Chapter 14

```
> with(DEtools):
> with(linalg):
Warning, the name adjoint has been redefined

Warning, the protected names norm and trace have been redefined and
unprotected
```

Question 1

```
> k:=(ln(6.)-ln(5.8))/10;
```

$$k := .0033901551$$

```
> estp:=5.8*exp(k*(t-1701));
```

$$estp := 5.8\,\mathbf{e}^{(.0033901551\,t - 5.766653825)}$$

```
> estpn:=subs(t=1701+10*n,estp);
```

$$estpn := 5.8\,\mathbf{e}^{(.0339015510\,n)}$$

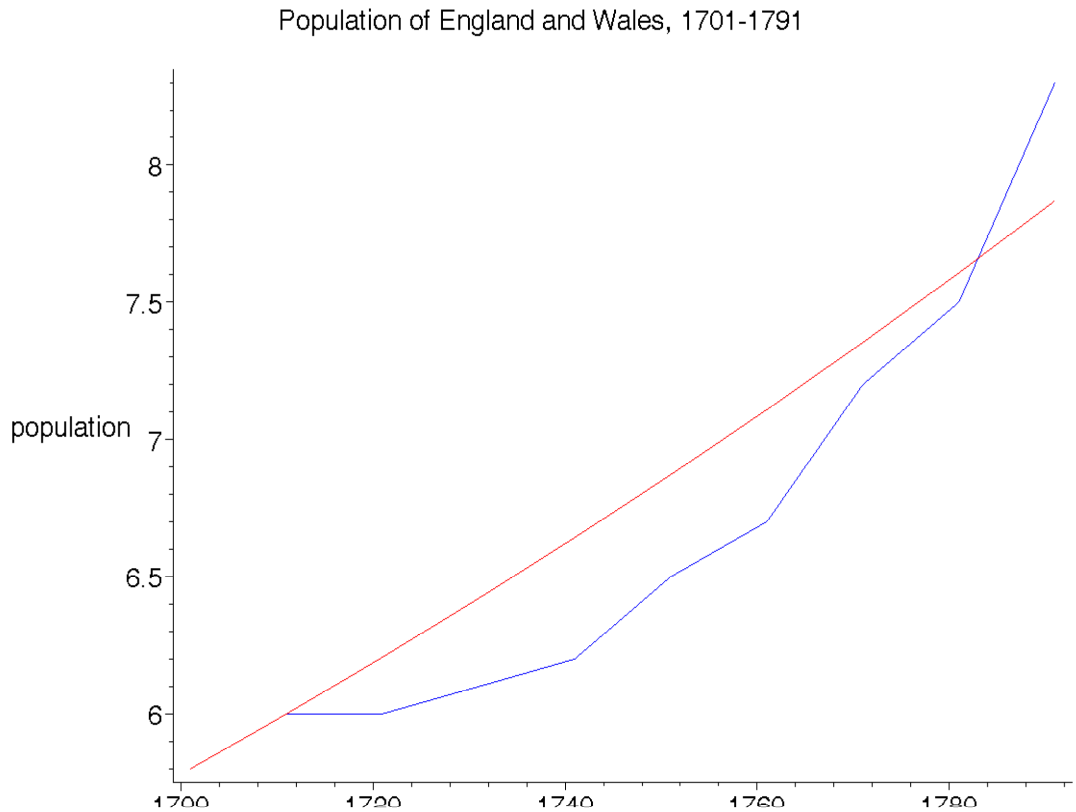
```
> pointsestp:=seq([1701+10*n,estpn],n=0..9);
```

pointsestp := [1701, 5.8], [1711, 5.999999996], [1721, 6.206896541], [1731, 6.420927456], [1741, 6.642338739], [1751, 6.871384903], [1761, 7.108329200], [1771, 7.353443998], [1781, 7.607011025], [1791, 7.869321744]

```
> pointsactp:=[1701,5.8],[1711,6.0],[1721,6.0],[1731,6.1],[1741
,6.2],[1751,6.5],[1761,6.7],[1771,7.2],[1781,7.5],[1791,8.3];
```

pointsactp := [1701, 5.8], [1711, 6.0], [1721, 6.0], [1731, 6.1], [1741, 6.2], [1751, 6.5], [1761, 6.7], [1771, 7.2], [1781, 7.5], [1791, 8.3]

```
> plot([[pointsestp],[pointsactp]],labels=["year","population"]
,title="Population of England and Wales,
1701-1791",colour=[red,blue]);
```



Question 2

We have the following information for each country after the epidemic:

Country A: $b = 3\%$, $d = 2\%$ with $(b\text{-}d) = 1\%$ $\lambda = .01$

Country B: $b = 5\%$, $d = 3\%$ with $(b\text{-}d) = 2\%$ $\lambda = .02$

```
> solve(2*p0*exp(0.01*t)=p0*exp(0.02*t),t);
```

$$69.31471806$$

Question 3

We have

$$\mathrm{p}(t) = p0\, \mathbf{e}^{((b-d-m)\,t)}$$

```
> solve(p0*exp(((b-d)-m)*t1)=(1/2)*p0*exp(((b-d)-m)*t0),t1);
```

$$\frac{t0\, b - t0\, d - t0\, m - \ln(2)}{b - d - m}$$

Or

$$t_1 - t_0 = -\frac{\ln(2)}{b - d - m}$$

$b = 5\%$, $d = 3\%$ and $m = 3\%$, then $b\text{-}d\text{-}m = \text{-}1\%$.

```
> evalf(-ln(2)/(-0.01));
```

$$69.31471806$$

Question 4

Given

$\mathrm{p}(t) = p_0\, \mathbf{e}^{(k\,t)}$

(i) Population trebbling in size

```
> solve(p0*exp(0.03*t1)=3*p0*exp(0.03*t0),t1);
```

$$36.62040962 + t0$$

i.e., $t_1 - t_0 = 36.62040962$.

(ii) General result

```
> p0:='p0': lambda:='lambda': k:='k':
> solve(p0*exp(k*t1)=lambda*p0*exp(k*t0),t1);
```

$$\frac{k\,t0 + \ln(\lambda)}{k}$$

Which can be expressed

$$t_1 - t_0 = \frac{\ln(\lambda)}{k}$$

Question 5

To derive the Taylor expansion about the point *a/b*, series command

```
> series(p*(a-b*p),p=a/b,2);
```

$$-a\left(p - \frac{a}{b}\right) + \mathrm{O}\left(\left(p - \frac{a}{b}\right)^2\right)$$

Neglecting the error term, then

$$\frac{dp}{dt} = -a\left(p - \frac{a}{b}\right)$$

```
> dsolve({diff(p(t),t)=-a*(p(t)-a/b),p(0)=p0},p(t));
```

$$\mathrm{p}(t) = \frac{a}{b} - \frac{\mathbf{e}^{(-a\,t)}\,(a - p0\,b)}{b}$$

```
> assume(a>0);
> limit(a/b-exp(-a*t)*(a-p0*b)/b,t=infinity);
```

$$\frac{a\sim}{b}$$

Then $\mathrm{p}(t)$->*a/b*, which implies that equilibrium is never achieved in a finite time period.

Question 6

(i)

```
> a:='a':
```

The fixed points are found from

```
> solve(p*(a+c*p)=0,p);
```

$$0, -\frac{a}{c}$$

The turning point is

```
> solve(diff(p*(a+c*p),p)=0,p);
```

$$-\frac{1}{2}\frac{a}{c}$$

which is negative since $a > 0$ and $c > 0$. Furthermore,

```
> diff(p*(a+c*p),p$2);
```

$$2\,c$$

which is positive, hence the turning point is a minimum.

