# A Student's Guide to the Ising Model

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## Homework Solutions


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|---|---|---|
| Chapter 1 | The Ising Model | *page* 1 |
| Chapter 2 | Finite Ising Systems | 6 |
| Chapter 3 | Partial Summations and Effective Interactions | 12 |
| Chapter 4 | Infinite Ising Systems in One Dimension | 20 |
| Chapter 5 | The Onsager Solution and Exact Series Expansions | 22 |
| Chapter 6 | The Mean-Field Approach | 24 |
| Chapter 7 | Position-Space Renormalization-Group Techniques | 25 |


## Chapter 1 The Ising Model

**1.1** **Figure 1.9** shows a $B$-versus-$T$ plot with various numbered points indicated for consideration. **(a)** At which of the points 1, 2, 3, 4, is the magnetization the greatest? **(b)** Is the magnetization at point 2 greater than, less than, or equal to the magnetization at point 4? **(c)** Is the magnetization at point 3 greater than, less than, or equal to the magnetization at point 4?

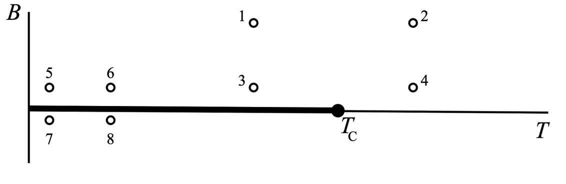


Figure 1.9 A $B$-versus-$T$ plot with various numbered points from 1 to 8.

**1.1** **Solution:** **(a)** Point 1 has the greatest magnetization because it has the lowest temperature and the highest magnetic field. **(b)** The magnetization at point 2 is greater than the magnetization at point 4 because the magnetic field is greater at point 2 and the temperature is the same at both points. **(c)** The magnetization at point 3 is greater than the magnetization at point 4 because the temperature is lower at point 3 and the magnetic field is the same at both points.

**1.2** **Figure 1.9** shows a $B$-versus-$T$ plot with various numbered points indicated for consideration. **(a)** At which of the points 5, 6, 7, 8, is the magnetization the greatest? **(b)** Is the difference in magnetization in going from point 6 to point 5 greater than, less than, or equal to the difference in magnetization in going from point 7 to point 5?

**1.2** **Solution:** **(a)** Point 5 has the greatest magnetization because it has the lowest temperature and the largest magnetic field. **(b)** The difference in magnetization is greater from points 7 to 5 than from points 6 to 5 because in going from 7 to 5 the magnetization jumps from a negative value to a positive value. In contrast, going from 6 to 5 simply results in the magnetization increasing a small amount as it gets closer to its maximum value of +1.

**1.3** Two nearest-neighbor Ising systems have identical coupling energies, $J$, but system 1 has a reduced coupling $K_1$ and system 2 has a reduced coupling $K_2$. If $K_1 > K_2$, which system has the higher temperature?

**1.3** **Solution:** System 2 has the smaller reduced coupling, and hence it has the higher temperature.

**1.4** The critical point of the nearest-neighbor, two-dimensional square lattice is given by

$$K_c = 0.4406\ldots$$

Do you expect $K_c$ for the nearest-neighbor, two-dimensional triangular lattice to be greater than, less than, or equal to 0.4406...? Explain.

**1.4** **Solution:** The value of $K_c$ for a triangular lattice will be less than $K_c$ for a square lattice because the triangular lattice has more nearest neighbor couplings per site, which means a higher critical temperature and hence a smaller critical coupling.

**1.5** Two Ising spins are connected by a nearest-neighbor bond, as indicated in **Figure 1.10**. The reduced Hamiltonian is

$$-\beta H = Ks_1s_2 + h(s_1 + s_2)$$

**(a)** What is the number of states in this system? **(b)** Calculate the partition function, $Z$, for this system. Express your result in terms of $K$ and $h$. **(c)** What is the probability that both spins have the value +1?


Figure 1.10 A group of two Ising spins. The reduced Hamiltonian indicates the spins have a nearest-neighbor coupling, *K*, and experience a magnetic field, *h*.

**1.5 Solution: (a)** Because this system has two sites, each of which has a variable that can take on 2 values, the number of states is $2^2 = 4$. **(b)** The Boltzmann factor for both spins equal to +1 is $e^{K+2h}$; the Boltzmann factor for spin 1 equal to +1 and spin 2 equal to –1 is $e^{-K}$; the Boltzmann factor for spin 1 equal to –1 and spin 2 equal to +1 is $e^{-K}$; the Boltzmann factor for both spins equal to –1 is $e^{K-2h}$. Summing these Boltzmann factors yields

$$Z = e^{K+2h} + 2e^{-K} + e^{K-2h}.$$

**(c)** If both spins are equal to +1 the Boltzmann factor is $e^{K+2h}$. Therefore, the probability that both spins are +1 is $P = \frac{1}{Z}e^{K+2h}$.

**1.6** A group of three Ising spins is arranged in an equilateral triangle, as shown in **Figure 1.11**. The reduced Hamiltonian is

$$-\beta H = K(s_1s_2 + s_2s_3 + s_3s_1) + h(s_1 + s_2 + s_3)$$

**(a)** What is the number of states in this system? **(b)** Calculate the partition function, *Z*, for this system. Express your result in terms of *K* and *h*. **(c)** What is the probability that two spins have the value +1 and the third spin has the value –1?

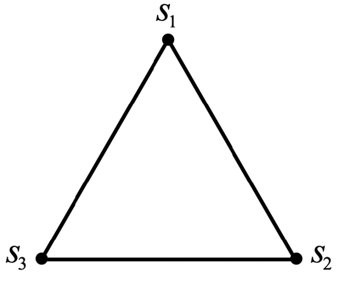


Figure 1.11 A group of three Ising spins. The reduced Hamiltonian indicates the spins have a nearest-neighbor coupling, *K*, and experience a magnetic field, *h*.

**1.6 Solution: (a)** This system has three sites, each of which has a variable that takes on 2 values, therefore the number of states is $2^3 = 8$. **(b)** The Boltzmann factor for all three spins equal to +1 is $e^{3K+3h}$. The Boltzmann factor for two spins equal to +1 and one spin equal to –1 is $e^{-K+h}$. This Boltzmann factor occurs 3 times in the partition function. The Boltzmann factor for two spins equal to –1 and one spin equal to +1 is $e^{-K-h}$. This Boltzmann factor also occurs three times in the partition function. The Boltzmann factor for all three spins equal to –1 is $e^{3K-3h}$. Summing all the Boltzmann factors yields

$$Z = e^{3K+3h} + 3e^{-K+h} + 3e^{-K-h} + e^{3K-3h}.$$

**(c)** If two spins are equal to +1 and the third is equal to –1, the Boltzmann factor is $e^{K+2h}$. This can happen in three different ways. Therefore, the probability for two spins +1 and one spin –1 is $P = \frac{1}{Z}(3e^{-K+h})$.

**1.7** Suppose an energy $E_0$ is added to the energy of each state of a system. **(a)** In this new system, show that the Boltzmann factor for a state with an original energy $E_i$ can be written as $e^{-E_0/k_BT}e^{-E_i/k_BT}$. **(b)** Show that the partition function for the new system can be written as $e^{-E_0/k_BT}Z$, where $Z$ is the original partition function. **(c)** Show that the probability for any given state is unchanged by the added energy.

**1.7 Solution: (a)** The Boltzmann factor for a state $i$, with energy $E_0 + E_i$, is $e^{-(E_0+E_i)/k_BT} = e^{-E_0/k_BT}e^{-E_i/k_BT}$. **(b)** In the original system

$$Z = \sum_{\text{all states, } i} e^{-E_i/k_BT}.$$

In the new system,

$$Z_{\text{new}} = \sum_{\text{all states, } i} e^{-(E_0+E_i)/k_BT} = e^{-E_0/k_BT}Z.$$

**(c)** In the original system $P_i = \frac{e^{-E_i/k_BT}}{Z}$. In the new system

$$P_i = \frac{e^{-(E_0+E_i)/k_BT}}{e^{-E_0/k_BT}Z} = \frac{e^{-E_i/k_BT}}{Z}.$$

This is the same as in the original system.

