CHAPTER 7

EXERCISES FOR VECTORS, MATRICES, AND GEOMETRY

Exercise 7.1 (Linear combination of random vectors)**.** Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_K$ be independent M-dimensional random vectors. However, the vectors are not drawn from the same population. Instead, the populations have individual means $\mathbb{E}[\mathbf{x}_k] = \boldsymbol{\mu}_k$ and covariance matrices $\text{cov}[\mathbf{x}_k] = \boldsymbol{\Sigma}_k$. Because the vectors are independent,

$$\text{cov}[\mathbf{x}_k, \mathbf{x}_{k'}^T] = \begin{cases} \mathbf{0} & k \neq k' \\ \boldsymbol{\Sigma}_k & k = k' \end{cases}. \tag{7.1}$$

Note that there are K distinct covariance matrices $\boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2, \ldots, \boldsymbol{\Sigma}_K$. Suppose $\mathbf{y} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_N\mathbf{x}_N$, where the c's are known scalars. What is the mean and covariance matrix of $\mathbf{y}$? □

Exercise 7.2 (Matrix combinations of random vectors)**.** Let $\mathbf{x} \sim \mathrm{N}_M(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Suppose $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is a constant matrix and $\mathbf{b}$ is a constant vector. What is the complete distribution of $\mathbf{y}$? □

Exercise 7.3 (Properties of the Trace and Determinant)**.** Let $\mathbf{A}$ be a square matrix, but not necessarily symmetric. Square matrices still have eigenvectors and eigenvalues that satisfy

$$\mathbf{A}\mathbf{s} = \lambda\mathbf{s}. \tag{7.2}$$

As a result, the matrix is close enough for practical purposes to having an eigen decomposition of the form

$$\mathbf{A} = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1}, \tag{7.3}$$

in which case A is said to diagonalizable and the properties below are easy to prove. If $\mathbf{A}$ is symmetric, then we know that there exists a decomposition of the above form where $\mathbf{S}$ is an orthogonal matrix and $\mathbf{\Lambda}$ has real diagonal elements, but these special properties are not needed in the following problems.

(a) Show that $\mathrm{tr}\,\mathbf{A} = \mathrm{tr}\,\mathbf{A}^T$ for any matrix $\mathbf{A}$.

(b) Show that $\mathrm{tr}\{\mathbf{AB}\} = \mathrm{tr}\{\mathbf{A}^T\mathbf{B}^T\}$ for any two matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathrm{tr}\{\mathbf{AB}\}$ is defined.

(c) Show that $\mathrm{tr}\,\mathbf{A} = \sum \lambda_i$, where the λ_i's are the eigenvalues of $\mathbf{A}$, and the sum is over all eigenvalues.

(d) What is $\mathrm{tr}\{\mathbf{A}^k\}$ in terms of the eigenvalues of $\mathbf{A}$?

(e) Show that $\mathrm{tr}[\mathbf{A}^{-1}] = \sum_{m=1}^{M} \lambda_m^{-1}$, where the λ_m's are the eigenvalues of $\mathbf{A}$.

(f) Show that $\mathrm{tr}\{\mathbf{AA}^T\} = \sum_i \sum_j a_{ij}^2$, where a_{ij} is the (i, j) element of $\mathbf{A}$.

(g) Show that $|\mathbf{A}| = \lambda_1\lambda_2 \dots \lambda_p$, where $\mathbf{A}$ is a $p \times p$ matrix and the λ_i's are the eigenvalues of $\mathbf{A}$.

(h) If $\mathbf{A}$ is symmetric and positive definite, show that $\mathrm{tr}\,\mathbf{A} > 0$ and $|\mathbf{A}| > 0$.

(i) Show that the trace of a positive semi-definite matrix is non-negative. Show that the determinant of a positive semi-definite matrix is non-negative.

□

Exercise 7.4 (Idempotent matrices)**.** An *idempotent* matrix is a matrix $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{A}$. Show that the eigenvalues of an idempotent matrix can have only two values, namely 0 and 1. Show that the number of eigenvalues equal to 1 is given by $\mathrm{tr}[\mathbf{A}]$. □

Exercise 7.5 (Properties of Orthogonal Matrices)**.**

(a) Show that orthogonal transformations preserve the Euclidean length of a vector. That is, if $\boldsymbol{\xi} = \mathbf{Ux}$, where $\mathbf{U}$ is an orthogonal matrix, then show that $\boldsymbol{\xi}^T\boldsymbol{\xi} = \mathbf{x}^T\mathbf{x}$.

