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A Student's Guide to the Navier-Stokes Equations: A Supplement on the Use of the Del Operator in Non-Cartesian Coordinates

The book used mostly used Cartesian coordinates when discussing the equations. In doing so, we performed the various operations using a "Cartesian-friendly" approach. Such an approach is fine to get your feet wet but the vector operations get somewhat more involved when non-Cartesian coordinates are used. This document serves as an introduction for applying some of the del operations we learned about in the book to a non-Cartesian coordinate system. Seeing these operations in detail may help you later on as you continue in your studies of fluid mechanics. One reason for this is that many times the fluid mechanics equations are written in what is called tensor notation. Tensor notation is a nice, succinct way to deal with the operations using the del operation (as well as other vector and tensor operations) without having to perform lines and lines of computations.

To start understanding how the operations can sometimes be different for non-Cartesian coordinate systems, it is best to first consider an example using Cartesian coordinates. For example, when taking the divergence of two vectors in Cartesian coordinates, we simply treated the operation in some ways like matrix multiplication. Thus, taking the divergence of the velocity vector, $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$, in Cartesian coordinates was simply the following operation:

$$\vec{\nabla} \cdot \vec{V} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

You would perform this operation much like you would matrix multiplication except that you would be applying the derivatives of the row vector to the column vector as opposed to multiplying. This approach worked well in Cartesian coordinates. However, the correct way to perform the divergence operation is to "foil" through the derivatives to all of the velocity terms. Therefore, written in full, the divergence operation should be written as:

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{V} &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(u\hat{i} + v\hat{j} + w\hat{k}\right) \\
&= \hat{i}\frac{\partial}{\partial x} \cdot \left(u\hat{i}\right) + \hat{i}\frac{\partial}{\partial x} \cdot \left(v\hat{j}\right) + \hat{i}\frac{\partial}{\partial x} \cdot \left(w\hat{k}\right) \\
&+ \hat{j}\frac{\partial}{\partial y} \cdot \left(u\hat{i}\right) + \hat{j}\frac{\partial}{\partial y} \cdot \left(v\hat{j}\right) + \hat{j}\frac{\partial}{\partial y} \cdot \left(w\hat{k}\right) \\
&+ \hat{k}\frac{\partial}{\partial z} \cdot \left(u\hat{i}\right) + \hat{k}\frac{\partial}{\partial z} \cdot \left(v\hat{j}\right) + \hat{k}\frac{\partial}{\partial z} \cdot \left(w\hat{k}\right)
\end{aligned}
$$

Notice how there are nine terms (as opposed to three). The derivatives of the individual velocity components multiplied by their base vectors need to be taken. We can do that for the first three terms here (utilizing the product rule from calculus):

$$
\begin{aligned}
\hat{i}\frac{\partial}{\partial x} \cdot \left(u\hat{i}\right) &= u\hat{i} \cdot \underbrace{\frac{\partial \hat{i}}{\partial x}}_{\text{equals } 0} + \underbrace{\hat{i} \cdot \hat{i}}_{\text{equals } 1} \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} \\
\hat{i}\frac{\partial}{\partial x} \cdot \left(v\hat{j}\right) &= v\hat{i} \cdot \underbrace{\frac{\partial \hat{j}}{\partial x}}_{\text{equals } 0} + \underbrace{\hat{i} \cdot \hat{j}}_{\text{equals } 0} \frac{\partial v}{\partial x} = 0 \\
\hat{i}\frac{\partial}{\partial x} \cdot \left(w\hat{k}\right) &= w\hat{i} \cdot \underbrace{\frac{\partial \hat{k}}{\partial x}}_{\text{equals } 0} + \underbrace{\hat{i} \cdot \hat{k}}_{\text{equals } 0} \frac{\partial w}{\partial x} = 0
\end{aligned}
\tag{1.1}
$$

