Chapter 3: Limits and Continuity

Part C: Consequences of Continuity

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Boundedness Theorem

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Define $c_1 = (a_1 + b_1)/2$.

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Repeating this process, we get

$$[a, b] = [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots ,$$

so that f is unbounded on each of them, and with $b_n - a_n = (b - a)/2^{n-1}$.

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Note that $a = a_1 \le a_2 \le \cdots \le b_2 \le b_1 = b$.

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so that f is unbounded on each of them, and with $b_n - a_n = (b - a)/2^{n-1}$.
Note that $a = a_1 \leq a_2 \leq \cdots \leq b_2 \leq b_1 = b$.
The Completeness Axiom gives $\alpha \in \mathbb{R}$ such that $a_n \leq \alpha \leq b_n$ for every n.

Boundedness Theorem

(*proof continued*)

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From $a_n \leq \alpha \leq b_n$ for every n, we find that α is in each $[a_n, b_n]$.

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From $a_n \leq \alpha \leq b_n$ for every n, we find that α is in each $[a_n, b_n]$.

Since f is continuous at α, there is a $\delta > 0$ such that $|x - \alpha| < \delta$ implies $|f(x) - f(\alpha)| < 1$.

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Since f is continuous at α, there is a $\delta > 0$ such that $|x - \alpha| < \delta$ implies $|f(x) - f(\alpha)| < 1$.

In particular, f is bounded on $(\alpha - \delta, \alpha + \delta)$.

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Since the length of the intervals $[a_n, b_n]$ is halved at each stage, for large enough n we will have $[a_n, b_n] \subset (\alpha - \delta, \alpha + \delta)$, implying that f is bounded on $[a_n, b_n]$.

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This is a contradiction. $\square$

Extreme Value Theorem

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Theorem 2

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there are $c, d \in [a, b]$ such that

$$f(c) = \max\{\, f(x) \mid x \in [a, b] \,\},$$
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We use $\sup_I f$ to denote $\sup\{\, f(x) \mid x \in I \,\}$.

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Let $a_1 = a$, $b_1 = b$ and define $c_1 = (a_1 + b_1)/2$.
Then $M = \sup_{[a_1,c_1]} f$ or $\sup_{[c_1,b_1]} f$.

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Let the subinterval that gives equality be called $[a_2, b_2]$.

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Repeating the process, we get

$$[a, b] = [a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots ,$$

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Completeness Axiom gives a point c which is in each $[a_n, b_n]$.

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Completeness Axiom gives a point c which is in each $[a_n, b_n]$.

If $f(c) < M$ then there is a $\delta > 0$ such that $x \in (c - \delta, c + \delta)$ implies $f(x) < \dfrac{M + f(c)}{2}$.

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For large enough n we will have $[a_n, b_n] \subset (\alpha - \delta, \alpha + \delta)$, implying that the values of f on $[a_n, b_n]$ are bounded above by $\dfrac{M + f(c)}{2} < M$.

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This contradicts the choice of $[a_n, b_n]$, hence $f(c) = M$. $\square$

Span

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. The **span** of f on $[a, b]$ is $\sup\{|f(x) - f(y)| : x, y \in [a, b]\}$.

Task 1: Find the spans of the following functions on $[0, 1]$: $\mathrm{sgn}(x)$, $\sin \pi x$.

Task 2: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that the span of f on $[a, b]$ equals

$$\max\{ f(x) \mid x \in [a, b] \} - \min\{ f(x) \mid x \in [a, b] \}.$$

Small Span Theorem

Theorem 3

Let $f : [a, b] \to \mathbb{R}$ be continuous. For every $\epsilon > 0$ there is a partition P of $[a, b]$ such that the span of f is less than ϵ on every subinterval of P.

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Repeating, we get a sequence of intervals

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none of which have such a partition, and with $b_n - a_n = (b - a)/2^{n-1}$.

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none of which have such a partition, and with $b_n - a_n = (b - a)/2^{n-1}$.
In particular, the span of f is at least ϵ on every $[a_n, b_n]$.

Small Span Theorem

(*proof continued*)

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By the Completeness Axiom, there is an $\alpha \in \mathbb{R}$ such that $a_i \leq \alpha \leq b_i$ for every i.

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By the Completeness Axiom, there is an $\alpha \in \mathbb{R}$ such that $a_i \leq \alpha \leq b_i$ for every i.

Hence α is in each $[a_n, b_n]$.

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Since f is continuous at α, there is a $\delta > 0$ such that $|x - \alpha| < \delta$ implies $|f(x) - f(\alpha)| < \epsilon/2$.