**1.8** Suppose we would like to write the Boltzmann factor in base 2, as follows:

$$2^{-E_i/kT}$$

Find the value of $k$.

**1.8 Solution:** Setting $2^{-E_i/kT} = e^{-E_i/k_BT}$, we find $k = k_B \ln 2$.

**1.9** Consider a state with the energy $E_i > 0$. The Boltzmann factor for this state is

$$e^{-E_i/k_BT}$$

Which of the three curves (1, 2, 3) in **Figure 1.12** is a plot of the Boltzmann factor as a function of temperature, $T$?

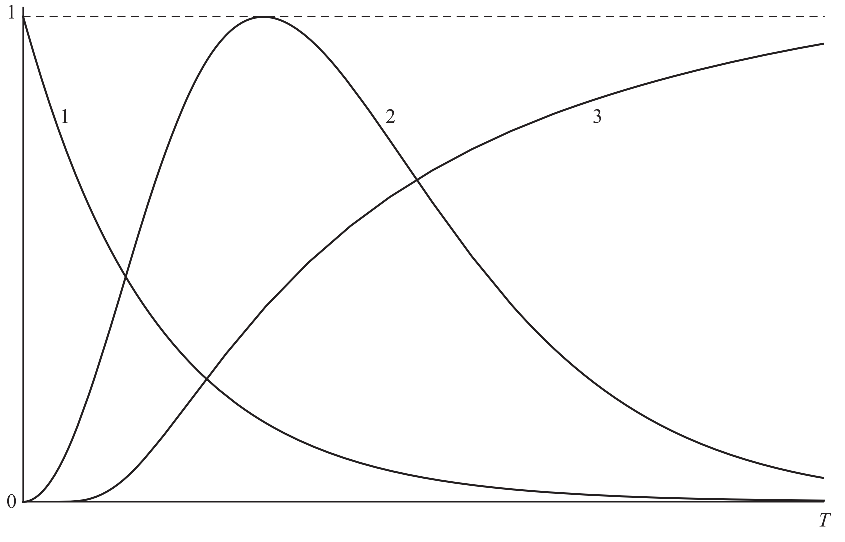


Figure 1.12 Three curves (1, 2, 3) as a function of temperature. One of the curves is the Boltzmann factor.

**1.9 Solution:** Curve 3. This curve starts out at zero for zero temperature, since $e^{-\infty} = 0$, and approaches 1 for infinite temperature, since $e^0 = 1$.

# Chapter 2 Finite Ising Systems

**2.1** Is the ground-state entropy of the one-spin system in **Figure 2.1** greater than, less than, or equal to the ground-state entropy of the two-spin system in **Figure 2.5**? Explain.


Figure 2.1 A single-spin system, with spin $s_1$.

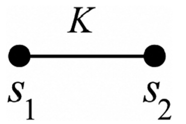


Figure 2.5 Two Ising spins with a nearest-neighbor coupling *K*.

**2.1 Solution:** The ground-state entropies are equal, because both systems have two ground states. The ground-state entropy *per site* is less for the two-spin system, however.

**2.2** Three independent spins are each acted on by a magnetic field $h$, as in **Figure 2.4** with $N = 3$. **(a)** Sum over the two configurations of any one of the spins to obtain its partition function, $Z_1$. **(b)** Sum over the eight configurations of the entire system to obtain the total partition function of the system, $Z_{\text{total}}$. **(c)** Raise your result for $Z_1$ to the third power to show that $Z_{\text{total}} = (Z_1)^3$.

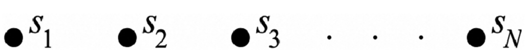


Figure 2.4 *N* independent Ising spins.

**2.2 Solution: (a)** For a single spin, the partition function is $Z_1 = e^h + e^{-h}$. The first term corresponds to the spin equal to +1, the second term corresponds to the spin equal to –1. **(b)** The partition function for the total system, with eight configurations, is

$$Z_{\text{total}} = e^{3h} + 3e^h + 3e^{-h} + e^{-3h}.$$

The first term corresponds to all spins equal to +1, the second term corresponds to the three configurations with two spins +1 and one spin –1, the third term corresponds to the three configurations with two spins –1 and one spin +1, and the last term corresponds to all spins –1. **(c)** Notice that the total partition function can be factored as follows:

$$Z_{\text{total}} = e^{3h} + 3e^{h} + 3e^{-h} + e^{-3h} = (e^{h} + e^{-h})^3 = (Z_1)^3.$$

**2.3** Consider the following series of numbers: $S = 0,\ 1,\ 2,\ 3$. **(a)** Is $\langle S\rangle^2$ greater than, less than, or equal to $\langle S^2\rangle$? Explain. **(b)** Calculate $\langle S\rangle^2$. **(c)** Calculate $\langle S^2\rangle$. **(d)** Calculate $\langle S^2\rangle - \langle S\rangle^2$.

**2.3 Solution: (a)** Less than. Squaring the numbers before averaging produces an average of larger numbers, and hence this result is greater than squaring the average of the original smaller numbers. **(b)** $\langle S\rangle^2 = 2.25$. **(c)** $\langle S^2\rangle = 3.5$. **(d)** $\langle S^2\rangle - \langle S\rangle^2 = 1.25$.

**2.4** Consider the following two series of numbers: $S_1 = 1, 2, 3$ and $S_2 = 1.9, 2, 2.1$. **(a)** Is $\langle S_1\rangle$ greater than, less than, or equal to $\langle S_2\rangle$? Explain. **(b)** Is $\langle {S_1}^2\rangle - \langle S_1\rangle^2$ greater than, less than, or equal to $\langle {S_2}^2\rangle - \langle S_2\rangle^2$? Explain. **(c)** Calculate $\langle {S_1}^2\rangle - \langle S_1\rangle^2$. **(d)** Calculate $\langle {S_2}^2\rangle - \langle S_2\rangle^2$.

**2.4 Solution: (a)** $\langle S_1\rangle = \langle S_2\rangle = 2$. Both sets of numbers are distributed symmetrically above and below 2. **(b)** $(\langle {S_1}^2\rangle - \langle S_1\rangle^2) > (\langle {S_2}^2\rangle - \langle S_2\rangle^2)$. The range of the spread in the numbers in  is greater than the range in $S_2$. **(c)** $\langle {S_1}^2\rangle - \langle S_1\rangle^2 = 0.666\ldots$. **(d)** $\langle {S_2}^2\rangle - \langle S_2\rangle^2 = 0.00666\ldots$.

**2.5** Suppose the two-spin Ising system in **Figure 2.5** has both a nearest-neighbor coupling $K$ and an external magnetic field $h$. The reduced Hamiltonian is

$$-\beta H = K s_1 s_2 + h(s_1 + s_2)$$

**(a)** Calculate the partition function for this system. **(b)** Calculate the reduced free energy per spin.

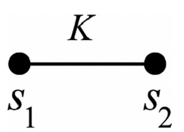


Figure 2.5 Two Ising spins with a nearest-neighbor coupling $K$.

**2.5 Solution: (a)** The partition function for the system, with four configurations, is

$$Z = e^{K+2h} + 2e^{-K} + e^{K-2h}.$$

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The first term corresponds to both spins +1, the second term corresponds to the two configurations with one spin +1 and one spin –1, and the last term corresponds to both spins equal to –1. **(b)** The reduced free energy per site is

$$f = \frac{1}{2}\ln Z = \frac{1}{2}\ln\left(e^{K+2h} + 2e^{-K} + e^{K-2h}\right).$$

**2.6** Add an external magnetic field to the three-spin system with periodic boundary conditions shown in **Figure 2.15**. **(a)** What is the Hamiltonian for this new system? **(b)** Calculate the partition function for this new system.

