February 2, 2015

Problems for Chapter 2 of *Advanced Mathematics for Applications*

Some Simple Preliminaries

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1 Variation of parameters

1. Find the general solution of the problem

$$x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - (x+2)\frac{\mathrm{d}u}{\mathrm{d}x} + u \;=\; f(x)$$

with $f(x)$ given. (One solution of the homogeneous equation is a polynomial; find a second solution by using (2.2.33) p. 35.)

2. Let v_1 be a solution of the homogeneous form of the differential equation of the Sturm-Liouville type (see section 15.1)

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u(x) \;=\; f(x)\,.$$

Determine the form of a second solution v_2 and write the solution of the non-homogeneous problem in $0 < x < 1$ satisfying the conditions $u(0) = u(1) = 0$.

3. In $0 < x < a$ let $u(x)$ and $v(x)$ be solutions, respectively, of

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + p(x)u \;=\; 0\,, \qquad \text{and} \quad \frac{\mathrm{d}^2v}{\mathrm{d}x^2} + q(x)v \;=\; 0\,,$$

with $p(x)$ and $q(x)$ given continuous functions and $u(0) = v(0) = 0$. Express the Wronskian $W(u, v)$ in terms of a suitable integral

4. Find the general solution of the equation

$$A\,\frac{\partial^2u}{\partial x^2} + \frac{\partial A}{\partial x}\frac{\partial u}{\partial x} + \frac{u}{A} = 0$$

where $A = A(x)$ is a given, strictly positive function.

5. Effect the change of dependent variable $u(x) = \phi(x)\,v(x)$ in the linear differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}u}{\mathrm{d}x} + q(x)u \;=\; 0$$

and choose ϕ so that the resulting equation does not contain $\mathrm{d}v/\mathrm{d}x$. Use this technique to find the general solution of the equation

$$x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} + n^2xu \;=\; \sin\omega x\,,$$

where n is a given integer and ω a given constant.

2 Distributions

1. Differentiate in the sense of distributions

$$[1 - H(x)]\,\cos x\,, \qquad [H(x+2) + 2H(x) - h(x-2)]\;e^{-2x}\,.$$

2. Find the second distributional derivatives of $\exp(-|x|)$ and $\sin|x|$.

3. Reduce the following expressions

$$x\,\delta'(x)\,, \qquad (\sin ax)\,\delta'(x)\,.$$

4. Calculate, by integration by parts, the integral

$$\int_{-1}^{1} |x| f''(x)\,\mathrm{d}x$$

5. Reduce $f(x)\delta''(x)$ to an expression involving the values of f and its derivatives at 0.

6. Calculate

$$I_1 = \int_{-\infty}^{\infty} \delta(ax^2 - b) f(x) dx, \qquad I_2 = \int_{0}^{\infty} \delta(ax^2 - b) f(x) dx\,.$$

   Consider all possible sign combinations of the constants a and b.

7. Show, by executing the derivatives as explained in section 2.3, that the expression

$$u(x) \;=\; \int_{-\infty}^{\infty} \exp\left(-k|x-y|\right) f(y)\,dy$$

   satisfies the equation

$$\frac{d^2u}{dx^2} + k^2 u \;=\; 2ik\,f(x)$$

   where $k > 0$ and f is a given function such that the integral exists.

8. Solve the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) \;=\; -\delta(x - x')$$

   in the range $0 < x < a$ with $0 < x' < a$. The solution is required to be bounded at $x = 0$ and to take on the value $u = A$ at $x = a$. (u is continuous at $x = x'$.)

9. Calculate the integral

$$I \;=\; \int_{0}^{b} H(x - a) f'(x) dx$$

   first directly and then by parts and show that the two results are equal. Consider both $a > b$ and $a < b$.

10. It might seem that solution of the differential equation

$$u'(x) \;=\; f'(x) H(a - x)$$

   with $f(x)$ given, is

$$u(x) \;=\; f(x)\,H(a - x) + \text{const.}\,.$$

   Show that this is in fact wrong and find the correct solution.

3 Green's functions

1. Find the Green's function for the problem

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = f(x)$$

in the range $0 < x < 1$ with $u(0) = u(1) = 0$.

2. Find the general solution of the problem

$$x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - (x+2)\frac{\mathrm{d}u}{\mathrm{d}x} + u = f(x)$$

in the range $0 < x < 1$ with $u(0) = u(1) = 0$. (One solution of the homogeneous equation is a polynomial.)

3. Find the Green's function for the problem

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) - a^2u = f(x)$$

in the range $0 < x < 1$ with $u(0) = u(1) = 0$. Are there special values of the given constant a where the solution breaks down?

4. Find the Green's function for the problem

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) - a^2u = f(x)$$

in the range $0 < x < 1$ with $u(0) = u(1) = 0$. Are there special values of the given constant a where the solution breaks down?

5. Find the Green's function for the problem

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) - a^2u = f(x)$$

in the range $0 < x < 1$ with $\mathrm{d}u/\mathrm{d}x|_{x=0}\, u = u(1) = 0$. Are there special values of the given constant a where the solution breaks down?