And then $x, y \in (\alpha - \delta, \alpha + \delta) \implies |f(x) - f(y)| < \epsilon$.

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By taking large enough n we can ensure that $[a_n, b_n] \subset (\alpha - \delta, \alpha + \delta)$.

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By taking large enough n we can ensure that $[a_n, b_n] \subset (\alpha - \delta, \alpha + \delta)$.

Then we would have span of f being less than ϵ on $[a_n, b_n]$, which contradicts our earlier observation about these intervals. $\square$

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Integrability of Continuous Functions

Theorem 4

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable on $[a, b]$.

Integrability of Continuous Functions

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Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable on $[a, b]$.

Proof. Let $\epsilon > 0$. The Small Span Theorem gives a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that

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Let
$$m_i = \min\{\, f(x) \mid x \in [x_{i-1}, x_i] \,\},$$
$$M_i = \max\{\, f(x) \mid x \in [x_{i-1}, x_i] \,\}.$$

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Then $M_i - m_i < \epsilon/(b-a)$ for every i.

Define step functions $s \in \mathcal{L}_f$ and $t \in \mathcal{U}_f$ by $s(x_i) = t(x_i) = f(x_i)$ for each i and

$$\begin{aligned} s(x) &= m_i \qquad \text{if } x_{i-1} < x < x_i, \\ t(x) &= M_i \qquad \text{if } x_{i-1} < x < x_i. \end{aligned}$$

Integrability of Continuous Functions

(*proof continued*)

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Then $s \le f \le t$ on $[a, b]$ and

$$\int_a^b t(x)\,dx - \int_a^b s(x)\,dx = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

$$< \frac{\epsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \epsilon.$$

Integrability of Continuous Functions

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$$< \frac{\epsilon}{b-a}\sum_{i=1}^{n}(x_i - x_{i-1}) = \epsilon.$$

By the Riemann Condition, f is integrable on $[a, b]$. □

Piecewise Continuous Functions

A function $f\colon [a,b] \to \mathbb{R}$ is called **piecewise continuous** if there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that f is continuous on each open interval (x_{i-1}, x_i) and the one-sided limits of f exist at the x_i's.

Piecewise Continuous Functions

A function $f : [a, b] \to \mathbb{R}$ is called **piecewise continuous** if there is a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ such that f is continuous on each open interval (x_{i-1}, x_i) and the one-sided limits of f exist at the x_i's.

Theorem 5

Let $f : [a, b] \to \mathbb{R}$ be a piecewise continuous function. Then f is integrable.

Piecewise Continuous Functions

A function $f : [a, b] \to \mathbb{R}$ is called **piecewise continuous** if there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ such that f is continuous on each open interval (x_{i-1}, x_i) and the one-sided limits of f exist at the x_i's.

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Proof. Mimic the proof of integrability of piecewise monotonic functions. □

Average of a function

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If $f : [a, b] \to \mathbb{R}$ is integrable, we define its **average** by

$$\bar{f} = \bar{f}_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

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Mean Value Theorem for Integration

Theorem 6

Let $f : [a, b] \to \mathbb{R}$ be continuous. There is $c \in (a, b)$ such that

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By the intermediate value theorem, there is a number c between a' and b' with $f(c) = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\, dx$. □

Weighted average

A **weighted mean** or **weighted average** of numbers $x_1, \ldots, x_n$ is a combination $\sum_{i=1}^{n} w_i x_i$ where each $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$.

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This definition requires fg to be integrable. For that, see Exercise 10 of §2.2.

Mean Value Theorem for Weighted Integration

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Consider functions $f, g : [a,b] \to \mathbb{R}$ where f is continuous, while g is integrable and $g \geq 0$ on $[a,b]$. There is $c \in (a,b)$ such that

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Then intermediate value theorem gives the desired c.

Table of Contents

Limit at infinity

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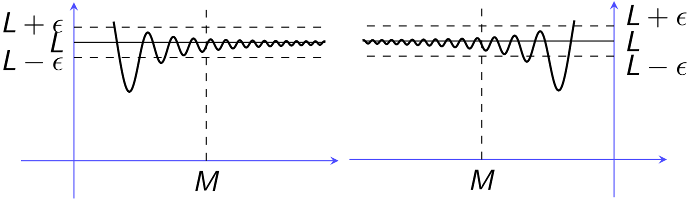

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Examples

Example: The basic limit for calculations at infinity is: $\lim_{x\to\infty} \frac{1}{x} = 0$.

