Solutions to the Tutorial Problems in the book "Magnetohydrodynamics of the Sun" by ER Priest (2014) CHAPTER 8

PROBLEM 8.1. A Fast Dynamo.
Demonstrate the stretch-fold-shear dynamo mechanism by considering the action of a ideal Beltrami flow $\mathbf{v} = 2\cos^2 t\ (0, \sin x, \cos x)$ in Cartesian coordinates, acting for a time $t = \pi$, on a field $\mathbf{B} = (1, 0, 0)\cos x$ in the large magnetic Reynolds limit.

SOLUTION. For this flow and field, the three components of the ideal induction equation become

$$\frac{\partial B_x}{\partial t} = 0,$$

$$\frac{\partial B_y}{\partial t} = \frac{\partial}{\partial x}(v_y B_x) = 2\cos^2 t \frac{\partial}{\partial x}(\sin x \cos x) = 2\cos^2 t \cos 2x,$$

$$\frac{\partial B_z}{\partial t} = \frac{\partial}{\partial x}(v_z B_x) = 2\cos^2 t \frac{\partial}{\partial x}(\cos x \cos x) = -2\cos^2 t \sin 2x.$$

Thus, B_x does not change in time but both B_y and B_z increase monotonically. For example, between $t = 0$ and $t = \pi$, the amplitudes of B_y and B_z increase by

$$\int_0^\pi 2\cos^2 t dt = \pi,$$

and the same is true for each subsequent interval of π.

PROBLEM 8.2. Differential Rotation.
Consider the effect of a toroidal velocity $\mathbf{v} = R\Omega(R, z)\hat{\boldsymbol{\phi}}$ in cylindrical polars on a poloidal field $(B_0\hat{\mathbf{z}})$, which remains steady and uniform.

(a) Show that, if there is no diffusion initially, then the toroidal field increases linearly in time and reaches a value of order $R_m B_0$ before diffusion sets in.

(b) Find the ultimate steady state when Ω is a function of $r = (R^2 + z^2)^{1/2}$ alone.

SOLUTION after Moffatt (1978) *Magnetic field generation in electrically conducting fluids.*

The action of a toroidal flow $\mathbf{v} = R\Omega(R, z)\hat{\boldsymbol{\phi}}$ on a poloidal field $\mathbf{B}_p = B_0\hat{\boldsymbol{z}}$ is to produce a toroidal field $B\hat{\boldsymbol{\phi}}$ by the induction equation

$$\frac{\partial B}{\partial t} = RB_0\frac{\partial \Omega}{\partial z} + \eta\left(\nabla^2 - \frac{1}{R^2}\right)B. \tag{1}$$

Thus, if Ω is constant on $\mathbf{B}_p$-lines and $B = 0$ initially, then $B = 0$ for $t > 0$ too (the law of isorotation), since each $\mathbf{B}_p$-line is rotated without distortion. However, if $\partial\Omega/\partial z \neq 0$, then $B(R, z, t)$ will grow in time.

(a) *Initial Phase*

In the initial phase, when diffusion is negligible, the solution of (1) is simply

$$B(R, z, t) = RB_0\frac{\partial \Omega}{\partial R}t,$$

so that the toroidal field grows linearly in time, as required.

The neglected diffusion term in (1) is of order $\eta B_0/L_0^2$, where L_0 is the scale over which Ω varies. This becomes comparable with the other term $RB_0\partial\Omega/\partial z$ when $t \approx L_0^2/\eta$, so that the toroidal field becomes of size $R_m B_0$, where $R_m = \Omega_0 L_0^2/\eta$, as required.

(b) *Steady State*

As $t \to \infty$ the solution of (1) approaches a steady state $B(R, z)$ satisfying

$$0 = RB_0\frac{\partial \Omega}{\partial z} + \eta\left(\nabla^2 - \frac{1}{R^2}\right)B. \tag{2}$$

In the particular case when $\Omega = \Omega(r)$, where $r = (R^2 + z^2)^{1/2}$, the solution of (2) is

$$B(r, \theta) = -\frac{B_0 \sin\theta\cos\theta}{3\eta r^3}\int_0^r r^4\Omega(r)dr,$$

as proved in detail in Moffatt's book.