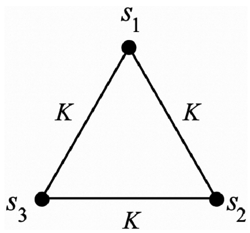


Figure 2.15 A three-spin system with periodic boundary conditions can also be pictured as three spins at the vertices of an equilateral triangle.

**2.6 Solution: (a)** The Hamiltonian contains two-spin interactions along each edge of the triangle and single-spin interactions for each vertex. The resulting Hamiltonian is the following:

$$-\beta H = K(s_1 s_2 + s_2 s_3 + s_3 s_1) + h(s_1 + s_2 + s_3).$$

**(b)** Summing over the eight configurations of the system yields the following partition function:

$$Z = e^{3K+3h} + 3e^{-K+h} + 3e^{-K-h} + e^{3K-3h}.$$

The first term corresponds to all spins +1, the second term corresponds to the three configurations with two spins +1 and one spin –1, the third term corresponds to the three configurations with one spin +1 and two spins –1, and the last term corresponds to all spins equal to –1.

**2.7** Add an external magnetic field to the three-spin system with free boundary conditions shown in **Figure 2.10**. **(a)** What is the Hamiltonian for this new system? **(b)** Consider the following three prospective partition functions:

$$Z_1 = e^{4K}(e^{3h} + e^{-3h}) + e^{-4K}(e^h + e^{-h}) + 2(e^h + e^{-h})$$
$$Z_2 = e^{2K}(e^{3h} + e^{-3h}) + e^{-2K}(e^h + e^{-h}) + 2(e^h + e^{-h})$$
$$Z_3 = 2e^{2K}(e^{3h} + e^{-3h}) + 2e^{-2K}(e^h + e^{-h}) + 4(e^h + e^{-h}).$$

Identify the one that is correct for this system, and give a reason why each of the others is incorrect.

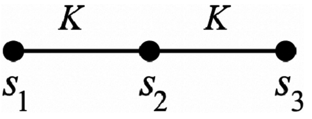


Figure 2.10 A three-spin system with periodic boundary conditions can also be pictured as three spins at the vertices of an equilateral triangle.

**2.7 Solution: (a)** The Hamiltonian for this system, with two nearest-neighbor interactions and three single-spin interactions, is

$$-\beta H = K(s_1 s_2 + s_2 s_3) + h(s_1 + s_2 + s_3).$$

**(b)** $Z_2$ is correct. $Z_1$ is incorrect because the Hamiltonian cannot give $4K$ for any configuration. $Z_3$ is incorrect because the total number of terms in it is 16, but the total number of configurations in the system is only 8.

**2.8 Figure 2.25** shows three curves (1, 2, 3) representing—in no particular order—the free energy, entropy, and specific heat of a finite Ising system plotted as a function of $K$. **(a)** Which curve is the reduced specific heat per site? Explain. **(b)** Which curve is the reduced free energy per site? Explain. **(c)** Which curve is the reduced entropy per site? Explain.

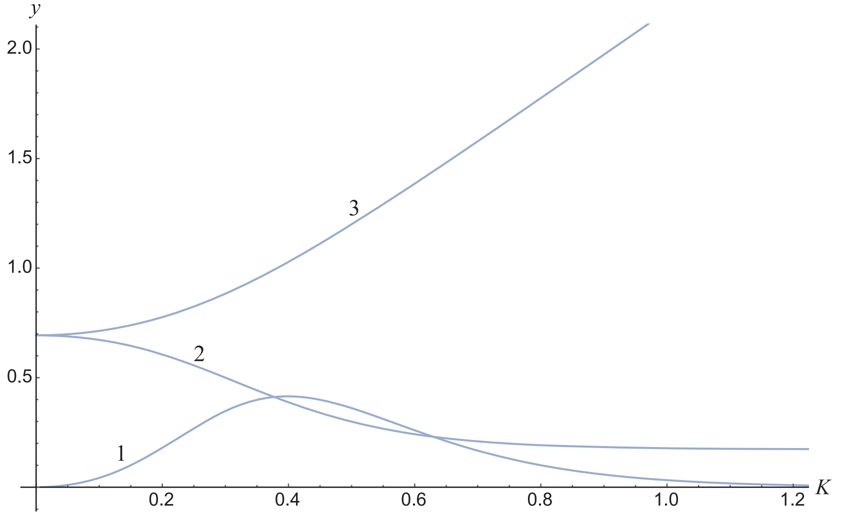


Figure 2.25 Free energy, specific heat, and entropy for an Ising system.

**2.8** **Solution: (a)** Curve 1 is the specific heat because it has a peak for finite $K$, and goes to zero for large and small $K$. **(b)** Curve 3 is the free energy because it grows larger with increasing $K$. **(c)** Curve 2 is the entropy because it is finite for $K = 0$ and decreases to a lower value for large $K$.

**2.9** Add an external magnetic field $h$ to the $2 \times 2$ system in **Figure 2.21**. **(a)** Write the Hamiltonian for this system. **(b)** Calculate the reduced free energy per spin, $f(K, h)$.

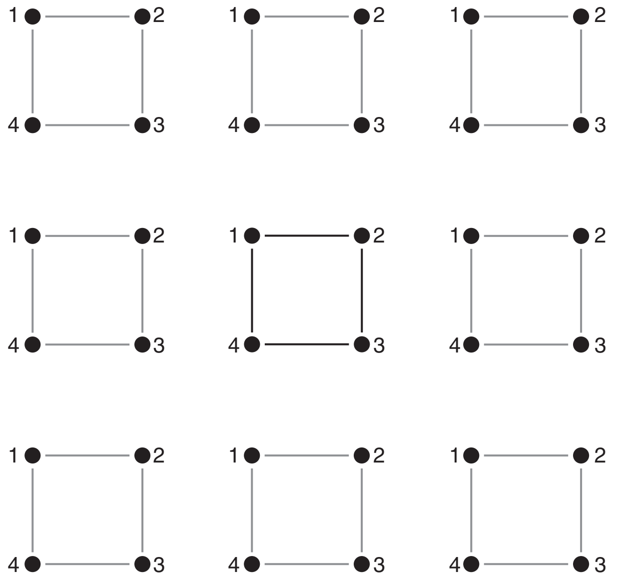


Figure 2.21 Periodic boundary conditions for a $2 \times 2$ square lattice correspond to replicating the $2 \times 2$ square to form an infinite lattice. The original $2 \times 2$ lattice is shown in the center in bold; replicas are shown ghosted.

**2.9** **Solution: (a)** The reduced Hamiltonian for this system is

$$-\beta H = K(s_1(s_4 + s_2) + s_2(s_3 + s_1) + s_3(s_2 + s_4) + s_4(s_1 + s_3)) + h(s_1 + s_2 + s_3 + s_4).$$

**(b)** The reduced free energy per site is

$$f(K, h) = \frac{1}{4} \ln(4 + 4e^{-2h} + 4e^{2h} + 2e^{-8K} + e^{8K-4h} + e^{8K+4h}).$$

**2.10** What is the ground-state ($K \to +\infty$) reduced entropy per spin for the $3 \times 3$ system shown in **Figure 2.23**?

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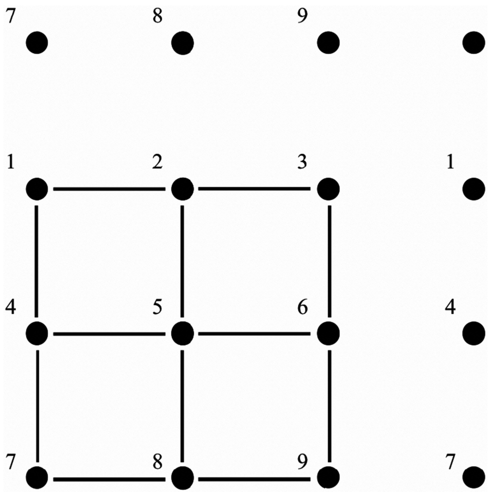


Figure 2.23 A 3 × 3 square lattice with periodic boundary conditions.

**2.10 Solution:** The ground state has two configurations—all spins +1 and all spins –1. The system consists of nine sites. It follows that the reduced entropy per site is $\frac{1}{9}\ln 2$.

# Chapter 3 Partial Summations and Effective Interactions

**3.1** Refer to **Figure 3.1** for the following questions. **(a)** Is $Z_+$ greater than, less than, or equal to $Z_-$ in the zero-coupling limit? Explain. **(b)** What are $Z_+$ and $Z_-$ in the zero-coupling limit?

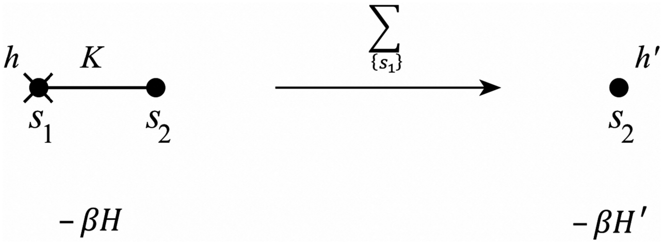


Figure 3.1 A simple partial-summation transformation. In this case, the spin $s_1$ is summed over, as indicated with the X. The result is an effective Hamiltonian for spin $s_2$.

**3.1 Solution: (a)** Equal. There is no coupling to spin 2 in this limit, and hence the partial partition functions don't depend on the state of spin 2. **(b)** In the limit of zero coupling–that is, $K \to 0$–we find $Z_+ = Z_- = e^h + e^{-h}$. In these expressions, the first term corresponds to $s_1 = +1$ and the second term corresponds to $s_1 = -1$.

**3.2** Consider the partial-summation transformation shown in Figure 3.11, where the configurations of spin $s_1$ are summed over to generate an effective Hamiltonian for spin $s_2$.

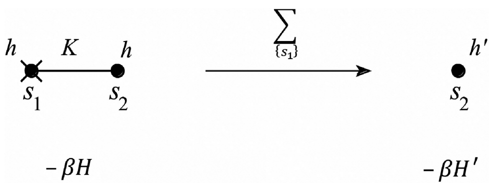


Figure 3.11 A partial-summation transformation from two spins to one spin.

The original (unprimed) and effective (primed) Hamiltonians for this system are as follows:

$$-\beta H = K s_1 s_2 + h(s_1 + s_2)$$
$$-\beta H' = K_0' + h' s_2$$

**(a)** What are $Z_+$ and $Z_-$ for this system? **(b)** Find $K_0'$ and compare your answer to the result given in Equation 3.11. **(c)** Find $h'$ and compare your answer to the result given in Equation 3.10.

**3.2 Solution: (a)** Setting $s_2 = +1$ yields the following partial partition function:

$$Z_+ = e^{K+2h} + e^{-K} = e^h(e^{K+h} + e^{-K-h}).$$

Setting $s_2 = -1$ yields

$$Z_- = e^{K-2h} + e^{-K} = e^{-h}(e^{K-h} + e^{-K+h}).$$

The quantities in brackets are the partial partition functions given in Equations 3.10 and 3.11. **(b)** We find

$$K_0' = \frac{1}{2}\ln(Z_+Z_-) = \frac{1}{2}\ln[(e^{K+h} + e^{-K-h})(e^{K-h} + e^{-K+h})].$$

This is the same result given in Equation 3.11, due to the fact that the term $e^h$ in $Z_+$ and $e^{-h}$ in $Z_-$ cancel one another out. **(c)** We find

$$h' = \frac{1}{2}\ln\left(\frac{Z_+}{Z_-}\right) = h + \frac{1}{2}\ln\left(\frac{e^{K+h} + e^{-K-h}}{e^{K-h} + e^{-K+h}}\right).$$

This is just $h$ plus the result in Equation 3.10. Thus, the added the magnetic field in the original system goes straight through to the primed system.

**3.3** Consider the partial-summation transformation shown in **Figure 3.12**, where the configurations of spins $s_1$ and $s_2$ are summed over to generate an effective Hamiltonian for the remaining spin, $s_3$.

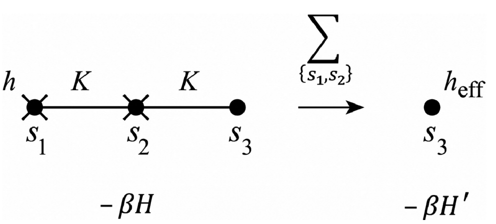


Figure 3.12 A partial-summation transformation from three spins to one spin.

The original (unprimed) and effective (primed) Hamiltonians for this system are as follows:

$$-\beta H = K(s_1 s_2 + s_2 s_3) + h s_1$$
$$-\beta H' = K_0' + h_{\text{eff}} s_3$$

**(a)** Do you expect $h_{\text{eff}}$ to be greater than, less than, or equal to $h''$ given in Equation 3.12? Explain. **(b)** Find $h_{\text{eff}}$ and compare with Equation 3.12. **(c)** Find $K_0'$ for this transformation.

**3.3 Solution: (a)** Equal. It's the same transformation, only done in one step instead of two, so the final result for $h_{\text{eff}}$ is the same. **(b)** The effective magnetic field is

$$h_{\text{eff}} = \frac{1}{2}\ln\left(\frac{e^{2K+h} + 2e^{-h} + e^{-2K+h}}{e^{2K-h} + 2e^{h} + e^{-2K-h}}\right).$$

This is exactly the same as in Equation 3.12. **(c)** The renormalized zero level is

$$K_0' = \frac{1}{2}\ln(Z_+ Z_-) = \frac{1}{2}\ln\left[\left(e^{2K+h} + 2e^{-h} + e^{-2K+h}\right)\left(e^{2K-h} + 2e^{h} + e^{-2K-h}\right)\right].$$

**3.4 Generalized three-to-two transformation** In **Figure 3.13** we show a generalized partial-summation transformation where the configurations of spin $s_2$ are summed over to generate an effective Hamiltonian for spins $s_1$ and $s_3$.

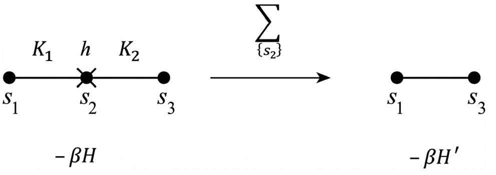


Figure 3.13 A generalized partial-summation transformation from three spins to two spins.

The Hamiltonian for the original system is

$$-\beta H = K_1 s_1 s_2 + K_2 s_2 s_3 + h s_2$$

**(a)** How many independent partial partition functions are there for this transformation? **(b)** Write out the effective Hamiltonian $-\beta H'$ in terms of primed quantities. **(c)** Determine each of the primed quantities in $-\beta H'$ in terms of $K_1$, $K_2$, and $h$.

**3.4 Solution: (a)** There are four independent partial partition functions for this system, $Z_{++}$, $Z_{+-}$, $Z_{-+}$ and $Z_{--}$. **(b)** The renormalized Hamiltonian must contain three independent interactions plus a renormalized zero level, for a total of four independent quantities—corresponding to the four independent partial partition functions. Hence, we can write

$$-\beta H' = K' s_1 s_3 \; + \; h_1' s_1 \; + \; h_2' s_3 \; + \; 2K_0'.$$

**(c)** The renormalized quantities in the effective Hamiltonian are as follows:

$$\begin{aligned} K' &= \frac{1}{4}\ln\left(\frac{Z_{++}Z_{--}}{Z_{+-}Z_{-+}}\right) \\ h_1' &= \frac{1}{4}\ln\left(\frac{Z_{++}Z_{+-}}{Z_{--}Z_{-+}}\right) \\ h_2' &= \frac{1}{4}\ln\left(\frac{Z_{++}Z_{-+}}{Z_{--}Z_{+-}}\right) \\ K_0' &= \frac{1}{8}\ln(Z_{++}Z_{+-}Z_{-+}Z_{--}). \end{aligned}$$

**3.5** Consider the partial-summation transformation shown in **Figure 3.14**, where the configurations of spins $s_2$, $s_3$, and $s_4$ are summed over to generate an effective Hamiltonian for the remaining spins, $s_1$ and $s_5$.

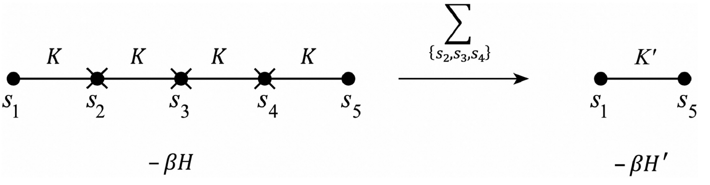


Figure 3.14 A partial-summation transformation from five spins to two spins.

The original (unprimed) and effective (primed) Hamiltonians for this system are as follows:

$$-\beta H = K(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_5)$$
$$-\beta H' = 2K_0' + K' s_1 s_5$$

**(a)** Which of the following expressions is the partial partition function $Z_{++}$ for this system?

$$Z_1 = e^{4K} + 4 + e^{-4K}$$
$$Z_2 = e^{3K} + 6 + e^{-3K}$$
$$Z_3 = e^{4K} + 6 + e^{-4K}$$
$$Z_4 = e^{3K} + 8 + e^{-3K}$$

**(b)** For each of the other expressions, give at least one reason why it cannot be $Z_{++}$. **(c)** What is $K_0'$ in the zero-coupling limit?

**3.5 Solution: (a)** $Z_3$ is the correct partial partition function. **(b)** $Z_1$ is incorrect because the total number of terms is 6, and it must be $2^3 = 8$; $Z_2$ is incorrect because it has $3K$ in the exponentials, whereas the maximum number multiplying $K$ should be 4; $Z_4$ is incorrect because the total number of terms is 10 rather than 8. **(c)** In the final system there are two sites, and the number of configurations summed over to yield each partial partition function is 8, therefore $K_0' = \frac{1}{2}\ln 8 = \frac{3}{2}\ln 2$.

**3.6 The Generalized Star-Triangle Transformation** In **Figure 3.15** we show a star-triangle transformation in which the strength of the couplings to spins $s_1$, $s_2$, and $s_3$ in the original system can be different.

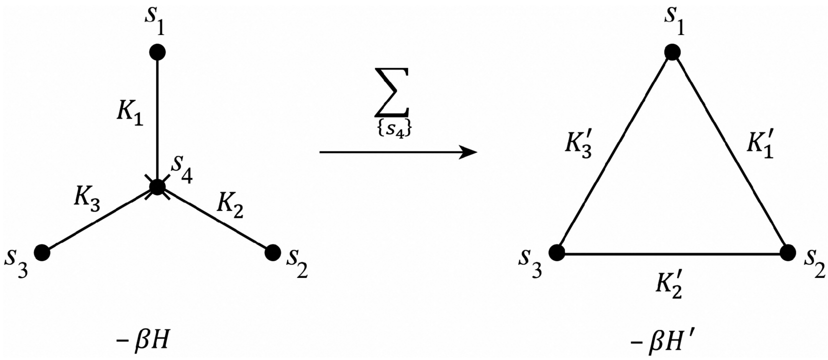


Figure 3.15 A star-triangle transformation with different couplings to each of the outer spins of the original system.

The original (unprimed) and effective (primed) Hamiltonians for this system are as follows:

$$-\beta H = K_1 s_1 s_4 + K_2 s_2 s_4 + K_3 s_3 s_4$$
$$-\beta H' = 3K_0' + K_1' s_1 s_2 + K_2' s_2 s_3 + K_3' s_3 s_1$$

**(a)** If $K_1 = 0$ and $K_2 = K_3 > 0$, is $K_1'$ greater than, less than, or equal to $K_2'$? Explain. **(b)** Assuming $K_1$, $K_2$, and $K_3$ all have different values, how many independent partial partition functions are there in this system? **(c)** Derive the expressions for $K_1'$, $K_2'$, and $K_3'$. **(d)** Show that $K_1 = 0$, $K_2 = K_3 = K$ yields the expected results for $K_1'$, $K_2'$, and $K_3'$.

**3.6 Solution: (a)** Less than. In fact, $K_1' = 0$, because there is no "through connection" between $s_1$ and $s_2$, while $K_2' > 0$, because there *is* a "through connection" between $s_2$ and $s_3$. **(b)** There are four independent partial partition functions, $Z_{+++}$, $Z_{++-}$, $Z_{+-+}$, and $Z_{-++}$. **(c)** The effective couplings are

$$K_1' = \frac{1}{4}\ln\left(\frac{Z_{+++}Z_{++-}}{Z_{+-+}Z_{-++}}\right)$$
$$K_2' = \frac{1}{4}\ln\left(\frac{Z_{+++}Z_{-++}}{Z_{++-}Z_{+-+}}\right)$$
$$K_3' = \frac{1}{4}\ln\left(\frac{Z_{+++}Z_{+-+}}{Z_{++-}Z_{-++}}\right)$$

**(d)** In this case, $K_1' = 0$, $K_3' = 0$, and $K_2' = \frac{1}{2}\ln(\cosh 2K)$, which is the same result as in Equation 3.19 for Figure 3.5, as expected.

**3.7 The Diamond Transformation** The diamond transformation is illustrated in **Figure 3.16**. In this case, two of the four original spins are summed over to generate an effective Hamiltonian for the remaining two spins.

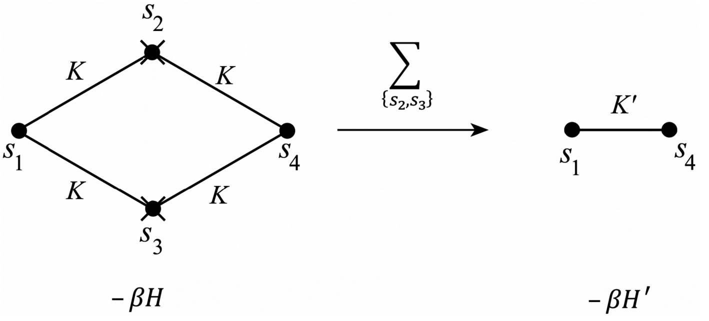


Figure 3.16 The diamond transformation.

The original and effective Hamiltonians are given below:

$$-\beta H = K(s_1 s_2 + s_1 s_3 + s_2 s_4 + s_3 s_4)$$

$$-\beta H' = 2K_0' + K' s_1 s_4$$

**(a)** Calculate $K'$ as a function of $K$. **(b)** Find the value of $K$ where $K'(K) = K$.

**3.7 Solution: (a)** The sum over $s_2$ yields $\frac{1}{2}\ln(\cosh 2K)$, as we see in Equation 3.19, and the same is true of $s_3$. Adding these results together gives the final coupling, $K' = \ln[\cosh(2K)]$. **(b)** The fixed point, $K'(K) = K$, for this transformation is $K = 0.6094\ldots$.

**3.8 The Sierpinski Transformation** The zero-field Sierpinski transformation is shown in **Figure 3.17**. This transformation is related to the process used to generate the fractal known as the Sierpinski Gasket.

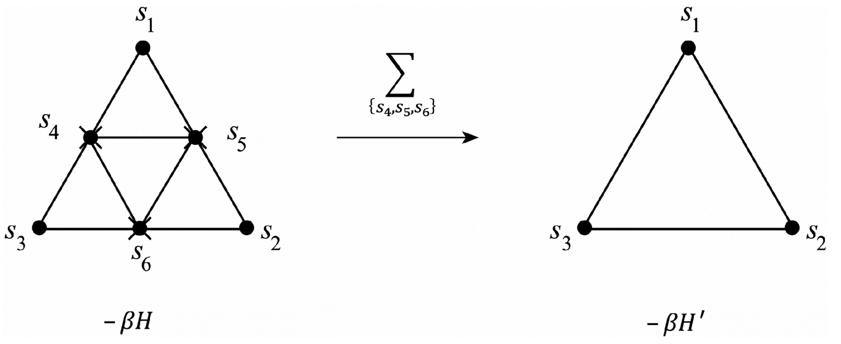


Figure 3.17 The Sierpinski transformation.

The original and effective Hamiltonians are given below:

$$-\beta H = K(s_1 s_4 + s_1 s_5 + s_4 s_5 + s_4 s_6 + s_5 s_6 + s_3 s_6 + s_6 s_2 + s_3 s_4 + s_2 s_5)$$

$$-\beta H' = 3K_0' + K'(s_1 s_2 + s_2 s_3 + s_3 s_1)$$

**(a)** Calculate $Z_{+++}$. **(b)** Calculate $Z_{++-}$.

**3.8 Solution: (a)** Summing over the eight configurations of $s_4$, $s_5$, and $s_6$, with $s_1 = s_2 = s_3 = +1$, yields

$$Z_{+++} = e^{9K} + 3e^{K} + 4e^{-3K}.$$

**(b)** Summing over the eight configurations of $s_4$, $s_5$, and $s_6$, with $s_1 = s_2 = +1$, and $s_3 = -1$, yields

$$Z_{++-} = e^{5K} + 4e^{K} + 3e^{-3K}.$$

## Chapter 4 Infinite Ising Systems in One Dimension

**4.1** In Equation 4.2, explain why 2 is raised to the power $N$, but cosh $K$ is only raised to the power $N - 1$.

**4.1 Solution:** There are $N$ sites in the system, each of which contributes a factor of 2, but only $N - 1$ bonds, each of which contributes a factor of cosh $K$.

**4.2** **(a)** Find the reduced entropy per site in the limit $K \to \infty$ for a 1-D lattice with $N$ sites. **(b)** What is this entropy in the thermodynamic limit? **(c)** Are your results in parts (a) and (b) different for periodic versus free boundary conditions?

**4.2 Solution: (a)** In this limit the temperature goes to zero. There are two ground states, and $N$ sites, hence the reduced entropy per site is $\frac{1}{N}\ln 2$. **(b)** In the thermodynamic limit we have $N \to \infty$, and hence the reduced entropy per site goes to 0. **(c)** No. The type of boundary condition doesn't change the fact that there are two ground states and $N \to \infty$ sites.

**4.3** Referring to **Figure 4.9**, approximately what value of $K$ yields a correlation length equal to 500 lattice spacings?

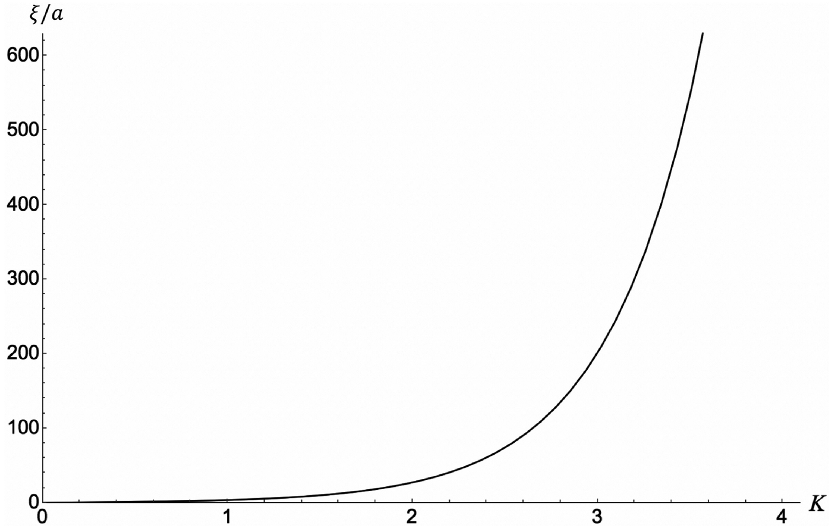


Figure 4.9 Correlation length, $\xi$, in units of the lattice spacing, $a$, as a function of $K$.

**4.3 Solution:** Referring to the graph in Figure 4.9, we find that $\xi/a = 500$ when $K \sim 3.45$.

**4.4** Consider the 1-D Ising model with nearest-neighbor couplings $K$. **(a)** Evaluate $\xi/a$ for $K = 3$. **(b)** If $K$ is increased, does $\xi/a$ increase, decrease, or stay the same? Explain.

**4.4 Solution: (a)** Substituting $K = 3$ in $\xi = \frac{a}{|\ln(\tanh K)|}$ (Equation 4.12), we find $\xi/a = 201.714\ldots$. **(b)** If $K$ is increased, tanh $K$ approaches 1, and ln (tanh $K$) approaches zero, hence $\xi/a$ increases.

**4.5** Evaluate the following sum:

$$\sum_{\{s_1\}} s_2 e^{K s_1 s_2}$$

Show that your result is independent of $s_1$.

**4.5 Solution:** Carrying out the summation, we find

$$\sum\nolimits_{\{s_1\}} s_2 e^{K s_1 s_2} = s_2 e^{K s_2} + s_2 e^{-K s_2} = s_2(2\,\cosh(K)).$$

We have summed over $s_1$ and hence there is no longer any dependence on it.

**4.6** Use the results from Equation 4.2 to calculate the partition function for a chain of 4 Ising spins with free boundary conditions and nearest-neighbor couplings $K$.

**4.6 Solution:** Direct substitution with $N = 4$ yields $Z_4 = 2^4 (\cosh\,K)^3$.

**4.7** Write the transfer matrix for the case of a ladder lattice with a magnetic field, $h$.

**4.7 Solution:** Adding a magnetic field $h$ to the Hamiltonian on the ladder lattice yields $K \to \infty$

$$T_{(s_i,\sigma_i;s_{i+1},\sigma_{i+1})} = e^{K\left[s_i s_{i+1} + \sigma_i \sigma_{i+1} + \frac{1}{2}(s_i \sigma_i + s_{i+1}\sigma_{i+1})\right] + \frac{1}{2}h(s_i + \sigma_i + s_{i+1} + \sigma_{i+1})}.$$

The factor of $^1/_2$ in front of $h$ accounts for the sites $i$ and $i + 1$ being shared equally with the transfer matrices on the left and right. Writing out $T$ in matrix form we have

$$T = \begin{pmatrix} e^{3K+2h} & e^h & e^h & e^{-K} \\ e^h & e^K & e^{-3K} & e^{-h} \\ e^h & e^{-3K} & e^K & e^{-h} \\ e^{-K} & e^{-h} & e^{-h} & e^{3K-2h} \end{pmatrix}.$$

# Chapter 5 The Onsager Solution and Exact Series Expansions

**5.1** Give the high-temperature expansion of the reduced free energy per site for the ladder lattice to order $v^6$, where $v = \tanh K$.

**5.1 Solution:** For the ladder lattice at high temperature we have

$$f = \ln 2 + \frac{3}{2}\ln[\cosh(K)] + \frac{1}{2}v^4 + \frac{1}{2}v^6 + \dots.$$

The $\frac{3}{2}\ln[\cosh(K)]$ term indicates there are 1.5 bonds per site on the infinite lattice, the $\frac{1}{2}v^4$ term indicates there is only one square for every two sites on the lattice, and the $\frac{1}{2}v^6$ term indicates the same for a rectangle of six sides.

**5.2** Give the low-temperature expansion of the reduced free energy per site for the simple cubic lattice to order $(e^{-2K})^{10}$.

**5.2 Solution:** For the simple cubic lattice at low temperature we have

$$f = 3K + (e^{-2K})^6 + 3(e^{-2K})^{10} + \dots.$$

The $3K$ term indicates there is one bond per site in each of the three coordinate directions, the $(e^{-2K})^6$ term indicates two broken bonds in each of the coordinate directions when one spin is flipped, and the $3(e^{-2K})^{10}$ term indicates 10 broken bonds when two neighboring spins are flipped, which can happen along each of the three coordinate directions.

**5.3** Give the low-temperature expansion of the reduced free energy per site for the triangular lattice to order $(e^{-2K})^{10}$.

**5.3 Solution:** For the triangular lattice at low temperature we have

$$f = 3K + (e^{-2K})^6 + 3(e^{-2K})^{10} + \dots.$$

The $3K$ term indicates there are three nearest-neighbor bonds per site on a triangular lattice, the $(e^{-2K})^6$ term indicates six broken bonds when one spin is flipped, and the $3(e^{-2K})^{10}$ term indicates 10 broken bonds when two neighboring spins are flipped, which can happen along each of the three nearest-neighbor directions.

**5.4** Give the low-temperature expansion of the reduced free energy per site for the hexagonal lattice to order $(e^{-2K})^4$.

**5.4 Solution:** For the hexagonal lattice at low temperature we have

$$f = \frac{3}{2}K + (e^{-2K})^3 + \frac{3}{2}(e^{-2K})^4 + \ldots.$$

The $\frac{3}{2}K$ term indicates there are 1.5 nearest-neighbor bonds per site on a hexagonal lattice, the $(e^{-2K})^3$ term indicates three broken bonds when one spin is flipped, and the $\frac{3}{2}(e^{-2K})^4$ term indicates 4 broken bonds when two neighboring spins are flipped.

# Chapter 6 The Mean-Field Approach

**6.1** Consider $T/T_c = 0.5$ in the single-spin mean-field calculation. Starting with $m_0 = 0.25$, find **(a)** $m_1$ and **(b)** $m_2$.

**6.1 Solution:** **(a)** We can find $m_1$ using $m_1 = \tanh\left[\left(\frac{T}{T_c}\right)^{-1} m_0\right]$. With $m_0 = 0.25$ and $\frac{T}{T_c} = 0.5$, the result is $m_1 = 0.4621\ldots$. **(b)** Similarly, we can find $m_2$ with $m_2 = \tanh\left[\left(\frac{T}{T_c}\right)^{-1} m_1\right]$. With $m_1 = 0.4621\ldots$, the result is $m_2 = 0.7278\ldots$.

**6.2** Find $m^*$ for $T/T_c = 0.75$ in the single-spin mean-field calculation.

**6.2 Solution:** Solving $m^* = \tanh\left[\left(\frac{T}{T_c}\right)^{-1} m^*\right]$ with $\frac{T}{T_c} = 0.75$, we find $m^* = 0.7755\ldots$.

**6.3** Find **(a)** the magnetization, $m$, and **(b)** the reduced specific heat per site, $c$, for $T/T_c = 0.85$ in the single-spin mean-field calculation.

**6.3 Solution:** **(a)** First, we find the magnetization at this temperature with $m^* = \tanh\left[\left(\frac{T}{T_c}\right)^{-1} m^*\right]$ and $\frac{T}{T_c} = 0.85$. The result is $m^* = m = 0.6295\ldots$.

**(b)** Next, we find the reduced specific heat with Equation 6.12,

$$c = \frac{C}{Nk_B} = \frac{m^2}{\frac{(T/T_c)^2}{(1-m^2)} - \left(\frac{T}{T_c}\right)}.$$

Substituting $m = 0.6295\ldots$ and $\frac{T}{T_c} = 0.85$ we find $c = 1.1428\ldots$.

# Chapter 7 Position-Space Renormalization-Group Techniques

**7.1** Consider the $b = 3$ decimation transformation for the 1-D chain lattice shown in **Figure 7.20**. The lattice has nearest-neighbor interactions of strength $K$. **(a)** Derive the transformation equations for $K'$ and $K_0'$ in terms of the partial partition functions $Z_{++}$ and $Z_{+-}$. **(b)** Derive expressions for $Z_{++}$ and $Z_{+-}$. **(c)** What is the transformation equation relating the free energy of the original lattice, $f(K)$, to the free energy of the primed lattice, $f(K')$?

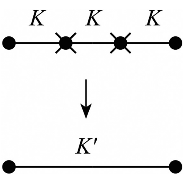


Figure 7.20 Problems 7.1 and 7.2.

**7.1 Solution: (a)** The transformation equations are as follows:

$$K' = \frac{1}{2}\log\left[\frac{Z_{++}}{Z_{+-}}\right]$$

$$K_0{}' = \frac{1}{2}\log[Z_{++}Z_{+-}].$$

This is the same as for $b = 2$ in Equation 7.1, though the results for $Z_{++}$ and $Z_{+-}$ are different for $b = 3$. **(b)** For $b = 3$ we find

$$Z_{++} = e^{3K} + 3e^{-K}$$

$$Z_{+-} = 3e^{K} + e^{-3K}.$$

**(c)** The transformation of the free energy is $f(K) = \frac{1}{3}K_0{}' + \frac{1}{3}f(K')$.

**7.2** Consider the $b = 3$ 1-$D$ transformation in Problem 7.1. **(a)** For a given finite value of $K$, do you expect $K'$ for $b = 3$ to be greater than, less than, or equal to $K'$ for $b = 2$? Explain. **(b)** For a given value of $K$, do you expect $K_0{}'$ for $b = 3$ to be greater than, less than, or equal to $K_0{}'$ for $b = 2$? Explain. **(c)** Check your results in parts (a) and (b) with explicit calculations for $K = 1.0$.

**7.2 Solution:** **(a)** Less than. The end spins are farther apart, and therefore their correlation—and hence their coupling $K'$—is smaller for $b = 3$ than for $b = 2$. **(b)** Greater than. More spins are summed away in each iteration, and therefore the contribution to the free energy—and hence the term $K_0{}'$—is larger for $b = 3$ than for $b = 2$. **(c)** For $K = 1$ we have $K' = 0.4743\ldots$ and $K_0{}' = 2.5790\ldots$ for $b = 3$, and $K' = 0.6625\ldots$ and $K_0{}' = 1.3556\ldots$ for $b = 2$. This agrees with the answers in parts (a) and (b).

**7.3** Consider Equation 7.4:

$$f(K) = \frac{1}{2}K_0{}' + \frac{1}{2}f(K')$$

In the text we verified this expression for $K = 0$; now we would like to verify it for a finite value of $K$. **(a)** Calculate $K_0{}'$ for $K = 2.0$. **(b)** Calculate $K'$ for $K = 2.0$. **(c)** Using $f(K) = \ln 2 + \ln(\cosh K)$, verify Equation 7.4 for $K = 2.0$.

