9 Chapter 9 Solution Set

Problems

9.1 Show that in the special case where the background is lossless so that the scattering potential V_0 is real valued that

$$\psi_+^*(\mathbf{r}, k_0\mathbf{s}_0) = e^{-ik_0\mathbf{s}_0\cdot\mathbf{r}} + \int d^3r'\, G_-(\mathbf{r}, \mathbf{r}')V_0(\mathbf{r}')e^{-ik_0\mathbf{s}_0\cdot\mathbf{r}'},$$

from which we conclude that for such backgrounds

$$\psi_+^*(\mathbf{r}', k_0\mathbf{s}_0) = \psi_-(\mathbf{r}', -k_0\mathbf{s}_0).$$

Take the complex conjugate of both sides of the LS equation satisfied by the plane wave scattering state $\psi_+(\mathbf{r}, k_0\mathbf{s}_0)$ for the case where V_0 is real valued to obtain

$$\psi_+^*(\mathbf{r}, k_0\mathbf{s}_0) = e^{-ik_0\mathbf{s}_0\cdot\mathbf{r}} + \int d^3r'\, G_{0_+}^*(\mathbf{r}, \mathbf{r}')V_0(\mathbf{r}')e^{-ik_0\mathbf{s}_0\cdot\mathbf{r}'}.$$

Now make use of the fact that for a real V_0 $G_{0_+}^* = G_{0_-}$ to obtain the required result.

9.2 Compute the plane wave scattering states in two space dimensions for an inhomogeneous medium consisting of a homogeneous plane parallel slab with constant wavenumber $k_1 \neq k_0$.

The plane wave scattering states for a plane parallel slab are the classical physical optics solutions of the field resulting from plane wave incidence onto a plane parallel plate. We assume that the slab is aligned perpendicular to the z axis and the incident plane wave is propagating in the positive z direction and given by

$$U^{(in)}(\mathbf{r}) = e^{ik_0\mathbf{s}_0\cdot\mathbf{r}} = e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}e^{i\gamma_0 z}$$

where $\boldsymbol{\rho}$ is the component of $\mathbf{r}$ in the (x, y) plane and $k_0\mathbf{s}_0 = \mathbf{K}_\rho + \hat{\mathbf{z}}\gamma_0$. We can also express the incident wave in terms of the "angle of incidence" θ_0 formed by $\mathbf{s}_0$ with the positive z axis (normal to the slab). In terms of θ_0 we have that $|\mathbf{K}_\rho| = K_\rho = k_0 \sin\theta_0$ and $\gamma_0 = k_0 \sin\theta_0$.

The incident plane wave will generate a reflected plane wave and a transmitted plane wave into the plane parallel slab that will suffer multiple reflections from both sides of the slab resulting in a standing wave in the interior of the slab. This standing wave will then generate both a reflected wave and transmitted wave from the slab. It is well knows and easily proven using continuity of field and normal derivative of the field at the two ends of the slab that *all of the plane waves resulting in the scattering process possess wave vectors having identical transverse components.* This then allows us to write the plane waves in the problem in the form

$$\psi_0(\mathbf{r}) = e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}e^{\pm i\gamma_0 z}, \quad \psi_1(\mathbf{r}) = e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}e^{\pm i\gamma_1 z}$$

where ψ_0 denotes plane waves propagating outside the slab and ψ_1 plane waves propagating inside the slab. In all cases

$$\gamma_0 = \sqrt{k_0^2 - K_\rho^2} = k_0 \sin\theta_0, \quad \gamma_1 = \sqrt{k_1^2 - K_\rho^2} = k_1 \sin\theta_1$$

where k_1 is the wavenumber of the material in the slab region and θ_1 the angle formed by the unit wave vector $\mathbf{s}_1$ with the positive z axis.

We will first solve the problem for an incident wave propagating in the positive z direction and for a slab whose left boundary is aligned on the $z = 0$ plane and right boundary is located at $z = 2a_0$. Once we obtain this solution it is easy to obtain the solution for a centered slab having boundaries at $z = \pm a_0$ and for incident plane waves from either the left or right half-spaces.

We can express the wavefields in the three regions left of the slab, within the slab, and to the right of the slab in the forms

$$\begin{aligned}
\psi_<(\mathbf{r}) &= e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}[e^{i\gamma_0 z} + Re^{-i\gamma_0 z}], \quad z < 0\\
\psi(\mathbf{r}) &= e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}[Ae^{i\gamma_1 z} + Be^{-i\gamma_1 z}], \quad 0 \le z < l_0\\
\psi_>(\mathbf{r}) &= e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}Te^{i\gamma_0 z} \quad z > l_0
\end{aligned}$$

where $l_0 = 2a_0$ is the width of the slab. Continuity of field and normal derivative to the slab yields

$$\begin{aligned}
[e^{i\gamma_0 z} + Re^{-i\gamma_0 z}]_{z=0} &= [Ae^{i\gamma_1 z} + Be^{-i\gamma_1 z}]|_{z=0}\\
[Ae^{i\gamma_1 z} + Be^{-i\gamma_1 z}]_{z=l_0} &= Te^{i\gamma_0 z}|_{z=l_0}\\
\gamma_0[e^{i\gamma_0 z} - Re^{-i\gamma_0 z}]_{z=0} &= \gamma_1[Ae^{i\gamma_1 z} - Be^{-i\gamma_1 z}]|_{z=0}\\
\gamma_1[Ae^{i\gamma_1 z} - Be^{-i\gamma_1 z}]_{z=l_0} &= \gamma_1 Te^{i\gamma_0 z}|_{z=l_0}
\end{aligned}$$