6. Find the Green's function for the problem

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}u}{\mathrm{d}x}\right) - a^2u = f(x)$$

in the range $1 < x < L$ with $u(1) = \mathrm{d}u/\mathrm{d}x|_{x=L} = 0$. Are there special values of the given constant a where the solution breaks down?

7. Find the Green's function for the problem

$$\frac{\mathrm{d}u}{\mathrm{d}x} + au = f(x)$$

in the range $0 < x < \infty$ with $u(0) = 0$ and a a positive constant.

8. Find the Green's function for the problem

$$-\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + k^2u \;=\; f(x)$$

in the interval $0 < x < L$ when $u' = 0$ at both $x = 0$ and $x = L$.

9. Find the Green's function for the problem

$$-\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + k^2u \;=\; f(x)$$

in the interval $0 < x < \infty$. The boundary conditions are $u(0) = 0$ and u bounded at infinity. The variable x ranges between 0 and infinity.

10. Let v_1 and v_2 be solutions of the homogeneous form of the differential equation of the Sturm-Liouville type (see section 15.1)

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right] + q(x)u(x) \;=\; f(x)\,.$$

Find the Green's function for this problem in $0 < x < 1$ if the boundary conditions are $u(0) = u(1) = 0$.

4 Solution by power series

1. For $x > 0$ find by the power series method the solution of the equation

$$\frac{\mathrm{d}u}{\mathrm{d}x} + xu \;=\; 1$$

satisfying $u(0) = 0$.

2. In the interval $a < x < b$ find by the power series method solutions of the equation

$$(1 + x^2)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - x^3\frac{\mathrm{d}u}{\mathrm{d}x} \;=\; x^2 + 4$$

satisfying $u(a) = 0$. Prove that, if $u(b) = 0$, $u(x) < 0$ for $a < x < b$.

3. For $x > 0$ find by the power series method the solutions of the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x} + x\frac{\mathrm{d}u}{\mathrm{d}x} + u \;=\; 0$$

satisfying general conditions at $x = 0$.

4. Find by the power series method one solution of the equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{u}{x} \;=\; 0\,,$$

What is the form of a second solution of this equation?

5. Determine the nature of the singular points of the differential equation

$$(1 - x^2)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - x\frac{\mathrm{d}u}{\mathrm{d}x} + n^2u \;=\; 0\,,$$

in which n is an integer. Find by the power series method one solution of the equation. What is the form of a second solution of this equation?

6. By the power series method find two independent solutions of Airy's equation (see section 4.3 p. 93)
$$u'' - xu = 0\,.$$
Determine the radius of convergence of the resulting series by the methods of section 8.4 p. 228.

7. Determine the first few terms of the two series satisfying the following special case of Whittaker's equation:
$$u'' + \left[\left(\frac{1}{4} - m^2\right)\frac{1}{x^2} - \frac{1}{4}\right] = 0$$
where m is not an integer. Verify your result in the simple case $m = 1/4$.

8. By the power series method determine one solution of the modified Bessel equation of order 0 (see section 12.4 p. 310)
$$u'' + \frac{1}{x}u' - u = 0$$
Determine the radius of convergence of the resulting series by the methods of section 8.4 p. 228. Then determine a second solution by using (2.2.33) p. 35.

9. By means of the power series method discuss the solutions of the equation
$$xu'' + (x+1)u' + (n+1)u = 0$$
where n is an integer, valid near $x = 0$ and for $x \to \infty$.

10. In the interval $0 < x < \pi$ find by the power series method the solution of the equation
$$x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + u\,\sin x = x$$
satisfying $u(\pi) = 1$, $\mathrm{d}u/\mathrm{d}x|_{x=\pi} = 0$.

11. Study the nature of the singular points of the equation
$$z\frac{\mathrm{d}^2u}{\mathrm{d}z^2} + (a - z + 1)\frac{\mathrm{d}u}{\mathrm{d}z} + Nu = 0\,,$$
where N is a non-negative integer and a an arbitrary real constant. Consider then the particular value $a = -N$. Give one explicit solution of the equation for this case and determine whether the other solution is singular or not.

12. For $x > 0$ find by the power series method the general solution of the equation
$$2x^2\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (3x - 2x^2)\frac{\mathrm{d}u}{\mathrm{d}x} - (x+1)u = 0\,.$$

13. For $x > 0$ find by the power series method the general solution of the equation
$$4x\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + 2\frac{\mathrm{d}u}{\mathrm{d}x} + u = 0\,.$$

14. For $x > 0$ find by the power series method the general solution of the equation
$$9x(1-x)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - 12\frac{\mathrm{d}u}{\mathrm{d}x} + 4u = 0\,.$$

15. Find by a power series expansion in t the solution to the initial-value problem
$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = u$$
subject to $u(x,0) = e^x$, $\partial u/\partial t|_{t=0} = 0$.