All we are doing here is utilizing the product rule with an added wrinkle, the dot product between base vectors (i.e. i, j, and k) is taken in each term. Since a Cartesian coordinate system is an orthogonal coordinate system (i.e. all of the base vectors are perpendicular to each other), then the base vectors dotted with each other are zero if we are dotting two different base vectors and one if the base vectors being dotted are the same. We can continue with this trend for the next six terms in the divergence:

$$
\begin{aligned}
\hat{j}\frac{\partial}{\partial y}\cdot\left(u\hat{i}\right) &= u\hat{j}\cdot\underbrace{\frac{\partial\hat{i}}{\partial y}}_{\text{equals }0}+\underbrace{\hat{j}\cdot\hat{i}}_{\text{equals }0}\frac{\partial u}{\partial y}=0\\
\hat{j}\frac{\partial}{\partial y}\cdot\left(v\hat{j}\right) &= v\hat{j}\cdot\underbrace{\frac{\partial\hat{j}}{\partial y}}_{\text{equals }0}+\underbrace{\hat{j}\cdot\hat{j}}_{\text{equals }1}\frac{\partial v}{\partial y}=\frac{\partial v}{\partial y}\\
\hat{j}\frac{\partial}{\partial y}\cdot\left(w\hat{k}\right) &= w\hat{j}\cdot\underbrace{\frac{\partial\hat{k}}{\partial y}}_{\text{equals }0}+\underbrace{\hat{j}\cdot\hat{k}}_{\text{equals }0}\frac{\partial w}{\partial y}=0\\
\hat{k}\frac{\partial}{\partial z}\cdot\left(u\hat{i}\right) &= u\hat{k}\cdot\underbrace{\frac{\partial\hat{i}}{\partial z}}_{\text{equals }0}+\underbrace{\hat{k}\cdot\hat{i}}_{\text{equals }0}\frac{\partial u}{\partial z}=0\\
\hat{k}\frac{\partial}{\partial z}\cdot\left(v\hat{j}\right) &= v\hat{k}\cdot\underbrace{\frac{\partial\hat{j}}{\partial z}}_{\text{equals }0}+\underbrace{\hat{j}\cdot\hat{k}}_{\text{equals }0}\frac{\partial v}{\partial z}=0\\
\hat{k}\frac{\partial}{\partial z}\cdot\left(w\hat{k}\right) &= w\hat{k}\cdot\underbrace{\frac{\partial\hat{k}}{\partial z}}_{\text{equals }0}+\underbrace{\hat{k}\cdot\hat{k}}_{\text{equals }1}\frac{\partial w}{\partial z}=\frac{\partial w}{\partial z}
\end{aligned}
\tag{1.2}
$$

We can add Equations 1.1 and 1.2 to get the divergence in Cartesian coordinates:

$$\vec{\nabla}\cdot\vec{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

That is a lot of computations just to get a result we already knew. However, for non-Cartesian coordinate systems, this more extensive operation is usually neeeded. We can see this by performing the divergence in cylindrical coordinates using the del operator. In cylindrical coordinates, the velocity vector is defined as:

$$\vec{V}=V_r\hat{e}_r++V_\theta\hat{e}_\theta+V_z\hat{e}_z \tag{1.3}$$

The del in cylindrical coordinates is defined as the following:

$$\vec{\nabla}=\hat{e}_r\frac{\partial}{\partial r}+\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+\hat{e}_z\frac{\partial}{\partial z} \tag{1.4}$$

To take the the divergence of $\vec{V}$ in cylindrical coordinates, you should avoid taking the "matrix algebra" approach and instead write it out with the base vectors included, i.e.:

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{V} &= \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \right) \cdot (V_r \hat{e}_r + + V_\theta \hat{e}_\theta + V_z \hat{e}_z) \\
&= \hat{e}_r \frac{\partial}{\partial r} \cdot (V_r \hat{e}_r) + \hat{e}_r \frac{\partial}{\partial r} \cdot (V_\theta \hat{e}_\theta) + \hat{e}_r \frac{\partial}{\partial r} \cdot (V_z \hat{e}_z) \\
&+ \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \cdot (V_r \hat{e}_r) + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \cdot (V_\theta \hat{e}_\theta) + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \cdot (V_z \hat{e}_z) \\
&+ \hat{e}_z \frac{\partial}{\partial z} \cdot (V_r \hat{e}_r) + \hat{e}_z \frac{\partial}{\partial z} \cdot (V_\theta \hat{e}_\theta) + \hat{e}_z \frac{\partial}{\partial z} \cdot (V_z \hat{e}_z)
\end{aligned}
$$

Now, the key thing here is that the derivatives of the base vectors are not going to be zero in all cases, unlike Cartesian coordinates. In particular, we have the following relationships:

$$
\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \quad \text{and} \quad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r
$$

All other derivatives of the base vectors are zero. With this information, the nine terms can be written out in full (using the product rule of calculus):

$$
\begin{aligned}
\hat{e}_r\frac{\partial}{\partial r}\cdot(V_r\hat{e}_r) &= \underbrace{\hat{e}_r\cdot\hat{e}_r}_{\text{equals 1}}\frac{\partial V_r}{\partial r} + V_r\hat{e}_r\cdot\underbrace{\frac{\partial \hat{e}_r}{\partial r}}_{\text{equals 0}} = \frac{\partial V_r}{\partial r}\\
\hat{e}_r\frac{\partial}{\partial r}\cdot(V_\theta\hat{e}_\theta) &= \underbrace{\hat{e}_r\cdot\hat{e}_\theta}_{\text{equals 0}}\frac{\partial V_\theta}{\partial r} + V_\theta\hat{e}_r\cdot\underbrace{\frac{\partial \hat{e}_\theta}{\partial r}}_{\text{equals 0}} = 0\\
\hat{e}_r\frac{\partial}{\partial r}\cdot(V_z\hat{e}_z) &= \underbrace{\hat{e}_r\cdot\hat{e}_z}_{\text{equals 0}}\frac{\partial V_z}{\partial r} + V_z\hat{e}_r\cdot\underbrace{\frac{\partial \hat{e}_z}{\partial r}}_{\text{equals 0}} = 0\\
\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\cdot(V_r\hat{e}_r) &= \underbrace{\hat{e}_\theta\cdot\hat{e}_r}_{\text{equals 0}}\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_r}{r}\underbrace{\hat{e}_\theta\cdot\underbrace{\frac{\partial \hat{e}_r}{\partial \theta}}_{\text{equals }\hat{e}_\theta}}_{\text{equals 1}} = \frac{V_r}{r}\\
\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\cdot(V_\theta\hat{e}_\theta) &= \underbrace{\hat{e}_\theta\cdot\hat{e}_\theta}_{\text{equals 1}}\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_\theta}{r}\underbrace{\hat{e}_\theta\cdot\underbrace{\frac{\partial \hat{e}_\theta}{\partial \theta}}_{\text{equals }-\hat{e}_r}}_{\text{equals 0}} = \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} \qquad (1.5)\\
\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\cdot(V_z\hat{e}_z) &= \underbrace{\hat{e}_\theta\cdot\hat{e}_z}_{\text{equals 0}}\frac{1}{r}\frac{\partial V_z}{\partial \theta} + \frac{V_z}{r}\hat{e}_\theta\cdot\underbrace{\frac{\partial \hat{e}_z}{\partial \theta}}_{\text{equals 0}} = 0\\
\hat{e}_z\frac{\partial}{\partial z}\cdot(V_r\hat{e}_r) &= \underbrace{\hat{e}_z\cdot\hat{e}_r}_{\text{equals 0}}\frac{\partial V_r}{\partial z} + V_r\hat{e}_z\cdot\underbrace{\frac{\partial \hat{e}_r}{\partial z}}_{\text{equals 0}} = 0\\
\hat{e}_z\frac{\partial}{\partial z}\cdot(V_\theta\hat{e}_\theta) &= \underbrace{\hat{e}_z\cdot\hat{e}_\theta}_{\text{equals 0}}\frac{\partial V_\theta}{\partial z} + V_\theta\hat{e}_z\cdot\underbrace{\frac{\partial \hat{e}_\theta}{\partial z}}_{\text{equals 0}} = 0\\
\hat{e}_z\frac{\partial}{\partial z}\cdot(V_z\hat{e}_z) &= \underbrace{\hat{e}_z\cdot\hat{e}_z}_{\text{equals 1}}\frac{\partial V_z}{\partial z} + V_z\hat{e}_z\cdot\underbrace{\frac{\partial \hat{e}_z}{\partial z}}_{\text{equals 0}} = \frac{\partial V_z}{\partial z}
\end{aligned}
$$

We can plug all of these values into our divergence calculation to get:

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{V} &= \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}\right) \cdot (V_r \hat{e}_r + +V_\theta \hat{e}_\theta + V_z \hat{e}_z) \\
&= \underbrace{\hat{e}_r \frac{\partial}{\partial r} \cdot (V_r \hat{e}_r)}_{=\frac{\partial V_r}{\partial r}} + \underbrace{\hat{e}_r \frac{\partial}{\partial r} \cdot (V_\theta \hat{e}_\theta)}_{=0} + \underbrace{\hat{e}_r \frac{\partial}{\partial r} \cdot (V_z \hat{e}_z)}_{=0} \\
&+ \underbrace{\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot (V_r \hat{e}_r)}_{=\frac{V_r}{r}} + \underbrace{\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot (V_\theta \hat{e}_\theta)}_{=\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}} + \underbrace{\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot (V_z \hat{e}_z)}_{=0} \\
&+ \underbrace{\hat{e}_z \frac{\partial}{\partial z} \cdot (V_r \hat{e}_r)}_{=0} + \underbrace{\hat{e}_z \frac{\partial}{\partial z} \cdot (V_\theta \hat{e}_\theta)}_{=0} + \underbrace{\hat{e}_z \frac{\partial}{\partial z} \cdot (V_z \hat{e}_z)}_{=\frac{\partial V_z}{\partial z}} \\
&= \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}
\end{aligned}
$$

This was a considerable amount of calculations for the cylindrical coordinates. However, this is what is necessary in order to fully calculate the divergence using the del operator.

We can do a similar operation for the advective term in cylindrical coordinates. For starters, the advective operator is:

$$
\begin{aligned}
\vec{V} \cdot \vec{\nabla} &= (V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{e}_z) \cdot \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}\right) \\
&= V_r \underbrace{\hat{e}_r \cdot \hat{e}_r}_{=1} \frac{\partial}{\partial r} + V_r \underbrace{\hat{e}_r \cdot \hat{e}_\theta}_{=0} \frac{1}{r} V_\theta \frac{\partial}{\partial \theta} + V_r \underbrace{\hat{e}_r \cdot \hat{e}_z}_{=0} \frac{\partial}{\partial z} \\
&+ V_\theta \underbrace{\hat{e}_\theta \cdot \hat{e}_r}_{=0} \frac{\partial}{\partial r} + V_\theta \underbrace{\hat{e}_\theta \cdot \hat{e}_\theta}_{=1} \frac{1}{r}\frac{\partial}{\partial \theta} + V_\theta \underbrace{\hat{e}_\theta \cdot \hat{e}_z}_{=0} \frac{\partial}{\partial z} \\
&+ V_z \underbrace{\hat{e}_z \cdot \hat{e}_r}_{=0} \frac{\partial}{\partial r} + V_z \underbrace{\hat{e}_z \cdot \hat{e}_\theta}_{=0} \frac{1}{r}\frac{\partial}{\partial \theta} + V_z \underbrace{\hat{e}_z \cdot \hat{e}_z}_{=1} \frac{\partial}{\partial z} \\
&= V_r \frac{\partial}{\partial r} + V_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + V_z \frac{\partial}{\partial z}
\end{aligned}
\tag{1.6}
$$

Foiling through for this case is relatively straightforward since all of the dot products are zero except when the base vectors are the same. This is always the case for orthogonal coordinate systems (such as a cylindrical coordinate system). Applying Equation 1.6 to the velocity vector gives us:

$$\vec{V}\cdot\vec{\nabla}\vec{V}=\left(V_r\frac{\partial}{\partial r}+V_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+V_z\frac{\partial}{\partial z}\right)(V_r\hat{e}_r+V_\theta\hat{e}_\theta+V_z\hat{e}_z)$$

We can apply the advective term operator to each velocity coordinate term separately (again, taking advantage of the product rule):

For V_r:

$$\begin{aligned}
&\left(V_r\frac{\partial}{\partial r}+V_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+V_z\frac{\partial}{\partial z}\right)V_r\hat{e}_r\\
&=V_r\frac{\partial(V_r\hat{e}_r)}{\partial r}+V_\theta\frac{1}{r}\frac{\partial(V_r\hat{e}_r)}{\partial\theta}+V_z\frac{\partial(V_r\hat{e}_r)}{\partial z}\\
&=V_r\left(V_r\underbrace{\frac{\partial\hat{e}_r}{\partial r}}_{=0}+\hat{e}_r\frac{\partial V_r}{\partial r}\right)+V_\theta\frac{1}{r}\left(V_r\underbrace{\frac{\partial\hat{e}_r}{\partial\theta}}_{=\hat{e}_\theta}+\hat{e}_r\frac{\partial V_r}{\partial\theta}\right)+V_z\left(V_r\underbrace{\frac{\partial\hat{e}_r}{\partial z}}_{=0}+\hat{e}_r\frac{\partial V_r}{\partial z}\right)\\
&=V_r\frac{\partial V_r}{\partial r}\hat{e}_r+\underbrace{V_\theta\frac{1}{r}V_r\hat{e}_\theta}_{\text{goes to }\theta\text{-direction}}+V_\theta\frac{1}{r}\frac{\partial V_r}{\partial\theta}\hat{e}_r+V_z\frac{\partial V_r}{\partial z}\hat{e}_r
\end{aligned}\tag{1.7}$$

For V_θ:

$$\begin{aligned}
&\left(V_r\frac{\partial}{\partial r}+V_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+V_z\frac{\partial}{\partial z}\right)V_\theta\hat{e}_\theta\\
&=V_r\frac{\partial(V_\theta\hat{e}_\theta)}{\partial r}+V_\theta\frac{1}{r}\frac{\partial(V_\theta\hat{e}_\theta)}{\partial\theta}+V_z\frac{\partial(V_\theta\hat{e}_\theta)}{\partial z}\\
&=V_r\left(V_\theta\underbrace{\frac{\partial\hat{e}_\theta}{\partial r}}_{=0}+\hat{e}_\theta\frac{\partial V_\theta}{\partial r}\right)+V_\theta\frac{1}{r}\left(V_\theta\underbrace{\frac{\partial\hat{e}_\theta}{\partial\theta}}_{=-\hat{e}_r}+\hat{e}_\theta\frac{\partial V_\theta}{\partial\theta}\right)+V_z\left(V_\theta\underbrace{\frac{\partial\hat{e}_\theta}{\partial z}}_{=0}+\hat{e}_\theta\frac{\partial V_\theta}{\partial z}\right)\\
&=V_r\frac{\partial V_\theta}{\partial r}\hat{e}_\theta\underbrace{-\frac{1}{r}V_\theta^2\hat{e}_r}_{\text{goes to }r-\text{direction}}+V_\theta\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}\hat{e}_\theta+V_z\frac{\partial V_\theta}{\partial z}\hat{e}_\theta
\end{aligned}\tag{1.8}$$