One approach to the problem is then to solve the above set of equations for the quantities R, A, B and T which then yields the plane wave scattering states $\psi_+(\mathbf{r}, k_0\mathbf{s}_0)$ with $\hat{\mathbf{z}}\cdot\mathbf{s}_0 > 0$ corresponding to an incident plane wave propagating in the positive z direction. As mentioned above the the plane wave scattering states for a centered slab and for incident plane waves propagating in the negative z direction are then obtained using a translation theorem and a simple argument of symmetry.

A simpler procedure and the one used in physical optics books is based on tracking the multiple reflections between the two ends of the slab. Using this procedure we find, for example, that the field interior to the slab is given by

$$\psi(\mathbf{r}) = [t_0 e^{i\gamma_1 z} + t_0 r_1 e^{2i\gamma_1 l_0} e^{-i\gamma z} + t_0 r_1^2 e^{4i\gamma_1 l_0} e^{i\gamma_1 z} + \cdots] e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}},$$

where t_0 is the Fresnel transmission coefficient going from the exterior medium into the interior of the slab and r_1 is the Fresnel reflection coefficient from the interior medium into the interior medium and the various terms represent multiple reflections between the two ends of the slab. It is clear that we can separate the interior field into the two components propagating to the right and left to obtain

$$A = t_0[1 + \alpha^2 + \alpha^4 + \cdots] = \frac{t_0}{1-\alpha^2}, \quad B = t_0 r_1[1 + \alpha^2 + \alpha^4 + \cdots] = \frac{t_0 r_1}{1-\alpha^2}$$

where $\alpha = r_1 \exp(2i\gamma_1 l_0)$. We thus find that the plane wave scattering state within the slab is given by

$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0) = \frac{t_0}{1-\alpha^2}[e^{i\gamma_1 z} + r_1 e^{-i\gamma_1 z}] e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}}, \quad 0 \leq z \leq l_0,$$

corresponding to plane waves propagating in the positive and negative z directions, respectively.

Using the same general procedure we find that the plane wave scattering state outside the slab is given by

$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0) = [e^{i\gamma_0 z} + [r_0 + \frac{t_0 t_1 r_1}{1-\alpha^2}] e^{-i\gamma_0 z}] e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}}, \quad z < 0,$$
$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0) = \frac{t_0 t_1}{1-\alpha^2} e^{i\gamma_0 z} e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}}, \quad z > l_0,$$

where t_1 is the transmission coefficient from the interior medium to the exterior medium and r_0 the reflection coefficient from the exterior medium back into the exterior medium.

The problem is completed by using the so-called Stokes relations and the expressions for the Fresnel coefficients in terms of the relative index of refraction $n_r = \sqrt{k_1/k_0}$ between the two media. The Stokes relations are given by

$$t_0 t_1 = 1 - r_0^2, \quad r_0 = -r_1$$

and when used in the above expressions for the plane wave scattering states yield

$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0) = [e^{i\gamma_0 z} + r_0[1 - \frac{1-r_0^2}{1-\alpha^2}] e^{-i\gamma_0 z}] e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}}, \quad z < 0,$$
$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0) = \frac{t_0}{1-\alpha^2}[e^{i\gamma_1 z} - r_0 e^{-i\gamma_1 z}] e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}}, \quad 0 \leq z \leq l_0,$$
$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0) = \frac{1-r_0^2}{1-\alpha^2} e^{i\gamma_0 z} e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}}, \quad z > l_0,$$

where

$$\alpha^2 = r_0^2 \exp(4i\gamma_1 l_0).$$

The final step is to substitute the expressions for the Fresnel coefficients t_0 and r_0 into the above equations. These quatities are given by

$$r_0 = \frac{\cos\theta_0 - \sqrt{n_r^2 - \sin^2\theta_0}}{\cos\theta_0 + \sqrt{n_r^2 - \sin^2\theta_0}},$$

$$t_0 = \frac{2\cos\theta_0}{\cos\theta_0 + \sqrt{n_r^2 - \sin^2\theta_0}}.$$

The plane wave scattering states obtained above can be transformed into the states for a centered slab and for plane waves propagating in either z direction by making use of a simple translation theorem similar to the translation theorem proven in Appendix A and the subject of the following problem. This theorem states (see following problem statement) that if the slab is translated to the left by a distance a_0 so that it is now centered at $z = 0$ the plane wave scattering states are transformed to

$$\psi_+^{(c)}(\mathbf{r}, k_0\mathbf{s}_0) = e^{-i\gamma_0 a_0}\psi_+(\mathbf{r} + a_0\hat{\mathbf{z}}, k_0\mathbf{s}_0)$$

where $\psi_+^{(c)}(\mathbf{r}, k_0\mathbf{s}_0)$ is the plane wave scattering state for the centered slab and $\psi_+(\mathbf{r}, k_0\mathbf{s}_0)$ the plane wave scattering state for the slab centered at $z = a_0$ that we just computed. On making the above substitution we obtain

$$\psi_+^{(c)}(\mathbf{r}, k_0\mathbf{s}_0) = [e^{i\gamma_0 z} + r_0 e^{-i2\gamma_0 a_0}[1 - \frac{1 - r_0^2}{1 - \alpha^2}]e^{-i\gamma_0 z}]e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}, \quad z < -a_0,$$

$$\psi_+^{(c)}(\mathbf{r}, k_0\mathbf{s}_0) = \frac{t_0}{1 - \alpha^2}e^{-i\gamma_0 a_0}[e^{i\gamma_1 a_0}e^{i\gamma_1 z} - r_0 e^{-i\gamma_1 a_0}e^{-i\gamma_1 z}]e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}, \quad -a_0 \le z \le a_0,$$

$$\psi_+^{(c)}(\mathbf{r}, k_0\mathbf{s}_0) = \frac{1 - r_0^2}{1 - \alpha^2}e^{i\gamma_0 z}e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}, \quad z > a_0.$$

The centered slab has the advantage that it is symmetrical about the $\boldsymbol{\rho}$ plane so that the plane wave scattering states $\psi_+^{(c)}(\mathbf{r}, k_0\mathbf{s}_0)$ with $\hat{\mathbf{z}}\cdot\mathbf{s}_0 < 0$ can be easily obtained from those with $\hat{\mathbf{z}}\cdot\mathbf{s}_0 > 0$ by simply replacing z by $-z$. More precisely this result, which is easily proven for centered and symmetrical scattering potentials using the LS equations, states that

$$\psi_+^{(c)}(\mathbf{r}, k_0\tilde{\mathbf{s}_0}) = \psi_+^{(c)}(\tilde{\mathbf{r}}, k_0\mathbf{s}_0),$$

where $\tilde{\mathbf{s}_0} = (s_{0_x}, s_{0_y}, -s_{0_z})$ and $\tilde{\mathbf{r}} = (x, y, -z)$. Using these two results we the obtain that the general form of the plane wave scattering states for a centered slab having sides at $z = \pm a_0$ and for incident plane waves propagating in any direction is given by

$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0) = [e^{\pm i\gamma_0 z} + r_0 e^{-i2\gamma_0 a_0}[1 - \frac{1 - r_0^2}{1 - \alpha^2}]e^{\mp i\gamma_0 z}]e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}, \quad z < -a_0,\ z > a_0$$

$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0) = \frac{t_0}{1 - \alpha^2}e^{-i\gamma_0 a_0}[e^{i\gamma_1 a_0}e^{\pm i\gamma_1 z} - r_0 e^{-i\gamma_1 a_0}e^{\mp i\gamma_1 z}]e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}, \quad -a_0 \le z \le a_0,$$

$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0) = \frac{1 - r_0^2}{1 - \alpha^2}e^{\pm i\gamma_0 z}e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}, \quad z > a_0,\ z < -a_0,$$

where we have now dropped the superscript c to denote the centered states and the upper sign applies for a plane wave incident from the l.h.s. and the lower sign for a plane wave incident from the r.h.s.

9.3 Prove that the plane wave scattering states satisfy the relationship

$$\psi_+(\mathbf{r}, k_0\mathbf{s}_0; \mathbf{X}_0) = e^{ik_0\mathbf{s}_0\cdot\delta\mathbf{X}_0}\psi_+(\mathbf{r} - \delta\mathbf{X}_0, k_0\mathbf{s}_0; \mathbf{X}'_0)$$

where $\delta\mathbf{X}_0 = \mathbf{X}_0 - \mathbf{X}'_0$ with $\mathbf{X}_0$ and $\mathbf{X}'_0$ being any two central locations of the background scattering potential.

This is accomplished in an identical manner as employed in proving the proof of the translation theorem in Appendix A.

9.4 Fill in the missing steps in the derivation of Eqs.(9.10) of Example 9.1.

Make the substitution

$$< Y_l^m, \psi_\pm >= 4\pi i^l Y_l^{m*}(\mathbf{s}_0) g_l^\pm(r)$$

in

$$< Y_l^m, \psi_\pm >= 4\pi i^l Y_l^{m*}(\mathbf{s}_0) j_l(k_0 r) \mp ik_0 \int_0^\infty r'^2 dr'\, j_l(k_0 r_<) h_l^\pm(k_0 r_>) V_0(r') < Y_l^m, \psi_\pm >,$$

to obtain the required result.

