High Frequency Linear Noisy Circuit Analysis

Outline

- Multi-port matrix representations
- The Smith Chart
- Why S-parameters?
- Differential S-parameters
- Two-port stability
- Two-port power gain definitions

Multi-port matrix representations

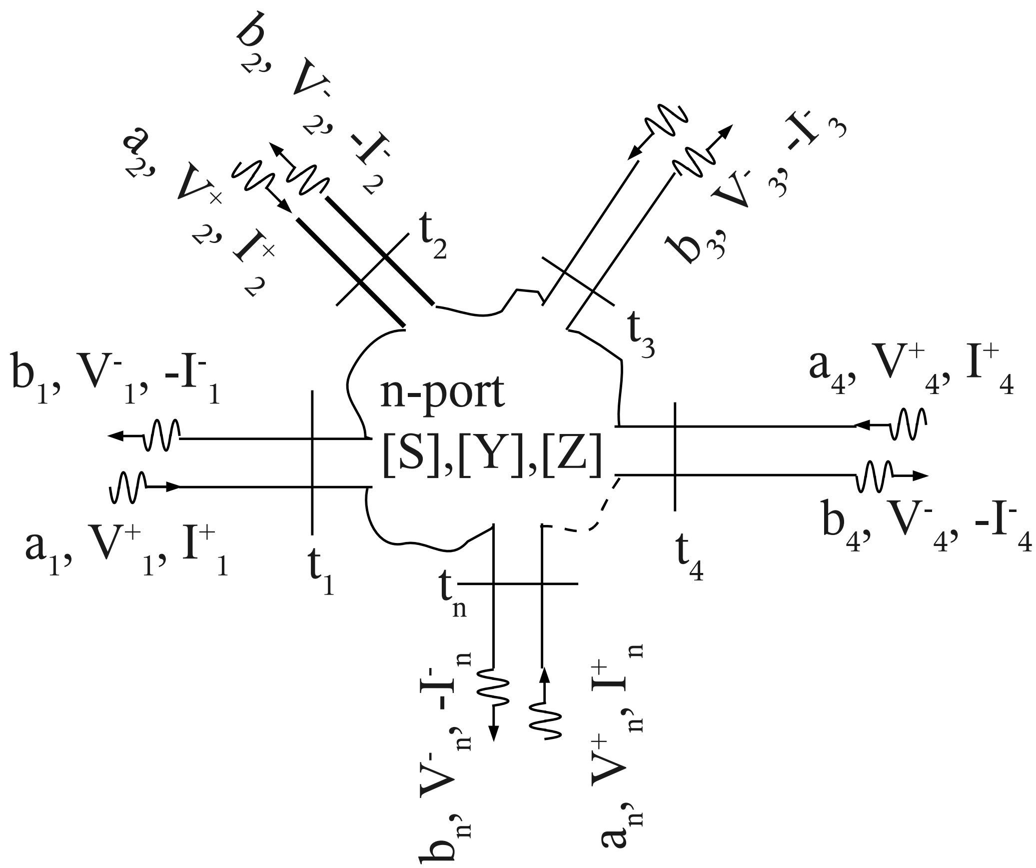
- N-port: [Y], [Z], [S]
- Two-port: [G],[H],

  [ABCD], [T]

$$V_n = V_n^+ + V_n^- \qquad I_n = I_n^+ + I_n^-$$

$$I_n = I_n^+ + I_n^- = \frac{V_n^+}{Z_0} - \frac{V_n^-}{Z_0}$$

- V^+_n, V^-_n = incident/reflected voltages
- I^+_n, I^-_n = incident/reflected currents
- a_n, b_n = incident/reflected power waves
- Z_0 = reference impedance
- t_n = reference plane location at port *n*



$$a_n = \frac{V_n^+}{\sqrt{Z_0}} \qquad b_n = \frac{V_n^-}{\sqrt{Z_0}}$$

$$P_{in,avg,\,i} = \frac{V_n^+ \times I_n^{+*}}{2} = \frac{|a_i|^2}{2} \qquad P_{out,avg,\,i} = \frac{V_n^- \times I_n^{-*}}{2} = \frac{|b_i|^2}{2}$$

[Z], [Y] n-PORT parameters

$$\begin{bmatrix} V_1 \\ V_2 \\ . \\ . \\ V_n \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & . & . & Z_{1n} \\ Z_{21} & Z_{22} & . & . & Z_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ Z_{n1} & Z_{n2} & . & . & Z_{nn} \end{bmatrix} \times \begin{bmatrix} I_1 \\ I_2 \\ . \\ . \\ I_n \end{bmatrix}$$

$$Z_{ij} = \frac{V_i}{I_j} \left[I_k = 0 \; for \; k \neq j \right]$$

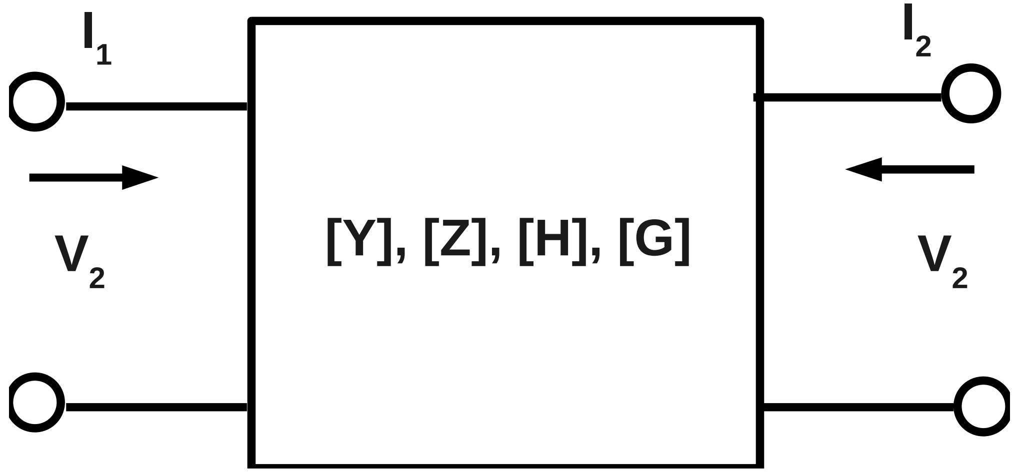
$$[Z] = [Y]^{-1}$$

$$\begin{bmatrix} I_1 \\ I_2 \\ . \\ . \\ I_n \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & . & . & Y_{1n} \\ Y_{21} & Y_{22} & . & . & Y_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ Y_{n1} & Y_{n2} & . & . & Y_{nn} \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \\ . \\ . \\ V_n \end{bmatrix}$$

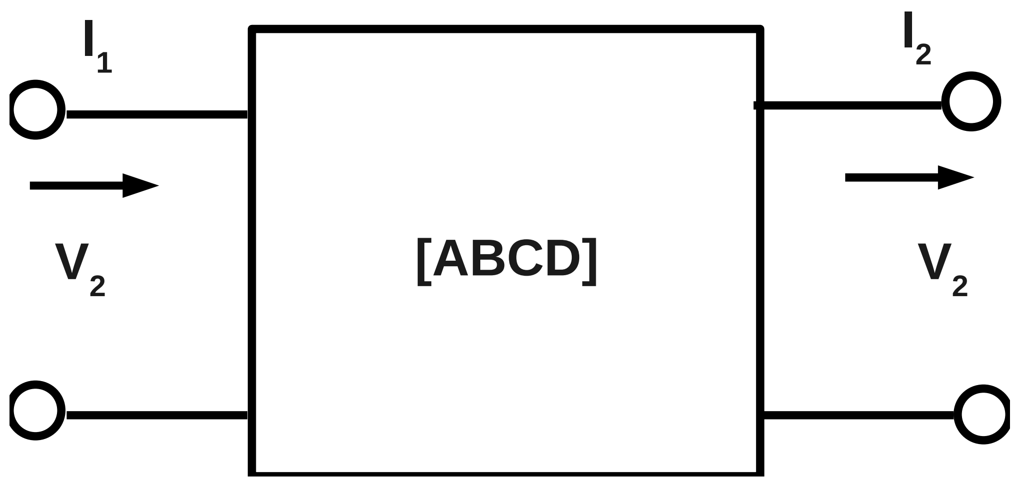
$$Y_{ij} = \frac{I_i}{V_j} \left[V_k = 0 \; for \; k \neq j \right]$$

Two-port matrices

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \times \begin{bmatrix} I_1 \\ V_2 \end{bmatrix}$$



$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \times \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$$



$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \times \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \times \ldots \times \begin{bmatrix} A_n & B_n \\ C_1 & D_n \end{bmatrix} = \prod_{i=1}^{n} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}$$

Reflection coefficient

- Describes impedance mismatch with respect to a reference impedance Z_0.
- Important when circuit dimensions become comparable to circuit size

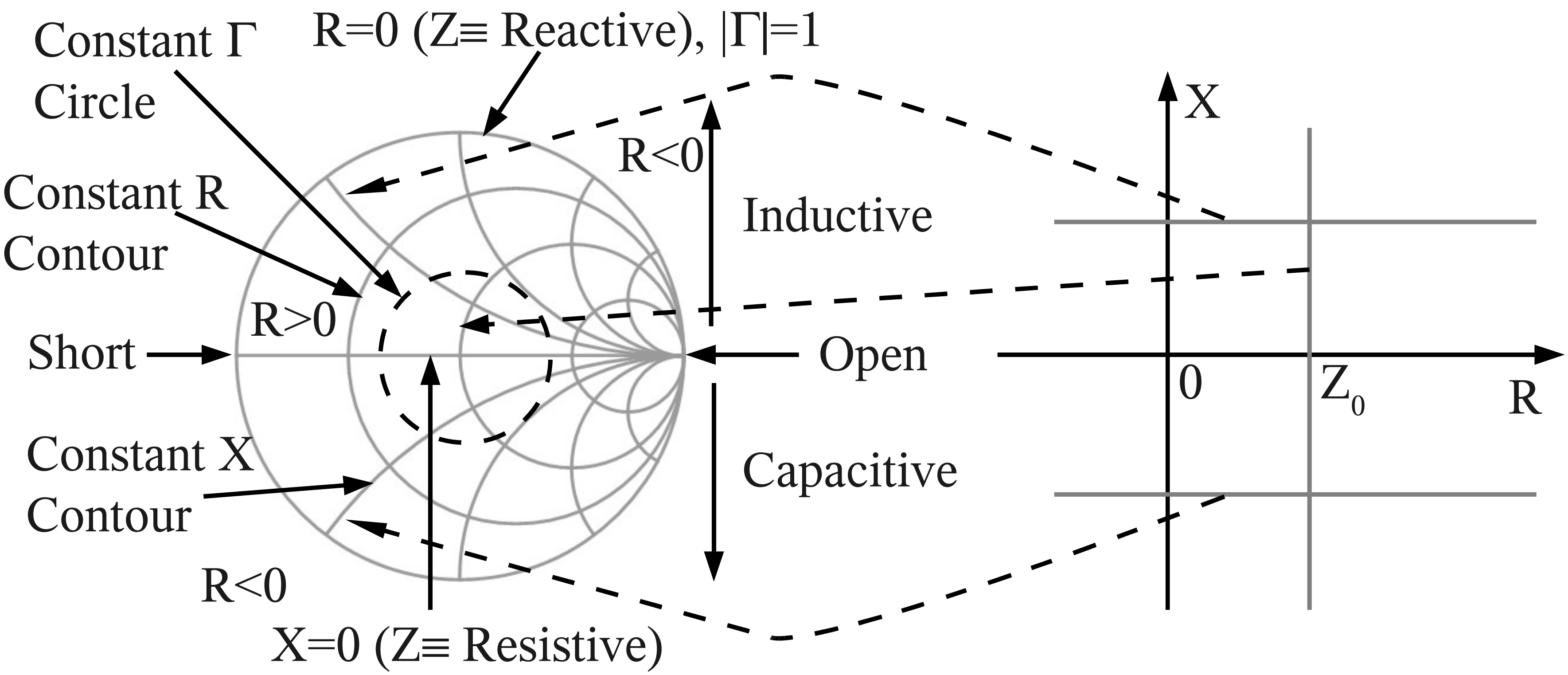
$$\Gamma = \frac{Z - Z_0}{Z + Z_0} = \frac{Y_0 - Y}{Y + Y_0}$$