For V_z:

$$\left(V_r\frac{\partial}{\partial r}+V_\theta\frac{1}{r}\frac{\partial}{\partial\theta}+V_z\frac{\partial}{\partial z}\right)V_z\hat{e}_z$$

$$= V_r\frac{\partial(V_z\hat{e}_z)}{\partial r}+V_\theta\frac{1}{r}\frac{\partial(V_z\hat{e}_z)}{\partial\theta}+V_z\frac{\partial(V_z\hat{e}_z)}{\partial z}$$

$$= V_r\left(V_z\underbrace{\frac{\partial\hat{e}_z}{\partial r}}_{=0}+\hat{e}_z\frac{\partial V_z}{\partial r}\right)+V_\theta\frac{1}{r}\left(V_z\underbrace{\frac{\partial\hat{e}_z}{\partial\theta}}_{=0}+\hat{e}_z\frac{\partial V_z}{\partial\theta}\right)+V_z\left(V_z\underbrace{\frac{\partial\hat{e}_z}{\partial z}}_{=0}+\hat{e}_z\frac{\partial V_z}{\partial z}\right)$$

$$= V_r\frac{\partial V_z}{\partial r}\hat{e}_z+V_\theta\frac{1}{r}\frac{\partial V_z}{\partial\theta}\hat{e}_z+V_z\frac{\partial V_z}{\partial z}\hat{e}_z \tag{1.9}$$

Notice that two of the terms "switch" to another direction, namely the $V_\theta\frac{1}{r}V_r\hat{e}_\theta$ from the V_r advective term and the $-\frac{1}{r}V_\theta^2\hat{e}_r$ from the V_θ advective term. Adding up Equations 1.7, 1.8, and 1.9 leads to the following for the material derivative in cylindrical coordinates:

$$\boxed{\frac{D\vec{V}}{Dt}=\begin{pmatrix}\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+V_\theta\frac{1}{r}\frac{\partial V_r}{\partial\theta}+V_z\frac{\partial V_r}{\partial z}-\frac{1}{r}V_\theta^2\\ \frac{\partial V_\theta}{\partial t}+V_r\frac{\partial V_\theta}{\partial r}+V_\theta\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}+V_z\frac{\partial V_\theta}{\partial z}+V_\theta\frac{1}{r}V_r\\ \frac{\partial V_z}{\partial t}+V_r\frac{\partial V_z}{\partial r}+V_\theta\frac{1}{r}\frac{\partial V_z}{\partial\theta}+V_z\frac{\partial V_z}{\partial z}\end{pmatrix}}$$

The first row of Equation 1 is the $r-$direction, the second row is the θ-direction, and the third row is the $z-$direction.

The same procedure can be done for the Laplacian operator in cylindrical coordinates:

$$\begin{aligned}\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} &= \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\cdot\left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\\ &= \hat{e}_r\frac{\partial}{\partial r}\cdot\left(\hat{e}_r\frac{\partial}{\partial r}\right) + \hat{e}_r\frac{\partial}{\partial r}\cdot\left(\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right) + \hat{e}_r\frac{\partial}{\partial r}\cdot\left(\hat{e}_z\frac{\partial}{\partial z}\right)\\ &+ \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left(\hat{e}_r\frac{\partial}{\partial r}\right) + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left(\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right) + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left(\hat{e}_z\frac{\partial}{\partial z}\right)\\ &+ \hat{e}_z\frac{\partial}{\partial z}\cdot\left(\hat{e}_r\frac{\partial}{\partial r}\right) + \hat{e}_z\frac{\partial}{\partial z}\cdot\left(\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right) + \hat{e}_z\frac{\partial}{\partial z}\cdot\left(\hat{e}_z\frac{\partial}{\partial z}\right)\end{aligned} \tag{1.10}$$