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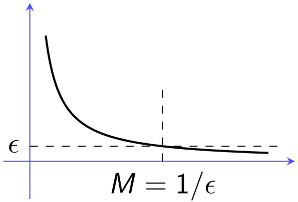

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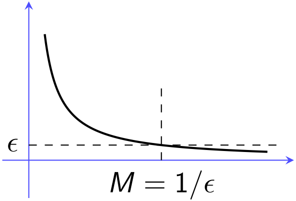

Task: Show that $\lim_{x\to\infty} e^{-x} = \lim_{x\to-\infty} e^{x} = 0.$

Monotone Convergence Theorem

Theorem 8

Suppose f is increasing on $[a, \infty)$. Then $\lim_{x \to \infty} f(x)$ exists if and only if f is bounded above, and then the limit equals the supremum of the values of f.
Similarly, if f is decreasing then $\lim_{x \to \infty} f(x)$ exists if and only if f is bounded below, and then the limit equals the infimum of the values of f.

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Monotone Convergence Theorem

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Hence, $x \geq M \implies L - \epsilon < M \leq f(x) \leq L < L + \epsilon$.

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Then $\epsilon > 0$ and for every $x \geq x_0$ we have $f(x) - L \geq \epsilon$, which contradicts L being the limit of f.

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The first part of the proof gives $L = \sup(A)$. $\square$

Standard results

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Theorem 9

The algebra of limits and the sandwich theorem hold for limits at infinity.

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Let $M = \max\{M_f, M_g\}$.
Then $x > M \implies L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon$.
Hence $\lim_{x \to \infty} g(x) = L$. $\square$

Examples

Example 1: An application of the algebra of limits:

$$\lim_{x\to\infty} \frac{3x^2 + x + 5}{x^2 - 5} = \lim_{x\to\infty} \frac{3 + 1/x + 5/x^2}{1 - 5/x^2} = \frac{3 + 0 + 5\cdot 0^2}{1 - 5\cdot 0^2} = 3.$$

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$$-1 \le \sin x \le 1 \implies -\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}.$$

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We use the Sandwich Theorem, after observing that we can take $x>0$ in this limit calculation.

$$-1\le\sin x\le 1\implies -\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}.$$

We know that $\lim_{x\to\infty}\pm\frac{1}{x}=0$.

Examples

Example 1: An application of the algebra of limits:

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Continuity and limit at infinity

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Theorem 10

Let f, g be real functions such that $g \circ f$ is defined on (a, ∞). Let $q = \lim_{x \to \infty} f(x)$ and suppose g is continuous at q. Then

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Example:
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Logarithmic growth

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Task: Show that for any $n \in \mathbb{N}$, $\lim_{x\to\infty} \dfrac{\log x}{x^{1/n}} = 0$.

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Thus, log grows more slowly than any positive power of x.

Limits at ∞ and 0

Theorem 11

Let $a > 0$ and $f : (a, \infty) \to \mathbb{R}$. Then $\lim_{x\to\infty} f(x) = L$ if and only if $\lim_{x\to 0+} f(1/x) = L$.

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Task: Find $\lim_{x \to 0+} x \log x$ and $\lim_{x \to 0+} x^x$.

Limits equal to infinity

- We say $\lim_{x \to a} f(x) = \infty$ if for every $N \in \mathbb{R}$ there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) > N$.

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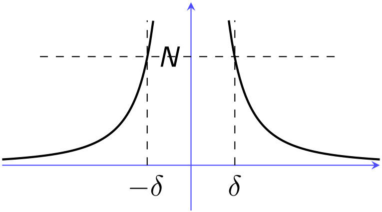

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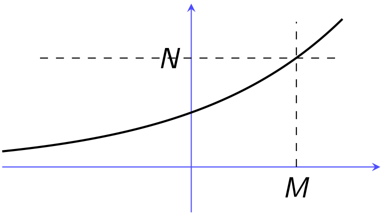

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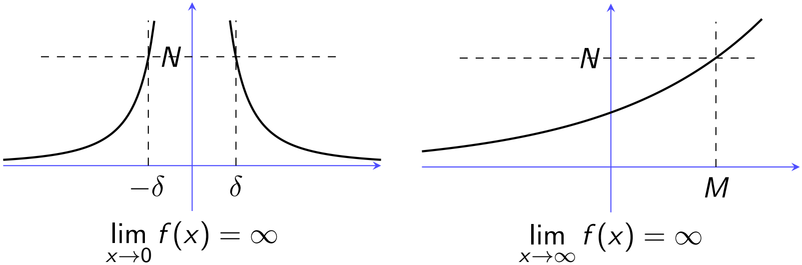

There are similar definitions for limits involving $-\infty$ and for one-sided limits.

