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Isentropic Efficiency in Engineering Thermodynamics

Introduction

This article is a summary of selected parts of chapters 4, 5 and 6 in the textbook by Moran and Shapiro (2008). The intent is primarily to clarify for geologists the nature of isentropic and isenthalpic processes as used in engineering, and the meaning of isentropic efficiency. Equation numbers are those in Moran and Shapiro, except for three equations in square brackets. We follow the common engineering practice of using lower case for thermodynamic properties, e.g., *h*, *u*, etc., as referring to specific (per gram) rather than molar quantities. The data used are from the NIST program REF-PROP (Lemmon et al., 2007), and are slightly different from the data used in Moran and Shapiro. In program REFPROP, the default reference states¹ are defined as having zero internal energy and zero entropy so that by definition delta notation is avoided, i.e., we can use *h*, *u* rather than Δh and Δu and similarly for other properties. Nevertheless, whatever the definitions and notation, numerical values of internal energy and all quantities containing it represent differences between two equilibrium states, real or hypothetical.

Moving fluids may have thermodynamic state properties (T, p, v, etc.) which do not change with time (steady state) so that at a fixed point in space (e.g., the inlet or outlet of a turbine) the moving fluid can be considered to be in a state of equilibrium. The fluid has kinetic and possibly gravitational energy as well as internal energy, but these can be included in the formulations. Or these variables may be changing with time, in which case the equations are differentiated with respect to time.

Equations for Control Volumes

Just as for a closed system, energy and mass can enter and leave a region in space (the control volume, CV), and energy transfer can be in the form of work and heat. But with the CV, another type of energy transfer occurs, the energy which accompanies mass transfer.

Conservation of Mass in a Control Volume

For each of the extensive properties, mass, energy, and entropy, the CV form of the property balance is obtained by transforming the corresponding closed system form. First, consider mass, which in a closed system is constant.

¹The reference state for water is the saturated liquid at the triple point, and for air is the saturated liquid at the normal boiling point.

Mass Rate balance

Consider that we have a CV with one inlet and one outlet. At time *t* the mass under consideration is the sum $m = m_{CV}(t) + m_i$, where $m_{CV}(t)$ is the mass in the CV and m_i is the mass in a small volume *i* at the inlet. In the time interval Δt the mass in region *i* enters the CV, and some of the mass m_e initially in the CV exits and occupies a small volume *e* at the outlet. The mass in regions *i*, CV, and *e* may differ from time *t* to time $t + \Delta t$, but the total mass is constant. So

$$m_{\rm CV}(t) + m_i = m_{\rm CV}(t + \Delta t) + m_e$$

or

$$m_{\rm CV}(t+\Delta t) - m_{\rm CV}(t) = m_i - m_e$$

Now divide by Δt ,

$$\frac{m_{\rm CV}(t+\Delta t) - m_{\rm CV}(t)}{\Delta t} = \frac{m_i}{\Delta t} - \frac{m_e}{\Delta t}$$
$$\frac{dm_{\rm CV}}{dt} = \dot{m}_i - \dot{m}_e \tag{4.1}$$

Then as $\Delta t \rightarrow 0$,

where dm_{CV}/dt is the time rate of change of mass in the CV, and \dot{m}_i and \dot{m}_e are the inlet and outlet mass flow rates, all at time *t*. If there is more than one inlet and outlet, summation signs are added to the right-hand terms.

Mass Flow Rate

Consider a small quantity of mass having velocity V and density ρ flowing across a small part of area A, dA, in time interval Δt . If V_n is the velocity normal to the area A, the mass that crosses dA in time Δt is $\rho(V_n\Delta t)dA$. Dividing by Δt and letting $\Delta t \rightarrow 0$ gives $\rho V_n dA$, the instantaneous mass flow rate across dA. Integrating over the area A through which mass passes gives

$$\dot{m} = \int_{A} \rho V_{n} dA \tag{4.3}$$

For flow in one dimension, which covers most cases, this becomes

$$\dot{m} = \rho \, \text{AV} \tag{4.4a}$$

or

$$\dot{m} = \frac{AV}{v} \tag{4.4b}$$

where *v* is the specific volume.