We can go term by term now with the reminder that the only base vectors affected by the derivatives are the θ-derivatives of $\hat{e}_\theta$ and $\hat{e}_r$:

$$\hat{e}_r\frac{\partial}{\partial r}\cdot\left(\hat{e}_r\frac{\partial}{\partial r}\right) = \underbrace{\hat{e}_r\cdot\hat{e}_r}_{=1}\frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r^2}$$

$$\hat{e}_r\frac{\partial}{\partial r}\cdot\left(\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right) = \underbrace{\hat{e}_r\cdot\hat{e}_\theta}_{=0}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) = 0$$

$$\hat{e}_r\frac{\partial}{\partial r}\cdot\left(\hat{e}_z\frac{\partial}{\partial z}\right) = \underbrace{\hat{e}_r\cdot\hat{e}_z}_{=0}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial z}\right) = 0$$

$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left(\hat{e}_r\frac{\partial}{\partial r}\right) = \hat{e}_\theta\frac{1}{r}\cdot\left(\hat{e}_r\frac{\partial^2}{\partial\theta\partial r} + \underbrace{\frac{\partial\hat{e}_r}{\partial\theta}}_{=\hat{e}_\theta}\frac{\partial}{\partial r}\right) = \frac{1}{r}\underbrace{\hat{e}_\theta\cdot\hat{e}_r}_{=0}\frac{\partial^2}{\partial\theta\partial r} + \frac{1}{r}\underbrace{\hat{e}_\theta\cdot\hat{e}_\theta}_{=1}\frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}$$

$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left(\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right) = \hat{e}_\theta\frac{1}{r^2}\cdot\left(\hat{e}_\theta\frac{\partial^2}{\partial\theta^2} + \underbrace{\frac{\hat{e}_\theta}{\partial\theta}}_{=-\hat{e}_r}\frac{\partial}{\partial\theta}\right) = \frac{1}{r^2}\underbrace{\hat{e}_\theta\cdot\hat{e}_\theta}_{=1}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\underbrace{\hat{e}_\theta\cdot(-\hat{e}_r)}_{=0}\frac{\partial}{\partial\theta} = \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left(\hat{e}_z\frac{\partial}{\partial z}\right) = \frac{1}{r}\underbrace{\hat{e}_\theta\cdot\hat{e}_z}_{=0}\frac{\partial^2}{\partial\theta\partial z} = 0$$

$$\hat{e}_z\frac{\partial}{\partial z}\cdot\left(\hat{e}_r\frac{\partial}{\partial r}\right) = \underbrace{\hat{e}_z\cdot e_r}_{=0}\frac{\partial^2}{\partial r\partial z} = 0$$

$$\hat{e}_z\frac{\partial}{\partial z}\cdot\left(\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right) = \underbrace{\hat{e}_z\cdot\hat{e}_\theta}_{=0}\frac{1}{r}\frac{\partial^2}{\partial\theta\partial z} = 0$$

$$\hat{e}_z\frac{\partial}{\partial z}\cdot\left(\hat{e}_z\frac{\partial}{\partial z}\right) = \underbrace{\hat{e}_z\cdot\hat{e}_z}_{=1}\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z^2}$$

We can add up the terms to get the Laplacian in cylindrical coordinates:

$$\boxed{\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}} \tag{1.11}$$

Sometimes the $r-$ derivatives are "grouped" into a single term to get an alternative for the Laplacian in cylindrical coordinates:

$$\boxed{\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}} \tag{1.12}$$

Going over these mathematical manipulations for some sample operations (divergence, material derivative, and the Laplacian) in cylindrical coordinates hopefully gives you a taste of how to use to del operator in its various forms for non-Cartesian coordinate systems. The examples used in this supplemental document used cylindrical coordinates, which is still an orthogonal coordinate system. Even greater computations are required for non-orthogonal coordinate systems. Luckily, much of these computations can be encapsulated and written more succinctly with the help of tensor notation. A great book on tensors can be found in the collection of Cambridge's students guides, namely: *A Student's Guide to Vectors and Tensors* by Daniel Fleisch.

In general, however, when using cylindrical or spherical coordinates, you normally can just "look" up the operations involving the del symbol.

The governing equations are given below in cylindrical coordinate systems.