PROBLEM 8.3. An Antidynamo Theorem.

Prove that a planar velocity $\mathbf{v} = v_x(x, y, z, t)\hat{\mathbf{x}} + v_y(x, y, z, t)\hat{\mathbf{y}}$ cannot produce dynamo action.

SOLUTION.

For this velocity the z-component of the induction equation is

$$\frac{\partial B_z}{\partial t} = -\frac{\partial}{\partial x}(v_x B_z) - \frac{\partial}{\partial y}(v_y B_z) + \eta\nabla^2 B_z.$$

However, this does not allow regeneration of B_z from B_x or B_y since neither B_x nor B_y appears on the right-hand side. Also, it implies that B_z decays in time. In other words, this planar velocity does not produce dynamo action for B_z.

PROBLEM 8.4. Parker Dynamo.
Examine the properties of Parker's dynamo in Cartesian geometry (Sec. 8.3.1.1) in the particular case when $B_z = 0$ and $\partial/\partial z = 0$.

SOLUTION.

Parker (1955) used a simple cartesian model to examine the properties of an oscillatory α-ω dynamo. Suppose a shear V_y acts on a poloidal field (B_x, B_z) to produce a toroidal field B_y. For simplicity, assume $B_z = 0$ and $\partial/\partial z = 0$, and suppose α and $\omega = \partial V_y/\partial x$ are both constant. Then, following Roberts (1967) *An Introduction to MHD*, the dynamo equations become

$$\frac{\partial B_y}{\partial t} = \frac{\partial^2 B_y}{\partial z^2} + R_\omega B_x,$$

$$\frac{\partial B_x}{\partial t} = \frac{\partial^2 B_x}{\partial z^2} + R_\alpha \frac{\partial B_y}{\partial z},$$

where $R_\omega = (L^3/\eta)\partial\omega/\partial r$ and $R_\alpha =$.

Solutions to these of the form

$$[B_x(z,t), B_y(z,t)] = [B_x, B_y] \exp(ikz + \lambda t),$$

give

$$(\lambda^2 + k^2) = ikD,$$

whose solution is

$$\lambda = -k^2 + [1 + i \ \mathrm{sgn}(kD)](\tfrac{1}{2}kD)^{1/2}.$$

Splitting λ into real and imaginary parts $\lambda_i + i\lambda_r$ gives

$$\lambda_r = -k^2 + (\tfrac{1}{2}kD)^{1/2},$$

$$\lambda_i = \mathrm{sgn}(kD)(\tfrac{1}{2}kD)^{1/2}.$$

Thus, there is a critical dynamo number $D = D_{crit}$ given by

$$|D_{crit}| = 2|k|^3,$$

such that when $D < D_{crit}$ the solution decays in time, whereas when $D > D_{crit}$ it grows. The marginal oscillatory state ($D = D_{crit}$) is oscillatory. When $D > D_{crit}$, the magnetic field propagates in the z-direction as a dynamo wave with phase velocity $-|k|(\text{sgn} D_{crit})$, whose direction depends on the sign of D.

The relative phases of B_x and B_y depend on R_α and R_ω. Thus

$$\frac{B_x}{B_y} = [1 + i \ \text{sgn}(kD)] \frac{k^2}{R_\omega},$$

and so the argument of B_x/B_y is positive if $\text{sgn}(kR_\alpha) > 0$ and negative if $\text{sgn}(kR_\alpha) < 0$.

PROBLEM 8.5. Flux Expulsion.
Consider the effect of a differential rotation $\mathbf{v} = R\Omega(R)\hat{\boldsymbol{\phi}}$ in cylindrical polars (R, ϕ, z) on a field $(B_R, B_\phi, 0)$ that is initially uniform $(B_0\hat{\mathbf{x}})$.

