4 Chapter 4 Solution Set

Problems

4.1 Prove that the Weyl expansion Eq.(4.4a) satisfies the homogeneous Helmholtz equation if $|Z| > 0$.

This follows immediately from the easily established result that the expansion converges uniformly $\forall \mathbf{R}$ with $Z > 0$. The orders of differentiation and integration can then be exchanged thus establishing the desired result.

4.2 Use the angular spectrum expansion to prove that the solution to the RS boundary value problem is unique; i.e., that Dirichlet or Neumann conditions over any infinite plane bounding a half-space over which a field satisfies the Sommerfeld radiation condition uniquely determine the field throughout this half-space.

This follows from the angular spectrum expansion of the RS boundary value problem presented in Section 4.3. For example, for Dirichlet conditions we showed that

$$U_+(\boldsymbol{\rho}, z, \omega) = \frac{1}{(2\pi)^2} \int d^2K_\rho\, \tilde{U}_+(\mathbf{K}_\rho, z_0, \omega) e^{\pm i\gamma(z-z_0)} e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}}$$

valid $\forall z > z_0$. The uniqueness of the solution then follows from the uniqueness of the inverse Fourier transformation mapping $U_+(\boldsymbol{\rho}, z, \omega) \to \tilde{U}_+(\mathbf{K}_\rho, z_0, \omega)$.

4.3 Derive the plane wave expansion of the radiated field given in Eq.(4.15a) by performing the K_z integration using contour integration and the calculus of residues in the Fourier integral representation of the radiated field

$$U_+(\mathbf{r}, \omega) = \frac{1}{(2\pi)^3} \int d^3K\, \frac{\tilde{Q}(\mathbf{K}, \omega)}{k^2 - K^2} e^{i\mathbf{K}\cdot\mathbf{r}}. \tag{4.1}$$

The key to this problem is the bound (cf., Problem 4.8)

$$|\tilde{Q}(\mathbf{K}, \omega)| \leq |Q_{max}| \int_{\tau_0} d^3r\, |e^{-i\mathbf{K}\cdot\mathbf{r}}| \leq |Q_{max}|\tau_\rho \int_{z^-}^{z^+} dz\, e^{\Im K_z z}$$
$$= |Q_{max}|\tau_\rho \frac{e^{\Im K_z z^+} - e^{\Im K_z z^-}}{\Im K_z}$$

which holds for all real values of $\mathbf{K}_\rho = (K_x, K_y)$ and where $|Q_{max}|$ is the maximum of $Q(\mathbf{r},\omega)$ in τ_0 and τ_ρ is the maximum area of τ_0 in the $\boldsymbol{\rho}$ plane. If we then rewrite Eq.(4.1) in the form

$$U_+(\mathbf{r},\omega) = -\frac{1}{(2\pi)^3}\int d^2K_\rho e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}\int_{-\infty}^{\infty} dK_z\,\frac{\tilde{Q}(\mathbf{K},\omega)}{(K_z-\gamma)(K_z+\gamma)}e^{iK_z z}$$

we can close the K_z contour in the u.h.p. if $z > z^+$ and in the l.h.p. if $z < z^-$ and we obtain using Cauchy's integral theorem the plane wave expansion of the radiated field given in Eq.(4.15a).

4.4 Derive a plane wave expansion involving only homogeneous plane waves valid for a field radiated by a causal source in a non-dispersive medium that is valid for all times after the source ceases to radiate.

See Problem 3.3 at the end of Chapter 3 where the expansion was found to be

$$u(\mathbf{r},t) = -\frac{ic}{2(2\pi)^3}\int_{-\infty}^{\infty} kdk\int_{4\pi} d\Omega_s\,\tilde{q}(k\mathbf{s},ck)e^{ik(\mathbf{s}\cdot\mathbf{r}-ct)},$$

where

$$\tilde{q}(k\mathbf{s},ck) = \int_{-\infty}^{\infty} dt'\int_{\tau_0} d^3r'\,q(\mathbf{r}',t')e^{-ik(\mathbf{s}\cdot\mathbf{r}'+ct')}$$

4.5 Comment on the relationship between the evanescent plane waves in the *time-domain* angular spectrum expansion of the field radiated by the source in the previous problem with the homogeneous waves in the plane wave expansion of this field for times exceeding the turn-off time of the source.

The time-domain angular spectrum expansion in angle variable form is found from Eq.(4.18a) to be given by

$$\begin{aligned} u_+(\mathbf{r},t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\,U_+(\mathbf{r},\omega)e^{-i\omega t} = \frac{c}{2\pi}\int_{-\infty}^{\infty} dk\,U_+(\mathbf{r},ck)e^{-ickt} \\ &= -\frac{ic}{2(2\pi)^3}\int_{-\infty}^{\infty} kdk\int_{-\pi}^{\pi} d\beta\int_{C_\pm}\sin\alpha d\alpha\,\tilde{q}(k\mathbf{s},ck)e^{ik(\mathbf{s}\cdot\mathbf{r}-ct)}, \end{aligned} \tag{4.2}$$

where we have set $\omega = ck$ and $\tilde{Q}(k\mathbf{s},\omega) = \tilde{q}(k\mathbf{s},ck)$. The plane wave expansion valid for times $t > T_0$ found in the previous problem is given by

$$u(\mathbf{r},t) = -\frac{ic}{2(2\pi)^3}\int_{-\infty}^{\infty} kdk\int_{4\pi} d\Omega_s\,\tilde{q}(k\mathbf{s},ck)e^{ik(\mathbf{s}\cdot\mathbf{r}-ct)},$$

which is formally identical to Eq.(4.2) with the exception that the complex contours $C_\pm$ have been converted to the real contour $0 \le \alpha \le 2\pi$. We thus conclude that *the evanescent waves in the angular spectrum expansion of the (time dependent) field are replaced by homogeneous waves propagating inward toward the source region τ_0 when $t > T_0$.*

4.6 Derive the radiation pattern of a uniform circular disk confined to the plane $z = 0$ employed in Section 4.5 for comparison with the wavelet radiation pattern.

The source of a uniform circular disk on the plane $z = 0$ can be expressed in the for

$$Q(\mathbf{r}, \omega) = \text{circ}(\rho/a)\delta(z)$$

where circ(x) is the "circ" function (equal to unity if $x < 1$ and zero otherwise.) The radiated field is then found to be

$$U_+(\mathbf{r}, \omega) = \int_0^a \rho d\rho \int_0^{2\pi} d\phi\, G_+(\mathbf{r} - \boldsymbol{\rho}) \sim \{-\frac{1}{4\pi}\int_0^a \rho d\rho \int_0^{2\pi} d\phi\, e^{-ik\mathbf{s}\cdot\boldsymbol{\rho}}\}\frac{e^{ikr}}{r}$$

from which we conclude that

$$f(\mathbf{s}) = -\frac{1}{4\pi}\int_0^a \rho d\rho \int_0^{2\pi} d\phi\, e^{-ik\mathbf{s}\cdot\boldsymbol{\rho}}$$
$$= -\frac{1}{4\pi}\int_0^a \rho d\rho \int_0^{2\pi} d\phi\, e^{-ik\rho\sin\alpha(\cos\phi\cos\beta + \sin\phi\sin\beta)}$$
$$= -\frac{1}{4\pi}\int_0^a \rho d\rho \int_0^{2\pi} d\phi\, e^{-ik\rho\sin\alpha\cos(\phi-\beta)}$$
$$= -\frac{1}{2}\int_0^a \rho d\rho\, J_0(k\rho\sin\alpha) = -\frac{a^2}{2}\frac{J_1(ka\sin\alpha)}{ka\sin\alpha}$$

Finally, since

$$\lim_{x\to 0}\frac{J_1(x)}{x} = \frac{1}{2}$$

we find the normalized radiation pattern to be

$$f(\alpha) = 2\frac{J_1(ka\sin\alpha)}{ka\sin\alpha}.$$

4.7 Express the back propagated field in Eq.(4.25) in terms of Neumann conditions.

This is obtained directly from Eq.(4.25) by making use of Eq.(4.23) in Example 4.4.

4.8 Derive the inequality given in Eq.(4.16).

See the solution to problem 4.3.

4.9 Use the Weyl expansion in the expression for the field radiated by a source located in the left-half space within the strip $z^- < z^+ < 0$ in the presence of a Dirichlet plane (plane over which the field vanishes) located at $z = 0$ to develop an angular spectrum expansion of the radiated field in the half-space $z < z^-$; i.e., to the left of the source. [cf., Problems 2.14 and 2.15 from Chapter 2.]

We found in Problem 2.15 that the field radiated in the presence of the Dirichlet plane at $z = 0$ can be expressed in the form

$$U(\mathbf{r}, \omega) = \int_{\tau_0} d^3r'\,[G_+(\mathbf{r} - \mathbf{r}', \omega) - G_+(\tilde{\mathbf{r}} - \mathbf{r}', \omega)]Q(\mathbf{r}'; \omega), \quad z < 0. \quad (4.3)$$

In the region $z < z^-$ lying to the left of the source strip $[z^-, z^+]$ we have that $z - z' < 0$ while $-z - z'$ is still positive. It then follows that

$$G_+(\mathbf{r}-\mathbf{r}',\omega) = \frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\frac{d^2K_\rho}{\gamma}\,e^{i\mathbf{k}^-\cdot(\mathbf{r}-\mathbf{r}')},$$
$$G_+(\tilde{\mathbf{r}}-\mathbf{r}',\omega) = \frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\frac{d^2K_\rho}{\gamma}\,e^{i\mathbf{k}^+\cdot(\tilde{\mathbf{r}}-\mathbf{r}')},$$

which, when used in Eq.(4.3) yield

$$U(\mathbf{r},\omega) = \frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\frac{d^2K_\rho}{\gamma}\,[\tilde{Q}(\mathbf{k}^-,\omega) - \tilde{Q}(\mathbf{k}^+,\omega)]e^{i\mathbf{k}^-\cdot\mathbf{r}}.$$

The field in the region to the left of the source is seen to consist of a superposition of left propagating and evanescent plane waves. The first expansion is the usual plane wave expansion resulting from the source Q while the second expansionrepresents the field reflected from the conducting plane.

4.10 Represent the radiated field in an angular spectrum expansion for the previous problem in the interior strip $z^+ < z < 0$ lying between the source and Dirichlet plane.

In this region both $z - z'$ as well as $-z - z'$ are positive and we find that

$$G_+(\mathbf{r}-\mathbf{r}',\omega) = \frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\frac{d^2K_\rho}{\gamma}\,e^{i\mathbf{k}^+\cdot(\mathbf{r}-\mathbf{r}')},$$
$$G_+(\tilde{\mathbf{r}}-\mathbf{r}',\omega) = \frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\frac{d^2K_\rho}{\gamma}\,e^{i\mathbf{k}^+\cdot(\tilde{\mathbf{r}}-\mathbf{r}')},$$

where , again, it is assumed that $z^+ < z < 0$. Using these expansions we find after a bit of algebra that

$$G_+(\mathbf{r}-\mathbf{r}',\omega) - G_+(\tilde{\mathbf{r}}-\mathbf{r}',\omega)$$
$$= \frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\frac{d^2K_\rho}{\gamma}\,e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}[e^{i\gamma z} - e^{-i\gamma z}]e^{-i(\mathbf{K}_\rho\cdot\boldsymbol{\rho}'+\gamma z')}.$$

On using the above result in Eq.(4.3) we find that for all $z \in [z^+, 0]$

$$U(\mathbf{r},\omega) = \frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\frac{d^2K_\rho}{\gamma}\,e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}[e^{i\gamma z} - e^{-i\gamma z}]\tilde{Q}(\mathbf{k}^+,\omega).$$

We can interpret the above expression for the field U within the strip region $z^+ < z < 0$ as two superpositions of plane wave fields with one propagating in the positive z direction and the second in the negative z direction. The first expansion is the usual plane wave expansion resulting from the source Q while the second expansion represents the field reflected from the conducting plane. Clearly the sum $U = U_+ + U_-$ vanishes over the conducting plane (when $z = 0$). Note also that each pair of plane waves in the expansions satisfy the "law of reflection" that states that a plane wave having wave vector $(\mathbf{K}_\rho, \gamma)$ incident on a planar surface that is perpendicular to the z axis will generate a reflected plane wave whose wave vector will be $(\mathbf{K}_\rho, -\gamma)$.