9.5 Fill in the missing steps in the derivation of Eq.(9.15).

The required missing step is to show that

$$\chi(\mathbf{r}) = \int_{\partial\tau} dS' \, [U_{0_+}(\mathbf{r}') \frac{\partial}{\partial n'} G_{0_+}(\mathbf{r}, \mathbf{r}') - G_{0_+}(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial n'} U_{0_+}(\mathbf{r}')]$$

vanishes in the limit when $\partial\tau \to \Sigma_\infty$ with Σ_∞ being the surface of a sphere of radius $R \to \infty$. In this limit we have

$$U_{0_+}(\mathbf{r}') \sim f(\mathbf{s}') \frac{e^{ik_0 r'}}{r'}, \quad G_{0_+}(\mathbf{r}, \mathbf{r}') \sim -\frac{1}{4\pi} \frac{e^{ik_0 r'}}{r'} \psi_+(\mathbf{r}, -k_0\mathbf{s}')$$

where $\mathbf{s}' = \mathbf{r}'/r'$ is the unit vector in the direction of $\mathbf{r}'$. We then find that

$$U_{0_+}(\mathbf{r}') \frac{\partial}{\partial n'} G_{0_+}(\mathbf{r}, \mathbf{r}') - G_{0_+}(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial n'} U_{0_+}(\mathbf{r}')$$

$$\sim f(\mathbf{s}') \frac{e^{ik_0 r'}}{r'} [-ik_0 \frac{1}{4\pi} \frac{e^{ik_0 r'}}{r'} \psi_+(\mathbf{r}, -k_0\mathbf{s}')] + \frac{1}{4\pi} \frac{e^{ik_0 r'}}{r'} \psi_+(\mathbf{r}, -k_0\mathbf{s}') [ik_0 f(\mathbf{s}') \frac{e^{ik_0 r'}}{r'}] = 0$$

to order $1/r'^2$ establishing the required result.

9.6 Complete the derivation of Eqs.(9.21).

We begin with the equation

$$U_0(\mathbf{r}) = \lim_{r' \to \infty} r'^2 \int_{4\pi} d\Omega_{r'} \, \{U_0(\mathbf{r}') \frac{\partial}{\partial r'} G_{0_+}(\mathbf{r}, \mathbf{r}') - G_{0_+}(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial r'} U_0(\mathbf{r}')\}$$

where the free field U_0 and background Green function satisfy the asymptotic conditions

$$U_0(\mathbf{r}') \sim u_+(\mathbf{s}') \frac{e^{ik_0 r'}}{r'} - u_-(\mathbf{s}') \frac{e^{-ik_0 r'}}{r'}, \quad G_{0_+}(\mathbf{r}, \mathbf{r}') \sim -\frac{1}{4\pi} \frac{e^{ik_0 r'}}{r'} \psi_+(\mathbf{r}, -k_0\mathbf{s}')$$

where $\mathbf{s}' = \mathbf{r}'/r'$ is the unit vector in the direction of $\mathbf{r}'$. We then find that

$$U_0(\mathbf{r}')\frac{\partial}{\partial r'}G_{0_+}(\mathbf{r},\mathbf{r}') \sim -\frac{ik_0}{4\pi}[u_+(\mathbf{s}')\frac{e^{ik_0r'}}{r'} - u_-(\mathbf{s}')\frac{e^{-ik_0r'}}{r'}]\frac{e^{ik_0r'}}{r'}\psi_+(\mathbf{r},-k_0\mathbf{s}')$$
$$G_{0_+}(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial r'}U_0(\mathbf{r}') \sim -\frac{ik_0}{4\pi}\frac{e^{ik_0r'}}{r'}\psi_+(\mathbf{r},-k_0\mathbf{s}')[u_+(\mathbf{s}')\frac{e^{ik_0r'}}{r'} + u_-(\mathbf{s}')\frac{e^{-ik_0r'}}{r'}],$$

so that

$$U_0(\mathbf{r}')\frac{\partial}{\partial r'}G_{0_+}(\mathbf{r},\mathbf{r}') - G_{0_+}(\mathbf{r},\mathbf{r}')\frac{\partial}{\partial r'}U_0(\mathbf{r}') \sim \frac{ik_0}{2\pi}\frac{1}{r'^2}\psi_+(\mathbf{r},-k_0\mathbf{s}')u_-(\mathbf{s}')$$

which then yields the required result

$$U_0(\mathbf{r}) = \lim_{r'\to\infty} r'^2\int_{4\pi} d\Omega_{s'}\,\frac{ik_0}{2\pi}\frac{1}{r'^2}\psi_+(\mathbf{r},-k_0\mathbf{s}')u_-(\mathbf{s}') = \frac{ik_0}{2\pi}\int_{4\pi} d\Omega_s\, u_-(\mathbf{s})\psi_+(\mathbf{r},-k_0\mathbf{s}).$$

The expansion given in Eq.(9.21b) is obtained in a completely parallel manner using G_{0_-} in place of G_{0_+}.

9.7 Complete the derivation of Eqs.(9.24).

We consider the free space propagator $G_{0_f}(\mathbf{r},\mathbf{r}')$ as a function of $\mathbf{r}$ with $\mathbf{r}'$ a fixed parameter. We then find using Eq.(9.21b) that

$$G_{0_f}(\mathbf{r},\mathbf{r}') = \frac{ik_0}{2\pi}\int_{4\pi} d\Omega_s\, u_+(\mathbf{s},\mathbf{r}')\psi_-(\mathbf{r},k_0\mathbf{s})$$

where on using Eqs.(9.9a)

$$G_{0_f}(\mathbf{r},\mathbf{r}') = G_{0_+}(\mathbf{r},\mathbf{r}') - G_{0_-}(\mathbf{r},\mathbf{r}') \sim -\frac{1}{4\pi}\psi_+(\mathbf{r}',-k_0\mathbf{s})\frac{e^{+ik_0r}}{r} + \frac{1}{4\pi}\psi_-(\mathbf{r},k_0\mathbf{s})\frac{e^{-ik_0r}}{r}$$

so that

$$u_+(\mathbf{s},\mathbf{r}') = -\frac{1}{4\pi}\psi_+(\mathbf{r}',-k_0\mathbf{s}), \quad u_-(\mathbf{s},\mathbf{r}') = -\frac{1}{4\pi}\psi_-(\mathbf{r}',k_0\mathbf{s}).$$

The plane wave expansion of the free field propagator is then obtained from either of Eqs.(9.21) and found to be

$$G_{0_f}(\mathbf{r},\mathbf{r}') = -\frac{ik_0}{8\pi^2}\int_{4\pi} d\Omega_s\, \psi_+(\mathbf{r}',-k_0\mathbf{s})\psi_-(\mathbf{r},k_0\mathbf{s}).$$

9.8 Derive Eq.(9.25b) from Eq.(9.25a) and the reciprocity condition satisfied by the background Green functions.

We begin with Eq.(9.25a):

$$G_{0_+}(\mathbf{r},\mathbf{r}') = -\frac{ik_0}{8\pi^2}\int_{-\pi}^{\pi} d\beta\int_{C_\pm}\sin\alpha d\alpha\,\psi_+(\mathbf{r},k_0\mathbf{s})\psi_-(\mathbf{r}',-k_0\mathbf{s}),$$

where the contour $C_+ = [0 : \pi/2 - i\infty]$ is used if $z > z'$ and $C_- = [\pi/2 + i\infty : \pi]$ if $z < z'$. Reciprocity requires that $G_{0_+}(\mathbf{r},\mathbf{r}') = G_{0_+}(\mathbf{r}',\mathbf{r})$ so that $G_{0_+}(\mathbf{r},\mathbf{r}')$ can also be written in the form

$$G_{0_+}(\mathbf{r},\mathbf{r}') = -\frac{ik_0}{8\pi^2}\int_{-\pi}^{\pi} d\beta\int_{C_\mp}\sin\alpha d\alpha\,\psi_+(\mathbf{r}',k_0\mathbf{s})\psi_-(\mathbf{r},-k_0\mathbf{s}), \qquad (9.1)$$

but where now, since z and z' have been interchanged, $C_- = [\pi/2 + i\infty : \pi]$ is used if $z > z'$ and $C_+ = [0 : \pi/2 - i\infty]$ if $z < z'$.

We now make the change of integration variables $\beta' = \beta - \pi$ and $\alpha' = \pi - \alpha$ so that

$$\begin{aligned} \mathbf{s} &= \sin\alpha\cos\beta\hat{\mathbf{x}} + \sin\alpha\sin\beta\hat{\mathbf{y}} + \cos\alpha\hat{\mathbf{z}} \\ &\rightarrow \sin(\pi-\alpha')\cos(\beta'+\pi)\hat{\mathbf{x}} + \sin(\pi-\alpha')\sin(\beta'+\pi)\hat{\mathbf{y}} + \cos(\pi-\alpha')\hat{\mathbf{z}} \\ &= -\sin\alpha'\cos\beta'\hat{\mathbf{x}} - \sin\alpha'\sin\beta'\hat{\mathbf{y}} - \cos\alpha'\hat{\mathbf{z}} = -\mathbf{s}'. \end{aligned}$$

Under this transformation $d\beta = d\beta'$ and $\sin\alpha d\alpha = -\sin\alpha' d\alpha'$ and β' ranges over a full 2π radians while the contours $C_\mp$ transform to

$$\begin{aligned} C_- = \alpha = [\pi/2 + i\infty : \pi] &\rightarrow \alpha' = [\pi/2 - i\infty : 0] = -C_+ \\ C_+ = \alpha = [0 : \pi/2 - i\infty : \pi] &\rightarrow \alpha' = [\pi : \pi/2 + i\infty] = -C_-. \end{aligned}$$

The expansion Eq.(9.1) then assumes the form

$$\begin{aligned} G_{0_+}(\mathbf{r},\mathbf{r}') &= -\frac{ik_0}{8\pi^2}\int_{-\pi}^{\pi} d\beta \int_{-C_\pm} (-\sin\alpha')d\alpha'\,\psi_+(\mathbf{r}', -k_0\mathbf{s}')\psi_-(\mathbf{r}, k_0\mathbf{s}') \\ &= -\frac{ik_0}{8\pi^2}\int_{-\pi}^{\pi} d\beta \int_{C_\pm} \sin\alpha d\alpha\,\psi_-(\mathbf{r}, k_0\mathbf{s})\psi_+(\mathbf{r}', -k_0\mathbf{s}), \end{aligned}$$

where now C_+ is again used if $z > z'$ and C_- if $z < z'$.