AV is the volumetric flow rate. Combining equations (4.1) and (4.4b),

$$\frac{dm_{\rm CV}}{dt} = \frac{A_i V_i}{v_i} - \frac{A_e V_e}{v_e}$$
(4.5)

For flow at steady state, $dm_{\rm CV}/dt = 0$, so

$$\dot{m}_i = \dot{m}_e \tag{4.6}$$

The identity of the matter in the CV changes continuously, but the amount present at any instant is constant. Note that steady state flow does not necessarily mean that the CV is at steady state. For a CV at steady state, *all* properties including T, p, etc. are constant.

Conservation of Energy in a Control Volume

Work and Heat

Energy is transferred to or from a system by work (*W*) and/or heat (*Q*). Work is done by a force moving through a distance (or its equivalent, as e.g. in the case of electrical work). Neither work nor heat is a property of the system (a state variable) so neither differential can be integrated without specifying a path. This is noted by using δ rather than *d* in the expression²

$$\int_1^2 \delta W = W$$

The rate of energy transfer by work is called *power*, denoted by \dot{W} , where in one dimension,

$$\dot{W} = FV \tag{2.13}$$

where F is force and V is velocity.

Similarly,

$$\int_{1}^{2} \delta Q = Q \tag{2.28}$$

The net *rate of energy transfer by heat* is \dot{Q} , and if it is known how \dot{Q} varies with time, then

$$Q = \int_{1}^{2} \dot{Q}dt \tag{2.29}$$

The net rate of energy transfer as heat is related to the *heat flux* \dot{q} , the rate of heat transfer per unit area, by

$$\dot{Q} = \int_{A} \dot{q} dA \tag{2.30}$$

Energy Rate Balance

The closed system energy balance is not the familiar $\Delta U = Q - W$, but

$$\Delta E = \Delta U + \Delta KE + \Delta PE \tag{2.35b}$$

$$=Q-W \tag{2.35a}$$

where KE and PE are the terms for kinetic and potential energy. The differential form is

$$dE = \delta Q - \delta W \tag{2.36}$$

²See section 4 in the Additional Material file for a discussion of this point.

and the instantaneous time rate form of the energy balance is

$$\frac{dE}{dt} = \dot{Q} - \dot{W} \tag{2.37}$$

or

$$\frac{dE}{dt} = \frac{dKE}{dt} + \frac{dPE}{dt} + \frac{dU}{dt}$$
$$= \dot{Q} - \dot{W}$$
(2.38)

For one inlet, one outlet, 1D flow, then

$$\frac{dE_{\rm CV}}{dt} = \dot{Q} - \dot{W} + \dot{m}_i \left(u_i + \frac{V_i^2}{2} + gz_i \right) - \dot{m}_e \left(u_e + \frac{V_e^2}{2} + gz_e \right)$$
(4.9)

where E_{CV} is the energy of the CV at time t, \dot{Q} and \dot{W} are the net rates of energy transfer as heat and work across the boundary of the CV at time t, u is specific internal energy, g is the acceleration due to gravity and z is the elevation of the CV. If there is no mass flow, the equation reduces to equation (2.37).

Evaluating Work for a CV

It is convenient to separate the net rate of energy transfer as work into or out of a CV (\dot{W}) into two parts. One is the rate of work done by the fluid pressure at the inlet and outlet as mass is transported in or out. The other, called \dot{W}_{CV} , is the rate of *all other work*, such as done by rotating shafts, electrical work, etc.