(b) Show that orthogonal transformations preserve the cosine-angle between two vectors. That is, show that the cosine-angle between $\mathbf{x}$ and $\mathbf{y}$, and the cosine-angle between $\boldsymbol{\xi} = \mathbf{Ux}$ and $\boldsymbol{\eta} = \mathbf{Uy}$, are equal. The cosine-angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\cos\theta = \frac{\mathbf{x}^T\mathbf{y}}{\sqrt{(\mathbf{x}^T\mathbf{x})(\mathbf{y}^T\mathbf{y})}}. \tag{7.4}$$

(c) Show that the determinant of an orthogonal matrix is $+1$ or -1.

(d) Show that $|\mathbf{UAU}^T| = |\mathbf{A}|$, where $\mathbf{U}$ is an orthogonal matrix and $\mathbf{A}$ is any matrix.

(e) Show that $\mathrm{tr}\{\mathbf{UAU}^T\} = \mathrm{tr}\{\mathbf{A}\}$, where $\mathbf{U}$ is orthogonal and $\mathbf{A}$ is a square matrix.

(f) If $\mathbf{U}_1$ and $\mathbf{U}_2$ are orthogonal matrices, show that $\mathbf{U}_1\mathbf{U}_2$ is also orthogonal. Hint: remember that orthogonal matrices satisfy *two* properties.

□

Exercise 7.6. Suppose $\mathbf{A}$ is a constant matrix and $\mathbf{y}$ is a random vector with mean $\boldsymbol{\mu}_Y$ and covariance matrix $\boldsymbol{\Sigma}_Y$. Show that the expected value of the quadratic form $\mathbf{y}^T\mathbf{A}\mathbf{y}$ is $\mathrm{tr}[\mathbf{A}\boldsymbol{\Sigma}_Y] + \boldsymbol{\mu}^T\mathbf{A}\boldsymbol{\mu}$. □

Exercise 7.7. Show that the inverse of a transpose equals the transpose of the inverse:

$$\left(\mathbf{A}^T\right)^{-1} = \left(\mathbf{A}^{-1}\right)^T. \tag{7.5}$$

Hint: start with $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. □

Exercise 7.8 (Frobenius Norm)**.** The "length" of a vector is defined as the square root of the sum square vector elements. A natural generalization of "length" to a matrix $\mathbf{X}$ is

$$\|\mathbf{X}\|_F = \sqrt{\sum_i \sum_j x_{ij}^2}. \tag{7.6}$$

This measure is called the *Frobenius Norm*. It is clear that the Frobenius norm vanishes if and only if $\mathbf{X} = \mathbf{0}$, consistent with a measure of "length."

(a) Show that the Frobenius Norm can be written equivalently as

$$\|\mathbf{X}\|_F^2 = \mathrm{tr}\left[\mathbf{X}\mathbf{X}^T\right]. \tag{7.7}$$

(b) Suppose $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices with dimensions such that the product $\mathbf{U}\mathbf{A}\mathbf{V}$ is defined. Show that

$$\|\mathbf{U}\mathbf{A}\mathbf{V}\|_F = \|\mathbf{A}\|_F\,. \tag{7.8}$$

□

Exercise 7.9. Consider a $M \times M$ matrix $\mathbf{A}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_M$. Show that the eigenvalues of $a\mathbf{I} + b\mathbf{A}$ are $a + b\lambda_m$, $m = 1, 2, \ldots, M$. □

Exercise 7.10. If $\mathbf{A}$ is positive definite, show that *each* diagonal element of $\mathbf{A}$ must be positive. □

Exercise 7.11 (Distribution of a random quadratic form)**.** Suppose $\mathbf{x} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma}$ is positive definite. In this problem you will determine the distribution of the quadratic form

$$Q = (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\,. \tag{7.9}$$

(a) Let the eigenvector decomposition of the covariance matrix be

$$\boldsymbol{\Sigma} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^T, \tag{7.10}$$

where $\mathbf{U}$ is an orthogonal matrix, and $\boldsymbol{\Lambda}$ is the diagonal eigenvalue matrix. What is the eigenvector decomposition of $\boldsymbol{\Sigma}^{-1}$?

(b) Using the results from part (a), construct a transformation matrix $\mathbf{A}$ such that

$$\boldsymbol{\zeta} = \mathbf{A}(\mathbf{x} - \boldsymbol{\mu}) \quad \text{and} \quad \boldsymbol{\zeta}^T\boldsymbol{\zeta} = Q\,. \tag{7.11}$$

What is $\mathbf{A}$?

(c) What is the distribution of $\boldsymbol{\zeta}$?

(d) A theorem in statistics states that if two random variables are uncorrelated *and* distributed as a multivariate Gaussian, then they are independent. Use this theorem to show that ζ_i and ζ_j are independent if $i \neq j$.

(e) What is the distribution of Q?

$\square$