9.9 Prove that the operators $\hat{T}$ and $\hat{T}^\dagger$ defined in Eqs.(9.30a) and (9.30b) are compact.

The proofs follow identical lines as employed in the proof for the associated operators in a uniform background presented in Example 5.4. In particular, employing the same method as used in that earlier proof we use the fact that the operator $\hat{T} : \mathcal{H}_Q \rightarrow \mathcal{H}_f$ is Hilbert-Schmidt if there exists a complete orthonormal sequence $e_n \in \mathcal{H}_Q$ such that

$$\sum_n ||\hat{T}e_n||^2_{\mathcal{H}_f} < \infty.$$

We then select any orthonormal sequence $e_n(\mathbf{r}) \in \mathcal{H}_Q$ so that

$$\hat{T}e_n = -\frac{1}{4\pi}\int_{\tau_0} d^3r'\,\psi_+(\mathbf{r}', -k_0\mathbf{s})e_n(\mathbf{r}') = -\frac{1}{4\pi} < \psi_+^*(\mathbf{r}', -k_0\mathbf{s}), e_n >_{\mathcal{H}_Q}.$$

It then follows that

$$||\hat{T}e_n||^2_{\mathcal{H}_f} = (\frac{1}{4\pi})^2 \int_{4\pi} d\Omega_s\,| < \psi_+^*(\mathbf{r}', -k_0\mathbf{s}), e_n >_{\mathcal{H}_Q} |^2$$

from which we find that

$$\begin{aligned} \sum_n ||\hat{T}e_n||^2_{\mathcal{H}_f} &= (\frac{1}{4\pi})^2 \sum_n \int_{4\pi} d\Omega_s\,| < \psi_+^*(\mathbf{r}', -k_0\mathbf{s}), e_n >_{\mathcal{H}_Q} |^2 \\ = (\frac{1}{4\pi})^2 \int_{4\pi} d\Omega_s \sum_n | < \psi_+^*(\mathbf{r}', -k_0\mathbf{s}), e_n >_{\mathcal{H}_Q} |^2 &\le (\frac{1}{4\pi})^2 \int_{4\pi} d\Omega_s\,||\psi_+(\mathbf{r}', -k_0\mathbf{s})||^2_{\mathcal{H}_Q}||^2, \end{aligned}$$

where we have used Bessel's inequality. We thus conclude that $\hat{T}$ will be Hilbert-Schmidt if the plane wave scattering states $\psi_+(\mathbf{r}', -k_0\mathbf{s})$ have finite norm in $\mathcal{H}_Q$. That this is the case follows immediately from the requirement that the source volume τ_0 is compact (finite volume) and that the plane wave scattering states are solutions to the homogeneous Helmholtz equation with wavenumber $k_0 n(\mathbf{r})$ and, hence, must be continuous. The proof for $\hat{T}^\dagger$ is proved in a similar fashion.

9.10 Complete the derivation of Eqs.(9.55).

This problem should pose no difficulty for the reader.

9.11 Prove the inhomogeneous medium "Field Uniqueness Theorem" which states that the field radiated by a source compactly supported in a space region τ_0 within a non-uniform medium is uniquely determined over all space points lying outside τ_0 by the radiated field or its normal derivative (Dirichlet or Neumann conditions) over any closed surface $\partial\tau$ that completely surrounds τ_0.

This can be proven in a completely parallel manner as was used in the proof of the corresponding theorem for the wave equation presented in Chapter 1. In particular, we assume that a second field is radiated by a source in τ_0 and that assumes identical Dirichlet or Neumann conditions on a closed surface $\partial\tau$ of a volume τ that completely contains τ_0. Then the difference field δU_{0_+} must be radiated by some source in τ_0 and assume zero Dirichlet or Neumann conditions over $\partial\tau$. The first Helmholtz identity given in Eq.(9.16a) then shows that the field δU_{0_+} must vanish identically outside of τ and, hence, must be an NR source supported in τ[1]. But, by hypothesis, the source of δU_{0_+} has to be supported in $\tau_0 \in \tau$ proving that the field δU_{0_+} vanishes throughout the exterior of τ_0.

A second more direct proof, that also has the advantage that it is constructive and, in principle, allows the field to be *computed* throughout the exterior of the source region from the boundary conditions, is via the angular spectrum expansion presented in Section 9.3. In particular, Eq.(9.27) allows the field to be back propagated throughout the exterior of the convex hull of τ_0 from the radiation pattern of the field. The radiation pattern, in turn, can be computed from Dirichlet or Neumann conditions over $\partial\tau$ leading to both a proof of the theorem as well as an algorithm for performing the field determination from the boundary conditions.