Examples of infinite limits

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Algebra of limits

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Theorem 12

Let a stand for a real number or for the symbol ∞. Let f, g be functions defined in an open interval around a (when $a \in \mathbb{R}$) or on an interval (b, ∞) (when $a = \infty$). Let $c \in \mathbb{R}$. The limits of f, g as $x \to a$ obey the following rules.

1. $f \to \infty$ *and* $c > 0$ *implies* $cf \to \infty$.
2. $f \to \infty$ *and* $c < 0$ *implies* $cf \to -\infty$.
3. $f \to \infty$ *and* $g \to c$ *implies* $f + g \to \infty$.
4. $f, g \to \infty$ *implies* $f + g \to \infty$.
5. $f, g \to \infty$ *implies* $fg \to \infty$.
6. $f \to \infty$ *and* $g \to c > 0$ *implies* $fg \to \infty$.
7. $f \to \infty$ *implies* $1/f \to 0$.

Algebra of limits

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Polynomials

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We repeatedly apply the Algebra of Limits to x^k with $k > 0$:

$$\lim_{x\to\infty} x^k = \infty \implies \lim_{x\to\infty} \frac{1}{x^k} = 0 \implies \lim_{x\to\infty} \frac{a_{n-k}}{x^k} = 0$$
$$\implies \lim_{x\to\infty} \left(1 + \frac{a_{n-1}}{x} + \cdots \frac{a_0}{x^n}\right) = 1$$
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Similarly, there is no general formula for the limit of fg when $f \to \infty$ and $g \to 0$.

Indeterminate forms

There is no general rule for $f - g$ or f/g when $f, g \to \infty$.
So these are called **indeterminate forms** of type $\infty - \infty$ and ∞/∞ respectively.
Each such limit has to be worked out individually without recourse to a general formula.

Example: Consider $\lim_{x\to\infty}(x - \log x)$.
This is an $\infty - \infty$ form.
We can use the properties of the log function:

$$x - \log x = x\big(1 - \frac{\log x}{x}\big) \to \infty.$$

Similarly, there is no general formula for the limit of fg when $f \to \infty$ and $g \to 0$.
This is called an indeterminate form of type $\infty \cdot 0$.

Comparison Theorem, Composition

Theorem 13 (Comparison Theorem)

If $f(x) \le g(x)$ for every x, and $f(x) \to \infty$ as $x \to a$, then $g(x) \to \infty$ as $x \to a$. Here a can be a real number or $\pm\infty$.

Proof. Exercise. □

Theorem 14

Let $\lim_{x\to\infty} g(x) = L$ where L is a real number or ∞, and $\lim_{x\to a} f(x) = \infty$ where a is a real number or ∞. Then $\lim_{x\to a} g(f(x)) = L$.

Proof. Exercise. □

Exponential growth

We start with the known limits $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} \frac{\log x}{x} = 0$.

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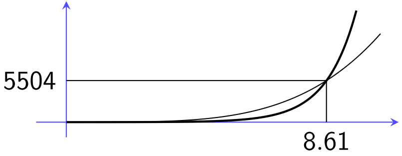
Here is a comparison with x^4:



Asymptotes

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If $\lim_{x \to \infty} f(x) = L$ then the graph of f merges with the line $y = L$ as x increases. This line is called a **horizontal asymptote** of f.

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The graph may also approach a slanted line as $x \to \pm\infty$. We say that $y = ax + b$ is a **slant asymptote** of $y = f(x)$ as $x \to \infty$ if

$$\lim_{x\to\infty} (f(x) - ax - b) = 0.$$

We have the analogous definition of a slant asymptote as $x \to -\infty$.

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We have the analogous definition of a slant asymptote as $x \to -\infty$.

The concept of slant asymptote includes that of horizontal asymptote as a special case ($a = 0$).

Slant asymptotes

Theorem 15

The line $y = ax + b$ is a slant asymptote of the function $y = f(x)$ as $x \to \infty$ if and only if $a = \lim_{x \to \infty} \frac{f(x)}{x}$ and $b = \lim_{x \to \infty} (f(x) - ax)$.

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Then $\lim_{x\to\infty} x\left(\frac{f(x)}{x} - a - \frac{b}{x}\right) = 0$.

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The converse is trivial. $\square$

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Consider the upper branch of the hyperbola $y^2 - x^2 = 1$.

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We have

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$$b = \lim_{x \to \infty} (\sqrt{x^2 + 1} - 1 \cdot x) = \lim_{x \to \infty} \frac{1}{\sqrt{x^2 + 1} + \cdot x} = 0.$$

Hence the line $y = x$ is a slant asymptote as $x \to \infty$.

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Hence the line $y = x$ is a slant asymptote as $x \to \infty$.
Similarly, $y = -x$ is a slant asymptote as $x \to -\infty$. The graph is: