SUPPLEMENTAL EXERCISES FOR
QUANTUM COMPUTING FOR COMPUTER SCIENTISTS

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1. Chapter 1

Supp-Ex. 1.1.1: Calculate $(3-2i)+(5.2+6.5i)$.

Supp-Ex. 1.1.2: Calculate $(3-2i)\times(5.2+6.5i)$.

Supp-Ex. 1.2.1: Calculate $(7,-4.1)\times(4,-3)$.

Supp-Ex. 1.2.2: Calculate $(7,-4.1)\div(4,-3)$.

Supp-Ex. 1.2.3: Prove that for all c_1, c_2, c_3, the following distributivity property holds

$$\frac{c_1+c_3}{c_2}=\frac{c_1}{c_2}+\frac{c_3}{c_2}.$$

Supp-Ex. 1.2.4: Find the modulus of $5+7i$.

Supp-Ex. 1.2.5: What is the relationship between $\overline{c_1}\div\overline{c_2}$ and $\overline{c_1\div c_2}$?

Supp-Ex. 1.2.6: Calculate $(5+2i)+[(6-4i)\times\overline{(3+5i)}]$.

Supp-Ex. 1.3.1: Draw $c_1=(4,-3), c_2=(6,2)$ and c_1+c_2.

Supp-Ex. 1.3.2: Consider $c_1=(4,-3), c_2=(6,2)$ and $c_3=(-2,3)$. Show the distributive property

$$c_1\times(c_2+c_3)=(c_1\times c_2)+(c_1\times c_3)$$

for these numbers.

Supp-Ex. 1.3.3: What is the polar representation of $12-4i$?

Supp-Ex. 1.3.4: What is the Cartesian representation of $(6.2, 35^o)$?

Supp-Ex. 1.3.5: Use the polar representations to calculate $(12 - 4i) \times (3 + 7i)$.

Supp-Ex. 1.3.6: Use the polar representations to calculate $(12 - 4i) \div (3 + 7i)$.

Supp-Ex. 1.3.7: Use the polar representations to calculate $(12 - 4i)^3$.

Supp-Ex. 1.3.8: Use the polar representations to calculate $(12 - 4i)^{\frac{1}{4}}$.

Supp-Ex. 1.3.9: Write the exponential form of $7 - 6i$.

Supp-Ex. 1.3.10: Give the Cartesian representations of all the six roots of unity.

2. Chapter 2

Supp-Ex. 2.1.1: Calculate $\begin{bmatrix} 3-2i \\ 7-3i \\ 7+2i \\ 12.1-8i \end{bmatrix} + \begin{bmatrix} -6i \\ 6-4i \\ 2 \\ 4+17i \end{bmatrix}$.

Supp-Ex. 2.1.2: Calculate $(3 - 4i) \cdot \begin{bmatrix} -6i \\ 6-4i \\ 2 \\ 4+17i \end{bmatrix}$.

Supp-Ex. 2.1.3: Formally show that for two vectors V, and W we have

$$c \cdot (V + W) = c \cdot V + c \cdot W.$$

Supp-Ex. 2.2.1: Consider the union of vectors in $\mathbb{C}^6$ and $\mathbb{C}^8$, i.e. $\mathbb{C}^6 \cup \mathbb{C}^8$. Does this form a complex vector space? Explain your answer.

Supp-Ex. 2.2.2: Find the transpose, conjugate, and adjoint of

$$\begin{bmatrix} 16+4i & 2+2i & 7-9i \\ 2+3i & 15+2i & 8-2i \\ 1+12i & 2 & 3-5i \end{bmatrix}.$$

Supp-Ex. 2.2.3: Formally prove the vector space $\mathbb{C}^{m\times n}$ satisfies the property

$$(c_1+c_2)\cdot V = c_1\cdot V + c_2\cdot V.$$

Supp-Ex. 2.2.4: Formally prove the vector space $\mathbb{C}^{n\times n}$ satisfies the property

$$(c\cdot A)^\dagger = \overline{c}\cdot A^\dagger$$

Supp-Ex. 2.2.5: Formally prove the vector space $\mathbb{C}^{n\times n}$ satisfies the property

$$\overline{A\star B} = \overline{A}\star\overline{B}$$

Supp-Ex. 2.2.6: Prove that a complex subspace is closed under inverse and that 0 is in the subspace.

Supp-Ex. 2.2.7: Prove that a linear map between two vecotr spaces satisfies

$$f(V_1 - V_2) = f(V_1) - f(V_2).$$

Supp-Ex. 2.2.8: Formally prove that the vector space $Fun([a,b],\mathbb{R})$ satisfies the property

$$(c\cdot A)^\dagger = \overline{c}\cdot A^\dagger$$

Supp-Ex. 2.3.1: Show that if a set $\{V_0, V_1, \ldots, V_{n-1}\}$ of vectors in $\mathbb{V}$ is linearly independent then for for any nonzero $V\in\mathbb{V}$, there are *unique* coefficients $c_0, c_1, \ldots, c_{n-1}$ in $\mathbb{C}$ such that

$$V = c_0\cdot V_0 + c_1\cdot V_1 + \cdots + c_{n-1}\cdot V_{n-1}.$$

Supp-Ex. 2.3.2: Find x, y and z to make the following into a basis for $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Supp-Ex. 2.3.3: Prove that for any given complex vector space every basis has the same number of elements. (Hint: write one basis in terms of another.)

Supp-Ex. 2.3.4: Let H be the Hadamard matrix. Show that $H^3 = H$.

Supp-Ex. 2.4.1: Calculate $\langle \begin{bmatrix} 3+2i \\ 4-3i \\ 7 \end{bmatrix}, \begin{bmatrix} i \\ 6-6i \\ 2 \end{bmatrix} \rangle$.

Supp-Ex. 2.4.2: Calculate $| \begin{bmatrix} 3+2i \\ 4-3i \\ 7 \end{bmatrix} |$.

Supp-Ex. 2.4.3: Calculate the distance $d(\begin{bmatrix} 3+2i \\ 4-3i \\ 7 \end{bmatrix}, \begin{bmatrix} i \\ 6-6i \\ 2 \end{bmatrix})$.

Supp-Ex. 2.4.4: What is the trace of

$$\begin{bmatrix} 16+4i & 2+2i & 7-9i \\ 2+3i & 15+2i & 8-2i \\ 1+12i & 2 & 3-5i \end{bmatrix}.$$

Supp-Ex. 2.5.1: The eigenvalues of

$$\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}.$$

are 3 and -2. Find their eigenvectors.

Supp-Ex. 2.5.2: The eigenvectors of

$$\begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}.$$

are $[-2,-1,1]^T$, $[-2,0,1]^T$, and $[-1,1,0]^T$. Find their eigenvalues.

Supp-Ex. 2.6.1: Show that

$$\begin{bmatrix} 1 & 1+i & 2i \\ 1-i & 5 & -3 \\ -2i & -3 & 0 \end{bmatrix}.$$

is a hermitian matrix.

Supp-Ex. 2.6.2: The matrix

$$\begin{bmatrix} 3 & i+2 \\ 2-i & 1 \end{bmatrix}$$

has eigenvalues

$$\left\{2-\sqrt{5-i^2}, \sqrt{5-i^2}+2\right\}.$$

Find their eigenvectors.

Supp-Ex. 2.6.3: Show that if U is a unitary matrix, so is $U^\dagger$.

Supp-Ex. 2.6.4: Show that if U is a unitary matrix, the columns of U form an orthonormal basis of $\mathbb{C}^n$.

Supp-Ex. 2.7.1: Calculate

$$[3,6,1,8]^T \otimes [2,8,9,0,5]^T.$$

Supp-Ex. 2.7.2: Calculate

$$\begin{bmatrix} 3 & i+2 \\ 2-i & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1+i & 2i \\ 1-i & 5 & -3 \\ -2i & -3 & 0 \end{bmatrix}.$$

Supp-Ex. 2.7.3: What is the unit of the tensor product of matrices multiplication?

Supp-Ex. 2.7.4: Formally prove that $(A \otimes B)^T = A^T \otimes B^T$.

3. Chapter 3

Supp-Ex. 3.1.1: A permutation matrix is a matrix that has exactly one 1 in every row and every column (all the other entries are 0). Prove that if P is a permutation matrix than $PP^T = I$.

Supp-Ex. 3.1.2: Show that if P is a permutation matrix and A is any matrix than PA is a permutation of the rows of A and AP is a permutation of the columns of A.

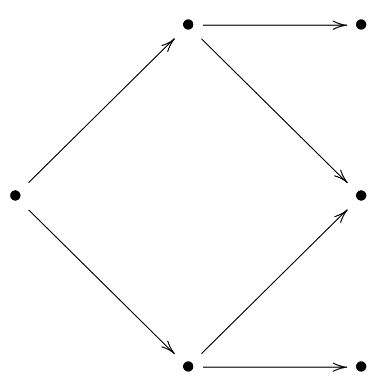
Supp-Ex. 3.1.1: What does a graph that corresponds to a permutation matrix look like?

Supp-Ex. 3.2.1: Prove that every permutation matrix is a doubly stochastic matrix.

Supp-Ex. 3.2.2: Show that if A and B are doubly stochastic matrices than so is $A \star B$.

Supp-Ex. 3.2.3: Is it true that if A and B are doubly stochastic matrices than so is $A \otimes B$?

Supp-Ex. 3.2.4: The probabilistic double slit experiment described in the text is unnecessarily complex. Formulate a version of the double slit experiment with the following diagram:



Supp-Ex. 3.3.1: Use the diagram in Supp-Ex. 3.2.4 to formulate a simpler version of a quantum double slit experiment.

4. Chapter 4

Supp-Ex. 4.1.1: Let $|\psi\rangle = [3+6i, 4-2i, 7+7i, 12-6i, -2i]^T$ correspond to a quantum system that corresponds to a particle being in one of five positions. Calculate the probability after a measurement the particle is in the second position.

Supp-Ex. 4.1.2: Let $|\psi\rangle = [3+6i, 4-2i, 7+7i, 12-6i, -2i]^T$. Calculate $|\psi\rangle + |\psi\rangle + |\psi\rangle = 3|\psi\rangle$.

Supp-Ex. 4.1.3: Let $|\psi\rangle = [3+6i, 4-2i, 7+7i, 12-6i, -2i]^T$. Normalize $|\psi\rangle$.

Supp-Ex. 4.1.4: Let $|\psi\rangle = [3+6i, 4-2i, 7+7i, 12-6i, -2i]^T$. Imagine that $(3-2i)|\psi\rangle$ corresponds to a quantum system that has a particle in one of five positions. Calculate the probability that the particle is in the second position after measurement.

Supp-Ex. 4.1.5: Consider a particle whose spin is described by the ket

$$|\psi\rangle = (7+2i)|\uparrow\rangle + (6-3i)|\downarrow\rangle.$$

Normalize $|\psi\rangle$.

Supp-Ex. 4.1.6: Consider a particle whose spin is described by the ket

$$|\psi\rangle = (7+2i)|\uparrow\rangle + (6-3i)|\downarrow\rangle.$$

After measurement, what is the chances of finding the particle in the spin up state?

Supp-Ex. 4.1.7: Let $|\psi\rangle = [3+6i, 4-2i, 7+7i, 12-6i, -2i]^T$ and $|\varphi\rangle = [-3i, 6+2i, 14, 2+i, 5-3i]^T$. Calculate $\langle\psi|\varphi\rangle$.

Supp-Ex. 4.2.1: Show that for three matrices we have that

$$[\Omega_1, \Omega_2 + \Omega_3] = [\Omega_1, \Omega_2] + [\Omega_1, \Omega_3].$$

Supp-Ex. 4.4.1: What is the physical significance of the fact that for two unitary matrices U_1 and U_2 we have that

$$(U_1 \star U_2)^\dagger = U_2^\dagger \star U_1^\dagger?$$

5. Chapter 5

Supp-Ex. 5.1.1: What is the norm of the vector

$$\begin{bmatrix} 7+2i \\ 6-i \end{bmatrix}?$$

Supp-Ex. 5.1.2: If one was to measure a qubit in state

$$(5+3i)\cdot\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (16-2i)\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

what is the probability of finding the qubit in state $|0\rangle$ after measurement?

Supp-Ex. 5.1.3: What, if anything, can we know about the second qubit if we measure the first qubit of the following state:

$$\frac{|11\rangle + |00\rangle + |01\rangle}{\sqrt{3}}?$$

What can we know about the first qubit if we measure the second qubit?

Supp-Ex. 5.2.1: Find the matrix that corresponds to the exclusive-or gate.

Supp-Ex. 5.2.2: Consider a gate that accepts three inputs and has one output. The single output should be on if exactly two of any of the three inputs are on. Find the matrix that corresponds to this gate.

Supp-Ex. 5.2.3: Consider the following logical expression:

$$(X \wedge \neg Y) \rightarrow \neg(Y \vee Z).$$

Write the logic diagram for this expression and find the matrix that corresponds to it.

Supp-Ex. 5.3.1: Show how to construct the OR gate using only Fredkin gates.

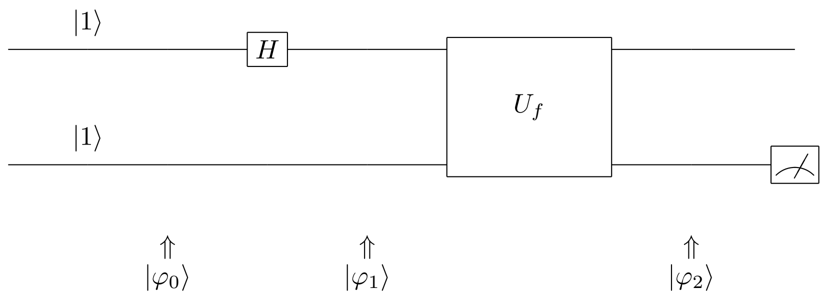
Supp-Ex. 5.4.1: Consider the qubit $|\psi\rangle = [3+2i, 5-i]^T$. Write this qubit in the form

$$|\psi\rangle = \cos(\theta)|0\rangle + e^{i\phi}\sin(\theta)|1\rangle.$$

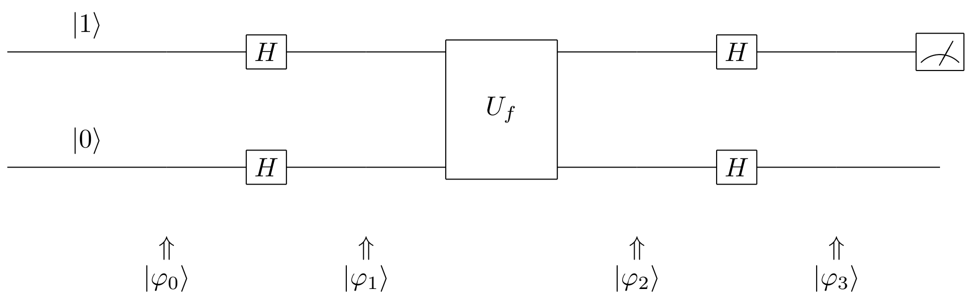
6. Chapter 6

Supp-Ex. 6.1.1: Calculate $H|1\rangle$.

Supp-Ex. 6.1.2: Consider the function f defined as $f(0) = 1$ and $f(1) = 0$. Describe $|\varphi_0\rangle, |\varphi_1\rangle$, and $|\varphi_2\rangle$ in the following circuit:



Supp-Ex. 6.1.3: Give the sequences of matrices that corresponds to following circuit



Supp-Ex. 6.1.4: Consider the function f defined as $f(0) = 1$ and $f(1) = 0$. Describe $|\varphi_0\rangle, |\varphi_1\rangle, |\varphi_2\rangle$, and $|\varphi_3\rangle$ for the quantum circuit given in Supp-Ex 6.1.3.

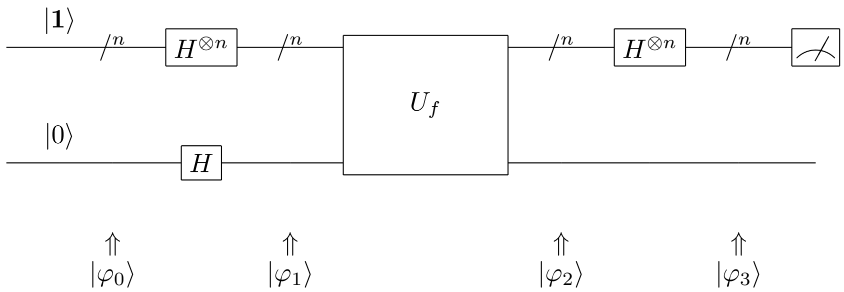
Supp-Ex. 6.2.1: Give all the balanced functions from $\{0,1\}^4$ to $\{0,1\}$.

Supp-Ex. 6.2.2: Show that the following is true for all $\mathbf{x}, \mathbf{y}, \mathbf{y}' \in \{0,1\}^n$

$$\langle \mathbf{x}, \mathbf{y} \oplus \mathbf{y}' \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \oplus \langle \mathbf{x}, \mathbf{y}' \rangle.$$

Supp-Ex. 6.2.3: Consider $|\mathbf{1}\rangle = |000\ldots001\rangle$. Calculate $H^{\otimes n}|\mathbf{1}\rangle$.

Supp-Ex. 6.2.4: Consider the function f defined as $f(00) = 0, f(01) = 1, f(10) = 1$ and $f(11) = 1$. Describe $|\varphi_0\rangle, |\varphi_1\rangle, |\varphi_2\rangle$, and $|\varphi_3\rangle$ for the quantum circuit



Supp-Ex. 6.3.1: Describe the requirements on f if we assume that f is periodic with the period "1011".

Supp-Ex. 6.4.1: Invert 100,32,65,81,39,51,91,27, and 67 around their mean.

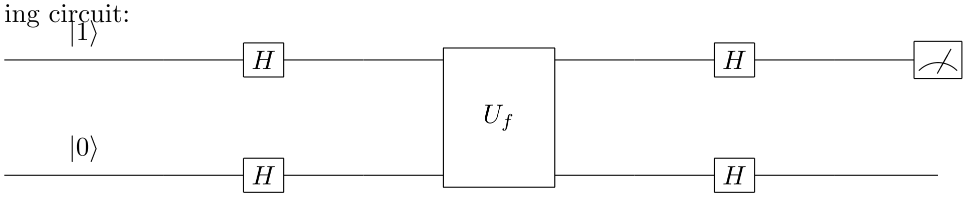
Supp-Ex. 6.5.1: Prove that

$$a \equiv a' \text{ Mod } N, \text{ if and only if } N|(a - a').$$

Supp-Ex. 6.5.2: Calculate the first seven values $f_{7,249}$.

7. Chapter 7

Supp-Ex. 7.2.1: Write the quantum assembler code that can preform the following circuit:



8. Chapter 8

Supp-Ex. 8.1.1: Show that **coNP** is a subset of **PSPACE**.

Supp-Ex. 8.2.1: Explain how quicksort with a randomized pivot uses is a form of a probabilistic algorithm. Explain why it works better than the deterministic version.

Supp-Ex. 8.3.1: As was said in the text, "there is nothing particularly quantum about the set $\mathfrak{C}$ of configurations and the matrix acting upon it. In fact, the same can be done for a deterministic Turing machine. In the deterministic case, we will only be concerned with vectors that have exactly one entry as 1 and all others as 0... The U_M will be such that every column has exactly one 1 with the remaining entries will be 0." Describe nondeterministic Turing machines in the same way.

Supp-Ex. 8.3.2: Write the deterministic Turing machine that searches for a "1" in an odd length string as a matrix that acts on a set of configurations $\mathfrak{C}$.

Supp-Ex. 8.3.3: Write the probabilistic Turing machine that searches for a "1" in an odd length string as a matrix that acts on a set of configurations $\mathfrak{C}$.

Supp-Ex. 8.3.4: Show that **BQP** is a subset of **PSPACE**.

9. Chapter 9

Supp-Ex. 9.1.1: Complete the following chart of an example of Alice and Bob communicating.

Step 1: Alice sends n random bits in random bases												
Bit number	1	2	3	4	5	6	7	8	9	10	11	12
Alice's random bits	1	1	0	1	1	0	1	1	0	1	1	0
Alice's random bases	+	X	X	X	+	X	+	+	X	+	X	X
Alice sends	□	□	□	□	□	□	□	□	□	□	□	□
Quantum channel	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$	$\Downarrow$
Bob's random bases	X	+	+	+	X	X	X	+	+	X	X	+
Bob observes	□	□	□	□	□	□	□	□	□	□	□	□
Bob's bits	□	□	□	□	□	□	□	□	□	□	□	□

10. Chapter 10

Supp-Ex. 10.1.1: Consider the following probabilities at transmitting the letters R,S,T,U,V,W

$$p(R) = \frac{1}{10} \quad p(R) = \frac{1}{5} \quad p(R) = \frac{1}{10} \quad p(R) = \frac{2}{5} \quad p(R) = \frac{1}{5}.$$

Calculate the Shannon entropy of this PDF.