Corrections and additions in the book "Bruhat-Tits theory: a new approach"

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Chapter 1

- In the last sentence of 1.3.29, delete "affine and extended".
- Replace "$W(\Psi)^{\mathrm{aff}}$" by "$W(\Psi)$" everywhere.

Chapter 2

- In Remark 2.5.11, replace "§7" by "§6".

Chapter 6

- In the line just before the proof of Proposition 6.6.2, replace "and is part of Theorem 7.5.1"with "this is Proposition 9.4.35".

Chapter 7

- At the end of the second paragraph of (the first page of) Chapter 7, add the following sentences "As discussed in §6.3, this endows the affine space $\mathcal{A}(S)$ with two (closely related) affine root systems $\Psi \subset \Psi' \subset \mathcal{A}(S)^*$ and the action of $N(k)$ on $\mathcal{A}(S)$ provides a map from $N(k)$ to the extended affine Weyl group $W(\Psi)^{\mathrm{ext}}$ whose image contains the affine Weyl group $W(\Psi)$ (cf. §1.3 for definitions). It is proved in §6.3 that assertions AS 1 – AS 4 of Axiom 4.1.6 hold. *We will assume in addition that assertion AS 5 holds*. Given Propositions 6.3.13 and 6.6.2, this is known when G is quasi-split, and for a general G amounts to the assumption that, when G is semi-simple and simply connected, the image of the map $N(k) \to W(\Psi)^{\mathrm{ext}}$ is precisely $W(\Psi)$. This is proved in this generality in Proposition 9.4.35, assuming that the residue field of k is perfect.
- In the second paragraph of the proof of Lemma 7.1.4, replace $-s$ with s, and in the fourth paragraph of the proof replace s by $-s$ and $\psi(\varphi)$ by $-\psi(\varphi)$.
- The sentence appearing just before Remark 7.4.2 should be replaced with "If $\Omega = \{x\}$, then instead of the subscript $\{x\}$ we will use x for simplicity."
- In Remark 7.4.2 replace "a chamber" by "the standard chamber (that is, the chamber fixed by the subgroup $G(k)_x^\#$ described in the remark)."

- Just after the first sentence of §7.5 add "Then $G(k)^0 = G(k)$." and replace $G(k)^0_{\mathcal{C}}$ with $G(k)_{\mathcal{C}}$ in the next sentence.

- Replace the second sentence of the proof of Theorem 7.5.1 with "According to Axiom 4.1.6 AS 5, the quotient $\mathcal{N}/\mathcal{Z}$ is isomorphic to W^{aff}"

- In the fourth paragraph of the proof of Theorem 7.5.1, replace ψ by $\alpha \in \Psi'$ and replace k at four places by r in the same paragraph. In the third paragraph from the bottom of the proof of this theorem, replace $\cap_{n\in\mathcal{N}} \mathcal{J}$ by $\cap_{n\in\mathcal{N}} n\mathcal{J}n^{-1}$.

- Replace the sentence appearing just before the last paragraph of the proof of Theorem 7.5.1, by "That the Weyl group of the Tits system is isomorphic to W^{aff} is simply our assumption that Axiom 4.1.6 AS 5 holds".

- In the proof of Lemma 7.5.2, replace Z_{sc} by Z', and replace "its preimage in $Z'(k)$" by "the preimage of $\mathcal{J}^{\#} \cap Z(k)$ in $Z'(k)$".

Chapter 8

- In the second paragraph of Chapter 8, replace "the first three sections" with "the first two sections".

- In the statement and proof of Lemma 8.1.4, replace $\bar{\mathfrak{f}}$ with $\mathfrak{f}_s$ everywhere.

- Replace the first sentence of §8.3 with the following:

  We will assume in this and the subsequent paragraphs that the residue field $\mathfrak{f}$ of k is perfect. Then the residue field of K is an algebraic closure of $\mathfrak{f}$ and by Corollary 2.3.8, G_K is quasi-split.

- Replace the second paragraph of 8.3.6 with the following:

  Let T be the centralizer of S in G. As G is quasi-split, T is a maximal K-torus of G. Let $\mathscr{T}^0$ be the connected Néron model of T. The inclusion $S \to T$ extends to a closed immersion $\mathscr{S} \to \mathscr{T}^0$ due to Lemma B.7.11.