4.11 Use the angular spectrum expansion of the outgoing wave multipole fields given in Eq.(4.66a) to derive the angular spectrum expansion of the radiated field from the multipole expansion of this field given in Eq.(4.52a).

The solution to this problem follows almost identical lines as the solution to problem 3.10 of the previous chapter. In particular, we use the plane wave (angular spectrum) expansion of the outgoing wave multipole fields $h_l^+(kr)Y_l^m(\hat{\mathbf{r}})$ given in Eq.(4.66a) in place of the plane wave expansions of the free multipole fields $j_l(kr)Y_l^m(\hat{\mathbf{r}})$ given in Eq.(3.36) of Example 3.4. We also consider the radiation problem rather than the interior boundary value problem considered in problem 3.10 but the manipulations are almost identical.

We begin with the multipole expansion

$$U_+(\mathbf{r},\omega) = -ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} q_l^m(\omega)h_l^+(kr)Y_l^m(\hat{\mathbf{r}})$$

which, upon substitution of the angular spectrum expansion of the outgoing wave multipole fields given in Eq.(4.66a) yields

$$\begin{aligned}U_+(\mathbf{r},\omega) &= -ik\sum_{l=0}^{\infty}\sum_{m=-l}^{l} q_l^m(\omega)\overbrace{\frac{(-i)^l}{2\pi}\int_{-\pi}^{\pi}d\beta\int_{C_\pm}\sin\alpha d\alpha\, Y_l^m(\mathbf{s})e^{ik\mathbf{s}\cdot\mathbf{r}}}^{h_l^+(kr)Y_l^m(\hat{\mathbf{r}})}\\ &= \frac{ik}{2\pi}\int_{-\pi}^{\pi}d\beta\int_{C_\pm}\sin\alpha d\alpha\, A(\mathbf{s},\omega)e^{ik\mathbf{s}\cdot\mathbf{r}}\end{aligned}$$

which is the required angular spectrum expansion with plane wave amplitude

$$A(\mathbf{s},\omega) = -\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^l q_l^m(\omega)Y_l^m(\mathbf{s}).$$

As a check we note that

$$A(\mathbf{s},\omega) = f(\mathbf{s},\omega)$$

as found in Section 4.8.4.

4.12 Derive the angle variable form of the expression for the wavelet field in Eq.(4.31) of Section 4.5 using the angle variable form of the angular spectrum expansion. Check your result by transforming your expression to Cartesian variable form.

We use the angle variable for the angular spectrum expansion with the angular spectrum equal to the wavelet radiation pattern to obtain

$$U_+(\mathbf{r},\omega) = \frac{ik}{2\pi}\int_{-\pi}^{\pi} d\beta \int_{C_\pm} \sin\alpha d\alpha \overbrace{\left[\frac{e^{ka\cos\alpha}}{e^{ka}}\right]}^{f(\mathbf{s},\omega)} e^{ik\mathbf{s}\cdot\mathbf{r}}$$

$$= \frac{ik}{2\pi}\int_{-\pi}^{\pi} d\beta \int_{C_\pm} \sin\alpha d\alpha \frac{e^{ka\cos\alpha}}{e^{ka}} e^{ik[\sin\alpha\cos\beta x + \sin\alpha\sin\beta y + \cos\alpha z]}$$

$$= \frac{ik}{2\pi}\int_{C_\pm} \sin\alpha d\alpha \frac{e^{ka\cos\alpha}}{e^{ka}} e^{ik\cos\alpha z} \int_{-\pi}^{\pi} d\beta\, e^{ik[\sin\alpha\cos\beta x + \sin\alpha\sin\beta y]}$$

$$= ik\int_{C_\pm} \sin\alpha d\alpha \frac{e^{ka\cos\alpha}}{e^{ka}} e^{ik\cos\alpha z} J_0(k\sin\alpha\rho)$$

which is the required result.