The rate of energy transfer by work is force \times velocity (equation (2.13)), so at the outlet, say,

$$\dot{W} = (p_e A_e) V_e \tag{4.10}$$

and similarly for the inlet, so the work rate term for equation (4.9) is

$$\dot{W} = \dot{W}_{\rm CV} + (p_e \mathbf{A}_e) \mathbf{V}_e - (p_i \mathbf{A}_i) \mathbf{V}_i \tag{4.11}$$

and because $AV = \dot{m}v$ (equation (4.4b))

$$\dot{W} = \dot{W}_{\rm CV} + \dot{m}_e(p_e v_e) - \dot{m}_i(p_i v_i) \tag{4.12}$$

The terms $\dot{m}_e(p_e v_e)$ and $\dot{m}_i(p_i v_i)$ account for the work associated with the pressure at the outlet and inlet, and are called *flow work*.

The Energy Rate Balance

Inserting this relation, equation (4.9) becomes

$$\frac{dE_{\rm CV}}{dt} = \dot{Q}_{\rm CV} - \dot{W}_{\rm CV} + \dot{m}_i \left(u_i + p_i v_i + \frac{V_i^2}{2} + gz_i \right) - \dot{m}_e \left(u_e + p_e v_e + \frac{V_e^2}{2} + gz_e \right)$$
(4.13)

Subscript CV is added to \dot{Q} to emphasize that this is the rate of heat transfer over the surface of the CV. And because h = u + pv where h is specific enthalpy, this becomes

$$\frac{dE_{\rm CV}}{dt} = \dot{Q}_{\rm CV} - \dot{W}_{\rm CV} + \dot{m}_i \left(h_i + \frac{V_i^2}{2} + gz_i \right) - \dot{m}_e \left(h_e + \frac{V_e^2}{2} + gz_e \right)$$
(4.14)

This is the master 1D, one inlet, one outlet form of the energy balance for a CV. It only remains to relate \dot{Q}_{CV} to entropy.

Steady State Form of the Energy Balance

When $\dot{m}_i = \dot{m}_e$ and $dm_{\rm CV}/dt = 0$, equation (4.14) becomes

$$0 = \frac{\dot{Q}_{CV}}{\dot{m}} - \frac{\dot{W}_{CV}}{\dot{m}} + (h_i - h_e) + \frac{V_i^2 - V_e^2}{2} + g(z_i - z_e)$$
(4.20b)

Nozzles and Diffusers

A nozzle is a tube of varying cross sectional area in which the fluid velocity increases in the direction of flow. In a diffuser, the velocity decreases in the direction of flow. In these, there is no work done other than *flow work*, and (as in a great many applications) change in potential energy is negligible. If in addition the heat loss is negligible, equation (4.20b) becomes

$$0 = (h_i - h_e) + \frac{\mathbf{V}_i^2 - \mathbf{V}_e^2}{2}$$
(4.21)

The Entropy Balance

The Entropy Balance for Closed Systems

The focus is on the *balance*, which means there is an explicit term σ representing the entropy difference between the real process and that process carried out reversibly, i.e., the amount of entropy produced in the system by *irreversibilities*. Thus

$$S_2 - S_1 = \int_1^2 \left(\frac{\delta Q}{T}\right)_b + \sigma \tag{6.24}$$

In words, this is

change in entropy in the sys- = amount of entropy *trans-* + entropy *produced* in the system during some time interval the time interval

and in differential form

$$dS = \left(\frac{\delta Q}{T}\right)_b + \delta\sigma \tag{6.25}$$

When there are no *internal irreversibilities*, equation (6.25) reduces to the *internally reversible* form

$$dS = \left(\frac{\delta Q}{T}\right)_{\text{int rev}} \tag{6.2b}$$

A distinction is made between internal irreversibilities, those taking place in the system, and external irreversibilities, those taking place in the environment. Engineering design thus focusses on identifying the sources of the irreversibilities and reducing them. Common sources are (p. 220):

- 1. Heat transfer due to a ΔT .
- 2. Unrestrained expansion of a fluid.
- 3. Spontaneous chemical reaction (including phase changes).
- 4. Spontaneous mixing.
- 5. Friction; sliding as well as within fluids.
- 6. Current flow through a resistance.
- 7. Magnetization or polarization with hysteresis.
- 8. Inelastic deformation.