9.12 Show that for a lossless inhomogeneous background that

$$\hat{T}^\dagger\hat{T} = \frac{i}{2k_0}\mathcal{M}_{\tau_0}\int_{\tau_0} d^3r'\, G_{0_f}(\mathbf{r}, \mathbf{r}'),$$

where $\hat{T}$ is the operator defined in Eq.(9.30a).

[1] The Dirichlet condition over $\partial\tau$ is uniquely determined from the Neumann condition and vice-versa via the second Helmholtz identity Eq.(9.16b).

From Eq.(9.33) we have

$$\hat{T}^{\dagger}\hat{T} = \frac{1}{(4\pi)^2}\mathcal{M}_{\tau_0}\int_{\tau_0} d^3r'\,\{\int_{4\pi} d\Omega\,\psi_+^*(\mathbf{r},-k_0\mathbf{s})\psi_+(\mathbf{r}',-k_0\mathbf{s})\}.$$

For a lossless medium we have that

$$\psi_+^*(\mathbf{r},-k_0\mathbf{s}) = e^{ik_0\mathbf{s}\cdot\mathbf{r}} + \int d^3r'\,G_{0_-}(\mathbf{r},\mathbf{r}')V_0(\mathbf{r}')e^{ik_0\mathbf{s}\cdot\mathbf{r}'} = \psi_-(\mathbf{r},\mathbf{s})$$

which yields

$$\begin{aligned}\hat{T}^{\dagger}\hat{T} &= \frac{1}{(4\pi)^2}\mathcal{M}_{\tau_0}\int_{\tau_0} d^3r'\,\{\int_{4\pi} d\Omega\,\psi_-(\mathbf{r},\mathbf{s})\psi_+(\mathbf{r}',-k_0k_0\mathbf{s})\}\\ &= \frac{i}{2k_0}\mathcal{M}_{\tau_0}\int_{\tau_0} d^3r'\,G_{0_f}(\mathbf{r},\mathbf{r}'),\end{aligned}$$

where we have made use of Eq.(9.24a).

9.13 Formulate and solve the 2D ISP for a source compactly supported within a homogeneous plane parallel slab with constant wavenumber $k \neq k_0$ and Dirichlet data over two bounding planes. Compare and contrast your solution with that obtained in Section 5.3 of Chapter 5.

This problem employs the plane wave scattering states obtained in Problem 9.2 and a formulation and solution that parallels that used in Problem 5.17. The ISP is defined using Eqs.(9.30a) and (9.30b) which, for this particular case, yield

$$\int_{\tau_0} d^3r\,\overbrace{e^{-i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}[A(\mathbf{K}_\rho)e^{-is\gamma_1 z} + B(\mathbf{K}_\rho)e^{is\gamma_1 z}]}^{\psi_+(\mathbf{r},-k_0\mathbf{s})}\,Q(\boldsymbol{\rho},z) = 2i\gamma_0\tilde{U}_+(\mathbf{K}_\rho,sa_0)e^{is\gamma_0 a_0},$$

with $s = \pm 1$ and where $k_1\mathbf{s} = \mathbf{K}_\rho + s\gamma_1\hat{\mathbf{z}}$ is the wavevector in the slab and $k_0\mathbf{s} = \mathbf{K}_\rho + s\gamma_0\hat{\mathbf{z}}$ the wavevector in the outside medium. Here, $\tilde{U}_+(\mathbf{K}_\rho, sa_0)$ is the spatial Fourier transform of the field on the two ends of the slab assumed to be $\pm a_0$ and the two quantities $A(\mathbf{K}_\rho)$ and $B(\mathbf{K}_\rho)$ are given in Problem 9.2.

Following our treatment of the ISP for a source in a plane parallel slab in a purely homogeneous medium presented in the solution to Problem 5.17 we re-write the above equation in the form

$$\begin{aligned}\int_{-a_0}^{+a_0} dz\,[A(\mathbf{K}_\rho)e^{-is\gamma_1 z} + B(\mathbf{K}_\rho)e^{is\gamma_1 z}]\overline{Q}(\mathbf{K}_\rho,z)&\\ = 2i\gamma_0\tilde{U}_+(\mathbf{K}_\rho,sa_0)e^{is\gamma_0 a_0},&\end{aligned}\tag{9.2}$$

where

$$\overline{Q}(\mathbf{K}_\rho,z) = \int d^2\rho\,Q(\mathbf{r})e^{-i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}$$

is the 2D spatial Fourier transform of the source w.r.t. $\boldsymbol{\rho}$. We base our solution of the ISP on Eq.(9.2) which is solved for $\overline{Q}(\mathbf{K}_\rho,z)$ considering the transverse wavevector $\mathbf{K}_\rho$ to be a fixed parameter. Once $\overline{Q}(\mathbf{K}_\rho,z)$ is computed the actual source is found using an inverse 2D Fourier transform.

