Appendix SA8.1 Alternative Additive Decompositions of x = LBf

Alternative views of an input-output equation like **x** = **LBf** will generate somewhat different additive decompositions. We explore three variations in this Appendix.

1. Using (8.10) directly on **x** = **LBf** gives

$$\begin{aligned}\Delta\mathbf{x} &= \underbrace{(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1)}_{\text{Effect of }\Delta\mathbf{L}} + \underbrace{(1/2)[(\mathbf{L}^0\Delta\mathbf{B})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0]}_{\text{Effect of }\Delta\mathbf{B}} \\ &\quad + \underbrace{(1/2)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})}_{\text{Effect of }\Delta\mathbf{f}}\end{aligned} \qquad \text{(A8.1.1)}$$

2. If we combine **L** and **B**, so that **M** = **LB** and **x** = **Mf**, and then use (8.7),

$$\Delta\mathbf{x} = (1/2)(\Delta\mathbf{M})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f})$$

Since **M** = **LB**,

$$\Delta\mathbf{M} = (1/2)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})$$

so that

$$\begin{aligned}\Delta\mathbf{x} &= (1/2)(1/2)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1) \\ &\quad + (1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f}) \\ &= \underbrace{(1/4)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1)(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of }\Delta\mathbf{L}} + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of }\Delta\mathbf{B}} \\ &\quad + \underbrace{(1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f})}_{\text{Effect of }\Delta\mathbf{f}}\end{aligned} \qquad \text{(A8.1.2)}$$

And, again since **M** = **LB**, the last term is $\underbrace{(1/2)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})}_{\text{Effect of }\Delta\mathbf{f}}$, as in (A8.1.1).

3. If we combine **B** and **f**, so that **y** = **Bf** and **x** = **Ly**, and then use (8.7) ,

$$\Delta\mathbf{x} = (1/2)(\Delta\mathbf{L})(\mathbf{y}^0 + \mathbf{y}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{y})$$

Since **y** = **Bf**,

$$\Delta\mathbf{y} = (1/2)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{B}^0 + \mathbf{B}^1)(\Delta\mathbf{f})$$

so that

$$
\begin{aligned}
\Delta\mathbf{x} &= (1/2)(\Delta\mathbf{L})(\mathbf{y}^0+\mathbf{y}^1)+(1/2)(\mathbf{L}^0+\mathbf{L}^1)[(1/2)(\Delta\mathbf{B})(\mathbf{f}^0+\mathbf{f}^1) \\
&\quad +(1/2)(\mathbf{B}^0+\mathbf{B}^1)(\Delta\mathbf{f})] \\
&= \underbrace{(1/2)(\Delta\mathbf{L})(\mathbf{y}^0+\mathbf{y}^1)}_{\text{Effect of }\Delta\mathbf{L}}+\underbrace{(1/4)(\mathbf{L}^0+\mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0+\mathbf{f}^1)}_{\text{Effect of }\Delta\mathbf{B}} \\
&\quad +\underbrace{(1/4)(\mathbf{L}^0+\mathbf{L}^1)(\mathbf{B}^0+\mathbf{B}^1)(\Delta\mathbf{f})}_{\text{Effect of }\Delta\mathbf{f}}
\end{aligned}
\qquad (A8.1.3)
$$

and since $\mathbf{y} = \mathbf{Bf}$, the first term is $\underbrace{(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)}_{\text{Effect of }\Delta\mathbf{L}}$, again as in (A8.1.1).

Table A8.1.1 summarizes these results. Terms that do not appear in (A8.1.1) are boxed. For example, in Equation (A8.1.2), $\Delta\mathbf{L}$ appears in two terms – $(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)$ and $(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^1+\mathbf{B}^1\mathbf{f}^0)$ – but each is weighted by (1/4) instead of the (1/2) in Equation (A8.1.1). The amount by which $(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)$ differs from $(1/4)[(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)+(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^1+\mathbf{B}^1\mathbf{f}^0)]$ depends entirely on the difference between $(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)$ and $(\mathbf{B}^0\mathbf{f}^1+\mathbf{B}^1\mathbf{f}^0)$. Similar observations can be made for the weightings on $\Delta\mathbf{B}$ in Equations (A8.1.2) and (A8.1.3) vs. Equation (A8.1.1) and on the weighting on $\Delta\mathbf{f}$ in Equation (A8.1.3) vs. Equations (A8.1.1) and (A8.1.2).

Table A8.1.1 Alternative Decompositions of $\mathbf{x} = \mathbf{LBf}$

Eq.	Effect of $\Delta\mathbf{L}$	Effect of $\Delta\mathbf{B}$	Effect of $\Delta\mathbf{f}$
(8.1.1)	$(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)$	$(1/2)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1+\mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0]$	$(1/2)(\mathbf{L}^0\mathbf{B}^0+\mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})$
(8.1.2)	$(1/4)(\Delta\mathbf{L})(\mathbf{B}^0+\mathbf{B}^1)(\mathbf{f}^0+\mathbf{f}^1)=$ $(1/4)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)+$ $\boxed{(1/4)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^1+\mathbf{B}^1\mathbf{f}^0)}$	$(1/4)(\mathbf{L}^0+\mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0+\mathbf{f}^1)=$ $(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1+\mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0]+$ $\boxed{(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^0+\mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^1]}$	$(1/2)(\mathbf{M}^0+\mathbf{M}^1)(\Delta\mathbf{f})=$ $(1/2)(\mathbf{L}^0\mathbf{B}^0+\mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})$
(8.1.3)	$(1/2)(\Delta\mathbf{L})(\mathbf{y}^0+\mathbf{y}^1)=$ $(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)$	$(1/4)(\mathbf{L}^0+\mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0+\mathbf{f}^1)=$ $(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1+\mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0]+$ $\boxed{(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^0+\mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^1]}$	$(1/4)(\mathbf{L}^0+\mathbf{L}^1)(\mathbf{B}^0+\mathbf{B}^1)(\Delta\mathbf{f})=$ $(1/4)(\mathbf{L}^0\mathbf{B}^0+\mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})+$ $\boxed{(1/4)(\mathbf{L}^0\mathbf{B}^1+\mathbf{L}^1\mathbf{B}^0)(\Delta\mathbf{f})}$