$$\Gamma = \frac{z-1}{z+1}$$ Where $z = Z/Z_0$

- Voltage Standing Wave Ratio

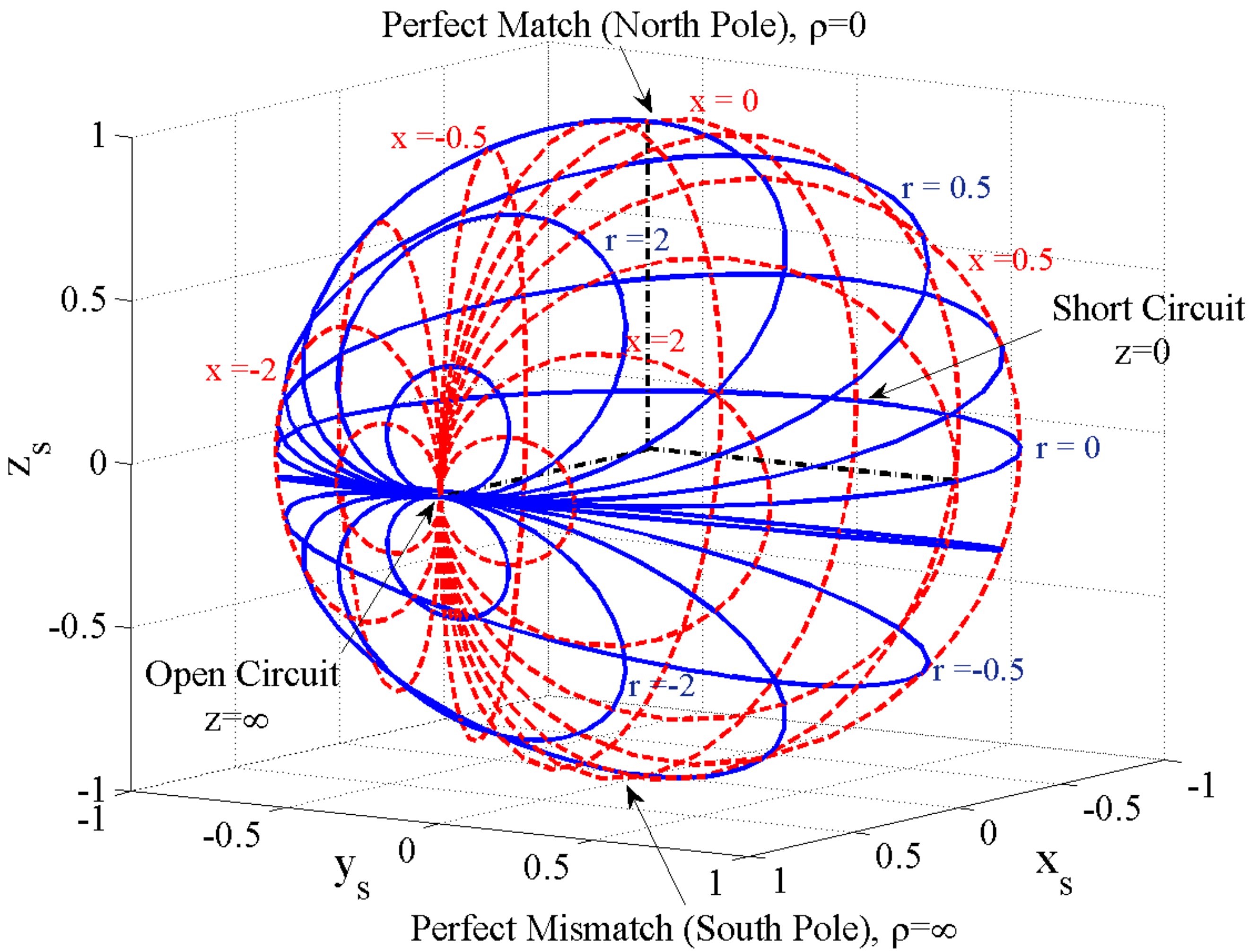
$$VSWR = \frac{1+|\Gamma|}{1-|\Gamma|}$$

Smith Chart



- Invented in 1939 to represent impedance by their reflection coefficient to solve transmission line problems

3-D Smith Chart



http://www.3dsmithchart.com/content/3d-smith-chart-demo

S Parameters

Why S parameters?

- Y, Z, H, G params require open-circuit or short-circuit terminations, difficult to reproduce at high frequencies
- S params are measured under 50-Ω terminations. 50-Ω t-lines and terminations can be fabricated beyond 500 GHz.
- Y/Z params are defined as ratios of voltages and currents whereas S params are ratios of incident and reflected travelling waves, measured accurately using directional couplers and power meters, typically combined in a *Network Analyzer*

N-Port S-parameters and matrix conversions

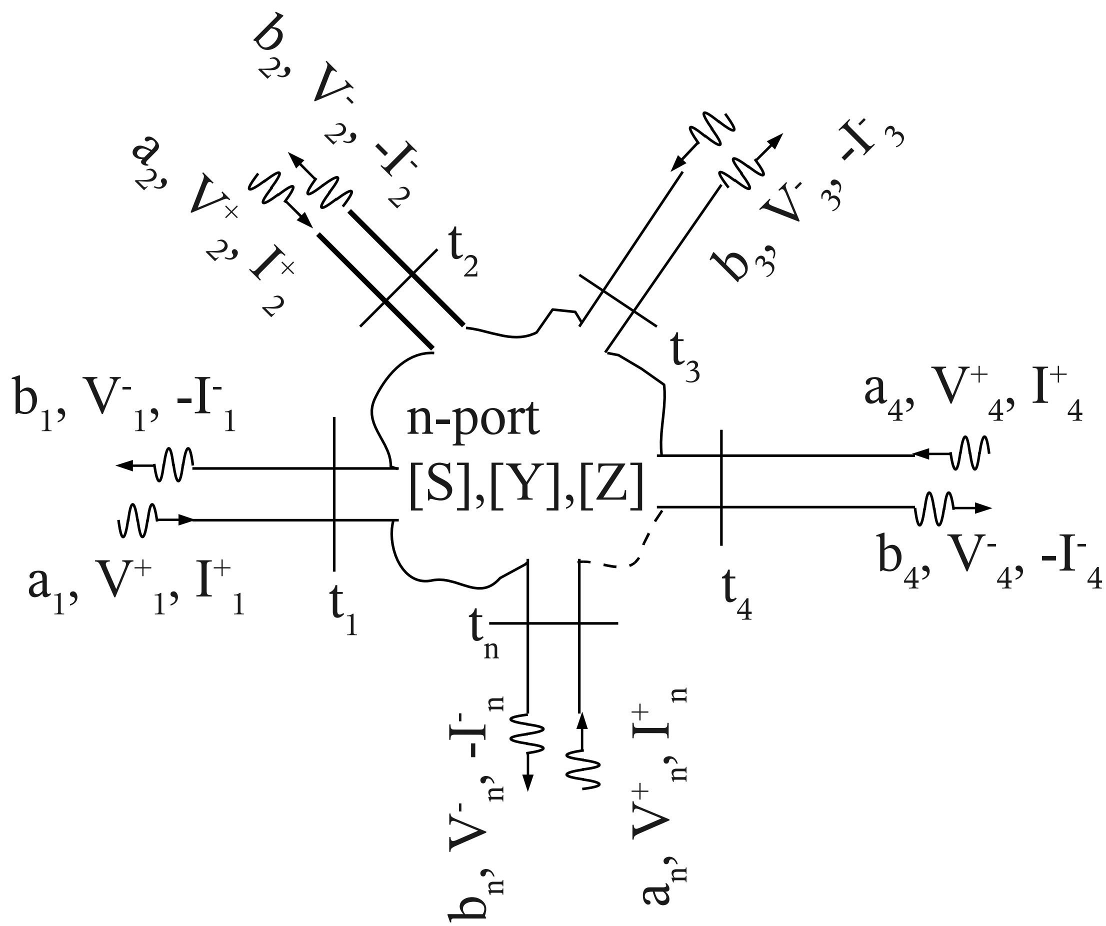
$$\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & . & . & S_{1n} \\ S_{21} & S_{22} & . & . & S_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ S_{n1} & S_{n2} & . & . & S_{nn} \end{bmatrix} \times \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix}$$

$$S_{ij} = \frac{b_i}{a_j} \left[a_k = 0 \ for \ k \neq j \right]$$

$$[S] = ([Z] - [U])([Z] + [U])^{-1}$$

$$[Z] = ([U] + [S])([U] - [S])^{-1}$$

- [U] is the *n*-dimensional identity matrix



S Parameters and Smith Chart

$$S_{11} = \frac{b_1}{a_1} \bigg|_{a_2=0}$$

$$S_{12} = \frac{b_1}{a_2} \bigg|_{a_1=0}$$

$$S_{21} = \frac{b_2}{a_1} \bigg|_{a_2=0}$$

$$S_{22} = \frac{b_2}{a_2} \bigg|_{a_1=0}$$

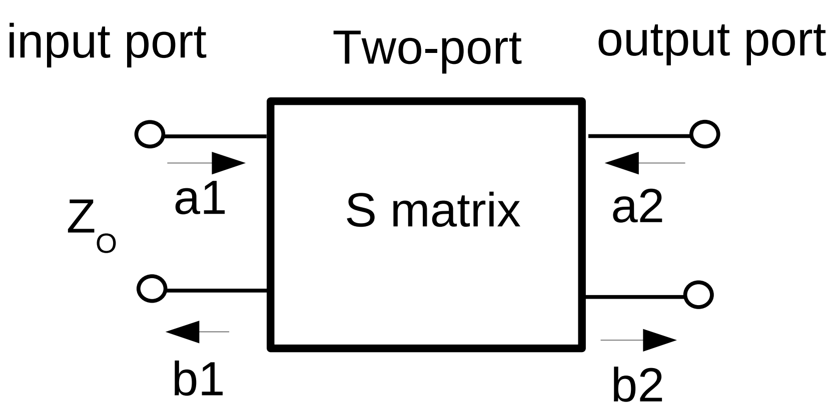
S_{11}/ S_{22} = input/output return loss