All actual processes are irreversible, i.e., they contain irreversibilities and hence produce entropy.

Entropy Rate balance for Closed Systems

If temperature is constant, equation (6.24) becomes

$$S_2 - S_1 = \frac{Q}{T_b} + \sigma$$

where Q/T_b represents the amount of entropy transferred through a portion of the system boundary at temperature T_b . Similarly, \dot{Q}/T_j represents the *time rate* of entropy transfer through a portion of the boundary whose instantaneous temperature is T_j . The closed system entropy rate balance is then

$$\frac{dS}{dT} = \sum_{j} \frac{Q_j}{T_j} + \dot{\sigma}$$
(6.28)

time rate of change in en- =	(sum of) time rate of en- +	time rate of entropy produc-
tropy in the system	tropy transfer through the	tion due to irreversibilities in
	portion(s) of the boundary	the system
	whose temperature is T_i	

Entropy Rate Balance for Control Volumes

Entropy is extensive, so it can be transferred in or out of systems by streams of matter. So modifying equation (6.28) gives

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$$\frac{dS_{\rm CV}}{dt} = \sum_{j} \frac{Q_j}{T_j} + \sum_{i} \dot{m}_i s_i - \sum_{e} \dot{m}_e s_e + \dot{\sigma}_{\rm CV}$$
(6.34)

where dS_{CV}/dt represents the time rate of change of entropy within the CV, \dot{Q}_j represents the time rate of heat transfer at the point on the boundary where the instantaneous temperature is T_j , \dot{Q}_j/T_j accounts for the accompanying rate of entropy transfer, $\dot{m}_i s_i$ and $\dot{m}_e s_e$ account for rates of *entropy transfer accompanying mass flow* into and out of the CV, and $\dot{\sigma}_{CV}$ denotes the time rate of entropy production due to irreversibilities within the CV.

Rate balance for Control Volumes at Steady State

The steady state form of (6.34) is obtained by setting $dS_{CV}/dt = 0$. The one inlet, one outlet form is then

$$0 = \sum_{j} \frac{Q_{j}}{T_{j}} + \dot{m}(s_{i} - s_{e}) + \dot{\sigma}_{CV}$$
(6.37)

or

$$s_e - s_i = \frac{1}{\dot{m}} \left(\sum_j \frac{\dot{Q}_j}{T_j} \right) + \frac{\dot{\sigma}_{\rm CV}}{\dot{m}}$$
(6.38)

The two terms on the right are now per unit mass flowing through the CV.

If there is no heat transfer,

$$s_e - s_i = \frac{\dot{\sigma}_{\rm CV}}{\dot{m}} \tag{6.39}$$

so when there are irreversibilities within the CV, unit mass entropy increases as it passes from inlet to outlet, and when no irreversibilities are present, $\dot{\sigma}_{CV} = 0$, $s_1 = s_2$, and the unit mass passes through isentropically. Calculation of $\dot{\sigma}_{CV}/\dot{m}$, the time rate of change of entropy, is illustrated in Example E6.6 in the box on page 10.

Isentropic Turbine Efficiency

For no loss of heat, velocity, or potential energy in a turbine, equation (4.20b) shows that the mass and energy rate balance becomes

$$\frac{W}{\dot{m}} = h_i - h_e \tag{1}$$

For a fixed inlet state, the work per unit mass flowing through the turbine depends only on h_e , and increases as h_e is reduced. The smallest allowed value of h_e will evidently give the maximum possible work output. Because there is no heat loss, equation (6.39) shows that this is the state having $\dot{\sigma}_{CV} = 0$ and $s_e = s_i$, i.e., an isentropic process. The only outlet states that can actually be attained are those having $s_e > s_i$.

In Figure 2, for an inlet state 1 at pressure p_1 , the outlet state 2s at pressure p_2 would be attained only in the limiting reversible case, and outlet state 2 represents a possible actual exit state. Because s_2 cannot be less than s_1 , the smallest allowed value of h_2 corresponds to state 2s, and the maximum turbine work is

$$\left(\frac{W_{\rm CV}}{\dot{m}}\right)_s = h_1 - h_{2s} \tag{2}$$



Figure 1: Enthalpy and entropy data for water from program REFPROP. The dashed line represents the expansion process in Example E6.6.



Figure 2: A schematic Mollier or *h-s* diagram to illustrate turbine efficiency. Isobars are blue, isotherms are red. The isotherm through state 2 is not shown for clarity. The change from state 1 to state 2h is isenthalpic and irreversible. The change from state 1 to state 2s is isentropic and reversible. The dashed lines $1\rightarrow 2$ and $1\rightarrow 2h$ represent disequilibrium states which cannot be represented on the diagram.

Entropy Production Example E6.6

See Figure 1

Steam enters a turbine at p = 30 bar, $T = 400^{\circ}$ C, and V=160 m/s. Saturated vapor exits at 100°C, V=100 m/s. At steady state, the turbine develops work equal to 540 kJ per kg of steam flowing through. Heat loss from the turbine to the surroundings occurs at an average surface temperature of 350 K. Find the rate of entropy production in the turbine per kg of steam flowing.

From (6.38)

$$s_e - s_i = \frac{1}{\dot{m}} \left(\sum_j \frac{\dot{Q}_j}{T_j} \right) + \frac{\dot{\sigma}_{\text{CV}}}{\dot{m}}$$

we evidently need the quantity $\frac{\dot{Q}_j/\dot{m}}{T_j}$, but work is involved so we must bring in (4.20b). Dropping the potential energy term and rearranging,

$$\frac{\dot{Q}_{\rm CV}}{\dot{m}} = \frac{\dot{W}_{\rm CV}}{\dot{m}} + (h_e - h_i) + \frac{\mathbf{V}_e^2 - \mathbf{V}_i^2}{2}$$

From the NIST program REFPROP, the enthalpy terms are $h_i = 3231.7 \text{ kJkg}^{-1}$ and $h_e = 2675.8 \text{ kJkg}^{-1}$ so

$$\frac{\dot{Q}_{\rm CV}}{\dot{m}} = 540 + (2675.8 - 3231.75) + \left[\frac{100^2 - 160^2}{2}\right] / 1000$$
$$= -23.75 \,\rm kJ \, kg^{-1}$$

where the factor of 1000 converts m^2/s^2 to $kJ\,kg^{-1}.$

From the NIST program REFPROP, the entropy terms are $s_i = 6.9234$ and $s_e = 7.3610$, so the rate of entropy production is

$$\frac{\dot{\sigma}_{\rm CV}}{\dot{m}} = -\frac{-23.75}{350} + (7.3610 - 6.9234)$$
$$= 0.5055 \,\rm kJ \, kg^{-1} \, K^{-1}$$

In the possible actual expansion through the turbine, $h_2 > h_{2s}$, and less work is done, and the generalized version of equation ([1]) for any states 1 and 2 is

$$\left(\frac{\dot{W}_{\rm CV}}{\dot{m}}\right) = h_1 - h_2 \tag{3}$$

The isentropic turbine efficiency is defined as

$$\eta_{t} = \frac{\dot{W}_{CV}/\dot{m}}{(\dot{W}_{CV}/\dot{m})_{s}} = \frac{h_{1} - h_{2}}{h_{1} - h_{2s}}$$
(6.46)

Values of η_t for turbines are typically 0.7 to 0.9 (70–90%). Nozzle efficiencies are calculated the same way, and are generally greater because they have no moving parts. Isentropic nozzle efficiencies of 95% or more are common, indicating that well designed nozzles are nearly free of internal irreversibilities. The calculation of isentropic efficiency for a turbine is shown in the box on page 12.