We can transform to Cartesian integration variables by setting

$$K_\rho = k\sin\alpha, \quad dK_\rho = k\cos\alpha d\alpha, \quad k\sin\alpha d\alpha = \frac{K_\rho dK_\rho}{\gamma}$$

with $\gamma = k\cos\alpha = \sqrt{k^2 - K_\rho^2}$. It is then an easy step to show that

$$U_+(\mathbf{r},\omega) \to i\int_0^\infty \frac{K_\rho dK_\rho}{\gamma} \frac{e^{\pm a\gamma}}{e^{ka}} J_0(K_\rho \rho) e^{\pm i\gamma z},$$

where the plus sign is used in the r.h.s. $z > a$ and the minus sign in the l.h.s. $z < -a$.

4.13 Use the scheme given in Example 4.11 to derive the 2D angular spectrum expansion of the 2D outgoing wave multipole fields.

To do this problem we will use the Cartesian x, y system employed in Section 4.9 which employs a polar angle ϕ relative to the positive x axis and an angular spectrum expansion with angle α relative to the positive y axis. The 2D angular spectrum expansion of a general outgoing wave field is then found from Eq.(4.39b) of Section 4.6.3 to be of the form

$$U_+(\mathbf{r},\omega) = \sqrt{\frac{k}{2\pi}} e^{i\pi/4} \int_{C_\pm} d\alpha\, A(\mathbf{s},\omega) e^{ik\mathbf{s}\cdot\mathbf{r}} \tag{4.4a}$$

where $\mathbf{s} = \sin\alpha\hat{\mathbf{x}} + \cos\alpha\hat{\mathbf{y}}$. Note that relative to the coordinate system used in Section 4.6.3 that the z axis has now been replaced by the y axis so that the plane waves propagate out from the x axis into half-spaces $y > 0$ or $y < 0$. The multipole expansion of a general outgoing wave field was obtained in Section 4.9 where it was shown to be given by

$$U_+(\mathbf{r},\omega) = -\frac{i}{4}\sum_{n=-\infty}^{\infty} q_n(\omega) H_n^+(kr) e^{in\phi} \tag{4.4b}$$

We now select U_+ in Eq.(4.4a) to be the 2D outgoing wave Green function

$G_+ = -i/4H_0(kR)$ and U_+ in Eq.(4.4b) to be the multipole field $H_n^+(kr)\exp(in\phi)$. We then find that

$$H_n^+(kr)e^{in\phi} = \sqrt{\frac{k}{2\pi}}e^{i\pi/4}\int_{C_\pm} d\alpha\, A_n(\mathbf{s},\omega)e^{ik\mathbf{s}\cdot\mathbf{r}}$$

where the angular spectrum A_n of the multipole field is to be determined. The multipole expansion of the Green function was derived in Section 3.6 of Chapter 3 where it was found to be

$$G_+(\mathbf{r}-\mathbf{r}',\omega) = -\frac{i}{4}\sum_{n=-\infty}^{\infty} J_n(kr')H_n^+(kr)e^{in(\phi-\phi')},$$

and where we have taken $r > r'$. On substituting the angular spectrum expansion of the multipole fields into the above multipole expansion of the Green function we obtain

$$G_+(\mathbf{r}-\mathbf{r}',\omega) = -\frac{i}{4}\sum_{n=-\infty}^{\infty} J_n(kr')e^{-in\phi'}\overbrace{\sqrt{\frac{k}{2\pi}}e^{i\pi/4}\int_{C_\pm} d\alpha\, A_n(\mathbf{s},\omega)e^{ik\mathbf{s}\cdot\mathbf{r}}}^{H_n^+(kr)e^{in\phi}}$$

$$= \sqrt{\frac{k}{2\pi}}e^{i\pi/4}\int_{C_\pm} d\alpha\,\{-\frac{i}{4}\sum_{n=-\infty}^{\infty} J_n(kr')e^{-in\phi'}A_n(\mathbf{s},\omega)\}e^{ik\mathbf{s}\cdot\mathbf{r}}. \tag{4.5a}$$

The angular spectrum expansion of the 2D outgoing wave Green function was found in Section 4.6.3 to be

$$G_+(\mathbf{r}-\mathbf{r}',\omega) = \frac{-i}{4\pi}\int_{C_\pm} d\alpha\, e^{ik\mathbf{s}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{4.5b}$$

On equating the above two expansions for the Green function we then require that

$$e^{-ik\mathbf{s}\cdot\mathbf{r}'} = \sqrt{\frac{k\pi}{2}}e^{i\pi/4}\sum_{n=-\infty}^{\infty} J_n(kr')e^{-in\phi'}A_n(\mathbf{s},\omega).$$