Continuity (conservation form):

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V_r)}{\partial r} + \rho\frac{V_r}{r} + \frac{1}{r}\frac{\partial (\rho V_\theta)}{\partial \theta} + \frac{\partial (\rho V_z)}{\partial z} = 0$$

Compressible Navier-Stokes (non-conservation form):

$r-$coordinate:

$$\begin{aligned}\rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right) &= -\frac{\partial p}{\partial r} - \frac{\partial}{\partial r}\left(\mu\frac{2}{3}\vec{\nabla} \cdot \vec{V}\right)\\ &+ \frac{\partial}{\partial r}\left(2\mu\frac{\partial V_r}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{\partial V_\theta}{\partial r}\right)\right)\\ &+ \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial V_r}{\partial z} + \frac{\partial V_z}{\partial r}\right)\right) + \frac{2\mu}{r}\left(\frac{\partial V_r}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_r}{r}\right) + \rho g_r\end{aligned}$$

θ-coordinate:

$$\rho\left(\frac{\partial V_\theta}{\partial t}+V_r\frac{\partial V_\theta}{\partial r}\ +\ \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{V_\theta V_r}{r}+V_z\frac{\partial V_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}-\frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu\frac{2}{3}\vec{\nabla}\cdot\vec{V}\right)$$
$$+\frac{\partial}{\partial r}\left(\mu\left(\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r}+\frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu\left(\frac{2}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{2V_r}{r}\right)\right)$$
$$+\frac{\partial}{\partial z}\left(\mu\left(\frac{\partial V_\theta}{\partial z}+\frac{1}{r}\frac{\partial V_z}{\partial \theta}\right)\right)+\frac{2\mu}{r}\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta}+\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r}\right)+\rho g_\theta$$

$z-$coordinate:

$$\rho\left(\frac{\partial V_z}{\partial t}\ +\ V_r\frac{\partial V_z}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}+V_z\frac{\partial V_z}{\partial z}\right)=-\frac{\partial p}{\partial z}-\frac{\partial}{\partial z}\left(\mu\frac{2}{3}\vec{\nabla}\cdot\vec{V}\right)$$
$$+\frac{\partial}{\partial r}\left(\mu\left(\frac{\partial V_z}{\partial r}+\frac{\partial V_r}{\partial z}\right)\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta}+\frac{\partial V_\theta}{\partial z}\right)\right)$$
$$+\frac{\partial}{\partial z}\left(2\mu\frac{\partial V_z}{\partial z}\right)+\frac{\mu}{r}\left(\frac{\partial V_r}{\partial z}+\frac{\partial V_z}{\partial r}\right)+\rho g_z$$

Energy Equation (non-conservation form):

$$\rho c_p\left(\frac{\partial T}{\partial t}\ +\ V_r\frac{\partial T}{\partial r}+\frac{V_\theta}{r}\frac{\partial T}{\partial \theta}+V_z\frac{\partial T}{\partial z}\right)=\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{k}{r}\frac{\partial T}{\partial \theta}\right)$$
$$+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\beta_T T\left(\frac{\partial p}{\partial t}+V_r\frac{\partial p}{\partial r}+\frac{V_\theta}{r}\frac{\partial p}{\partial \theta}+V_z\frac{\partial p}{\partial z}\right)+\dot{q}_{gen}+\Phi$$

Where:

$$\Phi=2\mu\left(\left(\frac{\partial V_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{V_r}{r}\right)^2+\left(\frac{\partial V_z}{\partial z}\right)^2\right)$$
$$+\mu\left(\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta}+\frac{\partial V_\theta}{\partial r}-\frac{V_\theta}{r}\right)^2+\left(\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial V_z}{\partial \theta}\right)^2+\left(\frac{\partial V_z}{\partial r}+\frac{\partial V_r}{\partial z}\right)^2\right)$$
$$-\frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial\left(rV_r\right)}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{\partial V_z}{\partial z}\right)^2$$