- Delete the last two sentences of the proof of Theorem 8.3.13 and in their place insert the following:

  As the group schemes $\mathscr{U}_{a,\Omega,0}$, $a \in \Phi$, and $\mathscr{T}^0$ have connected fibers, the homomorphisms of these group schemes into $\mathscr{G}^1_\Omega$ factor through $\mathscr{G}^0_\Omega$. Therefore, the subgroup $G(K)^0_\Omega$ generated by $U_{a,\Omega,0}\,(= \mathscr{U}_{a,\Omega,0}(\mathcal{O}))$ for $a \in \Phi$, and $T(K)^0\,(= \mathscr{T}^0(\mathcal{O}))$ is contained in $\mathscr{G}^0_\Omega(\mathcal{O})$. Thus

  $$G(K)^0_\Omega \subset \mathscr{G}^0_\Omega(\mathcal{O}) \subset \mathscr{G}^1_\Omega(\mathcal{O}) = G(K)^1_\Omega.$$

As $G(K)^0_\Omega$ is of finite index in $G(K)^1_\Omega$, we see that $G(K)^0_\Omega$ is an open subgroup of finite index in $\mathscr{G}^0_\Omega(\mathcal{O})$. Now Lemma A.4.26 shows that $G(K)^0_\Omega = \mathscr{G}^0_\Omega(\mathcal{O})$.

- The first sentence of §8.4 should be replaced with:

  As in the preceding section, we assume here that the residue field $\mathfrak{f}$ of k is perfect.

- In the second paragraph of §8.4, replace $\mathfrak{o}$ with $\mathcal{O}$ at two places.

- At the beginning of the third paragraph of 8.4.2, insert the following sentence:

  For $a \in \Phi$, let $\mathscr{U}_{a,\Omega,0}$ be the a-root group of $\mathscr{G}^0_\Omega$.

- Delete the first sentence of the proof of Lemma 8.4.4 and replace k with K at two places in the proof.

- Just before the statement of Proposition 8.4.15, insert the following:

  Let $\Omega \prec \Omega'$ be two nonempty bounded subsets of $\mathcal{A}$. Then the inclusion $G(K)^0_{\Omega'} \subset G(K)^0_\Omega$ gives rise to a $\mathcal{O}$-group scheme homomorphism $\rho_{\Omega,\Omega'} : \mathscr{G}^0_{\Omega'} \to \mathscr{G}^0_\Omega$. We denote by $\overline{\rho}_{\Omega,\Omega'}$ the induced map $\overline{\mathscr{G}}^0_{\Omega'} \to \overline{\mathscr{G}}^0_\Omega$ between the special fibers.

- Delete the first sentence of Proposition 8.4.15.

Chapter 9

- In the second sentence of the statement of Proposition 9.3.12 add "contained in an apartment of $\mathcal{B}(G)$" towards the end of that sentence.

- Just before the statement of Proposition 9.3.13 add the following new paragraph:

  There is a natural action of $V(S')$ on $\mathcal{B}(G_K)_S$. This is seen by noting that as every apartment contained $\mathcal{B}(G_K)_S$ corresponds to a maximal K-split torus containing S, there is a natural action of $V(S')$ on each such apartment. Now it needs to be shown that given a point $x \in \mathcal{B}(G_K)_S$ and $v \in V(S')$, and apartments A_i, $i = 1, 2$, in $\mathcal{B}(G_K)_S$ containing x, the point $v + x$ of A_1 equals the point $v + x$ of A_2. But the desired equality follows from the fact that there is a $g \in G(K)^1_x$ which commutes with S, and so with $V(S')$, and maps A_1 onto A_2.

- Delete the last paragraph of the proof of Proposition 9.3.13.

- From the last sentence of the fifth paragraph of the proof of Proposition 9.4.8, delete the word ”non-divisible”.

• Immediately after the proof of Corollary 9.9.4 add the following example that shows that a non-quasi-split (even an anisotropic) group may admit a reductive model.

Example Let $k = f((t))$, where f is a field. Let G be a connected semi-simple f-group and T be a maximal f-torus of G containing a maximal f-split torus of G. Let F be the splitting field of T. Then F/f is a finite Galois extension. Let Γ be the Galois group of F/f. Let $\ell = F((t))$. Then ℓ/k is an unramified Galois extension of k with Galois group Γ. Moreover, as T splits over F so does G. Therefore, G_ℓ is a ℓ-split semi-simple group and hence Bruhat–Tits theory is available for this group. As ℓ/k is an unramified extension, by unramified descent (which holds also when the residue field f is imperfect, see [BT84a] or [Pra20b]) Bruhat–Tits theory is also available for G_k.

We will denote $f[[t]]$ and $F[[t]]$ by $\mathfrak{o}$ and $\mathfrak{o}_\ell$ respectively. As G_F is a Chevalley group, $\mathscr{G}_{\mathfrak{o}_\ell} := G_F \times_F F[[t]] = G \times_f \mathfrak{o}_\ell$ is a Chevalley $\mathfrak{o}_\ell$-group scheme. Hence, $G(F[[t]])$ is a hyperspecial parahoric subgroup of $G(\ell)$ and the corresponding point x in the Bruhat–Tits building $\mathcal{B}(G_\ell)$ of $G(\ell)$ is a hyperspecial vertex. The $\mathfrak{o}$-group scheme $\mathscr{G} := G \times_f f[[t]]$ is clearly the descent of the Chevalley $\mathfrak{o}_\ell$-group scheme $\mathscr{G}_{\mathfrak{o}_\ell}$. Hence, $\mathscr{G}$ is a reductive $\mathfrak{o}$-group scheme, and $\mathscr{G}(\mathfrak{o}) = G(f[[t]])$ is a hyperspecial parahoric subgroup of $G(k)$. Moreover, the point x is a hyperspecial vertex in the building $\mathcal{B}(G_k) = \mathcal{B}(G_\ell)^\Gamma$.