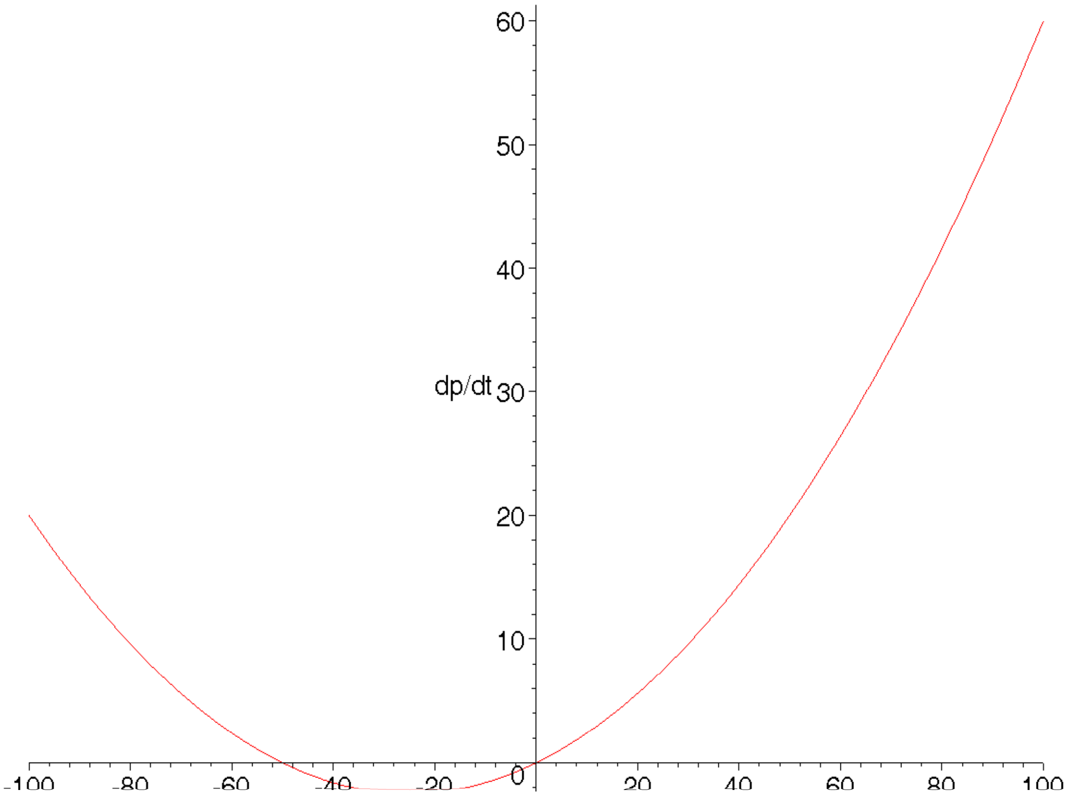
(ii)

$a = 0.2$ and $c = 0.004$.

```
> a:='a': c:='c': p:='p':
> exam:=subs({a=0.2,c=0.004},p*(a+c*p));
```

$$exam := p\,(.2 + .004\,p)$$

```
> plot(exam,p=-100..100,labels=["p","dp/dt"]);
```



Hence, for positive population p0, the population grows indefinitely.

(iii)

```
> dsolve(diff(p(t),t)=p(t)*(a+c*p(t)),p(t));
```

$$\mathrm{p}(t) = \frac{a}{-c + \mathbf{e}^{(-a\,t)}\,\_C1\,a}$$

This expression is zero if $c - \mathbf{e}^{(-a\,t)}\,\_C1\,a = 0$

```
> solve(-c+exp(-a*t)*_C1*a=c,t);
```

$$-\frac{\ln\left(2\,\frac{c}{\_C1\,a}\right)}{a}$$

Hence, an infinite population is reached in a finite time period.

Question 7

(i)

The two points are

```
> solve(r*(M-p)*(p-m)=0,p);
```

$$m, M$$

The turning point is

```
> solve(diff(r*(M-p)*(p-m),p)=0,p);
```

$$\frac{1}{2}m+\frac{1}{2}M$$

```
> subs(p=(m+M)/2,diff(r*(M-p)*(p-m),p$2));
```

$$-2\,r$$

which is negative, hence the turning point is a maximum.

```
> subs(p=m,diff(r*(M-p)*(p-m),p));
```

$$r\,(M-m)$$

which is positive since $M > m$, so the fixed point m is locally unstable.

```
> subs(p=M,diff(r*(M-p)*(p-m),p));
```

$$-r\,(M-m)$$

which is negative since $M > m$, so the fixed point M is locally stable.

The parameter m is associated with such factors as:

(a) reproduction system

(b) density of species (i.e., the area over which it operates).

The parameter M is associated with such factors as:

(a) food availability

(b) predation

(c) catch

(ii)

$p = 0$, i.e., at $p = m$. It is important, therefore, to establish the value of m, e.g., the blue whale.

Question 8

```
> dsolve({diff(p(t),t)=r*p(t)*(a-ln(p(t))),p(0)=p0},p(t));
```

$$\mathrm{p}(t)=\mathbf{e}^{\left(-\mathbf{e}^{(-t\,r+\ln(a-\ln(p0)))}+a\right)}$$

```
> simplify(exp(-exp(-t*r+ln(a-ln(p0)))+a));
```

$$p0^{\left(\mathbf{e}^{(-t\,r)}\right)}\,\mathbf{e}^{\left(-a\,\left(\mathbf{e}^{(-t\,r)}-1\right)\right)}$$

(ii) and (iii)

The fixed points are,

```
> solve(r*p*(a-ln(p))=0,p);
```

$$\mathbf{e}^{a}$$

```
> limit(r*p*(a-ln(p)),p=0);
```

$$0$$

Hence, there are two fixed points one at $p = 0$ and the other at $\mathbf{e}^a$.

```
> subs(p=exp(a),diff(r*p*(a-ln(p)),p));
```

$$r\,(a - \ln(\mathbf{e}^a)) - r$$

But $\ln(\mathbf{e}^a) = a$, so this expression is equal to $-r$. Since r is positive, then the slope of $\frac{dp}{dt}$ at the point $\mathbf{e}^a$ is negative, and so the point $p = \mathbf{e}^a$ is locally stable.

Also $\frac{dp}{dt}$ is at a maximum at the value

```
> solve(diff(r*p*(a-ln(p)),p)=0,p);
```

$$\mathbf{e}^{(a-1)}$$

(iv)

```
> assume(a>0,r>0);
> limit(p0^exp(-t*r)*exp(-a*(exp(-t*r)-1)),t=infinity);
```

$$\mathbf{e}^{a\sim}$$

```
> a:='a': r:='r':
```

Question 9

```
> dsolve({diff(x(t),t)=-3*y(t),diff(y(t),t)=-9*x(t)},{x(t),y(t)});
```

$$\{\mathrm{x}(t) = \_C1\,\mathbf{e}^{(-3\sqrt{3}\,t)} + \_C2\,\mathbf{e}^{(3\sqrt{3}\,t)},\ \mathrm{y}(t) = \sqrt{3}\,(\_C1\,\mathbf{e}^{(-3\sqrt{3}\,t)} - \_C2\,\mathbf{e}^{(3\sqrt{3}\,t)})\}$$

Since

$$\frac{dy}{dx} = \frac{\frac{\partial}{\partial t}y}{\frac{\partial}{\partial t}x} = -\frac{9\,x}{-3\,y} = \frac{3\,x}{y}$$

then we can integrate by parts for we have

```
> int(y, y);
```

$$\frac{1}{2}y^2$$

```
> int(3*x, x);
```

$$\frac{3}{2}x^2$$

```
> solve((1/2)*y^2=3*(x^2/2)+c/2,y);
```

$$\sqrt{3\,x^2 + c},\ -\sqrt{3\,x^2 + c}$$

where $c/2$ is the constant of integration.

Alternatively, we can solve directly

```
> dsolve(diff(y(x),x)=3*x/y(x),y(x));
```

$$\mathrm{y}(x) = \sqrt{3\,x^2 + \_C1},\ \mathrm{y}(x) = -\sqrt{3\,x^2 + \_C1}$$

Question 10

The $\frac{dT}{dt} = 0$ and $\frac{dB}{dt} = 0$ phase lines are

```
> solve(a*(1-T/k1)*T-b*T*B=0,B);
```

$$-\frac{a\,(-k1+T)}{k1\,b}$$

```
> convert(-a*(-k1+T)/(k1*b),parfrac,T);
```

$$-\frac{a\,T}{k1\,b}+\frac{a}{b}$$

```
> solve(c*(1-B/k2)*B-d*T*B=0,B);
```

$$0,\ -\frac{k2\,(-c+d\,T)}{c}$$

```
> convert(k2*(c-d*T)/c,parfrac,T);
```

$$-\frac{k2\,d\,T}{c}+k2$$

The two phase lines are then

$$B = \frac{a}{b} - \frac{a\,T}{b\,k1} \quad \text{for } \frac{dT}{dt} = 0$$

$$B = k2 - \frac{d\,k2\,T}{c} \quad \text{for } \frac{dB}{dt} = 0$$

which intersect at the point

```
> solve({B=-a*T/(k1*b)+a/b,B=-k2*d*T/c+k2},{B,T});
```

$$\left\{B = -\frac{a\,k2\,(-c+d\,k1)}{a\,c-k2\,d\,k1\,b},\ T = \frac{k1\,c\,(-k2\,b+a)}{a\,c-k2\,d\,k1\,b}\right\}$$

which is as far as we can take the general result with *Maple*.

Question 11

(i)

Given

$$\frac{\mathrm{N}(t)}{K} = \frac{N0}{K\left(\frac{N0}{K}+\left(1-\frac{N0}{K}\right)\mathbf{e}^{(-r\,t)}\right)}$$

then $\frac{\mathrm{N}(2)\ is}{K}$

```
> evalf(0.25/(0.25+(1-0.25)*exp(-0.71*2)));
```

$$.5796624108$$

and hence $\frac{\mathrm{N}(2)}{2}$ is

```
> evalf(.5796624108*80.5*10^6);
```

$$.4666282407\ 10^{8}$$

or 46.66282407 $x10^{6}$ *Kg*

(ii)

In solving for t, we write N(t) as Nt.

```
> solve(Nt/K=(N0/K)/((N0/K)+(1-(N0/K))*exp(-r*t)),t);
```

$$-\frac{\ln\left(\frac{N0\,(-Nt+K)}{Nt\,(K-N0)}\right)}{r}$$

which can be expressed

$$t=-\frac{1\,\ln\left(\frac{N0\left(1-\frac{Nt}{K}\right)}{\frac{K\,Nt\left(1-\frac{Nt}{K}\right)}{K}}\right)}{r}$$

(a) t at which N(t) = .5 K or $\frac{\mathrm{N}(t)}{K}=.5$.

```
> evalf(-(1/0.71)*ln((0.25*(1-0.5))/(0.5*(1-0.25))));
```

$$1.547341252$$

(b) t at which N(t) = .75 K or $\frac{\mathrm{N}(t)}{K}=.75$.

```
> evalf(-(1/0.71)*ln((0.25*(1-0.75))/(0.75*(1-0.25))));
```

$$3.094682502$$

Question 12

(i) (a)

```
> sol1:=solve({-x+(x*y/100)=0,2*y-(2*x*y)/25=0},{x,y});
```

$$sol1 := \{x=0, y=0\}, \{y=100, x=25\}$$

```
> sol11:=sol1[1];
```

$$sol11 := \{x=0, y=0\}$$

```
> sol12:=sol1[2];
```

$$sol12 := \{y=100, x=25\}$$

(ii) (a)

```
> fx11:=subs(sol11,diff(-x+(x*y/100),x));
```

$$fx11 := -1$$

```
> fy11:=subs(sol11,diff(-x+(x*y/100),y));
```

$$fy11 := 0$$

```
> gx11:=subs(sol11,diff(2*y-(2*x*y)/25,x));
```

$$gx11 := 0$$

```
> gy11:=subs(sol11,diff(2*y-(2*x*y)/25,y));
```

$$gy11 := 2$$

```
> A11:=matrix([[-1, 0], [0, 2]]);
```

$$A11 := \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

```
> fx12:=subs(sol12,diff(-x+(x*y/100),x));
```

$$fx12 := 0$$

```
> fy12:=subs(sol12,diff(-x+(x*y/100),y));
```

$$fy12 := \frac{1}{4}$$

```
> gx12:=subs(sol12,diff(2*y-(2*x*y)/25,x));
```

$$gx12 := -8$$

```
> gy12:=subs(sol12,diff(2*y-(2*x*y)/25,y));
```

$$gy12 := 0$$

```
> A12:=matrix([[0, 1/4], [-8, 0]]);
```

$$A12 := \begin{bmatrix} 0 & \frac{1}{4} \\ -8 & 0 \end{bmatrix}$$

(iii) (a)

```
> eigenvalues(A11);
```

$$-1, 2$$

```
> eigenvectors(A11);
```

$$[-1, 1, \{[1, 0]\}], [2, 1, \{[0, 1]\}]$$

```
> eigenvalues(A12);
```

$$I\sqrt{2}, -I\sqrt{2}$$

```
> eigenvectors(A12);
```

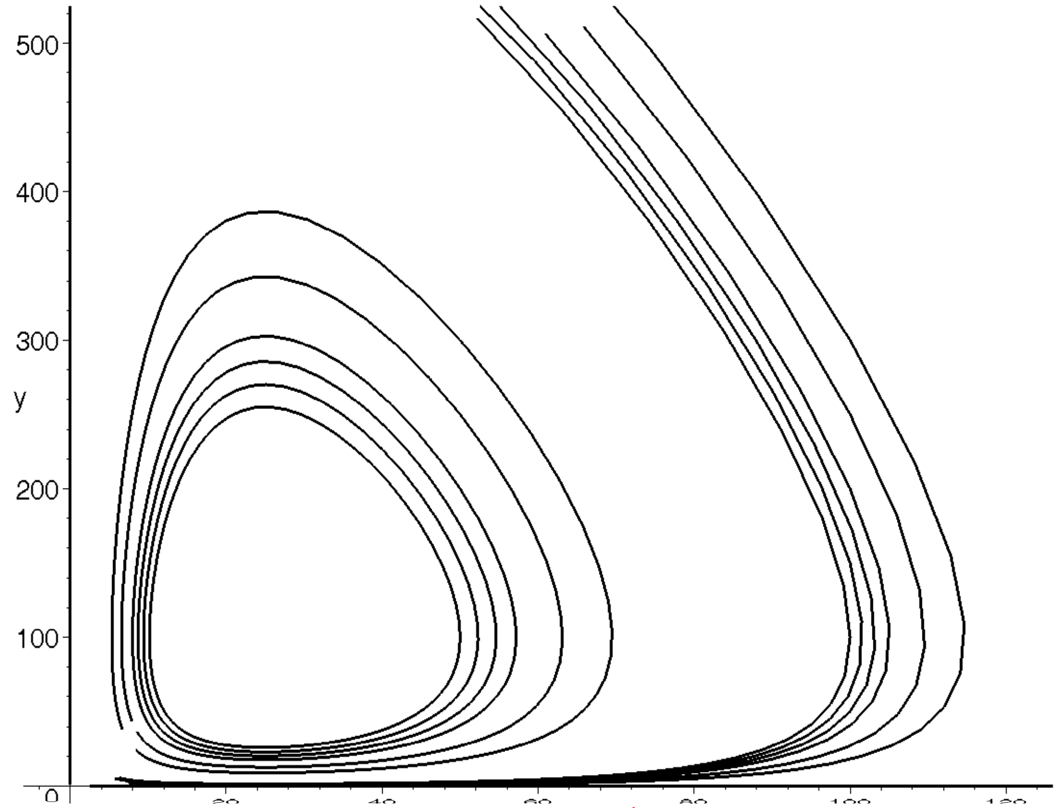
$$\left[I\sqrt{2}, 1, \left\{\left[\frac{-1}{8} I\sqrt{2}, 1\right]\right\}\right], \left[-I\sqrt{2}, 1, \left\{\left[\frac{1}{8} I\sqrt{2}, 1\right]\right\}\right]$$

(iv) (a)

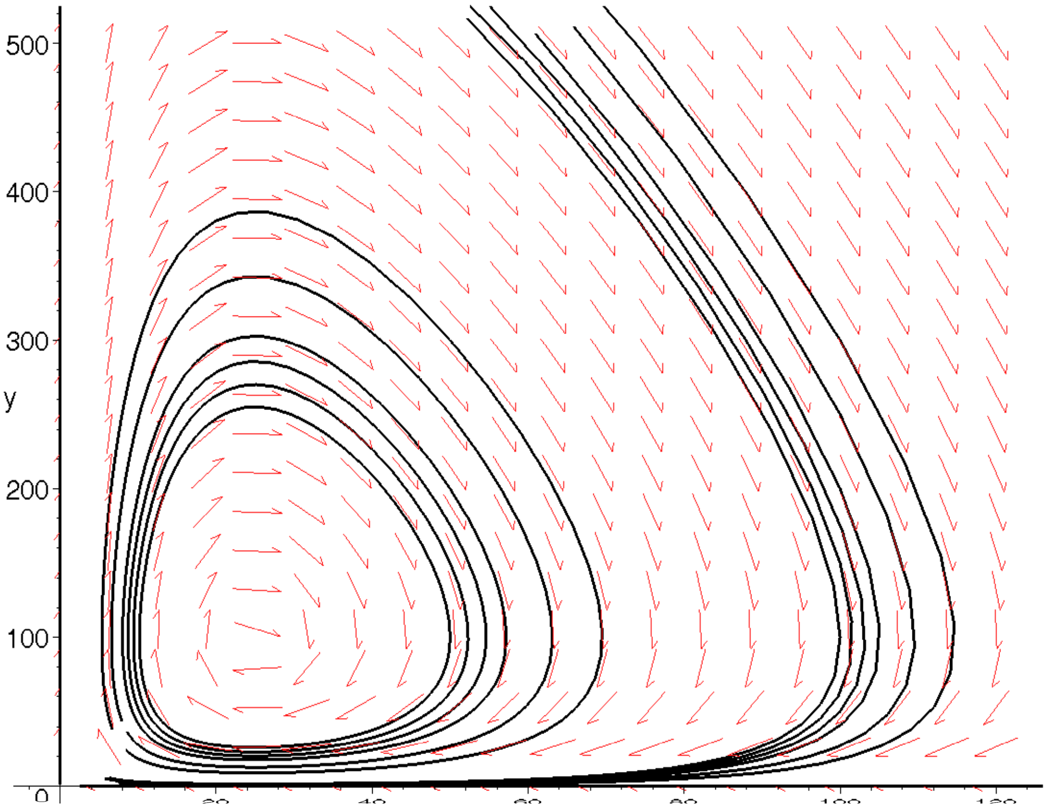
Note

The following procedure for plotting multiple trajectories along with the direction field is adapted from Coombes, K., R. et al (2nd ed. 1997) Differential Equations with Maple. John Wiley & Sons Inc., Chapter 12.

```
> des:=diff(x(t),t)=-x(t)+x(t)*y(t)/100,diff(y(t),t)=2*y(t)-
  2*x(t)*y(t)/25:
  iniset:={seq(seq([x(0)=a,y(0)=b],a=[0,50,100]),
  b=[0,50,100,150,200,250,300])}:
  pphase:=trange->DEplot([des],[x(t),y(t)],t=trange,iniset,x
  =0..120,y=0..500,stepsize=.05,
  method=rkf45,linecolour=black,arrows=NONE):
> pphase(-2..3);
```




```
> des:=diff(x(t),t)=-x(t)+x(t)*y(t)/100,diff(y(t),t)=2*y(t)-
  2*x(t)*y(t)/25:
  iniset:={seq(seq([x(0)=a,y(0)=b],a=[0,50,100]),
  b=[0,50,100,150,200,250,300])}:
  pphase:=trange->DEplot([des],[x(t),y(t)],t=trange,iniset,x
  =0..120,y=0..500,stepsize=.05,
  method=rkf45,linecolour=black,arrows=THIN):
> pphase(-2..3);
```



Which shows oscillations around the fixed point $(x, y) = (25, 100)$.

(i) (b)

```
> sol2:=solve({(-x/2)+((x*y/((1/4)+y)))=0,y-y^2-x*y/((1/4)+y
)=0},{x,y});
```

$$sol2 := \{x = 0, y = 0\}, \{y = 1, x = 0\}, \{y = \frac{1}{4}, x = \frac{3}{8}\}$$

```
> sol21:=sol2[1];
```

$$sol21 := \{x = 0, y = 0\}$$

```
> sol22:=sol2[2];
```

$$sol22 := \{y = 1, x = 0\}$$

```
> sol23:=sol2[3];
```

$$sol23 := \{y = \frac{1}{4}, x = \frac{3}{8}\}$$

(ii) (b)

```
> fx21:=subs(sol21,diff((-x/2)+((x*y/((1/4)+y))),x));
```

$$fx21 := \frac{-1}{2}$$

```
> fy21:=subs(sol21,diff((-x/2)+((x*y/((1/4)+y))),y));
```

$$fy21 := 0$$

```
> gx21:=subs(sol21,diff(y-y^2-x*y/((1/4)+y),x));
```

$$gx21 := 0$$

```
> gy21:=subs(sol21,diff(y-y^2-x*y/((1/4)+y),y));
```

$$gy21 := 1$$

```
> A21:=matrix([[-1/2, 0], [0, 1]]);
```

$$A21 := \begin{bmatrix} \frac{-1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

```
> fx22:=subs(sol22,diff((-x/2)+((x*y/((1/4)+y))),x));
```

$$fx22 := \frac{3}{10}$$

```
> fy22:=subs(sol22,diff((-x/2)+((x*y/((1/4)+y))),y));
```

$$fy22 := 0$$

```
> gx22:=subs(sol22,diff(y-y^2-x*y/((1/4)+y),x));
```

$$gx22 := \frac{-4}{5}$$

```
> gy22:=subs(sol22,diff(y-y^2-x*y/((1/4)+y),y));
```

$$gy22 := -1$$

```
> A22:=matrix([[0, 3/8], [-1/2, 1/8]]);
```

$$A22 := \begin{bmatrix} 0 & \frac{3}{8} \\ \frac{-1}{2} & \frac{1}{8} \end{bmatrix}$$

```
> fx23:=subs(sol23,diff((-x/2)+((x*y/((1/4)+y))),x));
```

$$fx23 := 0$$

```
> fy23:=subs(sol23,diff((-x/2)+((x*y/((1/4)+y))),y));
```

$$fy23 := \frac{3}{8}$$

```
> gx23:=subs(sol23,diff(y-y^2-x*y/((1/4)+y),x));
```

$$gx23 := \frac{-1}{2}$$

```
> gy23:=subs(sol23,diff(y-y^2-x*y/((1/4)+y),y));
```

$$gy23 := \frac{1}{8}$$

```
> A23:=matrix([[3/10, 0], [-4/5, -1]]);
```

$$A23 := \begin{bmatrix} \frac{3}{10} & 0 \\ \frac{-4}{5} & -1 \end{bmatrix}$$

(iii) (b)

```
> eigenvalues(A21);
```

$$\frac{-1}{2}, 1$$

```
> eigenvectors(A21);
```

$$\left[\frac{-1}{2}, 1, \{[1, 0]\}\right], [1, 1, \{[0, 1]\}]$$

```
> eigenvalues(A22);
```

$$\frac{1}{16}+\frac{1}{16} I \sqrt{47}, \frac{1}{16}-\frac{1}{16} I \sqrt{47}$$

```
> eigenvectors(A22);
```

$$\left[\frac{1}{16}+\frac{1}{16} I \sqrt{47}, 1, \left\{\left[1, \frac{1}{6}+\frac{1}{6} I \sqrt{47}\right]\right\}\right], \left[\frac{1}{16}-\frac{1}{16} I \sqrt{47}, 1, \left\{\left[1, \frac{1}{6}-\frac{1}{6} I \sqrt{47}\right]\right\}\right]$$

```
> eigenvalues(A23);
```

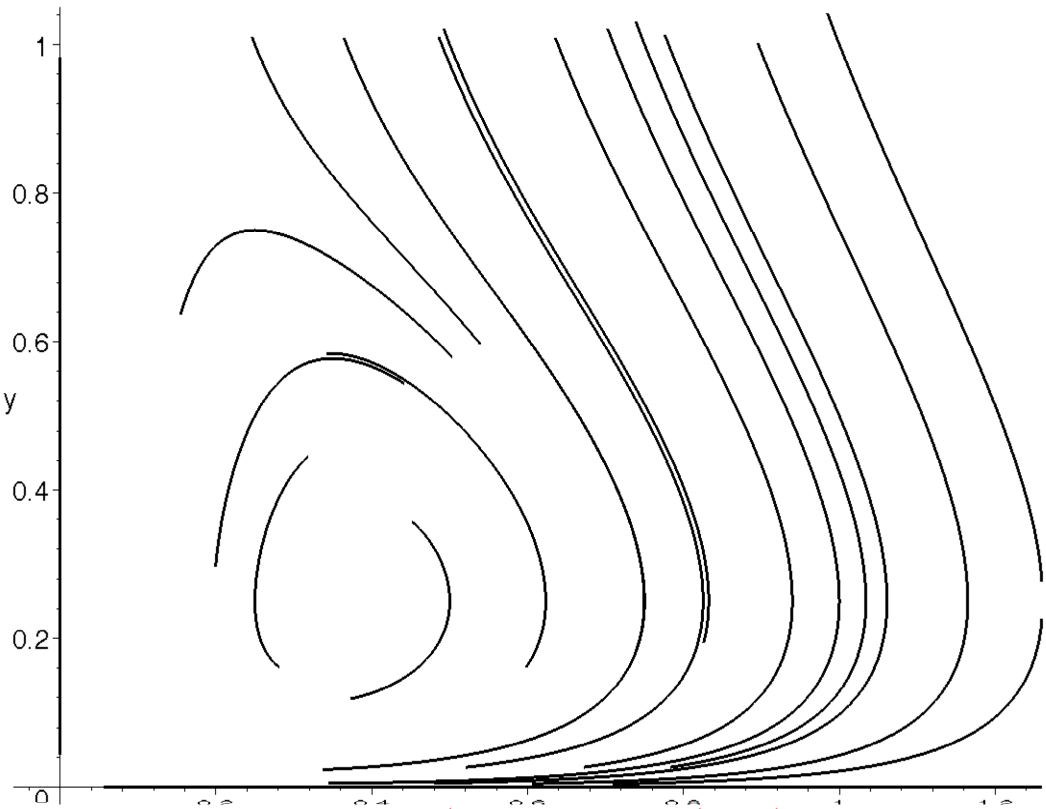
$$\frac{3}{10}, -1$$

```
> eigenvectors(A23);
```

$$\left[\frac{3}{10}, 1, \left\{\left[\frac{-13}{8}, 1\right]\right\}\right], [-1, 1, \{[0, 1]\}]$$

(iv) (b)

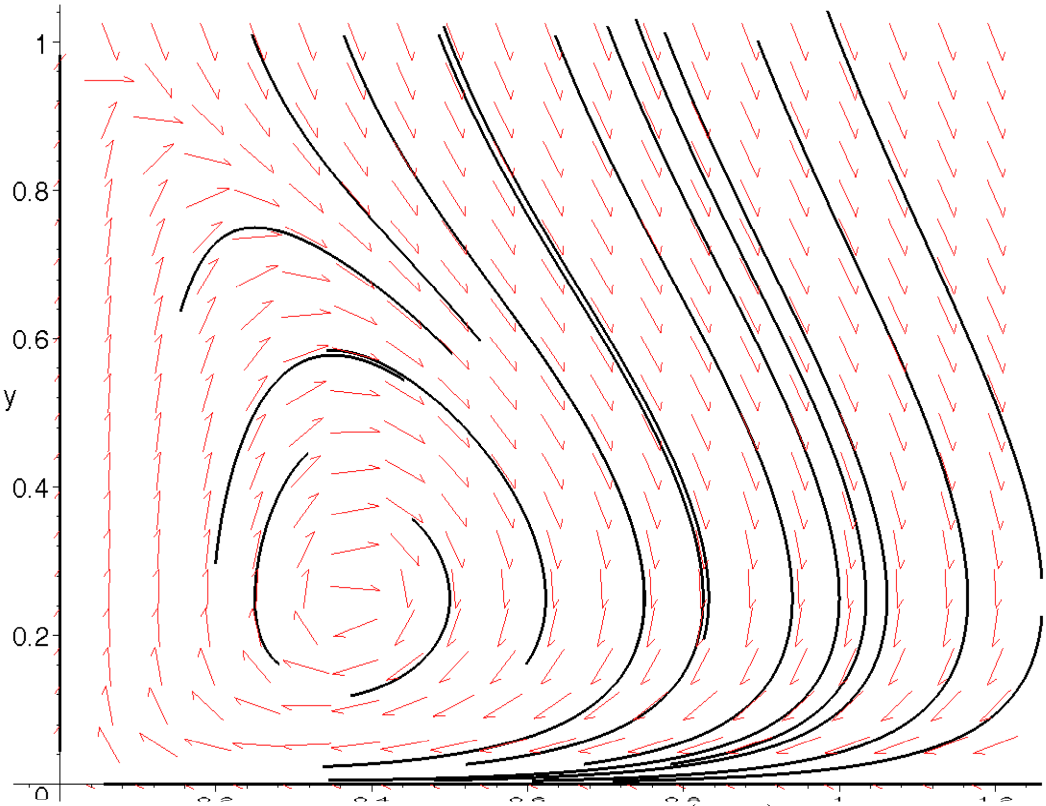
```
> des:=diff(x(t),t)=-(x(t)/2)+x(t)*y(t)/((1/4)+y(t)),diff(y(
t),t)=y(t)-y(t)^2-(x(t)*y(t))/((1/4)+y(t)):
iniset:={seq(seq([x(0)=a,y(0)=b],a=[0,.25,.5,.75,1]),
b=[0,.25,.5,.75,1])}:
pphase:=trange->DEplot([des],[x(t),y(t)],t=trange,iniset,x
=0..1.2,y=0..1,stepsize=.05,
method=rkf45,linecolour=black,arrows=NONE):
> pphase(-2..3);
```




```
> des:=diff(x(t),t)=-(x(t)/2)+x(t)*y(t)/((1/4)+y(t)),diff(y(
```

```
t),t)=y(t)-y(t)^2-(x(t)*y(t))/((1/4)+y(t)):
iniset:={seq(seq([x(0)=a,y(0)=b],a=[0,.25,.5,.75,1]),
b=[0,.25,.5,.75,1])}:
pphase:=trange->DEplot([des],[x(t),y(t)],t=trange,iniset,x
=0..1.2,y=0..1,stepsize=.05,
method=rkf45,linecolour=black,arrows=THIN):
```

```
> pphase(-2..3);
```



Which shows a limit cycle around the fixed point $(x, y) = \left(\frac{3}{8}, \frac{1}{4}\right)$.

(i) (c)

```
> sol3:=solve({x-x^2-x*y=0,y-2*x*y-2*y^2=0},{x,y});
```

$$sol3 := \{x = 0, y = 0\}, \{x = 1, y = 0\}, \{x = 0, y = \frac{1}{2}\}$$

(ii) (c)

```
> sol31:=sol3[1];
```

$$sol31 := \{x = 0, y = 0\}$$

```
> sol32:=sol3[2];
```

$$sol32 := \{x = 1, y = 0\}$$

```
> sol33:=sol3[3];
```

$$sol33 := \{x = 0, y = \frac{1}{2}\}$$

```
> fx31:=subs(sol31,diff(x-x^2-x*y,x));
```

$$fx31 := 1$$

```
> fy31:=subs(sol31,diff(x-x^2-x*y,y));
```

$fy31 := 0$

```
> gx31:=subs(sol31,diff(y-2*x*y-2*y^2,x));
```

$gx31 := 0$

```
> gy31:=subs(sol31,diff(y-2*x*y-2*y^2,y));
```

$gy31 := 1$

```
> A31:=matrix([[1, 0], [0, 1]]);
```

$$A31 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> fx32:=subs(sol32,diff(x-x^2-x*y,x));
```

$fx32 := -1$

```
> fy32:=subs(sol32,diff(x-x^2-x*y,y));
```

$fy32 := -1$

```
> gx32:=subs(sol32,diff(y-2*x*y-2*y^2,x));
```

$gx32 := 0$

```
> gy32:=subs(sol32,diff(y-2*x*y-2*y^2,y));
```

$gy32 := -1$

```
> A32:=matrix([[-1, -1], [0, -1]]);
```

$$A32 := \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

```
> fx33:=subs(sol33,diff(x-x^2-x*y,x));
```

$$fx33 := \frac{1}{2}$$

```
> fy33:=subs(sol33,diff(x-x^2-x*y,y));
```

$fy33 := 0$

```
> gx33:=subs(sol33,diff(y-2*x*y-2*y^2,x));
```

$gx33 := -1$

```
> gy33:=subs(sol33,diff(y-2*x*y-2*y^2,y));
```

$gy33 := -1$

```
> A33:=matrix([[1/2,0],[-1,-1]]);
```

$$A33 := \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & -1 \end{bmatrix}$$

(iii) (c)

```
> eigenvalues(A31);
```

1, 1

```
> eigenvectors(A31);
```

[1, 2, { [0, 1], [1, 0] }]

```
> eigenvalues(A32);
```

-1, -1

```
> eigenvectors(A32);
```

[-1, 2, { [1, 0] }]

```
> eigenvalues(A33);
```

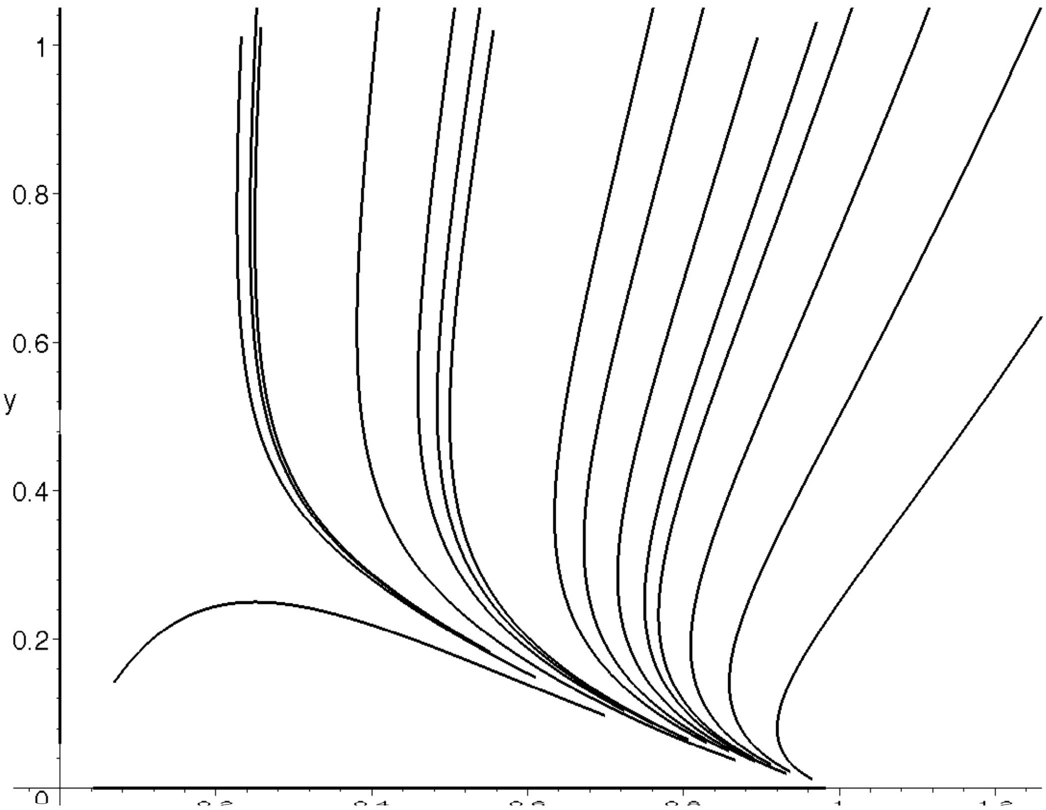
$$\frac{1}{2}, -1$$

```
> eigenvectors(A33);
```

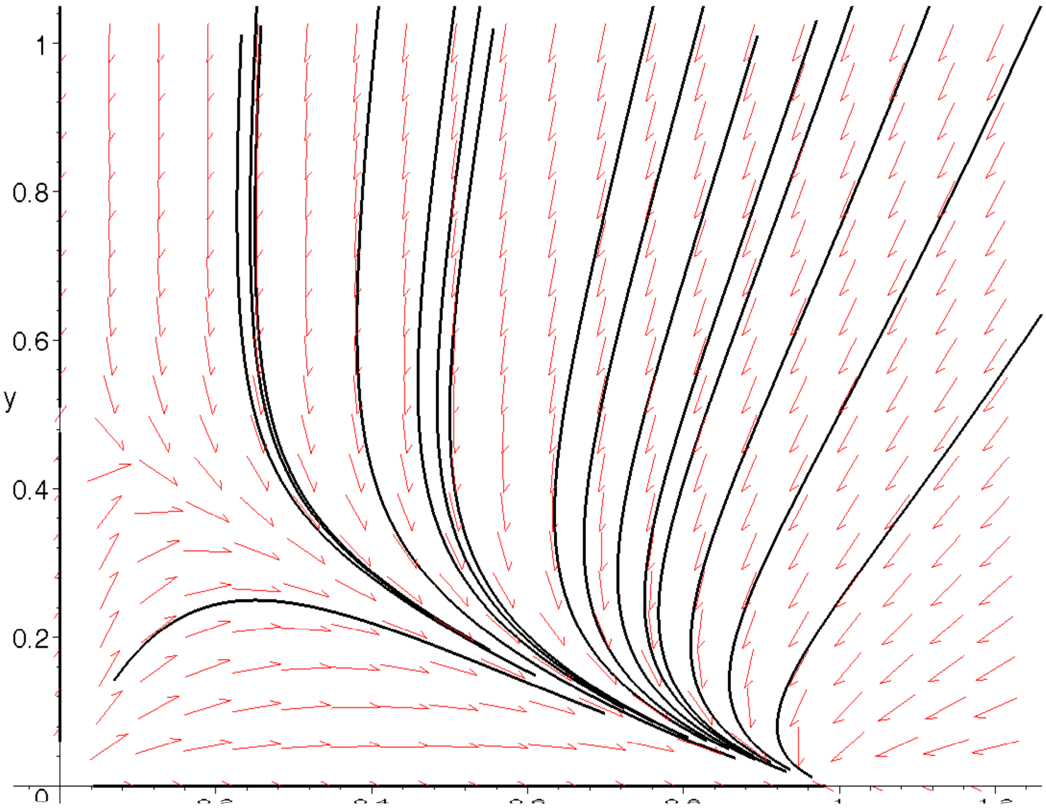
$$[-1, 1, \{[0, 1]\}], \left[\frac{1}{2}, 1, \left\{\left[\frac{-3}{2}, 1\right]\right\}\right]$$

(iv) (c)

```
> des:=diff(x(t),t)=x(t)-x(t)^2-x(t)*y(t),diff(y(t),t)=y(t)-2*x(t)*y(t)-2*y(t)^2:
iniset:={seq(seq([x(0)=a,y(0)=b],a=[0,.25,.5,.75,1]),b=[0,.25,.5,.75,1,1.25,1.5])}:
pphase:=trange->DEplot([des],[x(t),y(t)],t=trange,iniset,x=0..1.2,y=0..1,stepsize=.05,
method=rkf45,linecolour=black,arrows=NONE):
> pphase(-2..3);
```




```
> des:=diff(x(t),t)=x(t)-x(t)^2-x(t)*y(t),diff(y(t),t)=y(t)-2*x(t)*y(t)-2*y(t)^2:
iniset:={seq(seq([x(0)=a,y(0)=b],a=[0,.25,.5,.75,1]),b=[0,.25,.5,.75,1,1.25,1.5])}:
pphase:=trange->DEplot([des],[x(t),y(t)],t=trange,iniset,x=0..1.2,y=0..1,stepsize=.05,
method=rkf45,linecolour=black,arrows=THIN):
> pphase(-2..3);
```



Which shows competing predators at the fixed point $(x, y) = (1, 0)$.

Question 13

(i)

Critical point found by solving

```
> sol4:=solve({0=1.4*(1-y)*x,0=0.6*(1-4*y+x)*y},{x,y});
```

$$sol4 := \{x = 0., y = 0.\}, \{y = 1., x = 3.\}, \{y = .2500000000, x = 0.\}$$

(ii)

```
> sol41:=sol4[1];
```

$$sol41 := \{x = 0., y = 0.\}$$

```
> sol42:=sol4[2];
```

$$sol42 := \{y = 1., x = 3.\}$$

```
> sol43:=sol4[3];
```

$$sol43 := \{y = .2500000000, x = 0.\}$$

```
> fx41=subs(sol41,diff(1.4*(1-y)*x,x));
```

$$fx41 = 1.4$$

```
> fy41=subs(sol41,diff(1.4*(1-y)*x,y));
```

$$fy41 = -0.$$

```
> gx41:=subs(sol41,diff(0.6*(1-4*y+x)*y,x));
```

$$gx41 := 0.$$

```
> gy41:=subs(sol41,diff(0.6*(1-4*y+x)*y,y));
```

$$gy41 := .6$$

```
> A41:=matrix([[0, -4.2], [0.6, -2.4]]);
```

$$A41 := \begin{bmatrix} 0 & -4.2 \\ .6 & -2.4 \end{bmatrix}$$

```
> fx42=subs(sol42,diff(1.4*(1-y)*x,x));
```

$$fx42 = 0.$$

```
> fy42=subs(sol42,diff(1.4*(1-y)*x,y));
```

$$fy42 = -4.2$$

```
> gx42:=subs(sol42,diff(0.6*(1-4*y+x)*y,x));
```

$$gx42 := .6$$

```
> gy42:=subs(sol42,diff(0.6*(1-4*y+x)*y,y));
```

$$gy42 := -2.4$$

```
> A42:=matrix([[1.05, 0], [0.15, -0.6]]);
```

$$A42 := \begin{bmatrix} 1.05 & 0 \\ .15 & -.6 \end{bmatrix}$$

```
> fx43=subs(sol43,diff(1.4*(1-y)*x,x));
```

$$fx43 = 1.050000000$$

```
> fy43=subs(sol43,diff(1.4*(1-y)*x,y));
```

$$fy43 = -0.$$

```
> gx43:=subs(sol43,diff(0.6*(1-4*y+x)*y,x));
```

$$gx43 := .1500000000$$

```
> gy43:=subs(sol43,diff(0.6*(1-4*y+x)*y,y));
```

$$gy43 := -.600000000$$

```
> A43:=matrix([[1.4, 0], [0, 0.6]]);
```

$$A43 := \begin{bmatrix} 1.4 & 0 \\ 0 & .6 \end{bmatrix}$$

(iii)

```
> eigenvalues(A41);
```

$$-1.200000000 + 1.039230485\, I,\ -1.200000000 - 1.039230485\, I$$

```
> eigenvectors(A41);
```

$$[-1.200000000 + 1.039230484\, I,\ 1,\ \{[-1.732050807 + 2.000000001\, I,\ 0. + 1.\, I]\}],$$
$$[-1.200000000 - 1.039230484\, I,\ 1,\ \{[-1.732050807 - 2.000000001\, I,\ 0. - 1.\, I]\}]$$

```
> eigenvalues(A42);
```

$$-.6000000000,\ 1.050000000$$

```
> eigenvectors(A42);
```

$$[1.05,\ 1,\ \{[1,\ .09090909091]\}],\ [-.6,\ 1,\ \{[0,\ 1]\}]$$

```
> eigenvalues(A43);
```

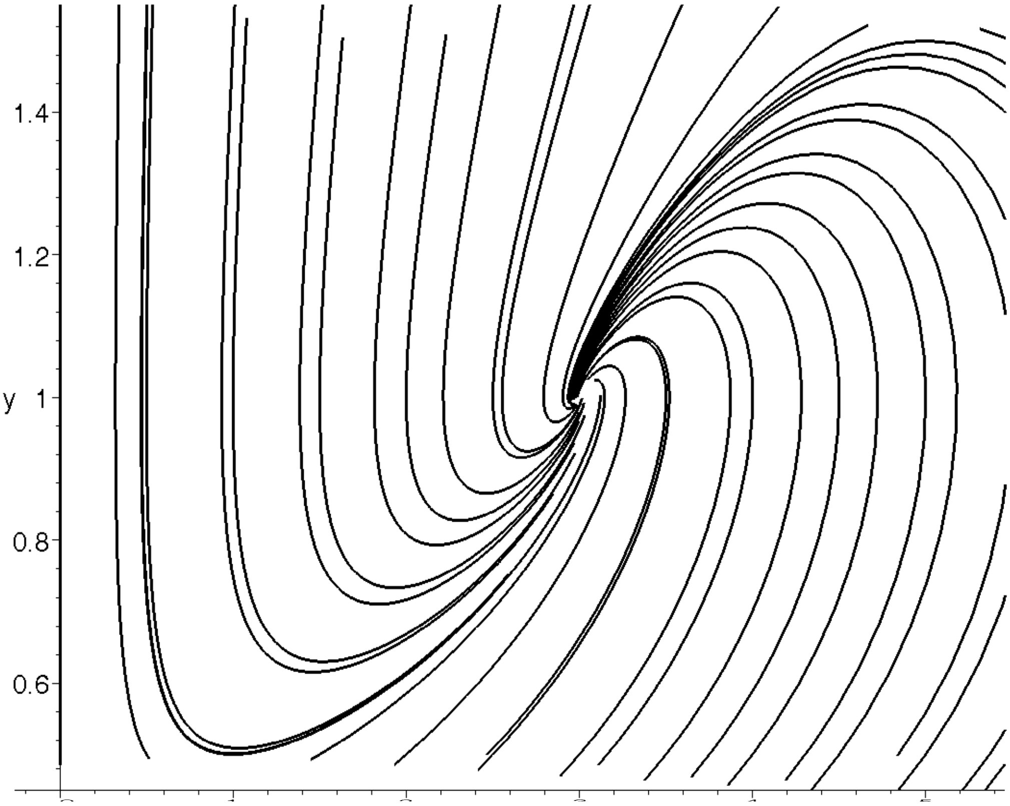
$$.6000000000,\ 1.400000000$$

```
> eigenvectors(A43);
```

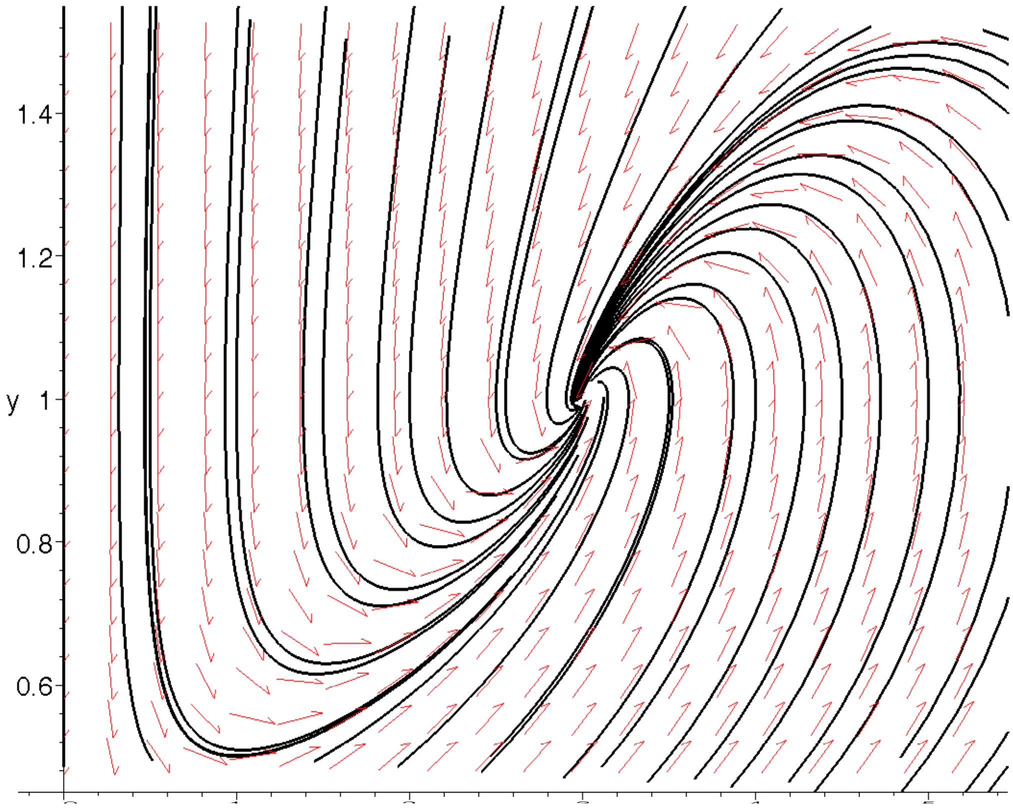
$$[1.4,\ 1,\ \{[1,\ 0]\}],\ [.6,\ 1,\ \{[0,\ 1]\}]$$

```
> des:=diff(x(t),t)=1.4*(1-y(t))*x(t),diff(y(t),t)=0.6*(1-4*
```

```
y(t)+x(t))*y(t):
iniset:={seq(seq([x(0)=a,y(0)=b],a=[0,.5,1,1.5,2,2.5,3,3.5
,4,4.5,5,5.5,6]), b=[0,.5,1,1.5,2])}:
pphase:=trange->DEplot([des],[x(t),y(t)],t=trange,iniset,x
=0..5.2,y=0.5..1.5,stepsize=.05,
method=rkf45,linecolour=black,arrows=NONE):
> pphase(-2..3);
```




```
> des:=diff(x(t),t)=1.4*(1-y(t))*x(t),diff(y(t),t)=0.6*(1-4*
y(t)+x(t))*y(t):
iniset:={seq(seq([x(0)=a,y(0)=b],a=[0,.5,1,1.5,2,2.5,3,3.5
,4,4.5,5,5.5,6]), b=[0,.5,1,1.5,2])}:
pphase:=trange->DEplot([des],[x(t),y(t)],t=trange,iniset,x
=0..5.2,y=0.5..1.5,stepsize=.05,
method=rkf45,linecolour=black,arrows=THIN):
> pphase(-2..3);
```



$(x, y) = (3, 1)$.

(iv)

A spreadsheet investigation of the system around point $(x, y) = (3, 1)$ reveals quite a different picture from the one above. There are two reasons for this. First, in the above analysis we have assumed the system is continuous. Discrete systems do not always behave in the same way that continuous systems behave, especially when they are non-linear. Second, the above stability was constructed from a linear approximation in the neighbourhood of the point $(x, y) = (3, 1)$. How small such a neighbourhood needs to be is not clear. This example illustrates the danger of using continuous linear approximations for non-linear discrete systems.

Question 14

Subtracting x_t from the first equation and y_t from the second we obtain

$$\Delta x_t = x_{t+1} - x_t = .3\, x_t - .3\, x_t^2 - .15\, x_t y_t$$

$$\Delta y_t = y_{t+1} - y_t = .3\, y_t - .3\, y_t^2 - .15\, x_t y_t$$

(i)

```
> sol5:=solve({x*((3/10)-(3/10)*x-(3/20)*y)=0,y*((3/10)-(3/10)*y-(3/20)*x)=0},{x,y});
```

$$sol5 := \{x = 0, y = 0\}, \{y = 1, x = 0\}, \{x = 1, y = 0\}, \{x = \frac{2}{3}, y = \frac{2}{3}\}$$

(ii)

```
> sol51:=sol5[1];
```

$$sol51 := \{x = 0, y = 0\}$$

```
> sol52:=sol5[2];
```

$$sol52 := \{y = 1, x = 0\}$$

```
> sol53:=sol5[3];
```

$$sol53 := \{x = 1, y = 0\}$$

```
> sol54:=sol5[4];
```

$$sol54 := \{x = \frac{2}{3}, y = \frac{2}{3}\}$$

```
> fx51:=subs(sol51,diff(x*((3/10)-(3/10)*x-(3/20)*y),x));
```

$$fx51 := \frac{3}{10}$$

```
> fy51:=subs(sol51,diff(x*((3/10)-(3/10)*x-(3/20)*y),y));
```

$$fy51 := 0$$

```
> gx51:=subs(sol51,diff(y*((3/10)-(3/10)*y-(3/20)*x),x));
```

$$gx51 := 0$$

```
> gy51:=subs(sol51,diff(y*((3/10)-(3/10)*y-(3/20)*x),y));
```

$$gy51 := \frac{3}{10}$$

```
> A51:=matrix([[3/10, 0], [0, 3/10]]);
```

$$A51 := \begin{bmatrix} \frac{3}{10} & 0 \\ 0 & \frac{3}{10} \end{bmatrix}$$

```
> fx52:=subs(sol52,diff(x*((3/10)-(3/10)*x-(3/20)*y),x));
```

$$fx52 := \frac{3}{20}$$

```
> fy52:=subs(sol52,diff(x*((3/10)-(3/10)*x-(3/20)*y),y));
```

$$fy52 := 0$$

```
> gx52:=subs(sol52,diff(y*((3/10)-(3/10)*y-(3/20)*x),x));
```

$$gx52 := \frac{-3}{20}$$

```
> gy52:=subs(sol52,diff(y*((3/10)-(3/10)*y-(3/20)*x),y));
```

$$gy52 := \frac{-3}{10}$$

```
> A52:=matrix([[3/20, 0], [-3/20, -3/10]]);
```

$$A52 := \begin{bmatrix} \frac{3}{20} & 0 \\ \frac{-3}{20} & \frac{-3}{10} \end{bmatrix}$$

```
> fx53:=subs(sol53,diff(x*((3/10)-(3/10)*x-(3/20)*y),x));
```

$$fx53 := \frac{-3}{10}$$

```
> fy53:=subs(sol53,diff(x*((3/10)-(3/10)*x-(3/20)*y),y));
```

$$fy53 := \frac{-3}{20}$$

```
> gx53:=subs(sol53,diff(y*((3/10)-(3/10)*y-(3/20)*x),x));
```

$$gx53 := 0$$

```
> gy53:=subs(sol53,diff(y*((3/10)-(3/10)*y-(3/20)*x),y));
```

$$gy53 := \frac{3}{20}$$

```
> A53:=matrix([[-3/10, -3/20], [0, 3/20]]);
```

$$A53 := \begin{bmatrix} \frac{-3}{10} & \frac{-3}{20} \\ 0 & \frac{3}{20} \end{bmatrix}$$

```
> fx54:=subs(sol54,diff(x*((3/10)-(3/10)*x-(3/20)*y),x));
```

$$fx54 := \frac{-1}{5}$$

```
> fy54:=subs(sol54,diff(x*((3/10)-(3/10)*x-(3/20)*y),y));
```

$$fy54 := \frac{-1}{10}$$

```
> gx54:=subs(sol54,diff(y*((3/10)-(3/10)*y-(3/20)*x),x));
```

$$gx54 := \frac{-1}{10}$$

```
> gy54:=subs(sol54,diff(y*((3/10)-(3/10)*y-(3/20)*x),y));
```

$$gy54 := \frac{-1}{5}$$

```
> A54:=matrix([[-1/5, -1/10], [-1/10, -1/5]]);
```

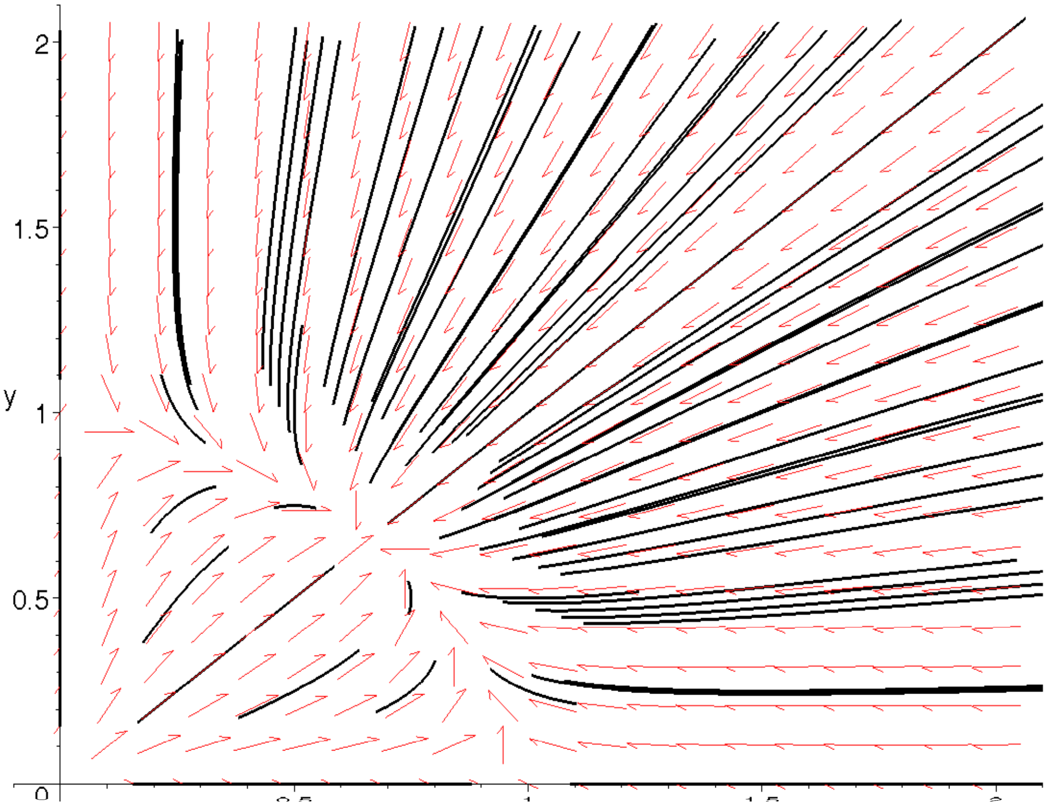
$$A54 := \begin{bmatrix} \frac{-1}{5} & \frac{-1}{10} \\ \frac{-1}{10} & \frac{-1}{5} \end{bmatrix}$$

(iii)

```
> des:=diff(x(t),t)=x(t)*(0.3-0.3*x(t)-0.15*y(t)),diff(y(t),
t)=y(t)*(0.3-0.3*y(t)-0.15*x(t)):
iniset:={seq(seq([x(0)=a,y(0)=b],a=[0,.25,.5,.75,1,1.25,1.
5,1.75,2]), b=[0,.25,.5,.75,1,1.25,1.5,1.75,2])}:
pphase:=trange->DEplot([des],[x(t),y(t)],t=trange,iniset,x
=0..2,y=0..2,stepsize=.05,method=rkf45,linecolour=black,ar
rows=NONE):
> des:=diff(x(t),t)=x(t)*(0.3-0.3*x(t)-0.15*y(t)),diff(y(t),
```

```
t)=y(t)*(0.3-0.3*y(t)-0.15*x(t)):
iniset:={seq(seq([x(0)=a,y(0)=b],a=[0,.25,.5,.75,1,1.25,1.5,1.75,2]), b=[0,.25,.5,.75,1,1.25,1.5,1.75,2])}:
pphase:=trange->DEplot([des],[x(t),y(t)],t=trange,iniset,x=0..2,y=0..2,stepsize=.05,method=rkf45,linecolour=black,arrows=THIN):
```

```
> pphase(-2..3);
```



Question 15

(i)

The system takes the form:

$x_1(t) = .42\, x_3(t)$

$x_2(t) = .6\, x_1(t)$

$x_3(t) = .75\, x_2(t) + .95\, x_3(t)$

(iii)

```
> mA:=matrix([[0,0,0.42],[0.6,0,0],[0,0.75,0.95]]);
```

$$mA := \begin{bmatrix} 0 & 0 & .42 \\ .6 & 0 & 0 \\ 0 & .75 & .95 \end{bmatrix}$$

```
> eigenvalues(mA);
```

-.07741719621 + .4062916928 I, -.07741719621 − .4062916928 I, 1.104834392

Note also,

```
> R:=sqrt((-0.07741719621)^2+(0.4062916928)^2);
```

$$R := .4136016948$$

So the system is stable.

(iv)

From eigenvalue 1.104834392

(v)

```
> eigenvectors(mA);
```

$[-.0774171964 + .4062916929\, I, 1, \{[-.5592136741 - .4093747157\, I,$
$-.4315265390 + .9080564211\, I, .4990911927 - .4655029237\, I]\}], [$
$-.0774171964 - .4062916929\, I, 1, \{[-.5592136741 + .4093747157\, I,$
$-.4315265390 - .9080564211\, I, .4990911927 + .4655029237\, I]\}],$
$[1.104834392, 1, \{[.3751975609, .2037577187, .9869789722]\}]$

```
> sumeig:=.3751975609+ .2037577187+ .9869789722;
```

$$sumeig := 1.565934252$$

```
> [.3751975609/sumeig, .2037577187/sumeig,
  .9869789722/sumeig];
```

$$[.2395998174, .1301189488, .6302812336]$$

Hence, calves settle down to 24%; yearlings to 13% and adults to 63%