S_{21}= transducer power gain

S_{12} = isolation

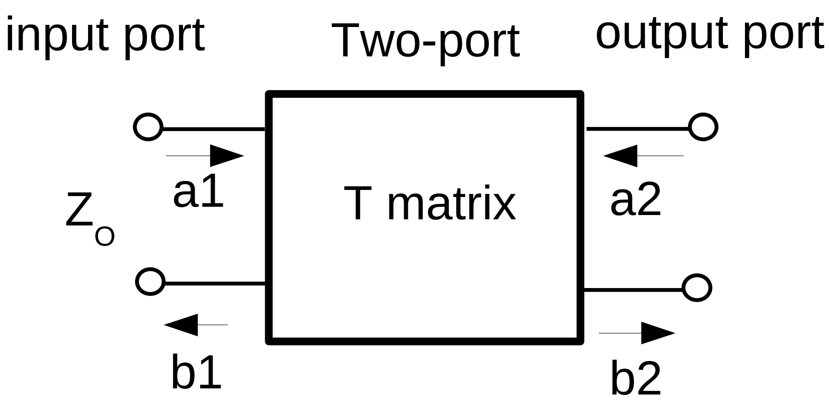
$$b_1 = S_{11} a_1 + S_{12} a_2$$

$$b_2 = S_{21} a_1 + S_{22} a_2$$



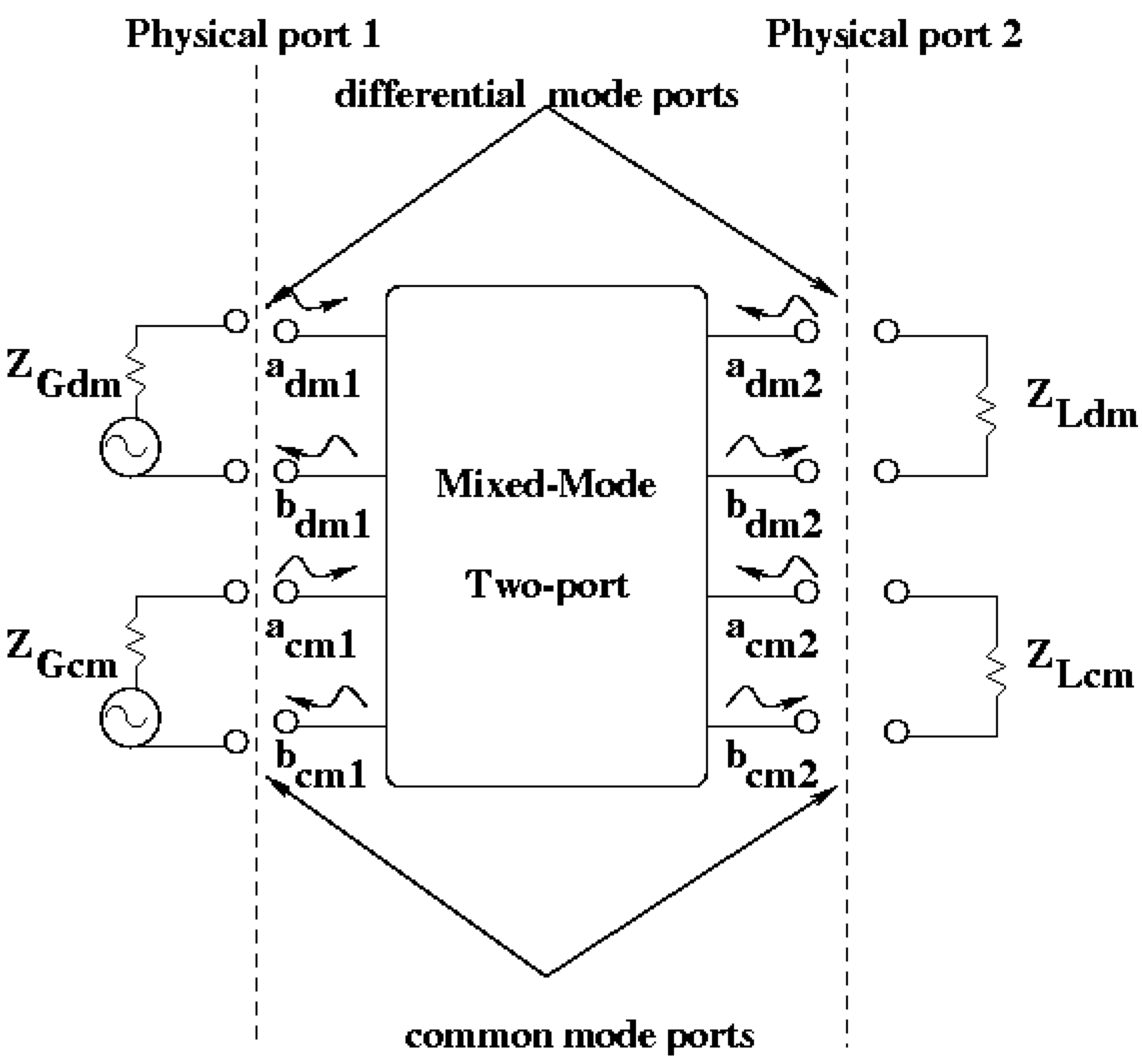
Chain transfer matrix of a 2-port

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \times \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$



Useful for a cascaded chain of two-ports

Differential 2-port definitions



$$\begin{vmatrix} b_{dm1} \\ b_{dm2} \\ b_{cm1} \\ b_{cm2} \end{vmatrix} = \begin{vmatrix} S_{dd} & S_{dc} \\ S_{cd} & S_{cc} \end{vmatrix} \times \begin{vmatrix} a_{dm1} \\ a_{dm2} \\ a_{cm1} \\ a_{cm2} \end{vmatrix}$$

Differential 2-port definitions (ii)

Examples:

•fully differential op-amp

•coupled transmission lines

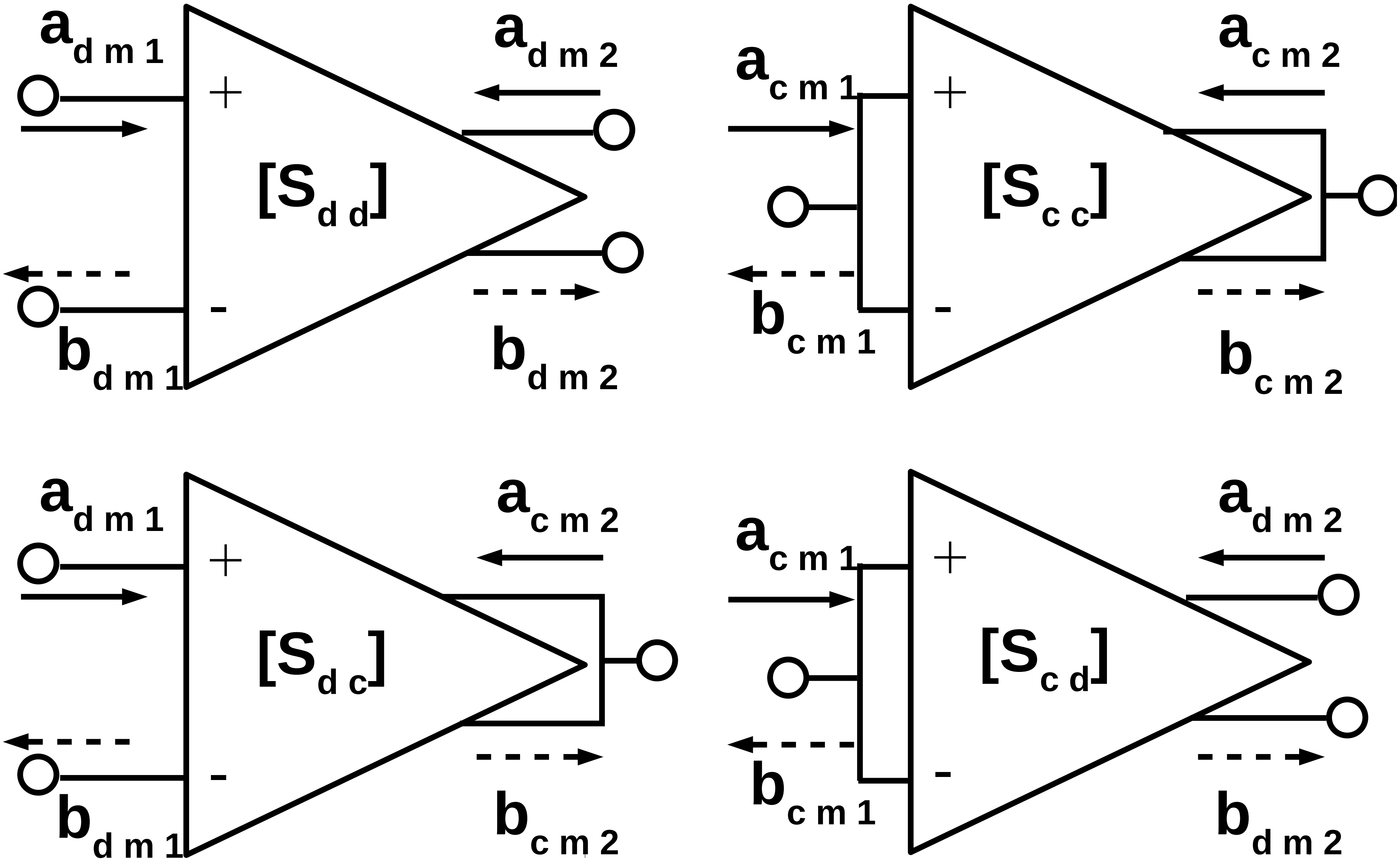
•S_{dd}, S_{cc}, S_{dc} and S_{cd} are 2x2 matrices

•Mixed-mode S matrix similar to differential amplifier gain matrix: A_{dd} , A_{cc}, A_{cd}, A_{dc}

•Can define stability conditions (*k*, etc.) and port impedances at each of the 4-ports

•Ideally, off-diagonal components should be 0

Mixed-mode differential 2-port gains



DM and CM definitions for 1-ports and 2-ports

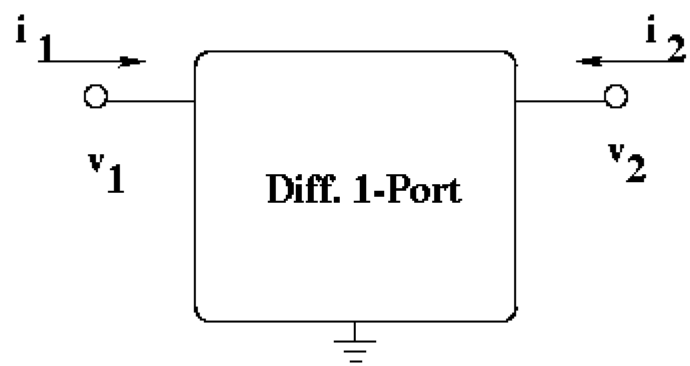
- Differential mode voltage, current and impedance
- Common-mode voltage, current and impedance

Diff. mode eqns. for coupled t-lines & inductors

$$v_{dm} \stackrel{\text{def}}{=} v_1 - v_2 \qquad i_{dm} \stackrel{\text{def}}{=} \frac{i_1 - i_2}{2}$$

$$v_{cm} \stackrel{\text{def}}{=} \frac{v_1 + v_2}{2} \qquad i_{cm} \stackrel{\text{def}}{=} i_1 + i_2$$

$$\begin{vmatrix} v_1 \\ v_2 \end{vmatrix} = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} \times \begin{vmatrix} i_1 \\ i_2 \end{vmatrix}$$



For an infinitely long line without reflection

L_{11}, L_{22}, C_{11}, C_{22} are self-inductance and self-capacitance

L_{12}, L_{21}, C_{12} and C_{21} are mutual inductance and mutual capacitance

DM and CM impedances for diff. 1-port

$$v_{dm} \stackrel{\text{def}}{=} v_1 - v_2 \qquad i_{dm} \stackrel{\text{def}}{=} \frac{i_1 - i_2}{2} \qquad \begin{vmatrix} 1/2 \\ -1/2 \end{vmatrix} = \begin{vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{vmatrix} \times \begin{vmatrix} i_1 \\ i_2 \end{vmatrix}$$

$$z_{dm} = \frac{v_{dm}}{i_{dm}} = 2(Z_{11} - Z_{12}) \quad \textit{since} \quad i_1 = -i_2 \ \textit{and} \ v_1 = -v_2$$

$$v_{cm} \stackrel{\text{def}}{=} \frac{v_1 + v_2}{2} \qquad i_{cm} \stackrel{\text{def}}{=} i_1 + i_2$$

$$z_{cm} = \frac{v_{cm}}{i_{cm}} = \frac{(Z_{11} + Z_{12})}{2} \quad \textit{since} \quad i_1 = i_2 \ \textit{and} \ v_1 = v_2$$

$$z_{se} = \frac{v_1}{i_1} = Z_{11} - \frac{Z_{12} Z_{21}}{Z_0 + Z_{22}} \ \textit{since} \ \frac{v_2}{i_2} = -Z_0 \ \textit{if port 2 is terminated on} \ Z_0$$

