Sedimentation

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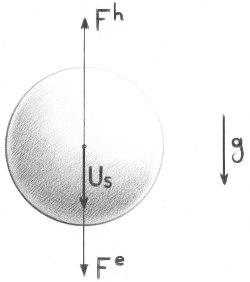
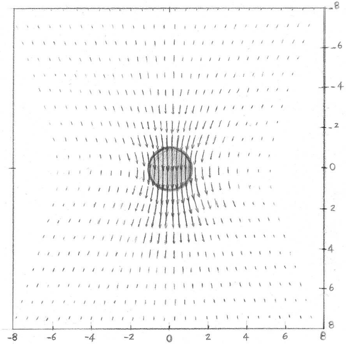
with illustrations by Sylvie Pic

Adapted from Chapter 6 of *A Physical Introduction to Suspension Dynamics*
Cambridge Texts in Applied Mathematics

Sedimentation of a single sphere

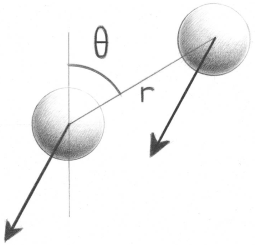


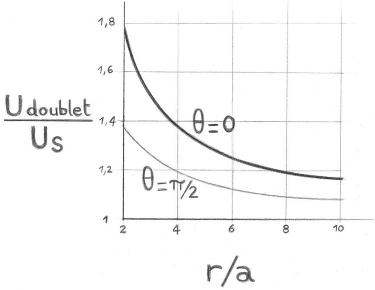
Stokes velocity
$\mathbf{U_S} = 2(\rho_p - \rho_f)a^2\mathbf{g}/9\mu$



Slow-decay disturbance $\sim O(\frac{aU_S}{r})$

Sedimentation of a pair of identical spheres



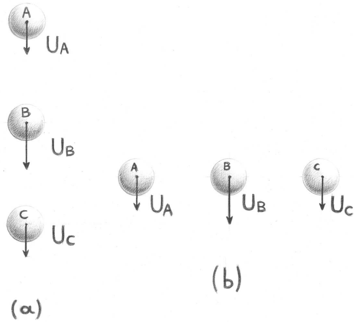


$$\frac{U_{\mathrm{doublet}}}{U_S} = 1 + \frac{3a}{2r} \quad \text{for} \quad \theta = 0,$$

$$\frac{U_{\mathrm{doublet}}}{U_S} = 1 + \frac{3a}{4r} \quad \text{for} \quad \theta = \frac{\pi}{2}$$

Two identical spheres fall at the same velocity and therefore do not change their orientation and separation

Sedimentation of a triplet



The particles do not maintain constant separation: the middle particle B falls faster

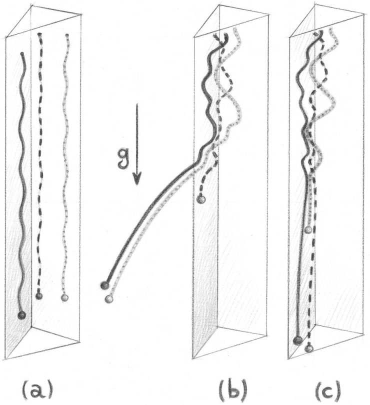
- case (a):

$$\frac{U_A}{U_S} = \frac{U_C}{U_S} = 1 + \frac{3}{2}\left(\frac{a}{r} + \frac{a}{2r}\right) = 1 + \frac{9a}{4r}$$
$$\frac{U_B}{U_S} = 1 + \frac{3}{2}\left(\frac{a}{r} + \frac{a}{r}\right) = 1 + \frac{3a}{r}$$

- case (b):

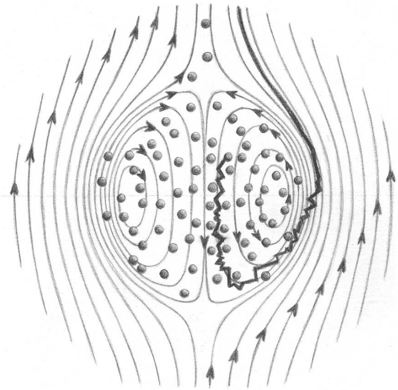
$$\frac{U_A}{U_S} = \frac{U_C}{U_S} = 1 + \frac{3}{4}\left(\frac{a}{r} + \frac{a}{2r}\right) = 1 + \frac{9a}{8r}$$
$$\frac{U_B}{U_S} = 1 + \frac{3}{4}\left(\frac{a}{r} + \frac{a}{r}\right) = 1 + \frac{3a}{2r}$$

Stokeslet simulation of a triplet



Sensitivity to initial configurations: signature of chaotic behavior originating in the long-range and many-body character of the hydrodynamic interactions

Settling of a spherical cloud of particles

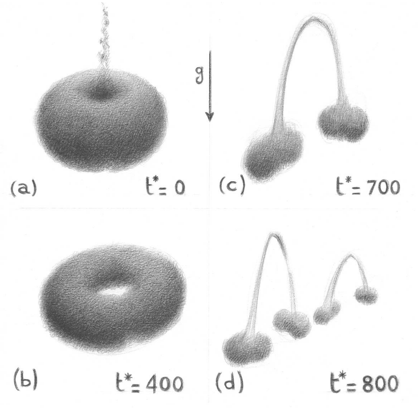


- Cloud velocity:

$$\mathbf{U}_{\mathbf{cloud}} = \frac{N\frac{4}{3}\pi a^3(\rho_p - \rho)\mathbf{g}}{2\pi\mu\frac{2+3\lambda}{\lambda+1}R} = N\frac{6a}{2\left(\frac{2+3\lambda}{\lambda+1}\right)R}\mathbf{U_S}$$

- Collective motion: toroidal circulation of the particles inside the cloud
- But chaotic fluctuations leading to particle leakage

Instability of a settling cloud of particles



Evolution of the cloud into a torus and subsequent breakup

Summing the effects between pairs of particles

- Velocity of a pair of spheres at a separation r:

  $\mathbf{U_S} + \mathbf{\Delta U}$ where $\mathbf{\Delta U}(r)$ incremental velocity due to a second particle

- Averaging over all possible separations which occur with conditional probability $P_{1|1}(r)$

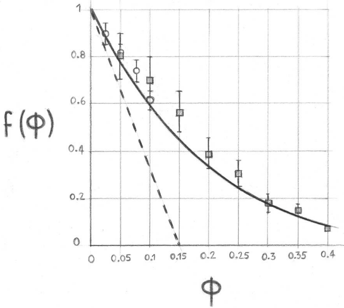
$$\mathbf{U_S} + \int_{r \geq 2a} \underbrace{\mathbf{\Delta U}}_{\frac{a\mathbf{U_S}}{r}} \underbrace{P_{1|1}(r)}_{ng(r)=n} \, dV$$

- Divergence with the size L of the vessel as

$$\int_{2a}^{L} r^{-1} r^2 dr \sim L^2$$

Strong divergence due to long-range hydrodynamic interactions

Hindered settling



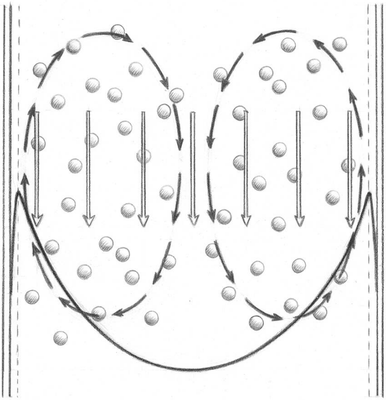
- Mean velocity:

$$\langle \mathbf{u} \rangle_p = \mathbf{U_S} f(\phi)$$

  Richardson-Zaki 1954: $f(\phi) = (1-\phi)^n$ with $n \approx 5$ at low Re

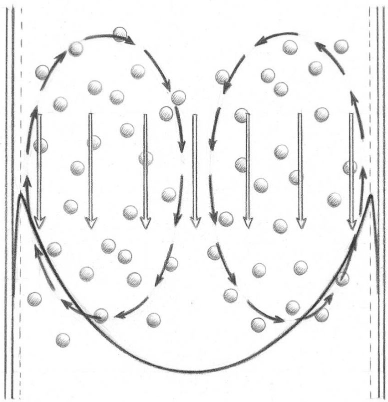
- Main effect = Back-flow
- Batchelor 1972: $f(\phi) = 1 + S\phi + O(\phi^2)$ with $S = -6.55$ assuming uniformly dispersed rigid spheres
- Results depend on microstructure in turn determined by hydrodynamics

Influence of the lateral walls of the vessel



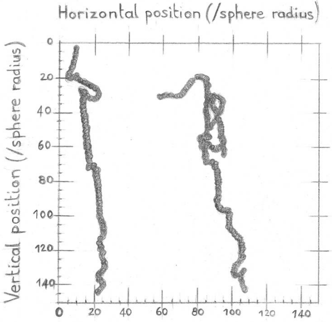
- Intrinsic convection = global convection of the suspension superimposed on the settling of the particles relative to the suspension
- Intrinsic convection originates in the buoyancy of the particle-depleted layer next to the side walls

Intrinsic convection



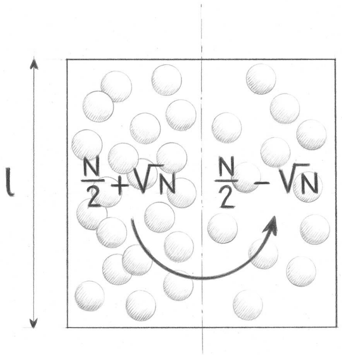
- Particle-depleted layer next to the side walls: the centers of the spheres cannot come closer than a radius a to the cell wall
- This buoyant particle-depleted layer located at one particle radius from the wall drives an upward flow near the wall
- Because no net flux condition across any horizontal section, downward return flow in the center
- Boundary layer formulation: Poiseuille flow with a slip velocity at the wall $w_* = \frac{9}{4}\phi U_S$

Velocity fluctuations



- Random walk through the suspension after a large enough number of hydrodynamic interactions
- Diffusive nature of the long-time fluctuating particle motion
- Anisotropic hydrodynamic self-diffusivities
- Large velocity fluctuations of the same order as the mean particle velocity
- Anisotropic fluctuations with a larger value in the direction of gravity

Divergence of velocity fluctuations?



Blob of size l ($a\phi^{-1/3} < l < L$)
containing $N_l = \phi l^3/a^3$ particles

- Random mixing of the suspension creates statistical fluctuations of $O(\sqrt{N_l})$
- Balance of the fluctuations in the weight $\sqrt{N_l}\frac{4}{3}\pi a^3(\rho_p - \rho)g$ by Stokes drag on the blob $6\pi\mu l w_p'$
- Convection currents, also called 'swirls,' on all length-scales l

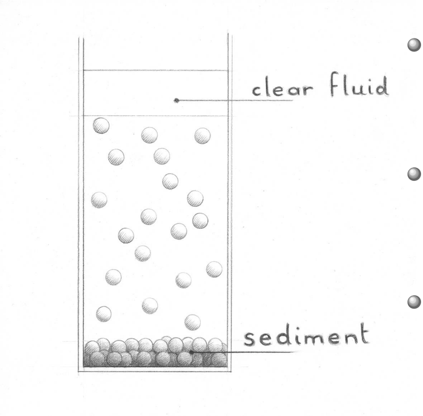
$$w_p'(l) \sim \frac{\sqrt{N_l}\frac{4}{3}\pi a^3(\rho_p - \rho)g}{6\pi\mu l} \sim U_S\sqrt{\phi\frac{l}{a}}$$

- Large-scale fluctuations are dominant

$$w_p' \sim U_S\sqrt{\phi\frac{L}{a}} \quad \text{diverge with L}$$

BUT no such divergence seen in experiments

Kinematic wave equation



Only variation in the direction of gravity z

- Conservation of particles

$$\frac{\partial \phi}{\partial t} + \frac{\partial (w_p \phi)}{\partial z} = 0$$

- Hyperbolic wave equation

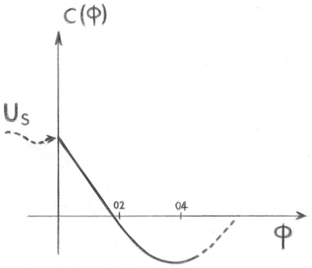
$$\frac{\partial \phi}{\partial t} + c(\phi)\frac{\partial \phi}{\partial z} = 0$$

- Wave speed

$$c(\phi) = \frac{d(w_p \phi)}{d\phi} = U_S[f(\phi) + \phi f'(\phi)]$$

using $w_p = U_S f(\phi)$

Kinematic wave speed



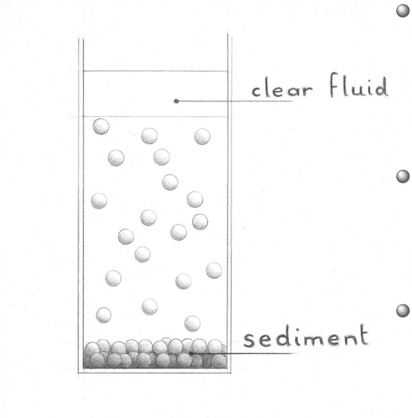
$$c(\phi) = \frac{d(w_p\phi)}{d\phi} = U_S[f(\phi)+\phi f'(\phi)]$$

- $f(\phi)$: decreasing function of ϕ thus $f'(\phi) < 0$
- $c(\phi) \leqslant w_p(\phi)$
- $c(\phi) \equiv U_S$ at $\phi = 0$ and then decreases rapidly to negative values before increasing to a small negative value at maximum packing
- Lower values of ϕ propagate faster than larger values

Self-sharpening

$\therefore$ Formation of sharp shocks

Shock speed



- Conservation of particle flux across the shock (subscript 1 ahead of the shock and 2 behind)

$$U_{\mathrm{shock}} = \frac{[w_p \phi]_1^2}{[\phi]_1^2} = \frac{w_p(\phi_2)\phi_2 - w_p(\phi_1)\phi_1}{\phi_2 - \phi_1}$$

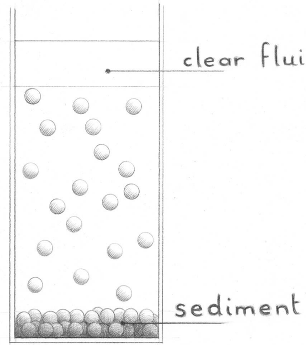
- Velocity of the sedimentation front $(\phi_2 = 0, \phi_1 = \phi_0)$

$$U_{\mathrm{sedimentation}} = w_p(\phi_0) = U_S f(\phi_0)$$

- Velocity of the growing-sediment front $(\phi_2 = \phi_0, \phi_1 \approx \phi_{\mathrm{max}})$

$$U_{\mathrm{sediment}} = \frac{\phi_0 w_p(\phi_0)}{\phi_0 - \phi_{\mathrm{max}}} = -U_S \frac{\phi_0 f(\phi_0)}{\phi_{\mathrm{max}} - \phi_0}$$

Front spreading



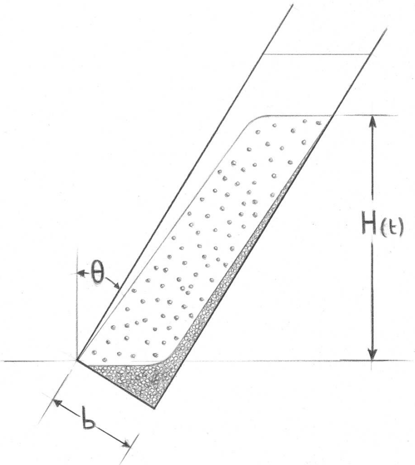
- Diffusive spreading of the sedimentation front
- Nonlinear convection-diffusion equation

$$\frac{\partial \phi}{\partial t} + c(\phi)\frac{\partial \phi}{\partial z} = \frac{\partial}{\partial z}(D^c \frac{\partial \phi}{\partial z})$$

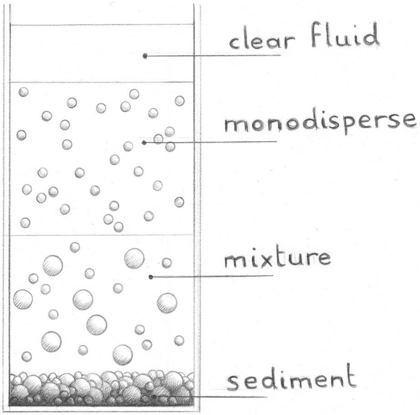
  with $D^c(\phi)$ gradient diffusivity
- But also convective spreading
  - Polydispersity in particle size leading to a distribution of sedimentation velocity
  - Differences in settling velocity of the density fluctuations created by the mixing

Sedimentation in an inclined channel

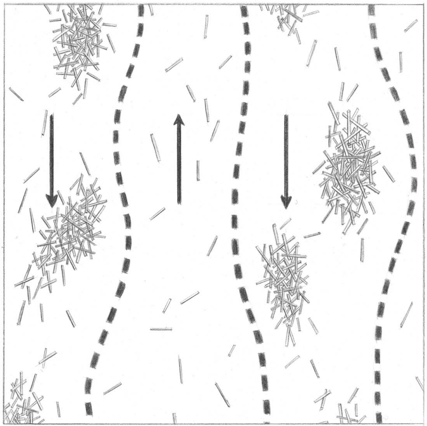
Enhancement in settling rate: $(H/b)\sin\theta$



Sedimentation of a suspension of bidisperse spheres



Instability of a sedimenting suspension of fibers



Movie references

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