Conclusions

Examples are also given in the text for the isentropic efficiencies of nozzles and compressors, but they are all similar to the turbine example shown. Once you accept that a flowing fluid can have the properties of an equilibrium state, the rest follows. We see that the isentropic approximation is perfectly valid in the sense that real work efficiency is simply compared to the maximum isentropic efficiency (which gives the maximum possible work), and in some cases such as nozzles and diffusers which have no moving parts, the efficiency can be high.

The problem then, if there is one, is not with engineering thermodynamics, but with the geological applications. The text makes clear the role of *irreversibilities* in reducing the efficiency and work output. Geological applications should therefore concentrate on evaluating these rather than assuming constant (or approximately constant) entropy. Venting volcanic fluids at high speed may well be adiabatic but with turbulence and tumbling, falling rock fragments the possible sources of irreversibility would seem to be very great. With a very low isentropic efficiency, the value of the isentropic assumption or comparison is not very useful.

As the effect of irreversibilities increases, the state represented by point 2 in Figure 2 moves farther up the p_2 isobar until finally $h_1 = h_2$, the isentropic efficiency is zero, and the adiabatic expansion is isenthalpic. It is sometimes claimed in the geological literature that no work is done in an adiabatic isenthalpic expansion. This is misleading. It means that there is no *useful* work done, i.e., work other than pv work. In the Joule-Thompson expansion, pv work is done before and after the expansion. In the volcanic environment, pv work is done inside the volcano, building up pressure until fluids escape, perhaps explosively. These fluids then do pv work on the environment.

Example E6.12 Turbine Efficiency

See Figure 2.

Air expands adiabatically through a turbine at steady state. The inlet air is at $p_1 = 3.0$ bar and $T_1 = 390$ K (~ 117°C). Air exits the turbine at $p_2 = 1.0$ bar. The work developed is 74 kJ per kg of air flow. What is the isentropic turbine efficiency?

In equation (6.46) the numerator is 74 kJ kg^{-1} . The denominator is

$$\left(\frac{\dot{W}_{\rm CV}}{\dot{m}}\right)_s = h_1 - h_{2s}$$

Program REFPROP gives $h_1 = 390.91 \text{ kJ kg}^{-1} \text{ and } s_1 = 6.8190 \text{ kJ kg}^{-1} \text{ K}^{-1}$. The properties of the outlet state including T_{2s} and h_{2s} can be found by finding the properties for air having the same entropy but at p = 1 bar. REFPROP makes this easy. At one bar and $s = 6.8190 \text{ kJ kg}^{-1} \text{ K}^{-1}$, $h_{2s} = 285.29 \text{ kJ kg}^{-1}$, and $T_{2s} = 285.07 \text{ K}$ ($\sim 12^{\circ}$ C), so

$$\left(\frac{\dot{W}_{\rm CV}}{\dot{m}}\right)_s = 390.91 - 285.29$$

= 105.62 kJ kg⁻¹

The isentropic efficiency is then

$$\eta_t = \frac{W_{\rm CV}/\dot{m}}{(\dot{W}_{\rm CV}/\dot{m})_s} \\ = \frac{74}{105.62} \\ = 0.701 \, (70.1\%)$$

So the inlet temperature is 117° C, and the outlet temperature would be 12° C if the turbine operated isentropically. What is the actual outlet temperature with the turbine operating at 70.1% efficiency? From equation ([3])

$$h_2 = h_1 - \left(\frac{\dot{W}_{CV}}{\dot{m}}\right) \\ = 390.91 - 74 \\ = 316.91 \,\text{kJ}\,\text{kg}^{-1}$$

For a state having a pressure of 1 bar and an enthalpy of $316.91 \text{ kJ kg}^{-1}$, REF-PROP shows the actual outlet temperature at state 2 to be 316.48 K or 43°C .

References

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