Weakly-coupled (widely spaced) t-lines

$$Z = \begin{vmatrix} 49.99 & 0.11 \\ 0.11 & 49.99 \end{vmatrix}$$

$$z_{cm} = \frac{49.99 + 0.11}{2} = 25.05\ Ohm$$

$$z_{dm} = 2 \times (49.99 - 0.11) = 99.76\ Ohm$$

$$z_{se} = 49.99 + 0.00012 = 49.9\ Ohm$$

Differential mode impedance is 2x the single-ended one

Strongly-coupled (closely spaced) t-lines

$$Z = \begin{vmatrix} 60.24 & 10.7 \\ 10.7 & 60.24 \end{vmatrix}$$

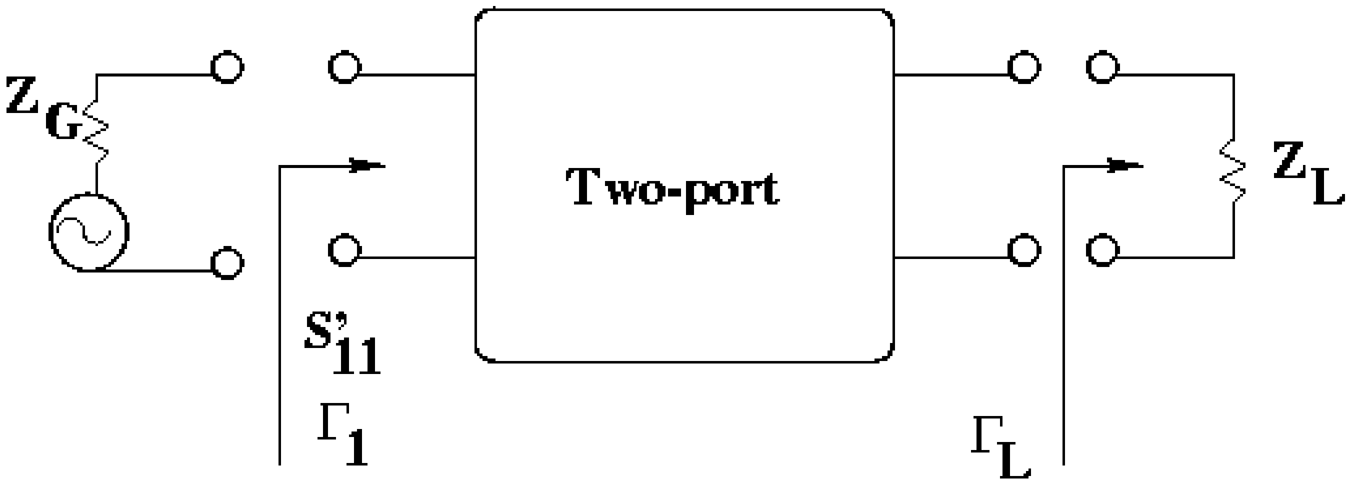
$$z_{cm} = \frac{(60.24 + 10.7)}{2} = 35.47 \quad Ohm$$

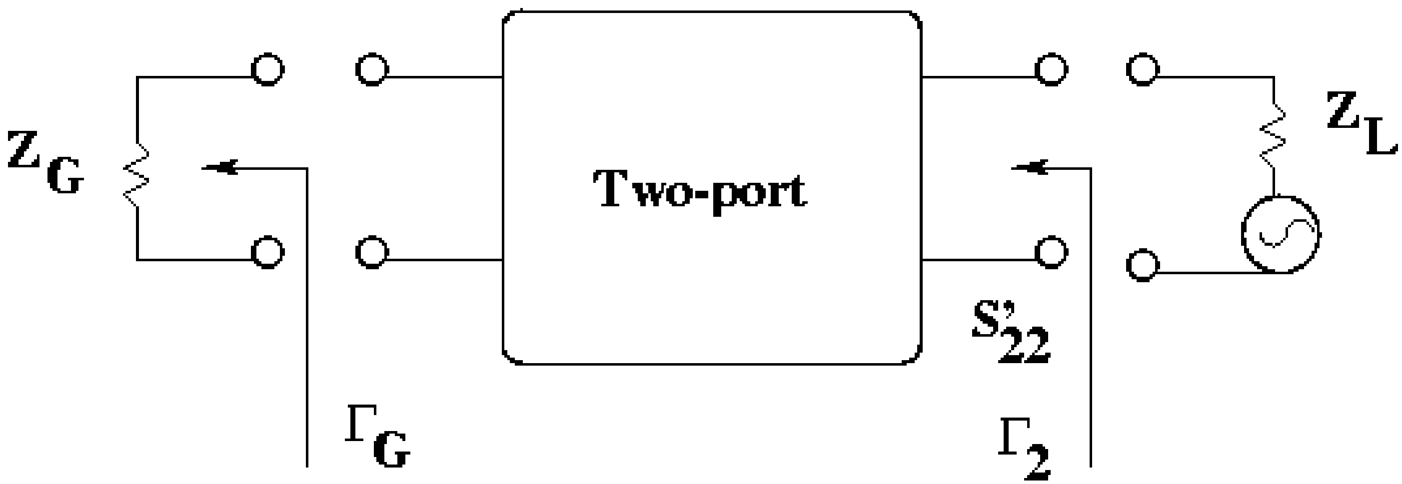
$$z_{dm} = 2(60.24 - 10.7) = 99.1 \quad Ohm$$

$$z_{se} = 60.24 - 1.038 = 59.20 \quad Ohm$$

Differential mode impedance is not 2x the single-ended one!

Stability conditions: 2-port definitions





Stability conditions for 2-Ports (ii)

The conditions for two port stability are that:

$$\left|S_{11}'\right| < 1 \qquad \left|S_{22}'\right| < 1$$

for all possible load terminations with a positive real part.

where:

$$S_{11}' = S_{11} + \frac{S_{12} S_{21} \Gamma_L}{1 - S_{22} \Gamma_L} \qquad S_{22}' = S_{22} + \frac{S_{12} S_{21} \Gamma_G}{1 - S_{11} \Gamma_G}$$

If either condition is violated we have only conditional stability

Stability Conditions for 2-Ports (iii)

The necessary and sufficient conditions for stability are:

$$k = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |D|^2}{2|S_{12}||S_{21}|} > 1 \text{ and } |D| < 1$$

where:

$$|S_{12}S_{21}| < 1 - |S_{11}|^2 \qquad |S_{12}S_{21}| < 1 - |S_{22}|^2$$

$$D = S_{22}S_{11} - S_{21}S_{12}$$

Since, normally, $|D| < 1$, one can show that

$k > 1$ is sufficient for unconditional stability

K = Rollet's stability factor

Stability (continued)

Alternate (more recent) unconditional 2-port stability criterion

$$\mu = \frac{1 - |S_{11}|^2}{|S_{22} - D\,S_{11}^*| + |S_{12}\,S_{21}|} > 1$$

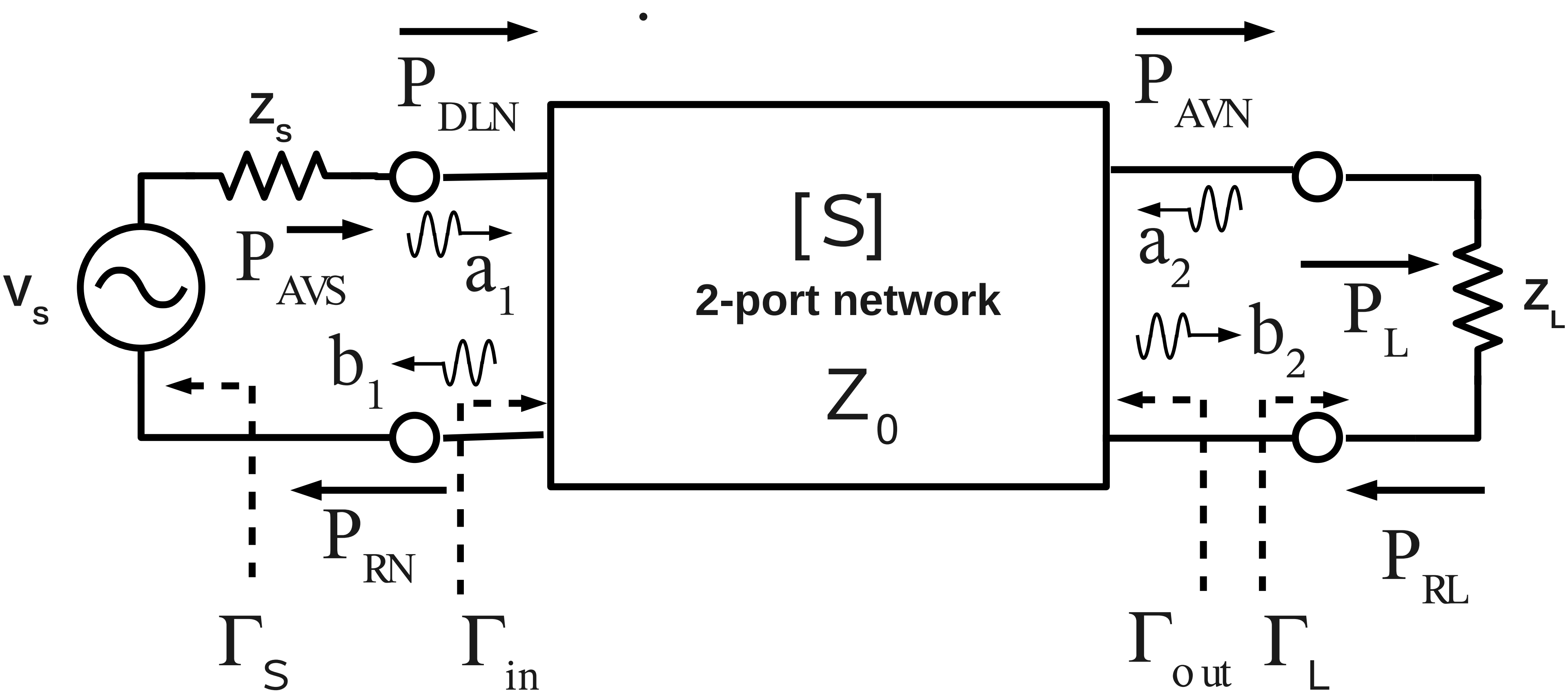
Unconditional stability criteria do not exist for

- a chain of cascaded two-ports
- n-ports

Two-port power gain definitions

$$P_{DLN} = P_{in} = P_{AVS} - P_{RN} = \frac{|a_1|^2}{2} - \frac{|b_1|^2}{2}$$

$$P_L = P_{AVN} - P_{RL} = \frac{|b_2|^2}{2} - \frac{|a_2|^2}{2}$$



Two-port power gain definitions (ii)

- Power Gain, $G_P = P_L/P_{DLN}$

$$G_P = \frac{|S_{21}|^2(1-|\Gamma_L|^2)}{(1-|\Gamma_{in}|^2)|1-S_{22}\Gamma_L|^2}$$

- Available power gain, $G_A = P_{AVN}/P_{AVS}$

$$G_A = \frac{|S_{21}|^2(1-|\Gamma_S|^2)}{|1-S_{11}\Gamma_S|^2(1-|\Gamma_{out}|^2)}$$

- Transducer power gain, $G_T = P_L/P_{AVS}$

$$G_T = \frac{|S_{21}|^2(1-|\Gamma_S|^2)(1-|\Gamma_L|^2)}{|1-\Gamma_S\Gamma_{in}|^2|1-S_{22}\Gamma_L|^2} = \frac{|S_{21}|^2(1-|\Gamma_S|^2)(1-|\Gamma_L|^2)}{|1-S_{11}\Gamma_S|^2|1-\Gamma_{out}\Gamma_L|^2}$$

- Maximum available power gain, MAG = G_A, G_T when conjugately matched

$$MAG = \left|\frac{S_{21}}{S_{12}}\right|(k-\sqrt{k^2-1})$$

Summary

- Multi-port network representations: Z, Y, S
- Two-port: G, H, ABCD, T
- Smith Chart and S-parameters
- Differential S-parameters
- Unconditional stability criteria established for two-ports, not for n-ports
- Diff. and common-mode stability analysis is critical
- Several power gain definitions exist