**7.3 Solution:** **(a)** For $K = 2.0$ we have

$$K_0' = \ln 2 + 0.5\ln[\cosh(2K)] = 2.34674\ldots.$$

**(b)** For $K = 2.0$ we have

$$K' = 0.5\ln[\cosh(2K)] = 1.65359\ldots.$$

**(c)** For $K = 2.0$ we have

$$f(K) = \ln 2 + \ln(\cosh K) = 2.01815\ldots.$$

Now, for Equation 7.4, we have

$$\begin{aligned} f(K) &= 0.5K_0'(K) + 0.5f(K') = 0.5(2.34674\ldots) + 0.5(1.68955) \\ &= 2.01815\ldots. \end{aligned}$$

This verifies the transformation of the free energy given in Equation 7.4.

**7.4** Calculate the reduced free energy per site for the 2-D square lattice with nearest-neighbor coupling $K = 0.5$ using the following methods: **(a)** MK ($b = 2$), **(b)** CDA, **(c)** cell-cluster. For comparison, the exact result is $f = 1.0257\ldots$.

**7.4 Solution:** **(a)** For the MK approximation, using Equation 7.12 and $K = 0.5$, we find $f = 1.24076\ldots$. **(b)** For the CDA approximation at $K = 0.5$ we find $f = 1.03417\ldots$, very close to the exact value of $1.0257\ldots$. **(c)** For the cell-cluster method at $K = 0.5$ we find $f = 1.02825\ldots$, even closer to the exact value.

**7.5 MK Transformation in 3D** Consider the MK transformation in Figures 7.6 and 7.7, only now for the case of a 3D simple cubic lattice. **(a)** What is the bond strengthening factor $\widetilde{K}(K)/K$ in this case? **(b)** Do you expect the value of the fixed point $K^*$ to be greater than, less than, or equal to the value found for the 2D lattice? Explain. (c) Use the reverse flow transformation to find $K^*$.

**7.5 Solution: (a)** In this case $\widetilde{K}$ is four times greater than $K$; that is, $\frac{\widetilde{K}}{K} = 4$. **(b)** The value of $K^*$ should be less for 3D MK than it is for 2D MK because the bonds are strengthened by a greater amount in 3D. In addition, we expect the critical value of $K$ to be smaller for a 3D lattice, and if 3D MK is a good approximation for a 3D lattice, then $K^*$ will be smaller as well. **(c)** We find $K^* = 0.06527\ldots$.

**7.6 Tyrant's Rule** In this problem, we consider a different type of projection from site spins to cells spins referred to as *Tyrants' Rule*. In this case, the value of the cell spin is simply equal to one of the site spins—the tyrant. As a specific example, let's say that the site spin in the upper left corner of the cell is the tyrant. **(a)** Give a table like Table 7.7 that defines this projection. **(b)** Find $Z_{++}$ and $Z_{+-}$ for this transformation. **(c)** Find the fixed point $K^*$ for this transformation.

**7.6 Solution: (a)** Here's the mathematical expression for the Tyrant's-Rule Projection applied to the first cell: $P(\{s_1, s_2, s_3, s_4\}, s') = \frac{1}{2}(1 + s_1 s')$. In this expression we note that  is chosen to be the upper left spin in the cell. For the second cell, we have $P(\{s_5, s_6, s_7, s_8\}, s') = \frac{1}{2}(1 + s_5 s')$, where $s_5$ is the upper left spin. In each case, note that the projection is 1 if  and the upper left spin are the same, and 0 if they are not. Here is the projection in the form of a table:

| $S_i$ | $s' = +1$ | $s' = -1$ |
|---|---|---|
| $++$<br>$++$ | 1 | 0 |
| $+-$ $++$ $++$<br>$++$ $+-$ $-+$ $+-$<br>$++$ $+-$ $+-$ $--$<br>$--$ $-+$ $+-$ | 1 | 0 |
| $-+$ $--$ $--$<br>$--$ $-+$ $+-$ $-+$<br>$--$ $-+$ $-+$ $++$<br>$++$ $+-$ $-+$ | 0 | 1 |
| $--$<br>$--$ | 0 | 1 |

**(b)** For the partition functions, we find:

$$Z_{++} = \sum_{\{s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8\}} P(\{s_1,s_2,s_3,s_4\},+1)P(\{s_5,s_6,s_7,s_8\},+1)e^{-\beta H}$$

$$= e^{16K} + 8e^{8K} + 8e^{4K} + 30 + 8e^{-4K} + 8e^{-8K} + e^{-16K}$$

$$Z_{+-} = \sum_{\{s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8\}} P(\{s_1,s_2,s_3,s_4\},+1)P(\{s_5,s_6,s_7,s_8\},-1)e^{-\beta H}$$

$$= 6e^{8K} + 16e^{4K} + 20 + 16e^{-4K} + 6e^{-8K}.$$

**(c)** Recall that

$$K' = \frac{1}{4}\ln\left[\frac{Z_{++}}{Z_{+-}}\right].$$

For the fixed point, we find $K'(K^*) = K^* = 0.4965\ldots$. This is greater than the exact value (0.440…), and with just a slightly greater error than the majority rule results.

**7.7 Egalitarian Rule** In this problem, we consider a different type of projection from site spins to cells spins referred to as *The Egalitarian Rule*. In this case, each site spin contributes equally to the cell spin. For example, if 4 site spins are +, the system maps to a + cell spin with weight 1 and to a – cell spin with weight 0. On the other hand, if 3 site spins are + and one site spin is –, the system maps to a + cell spin with weight $^3/_4$, and to a – cell spin with weight $\frac{1}{4}$. **(a)** Give a table like Table 7.7 that defines this projection. **(b)** Find $Z_{++}$ and $Z_{+-}$ for this transformation. **(c)** Find the fixed point $K^*$ for this transformation.

**7.7 Solution: (a)** Here's the mathematical expression for the Egalitarian-Rule Projection applied to the first cell:

$$P(\{s_1,s_2,s_3,s_4\},s') = \frac{1}{8}(4 + (s_1+s_2+s_3+s_4)s').$$

For the second cell, we have

$$P(\{s_5,s_6,s_7,s_8\},s') = \frac{1}{8}(4 + (s_5+s_6+s_7+s_8)s').$$

Here is the projection in the form of a table:

| $S_i$ | $s' = +1$ | $s' = -1$ |
|---|---|---|
| $++$<br>$++$ | 1 | 0 |
| $-+\ \ +-\ \ ++\ \ ++$<br>$++\ \ ++\ \ +-\ \ -+$ | ¾ | ¼ |
| $--\ \ +-\ \ ++$<br>$++\ \ +-\ \ --$<br>$-+\ \ -+\ \ +-$<br>$-+\ \ +-\ \ -+$ | ½ | ½ |
| $+-\ \ -+\ \ --\ \ --$<br>$--\ \ --\ \ -+\ \ +-$ | ¼ | ¾ |
| $--$<br>$--$ | 0 | 1 |

**(b)** For the partition functions, we find:

$$Z_{++} = \sum_{\{s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8\}} P(\{s_1,s_2,s_3,s_4\},+1)\, P(\{s_5,s_6,s_7,s_8\},+1) e^{-\beta H}$$

$$= e^{16K} + \frac{17}{2}e^{8K} + \frac{21}{2}e^{4K} + 25 + \frac{23}{2}e^{-4K} + 7e^{-8K} + \frac{1}{2}e^{-16K}$$

$$Z_{+-} = \sum_{\{s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8\}} P(\{s_1,s_2,s_3,s_4\},+1)\, P(\{s_5,s_6,s_7,s_8\},-1) e^{-\beta H}$$

$$= \frac{11}{2}e^{8K} + \frac{27}{2}e^{4K} + 25 + \frac{25}{2}e^{-4K} + 7e^{-8K} + \frac{1}{2}e^{-16K}.$$

**(c)** Recall that

$$K' = \frac{1}{4}\ln\left[\frac{Z_{++}}{Z_{+-}}\right].$$

For the fixed point, we find $K'(K^*) = K^* = 0.4699\ldots$. This is roughly 7% greater than the exact value (0.440…), and the best result of the various rules we have considered.