Since f is the residue field of k, we see that the special fiber of the hyperspecial parahoric group scheme $\mathscr{G} = G \times_f f[[t]]$ is G, and its generic fiber is G_k. As $f \hookrightarrow k = f((t))$, f-rank $G \leqslant k$-rank G_k. We assert that f-rank $G = k$-rank G_k. To see this, we recall from 8.3.6 and 9.2.5 that every parahoric $\mathfrak{o}$-group scheme associated to G_k contains an $\mathfrak{o}$-split torus whose generic fiber is a maximal k-split torus of G_k. Now let $\mathscr{S}$ be a $\mathfrak{o}$-split torus of $\mathscr{G}$ such that the generic fiber $\mathscr{S} \times_\mathfrak{o} k$ is a maximal k-split torus of G_k. The special fiber S of $\mathscr{S}$ is a f-split torus of G. This implies that f-rank $G \geqslant k$-rank G_k and our assertion is proved.

Therefore, if G is anisotropic over f, then G_k is anisotropic over k. In this case, the building $\mathcal{B}(G_k) = \{x\}$, and according to Theorem 2.2.9, $G(f((t)))$ is bounded. Since the hyperspecial subgroup $G(f[[t]])$ is a maximal bounded subgroup of $G(k) = G(f((t)))$, we conclude that if G is f-anisotropic, then $G(f((t))) = G(f[[t]])$, and replacing t with $1/t$, we see that $G(f((1/t))) = G(f[[1/t]])$. Now, in case G is f-anisotropic, we conclude the following:

$$G(f[t]) = G(f[t]) \cap G(f((1/t))) = G(f[t]) \cap G(f[[1/t]]) = G(f).$$

Chapter 12

- In item (3) on p. 441, replace §11.5 by Proposition 12.6.1.
- Remove the last sentence of the second paragraph of §12.4.
- Proposition 12.4.1 and its proof should be replaced with the following:

  Proposition 12.4.1. *Let S and S' be as above. Let x be a Θ-fixed point of $\widetilde{\mathcal{B}}(Z_{H'}(S))$ and A be any apartment of this building containing x. Then*
  $$A^{\Theta} = V(S') + x = \widetilde{\mathcal{B}}(Z_{H'}(S))^{\Theta}.$$

  Proof. Let $Z\,(\supset S')$ and $Z_{H'}(S)'$ be the maximal K-split central torus and the derived subgroup of $Z_{H'}(S)$ respectively. Then Z and $Z_{H'}(S)'$ are stable under Θ; moreover, $(Z^{\Theta})^0 = S'$.

  There is an action of $V(Z)$ on each apartment of $\widetilde{\mathcal{B}}(Z_{H'}(S))$ by translations since every maximal K-split torus of $Z_{H'}(S)$ contains Z. Moreover, the parahoric subgroup associated to x acts transitively on the set of apartments of $\widetilde{\mathcal{B}}(Z_{H'}(S))$ containing x, we see that the action of $V(Z)$ on the apartments of $\widetilde{\mathcal{B}}(Z_{H'}(S))$ combine to give a natural action of $V(Z)$ on the entire $\widetilde{\mathcal{B}}(Z_{H'}(S))$. Hence, there is a natural action of $V(S') = V(Z)^{\Theta}$ on $\widetilde{\mathcal{B}}(Z_{H'}(S))^{\Theta}$.

  Let T be the maximal K-split torus of H' containing S corresponding to the apartment A. Then $A = V(T)+x$, hence $A^{\Theta} = V(T)^{\Theta}+x = V(S')+x$. As the building $\widetilde{\mathcal{B}}(Z_{H'}(S))$ is the union of apartments containing x, we conclude that $\widetilde{\mathcal{B}}(Z_{H'}(S))^{\Theta} = V(S') + x$. □

- Replace the first sentence of the statement of Proposition 12.4.2 with the following:

  "Let S_1 and S_2 be maximal K-split tori of G and Ω be a non-empty bounded subset of $\widetilde{\mathcal{B}}(Z_{H'}(S_1))^{\Theta} \cap \widetilde{\mathcal{B}}(Z_{H'}(S_2))^{\Theta}$."

- The last sentence of the first paragraph of the statement of Proposition 12.8.5 should be placed immediately after the first sentence of the second paragraph.

Appendix A

- In line 10 (from the top) of p. 604, insert the word "perfect" before the word "residue".