Defining

$$\psi(z,s) = A(\mathbf{K}_\rho)e^{-is\gamma_1 z} + B(\mathbf{K}_\rho)e^{is\gamma_1 z}$$

we can write Eq.(9.2) in the form

$$\hat{T}\overline{q}(z) = f(s)$$

where

$$\hat{T} = \int_{-a_0}^{+a_0} dz\,\psi(z,s), \quad \overline{q}(z) = \overline{Q}(\mathbf{K}_\rho, z), \quad f(s) = 2i\gamma_0\tilde{U}_+(\mathbf{K}_\rho, sa_0)e^{is\gamma_0 a_0}$$

and $\mathbf{K}_\rho$ is assumed to be a fixed parameter.

With the above formulation the problem is solved using a procedure that parallels that used in Problem 5.17.

9.14 Compute the scattering amplitude for a set of discrete point scatterers embedded in an inhomogeneous medium using the Foldy-Lax scattering model.

the scattering amplitude for this case is found from Eq.(**??**)

$$U_+^{(s)}(\mathbf{r},\nu) = \sum_{m=1}^{M} \mathcal{V}_m G_{0_+}(\mathbf{r},\mathbf{X}_m)U_+(\mathbf{X}_m,\nu).$$

On making use of the asymptotic approximation for the Green functions given in Eq.(9.9a) we find that

$$U_+^{(s)}(\mathbf{r},\nu) =\sim -\frac{1}{4\pi}\sum_{m=1}^{M} \mathcal{V}_m \psi_+(\mathbf{X}_m, -k_0\mathbf{s})U_+(\mathbf{X}_m,\nu)\frac{e^{ik_0 r}}{r}.$$

yielding

$$f(\mathbf{s},\nu) = -\frac{1}{4\pi}\sum_{m=1}^{M} \mathcal{V}_m \psi_+(\mathbf{X}_m, -k_0\mathbf{s})U_+(\mathbf{X}_m,\nu).$$

9.15 Complete the derivation of Eq.(9.70) in Section 9.10.3.

This follows immediately from the result that

$$\int_0^{a_0} rdr\int_0^{2\pi} d\phi\, e^{-i\mathbf{K}\cdot\mathbf{r}} = \int_0^{a_0} rdr\int_0^{2\pi} d\phi\, e^{iiKr\cos\phi} = 2\pi\int_0^{a_0} rdr\, J_0(Kr) = 2\pi a_0\frac{J_1(Ka_0)}{K}$$

9.16 Derive Eq.(9.66).

We have from Eq.(9.64b)

$$\hat{T}\hat{T}^\dagger u_p = \int d^3r\,\chi_n^*(\mathbf{r}) < \chi_{n'}^*(\mathbf{r}), u_p >_{\mathcal{H}_f} = \sigma_p^2 u_p.$$

We can write the inner product in the above equation as

$$< \chi_{n'}^*(\mathbf{r}), u_p >_{\mathcal{H}_f} = \sum_{n'=1}^{N} \chi_{n'}(\mathbf{r})u_p(n')$$

which then yields

$$\int d^3r\, \chi_n^*(\mathbf{r}) \sum_{n'=1}^{N} \chi_{n'}(\mathbf{r}) u_p(n') = \sum_{n'=1}^{N} u_p(n') \int d^3r\, \chi_n^*(\mathbf{r}) \chi_{n'}(\mathbf{r})$$
$$= \sum_{n'=1}^{N} < \chi_n, \chi_{n'} >_{\mathcal{H}_V} u_p(n') = \sigma_p^2 u_p(n),$$

which completes the derivation.

9.17 Derive Eq.(9.74).

We start with the LS equation for the Green function $G_+(\mathbf{r}, \mathbf{r}')$ given in Eq.(9.45b):

$$G_+(\mathbf{r}, \mathbf{r}') = G_{0_+}(\mathbf{r}, \mathbf{r}') + \int d^3r'' G_+(\mathbf{r}, \mathbf{r}'') V(\mathbf{r}'') G_{0_+}(\mathbf{r}'', \mathbf{r}').$$

For point transmitters and receivers this equation reduces to Eq.(9.71) which then yields the expression for the multistatic data matrices given in Eq.(9.72).

For an extended transmitter and extended receiver we need to convolve the above Green function with the transmitter and receiver transmission functions according to Eqs.(9.73). We then find that

$$K_{j,k}(\omega) = \int d^2r \int d^2r'\, \mathcal{R}_r(\mathbf{r}, \boldsymbol{\beta}_j) \mathcal{R}_t(\mathbf{r}', \boldsymbol{\alpha}_k) [G_+(\mathbf{r}, \mathbf{r}') - G_{0_+}(\mathbf{r}, \mathbf{r}')]$$
$$= \int d^3r'' V(\mathbf{r}'') \int d^2r \int d^2r'\, G_+(\mathbf{r}, \mathbf{r}'') \mathcal{R}_r(\mathbf{r}, \boldsymbol{\beta}_j) G_{0_+}(\mathbf{r}'', \mathbf{r}') \mathcal{R}_t(\mathbf{r}', \boldsymbol{\alpha}_k)$$
$$= \int d^3r'' \psi_r(\boldsymbol{\beta}_j, \mathbf{r}'') V(\mathbf{r}'') \psi_t(\mathbf{r}'', \boldsymbol{\alpha}_k)$$

where we have made use of Eqs.(9.73).