^2 + \left(\frac{\partial c}{\partial c_0}\right)^2 \sigma_{c_0}^2$. Using the specific function for the concentration $c(x) = c_0 exp(-x/\lambda)$, we obtain:

$\frac{\partial c}{\partial \lambda} = c_0 exp(-x/\lambda)\frac{x}{\lambda^2} = \frac{c(x)x}{\lambda^2}$

$\frac{\partial c}{\partial c_0} = exp(-x/\lambda) = \frac{c(x)}{c_0}$

Hence: $\sigma_c^2 = c(x)^2 \left(\frac{\sigma_{c_0}^2}{c_0^2} + \frac{x^2}{\lambda^2}\frac{\sigma_\lambda^2}{\lambda^2}\right)$. We therefore obtain for the relative error: $\frac{\sigma_c^2}{c(x)^2} = \left(\frac{\sigma_{c_0}^2}{c_0^2} + \frac{x^2}{\lambda^2}\frac{\sigma_\lambda^2}{\lambda^2}\right)$.

Using numerical values we obtain: $\frac{\sigma_{c_0}^2}{c_0^2} = 0.05^2 = 0.0025$, $\frac{\sigma_\lambda^2}{\lambda^2} = (20/120)^2 = 1/36 = 0.0277$ and $\frac{x^2}{\lambda^2} = (200/120)^2 = 100/36 = 2.777$, which yields: $\frac{\sigma_c^2}{c(x)^2} = 0.0025 + 2.777 * 0.02777 = 0.077$. and finally: $\frac{\sigma_c}{c} = 0.277$

b) Solving the equation $c(x_T) = c_T$ yields: $ln(c_T/c_0) = -x_T/\lambda$, which we can transform into: $x_T = \lambda ln(c_0/c_T)$. Numerically, this yields a value of $x_T = 120\mu$m $ln(5) \simeq 200\mu$m

Error propagation: $\sigma_{x_T}^2 = \left(\frac{\partial x_T}{\partial \lambda}\right)^2 \sigma_\lambda^2 + \left(\frac{\partial x_T}{\partial c_0}\right)^2 \sigma_{c_0}^2$.

$\frac{\partial x_T}{\partial \lambda} = ln(c_0/c_T) = \frac{x_T}{\lambda}$

$\frac{\partial x_T}{\partial c_0} = \lambda \frac{c_T}{c_0}\frac{1}{c_T} = \frac{\lambda}{c_0}$

This yields: $\sigma_{x_T}^2 = \frac{x_T^2}{\lambda^2}\sigma_\lambda^2 + \frac{\lambda^2}{c_0^2}\sigma_{c_0}^2 = \frac{\sigma_\lambda^2}{\lambda^2}x_T^2 + \frac{\sigma_{c_0}^2}{c_0^2}\lambda^2$.

Numerically: $\sigma_{x_T}^2 = (200^2 \times 1/36 + 120^2 \times 0.0025)\mu m^2 = (33.3^2 + 6^2)\mu m^2 \simeq 33.8^2\mu m^2$. Also ist $x_T = 200 \pm 34\mu m$

c) To take into account an uncertainty in the threshold, we need an extra term in the error propagation: $\frac{\partial x_T}{\partial c_T}$. Hence, we obtain:

$\sigma_{x_T}^2 = \left(\frac{\partial x_T}{\partial \lambda}\right)^2 \sigma_\lambda^2 + \left(\frac{\partial x_T}{\partial c_0}\right)^2 \sigma_{c_0}^2 + \left(\frac{\partial x_T}{\partial c_T}\right)^2 \sigma_{c_T}^2$.

where $\frac{\partial x_T}{\partial c_T} = \lambda \frac{c_T}{c_0}\frac{c_0}{c_T^2} = \frac{\lambda}{c_T}$

This yields: $\sigma_{x_T}^2 = \frac{x_T^2}{\lambda^2}\sigma_\lambda^2 + \frac{\lambda^2}{c_0^2}\sigma_{c_0}^2 + \frac{\lambda^2}{c_T^2}\sigma_{c_T}^2 = \frac{\sigma_\lambda^2}{\lambda^2}x_T^2 + \left(\frac{\sigma_{c_0}^2}{c_0^2} + \frac{\sigma_{c_T}^2}{c_T^2}\right)\lambda^2$.

d)If we normalize concentrations by the initial concentration, an uncertainty in c_0 becomes reflected in an uncertainty in c_T, since the ratio c_0/c_T is decisive in the determination of x_T. The uncertainty in concentration is σ_c and for normalized concentrations, this corresponds to $\sigma_{c_0} = 0$ and $\sigma_{c_T} = \sigma_c$. If we use this in the result of part c) using $\sigma_c = 0.05c_0$, we obtain:

$\sigma_{x_T}^2 = \frac{x_T^2}{\lambda^2}\sigma_\lambda^2 + \frac{\lambda^2}{c_T^2}\sigma_c^2 = \frac{\sigma_\lambda^2}{\lambda^2}x_T^2 + \frac{\sigma_c^2}{c_T^2}\lambda^2$.

Or numerically: $\sigma_{x_T}^2 = (200^2 \times 1/36 + 120^2 \times 0.0025 \times 25)\mu m^2 = (33.3^2 + 30^2)\mu m^2 \simeq 45^2\mu m^2$. The threshold position including its uncertainty would thus be given by $x_T = 200 \pm 45\mu m$, which is a much larger uncertainty than is biologically observed.

6. **Dimensional analysis**

a) Diffusivity times mobility has the units of m^2/s × N s / m = N × m = J, thus corresponds to an energy. The physical property corresponding the the product of diffusivity times mobility thus has to correspond to an energy, which we will se in chapter 8 is equal to the thermal energy $k_B T$. This relation is also known as the Einstein relation.

b) We are looking for a force (air resistance) (F, Newton) given by viscosity (η, Pa s), speed (v, m/s) and area (A, m^2). Our Ansatz is: $F = \eta^a v^b A^c$, which yields an equation in units: N = Pa^a s^{a-b} m^{b+2c}. Using the relations for N and Pa in terms of basic units, we obtain: kg m s^{-2} = kg^a s^{-b-a} m^{b+2c-a}, which gives three separate equations (corresponding to the three different base units) for the constants a,b, and c:

(i) a=1;

(ii) b+a=2;

(iii) b+2c-a=1

Inserting (i) in (ii) gives b=1. Inserting this and (i) in (iii) gives c=1/2, which gives the final result: $F \propto \eta v A^{0.5}$. This is also known as Stokes friction.

7. **Dimensional analysis 2**

a) Dimensional analysis uses the Ansatz: $[E] = [R]^a \cdot [T]^b \cdot [\rho]^c$, thus:

$kgm^2/s^2 = m^a \cdot s^b \cdot kg^c/m^{3c}$

This gives equation for the kg giving: $c = 1$, for the s giving $b = -2$ and for the m giving: $2 = a - 3c = a - 3$ or $a = 5$ after inserting the result for the kg. We therefore obtain for the energy:

$E = \rho R^5/T^2$

b) We read the diameter of the blast from the image using the scale for 100 m, image 1: 105 m and image 2: 128 m. Using a) with a dimensionless prefactor equal to one, we obtain:

$$E_1 = \frac{1kg/m^3(105)^5 m^5}{0.016^2 s^2} = \frac{(1.05)^5 \cdot 10^{10}}{(1.6)^2 \cdot 10^{-4}} J = \frac{1.05^5}{1.6^2} 10^{14} J = 5 \cdot 10^{13} J.$$

$$E_2 = \frac{1kg/m^3(128)^5 m^5}{0.025^2 s^2} = \frac{(1.28)^5 \cdot 10^{10}}{(2.5)^2 \cdot 10^{-4}} J = \frac{1.28^5}{2.5^2} 10^{14} J = 5.5 \cdot 10^{13} J.$$

c) Error propagation using relative errors yields: $r_E^2 = r_\rho^2 + 25 r_R^2 + 4 r_T^2$.

d) The error in reading off distances from the figure is roughly 0.5 mm, which is valid for the diameter as well as the scale-bar. Therefore, the relative error in R is: $r_R = \sqrt{r_D^2 + r_S^2} \simeq 0.03$. Given the time stamp, we can estimate a time uncertainty of 1 ms, such that $r_t \simeq 0.05$. From the problem setting we get the error in the density of the air as $r_\rho = 0.1$. Error propagation from c) therefore gives $r_E = \sqrt{100 + 25 \cdot 9 + 4 \cdot 25}\% = \sqrt{425}\% \simeq 20\%$ or $\sigma_E = 10^{13} J$.

e) The two results from b) therefore are in agreement with each other within the uncertainty. Similarly, a web-search gives that the strength of the explosion in Alamogordo is typically estimated as $E = 7 \cdot 10^{13}$ J or 18 kt TNT. This is surprisingly close to the estimate and only two standard errors away, hence still compatible with our simple estimate. This is because the pre-factor for the energy of a shock wave is in fact very close to one, which however cannot be obtained from dimensional analysis. Experimentally this could be done using a process that we can image, where we know the energy involved, such as the shockwave of a busting balloon filled with water, where high speed movies can be easily found on youtube. With this we could in fact determine the prefactor and then obtain an even better estimate.

8. **Power laws and logarithmic scales**

The slope of the straight line is given by Δ ln R / Δ ln M, which we can numerically determine to be 0.36(2). This slope is compatible with a dependence of the size on the mass as R ∝ M$^{1/3}$. This means that the mass grows like a volume, which is sensible for an object of constant density.

9. **Power laws and logarithmic scales 2**

   a) The slope of the red line is $a = 5/4$. This means that there is a scaling law of the form: $d \propto L^{5/4}$.

   b) For the uncertainty, we determine two different slopes. One from the lowest small value to the largest big value and a second from the largest small value to the lowest big value. The real result must be between these two values and the distance to the average thus determines the uncertainty interval. The two slopes are: $Min = 3/(log(1500) - log(3.5)) \simeq 1.14$; $Max = 3/(log(1000) - log(5.5)) \simeq 1.33$. So the average exponent is $a = 1.24(9)$.

   c) If we only had to look at an absolute error, this would be important only for the small values. Therefore, the slope (and its uncertainty) is mostly determined by the big values, meaning that the two slopes have to go through the same points in the upper right of the figure. This means that the uncertainty of the value we determine will be smaller, as will the average slope.

   If the relative errors for the measurements are roughly constant, the errors of all points on a logarithmic scale are equally sized. Therefore, we have to treat all point equally when laying different straight lines. This basically gives the treatment of part b) above.

   A roughly constant relative error is more probable, since the shape of the bone and where the circumference is measured determine the systematic error, which will be the determining source of error. these systematic errors increase in accordance with the size of the bone, such that the relative errors are roughly constant.

10. **Power laws and logarithmic scales 3**

   a) The slope is given by $a = \frac{\log(10^1)-\log(10^{-2})}{\log(10^0)-\log(10^9)} = \frac{1-(-2)}{0-9} = -\frac{1}{3}$.

   b) Maximum slope: $a = \frac{\log(2.5\cdot10^1)-\log(7\cdot10^{-3})}{\log(10^0)-\log(10^9)} = \frac{1.4-(-2.15)}{0-9} = -\frac{3.55}{9} = -0.39$

   minimum slope: $a = \frac{\log(0.7\cdot10^1)-\log(2\cdot10^{-2})}{\log(10^0)-\log(10^9)} = \frac{0.85-(-1.7)}{0-9} = -\frac{2.55}{9} = -0.28$

   So the average exponent is $a = -0.33(5)$

   c) However, here values with large N have a much smaller relative error. Therefore, we have to determine the slopes differently with bigger weight given to those values. This gives:

   Maximum slope: $a = \frac{\log(3\cdot10^1)-\log(10^{-2})}{\log(10^0)-\log(10^9} = \frac{1.48-(-2)}{0-9} = -\frac{3.48}{9} = -0.385$

   minimum slope: $a = \frac{\log(0.7\cdot10^1)-\log(10^{-2})}{\log(10^0)-\log(10^9} = \frac{0.85-(-2)}{0-9} = -\frac{2.85}{9} = -0.32$

   So the average exponent is $a = -0.35(3)$

11. **Scaling laws and the Body Mass Index**

   (a) If fish have the same density (are made from the same material), their mass is given by their volume. Since the unit of a volume is m^3 and the length has units of m, we expect a scaling law of $M \propto L^3$. If you know this law (and its pre-factor) you do not need a scale on the fish market, but can simple measure the length of the fish to obtain its weight. The pre-factor actually is given by the density (roughly 10^3 kg/m^3) as well as a number describing the geometric shape of an average fish.

   (b) The same argument as above would give a scaling law of the form $M \propto L^3$. Apart from the density, the pre-factor is a simple number that characterizes the shape of a body. For instance as sphere would have a pre-factor of $\pi/6$, whereas it would be $\pi/25$ for a cylinder with a radius of one fifth of its length or $\pi/16$ with a radius of a quarter of the length. The pre-factor for humans is typically between 0.13 (=$\pi/25$) and 0.16 (=$\pi/20$).

   (c) If the BMI is a good measure for the average population, this means that everybody should have roughly the same value of M/L^2. This means that the pre-factor that we have looked at in b) is dependent on the size of the person. To be exact, it decreases as $1/L$. This means that short people should be sturdier, whereas tall people should be skinnier on average.

   (d) We determine the slope of the red line, which corresponds to the linear part of the curve (masses below about 60 kg). To get an estimate for the uncertainty, we actually look at two different straight lines, with maximum and minimum possible slope still compatible with the data. For the factor of ten increase between 10 and 100 kg, the lines cover the length interval between 80 to 180 cm and 65 to 200 cm respectively. Thus the slopes will be $\log(200/65) = 0.48$ and $\log(180/80) = 0.35$ respectively. Thus our estimate for the exponent becomes $\beta = 0.42(7)$. This is actually the exponent for the power law $L \propto M^{\beta}$. In terms of the normal version of the BMI, we have to look for $\alpha = 1/\beta = 2.4(4)$. The empirical value is therefore a little bit closer to that used for the BMI rather than the geometrically expected exponent. However also the usual exponent is only marginally fitting the data.