As a final step we use the Jacobi Anger expansion of the 2D plane wave field obtained in Example 3.9 of Chapter 3:

$$e^{-ik\mathbf{s}\cdot\mathbf{r}'} = \sum_{n=-\infty}^{\infty}(-i)^n J_n(kr')e^{-in(\phi'-\alpha)},$$

from which we conclude that

$$\sqrt{\frac{k\pi}{2}}e^{i\pi/4}A_n(\mathbf{s},\omega) = (-i)^n e^{in\alpha} \rightarrow A_n(\mathbf{s},\omega) = \sqrt{\frac{2}{k\pi}}e^{-i\pi/4}(-i)^n e^{in\alpha}$$

which yields the angular spectrum expansion of the 2D multipole fields

$$H_n^+(kr)e^{in\phi} = \frac{(-i)^n}{\pi}\int_{C_\pm} d\alpha\, e^{in\alpha}e^{ik\mathbf{s}\cdot\mathbf{r}}. \tag{4.6}$$

As a check we take $n=0$ to find that

$$G_+(\mathbf{r}-\mathbf{r}',\omega) = \frac{-i}{4}H_0(kR) = \frac{-i}{4\pi}\int_{C_\pm} d\alpha\, e^{ik\mathbf{s}\cdot\mathbf{R}},$$

which agrees with Eq.(4.5b).

4.14 Use the 2D angular spectrum expansion of the 2D outgoing wave multipole fields found in the previous problem to derive the 2D multipole expansion of a radiated field from its angular spectrum expansion.

We start with the 2D angular spectrum expansion

$$U_+(\mathbf{r},\omega) = \sqrt{\frac{k}{2\pi}}e^{i\pi/4}\int_{C_\pm} d\alpha\, A(\mathbf{s},\omega)e^{ik\mathbf{s}\cdot\mathbf{r}}$$

and expand that angular spectrum using Eq.(4.64a) from Section 4.9 to obtain

$$U_+(\mathbf{r},\omega) = \sqrt{\frac{k}{2\pi}}e^{i\pi/4}\int_{C_\pm} d\alpha\,\{\frac{1}{2\pi}\sum_{n-\infty}^{\infty} f_n(\omega)e^{in\alpha}\}e^{ik\mathbf{s}\cdot\mathbf{r}}$$

$$= \sqrt{\frac{k}{8\pi^3}}e^{i\pi/4}\sum_{n-\infty}^{\infty} f_n(\omega)\int_{C_\pm} d\alpha\, e^{in\alpha}e^{ik\mathbf{s}\cdot\mathbf{r}}$$

where the expansion coefficients are related to the multipole moments of the radiated field via Eq.(4.64b) of Section 4.9:

$$f_n(\omega) = -\frac{i}{4}\sqrt{\frac{8\pi}{k}}e^{-i\pi/4}(-i)^n q_n(\omega). \tag{4.7}$$

We now make use the multipole expansion found in the previous problem

$$H_n^+(kr)e^{in\phi} = \frac{(-i)^n}{\pi}\int_{C_\pm} d\alpha\, e^{in\alpha}e^{ik\mathbf{s}\cdot\mathbf{r}}.$$

to find that

$$U_+(\mathbf{r},\omega) = \sqrt{\frac{k}{8\pi}}e^{i\pi/4}\sum_{n-\infty}^{\infty} i^n f_n(\omega)H_n^+(kr)e^{in\phi} = -\frac{i}{4}\sum_{n-\infty}^{\infty} q_n(\omega)H_n^+(kr)e^{in\phi}$$

where we have made use of Eq.(4.7).

4.15 Use the 2D angular spectrum found in Problem 4.13 to derive the angular spectrum expansiion of a 2D outgoing wave field from its 2D multipole expansion.

This is simply the previous problem done in reverse order.

4.16 Derive the expression for the radiation pattern expansion coefficients $f_l^m(\omega)$ given in Problem 3.11 of the last chapter directly from the multipole expansion of the radiated field.

This is done in one step by projecting both sides of the multipole expansion of the radiated field given in Eq.(4.52) onto the spherical harmonics.