(a) Show that, if there is initially no diffusion and $d\Omega/dR \neq 0$, then the azimuthal field (B_ϕ) grows in time, and that it reaches a value of order $R_m^{1/2} B_0$ before diffusion sets in (assuming $R_m \equiv \Omega_0 R_0^2/\eta \gg 1$).

(b) Prove that, if $\Omega(R)$ is constant inside $R = R_0$ and vanishes outside, then the ultimate steady state has a vanishing field inside $R = R_0$, a state called *flux expulsion*.

SOLUTION after Moffatt (1978) *Magnetic field generation in electrically conducting fluids.*

For $v_\phi(R) = R\Omega(R)$, $B_R(R, \phi, t)$ and $B_\phi(R, \phi, t)$, and $\mathbf{B} = -\hat{\mathbf{z}} \times \boldsymbol{\nabla} A$, the ideal induction equation becomes

$$\frac{\partial A}{\partial t} + \Omega(R) \frac{\partial A}{\partial \phi} = \eta \nabla^2 A,$$

where the condition at $t = 0$ that $B_R = B_0 \cos\phi$ and $B_\phi = -B_0 \sin\phi$ implies that

$$A(R, \phi, 0) = B_0 R \sin\phi.$$

The appropriate solution for $A(R, \phi, t)$ is therefore of the form

$$A(R, \phi, t) = Im[B_0 f(R, t) e^{i\phi}],$$

where

$$\frac{\partial f}{\partial t} + i\Omega(R) f = \eta \left(\frac{1}{R} \frac{\partial}{\partial R} R \frac{\partial f}{\partial R} - \frac{1}{R^2} \right) f \tag{3}$$

and

$$f(R,0) = R.$$

(a) *Initial Phase*

When $t = 0$ we have a uniform field with no diffusion, so, in the initial phase, diffusion is negligible and the field evolves by the ideal induction equation, so that the solution to Eq.(3) with $\eta = 0$ is

$$f(R,t) = R^{-i\omega(R)t},$$

which gives

$$A(R,\phi,t) = B_o R \sin[\phi - \Omega(R)t]. \tag{4}$$

The resulting field components are

$$B_R = \frac{1}{R}\frac{\partial A}{\partial \phi} = B_0 \cos[\phi - \Omega(R)t],$$

$$B_\phi = -\frac{\partial A}{\partial R} = -B_0 \sin[\phi - \Omega(R)t] + B_0 R \frac{d\Omega}{dR} t \cos[\phi - \Omega(R)t]. \tag{5}$$

Thus, when $d\Omega/dR = 0$, we have solid-body rotation and the field is rotated without distortion. But, $d\Omega/dR \neq 0$, B_ϕ grows linearly with time due to the stretching of the field lines by the differential rotation.

How long is it before diffusion becomes important? The diffusion term during the above ideal motion (4) is

$$\eta\nabla^2 A = \eta B_0 \frac{1}{R^2}\frac{d}{dR}\left(R^3 \frac{d\Omega}{dR}\right) t \cos[\phi - \Omega(R)t] - \eta B_0 R \left(\frac{d\Omega}{dR}\right)^2 t^2 \sin[\phi - \Omega(R)t], \tag{6}$$

whereas the advection term is

$$\Omega \frac{\partial A}{\partial \phi} = B_0 \Omega R \cos[\phi - \Omega(R)t]. \tag{7}$$

Thus, writing in order of magnitude $\Omega = \Omega_0$ and $d\Omega/dR = \Omega_0/R_0$, diffusion is negligible provided both coefficients of the trig terms in (6) are much smaller than the corresponding coefficient in (7), namely, provided

$$\Omega_0 t \ll R_m \qquad \text{and} \qquad \Omega_0 t \ll R_m^{1/2},$$

where $R_m = \Omega_0 R_0^2/\eta$.

Since $R_m \gg 1$ the second condition is the appropriate condition, as required. Moreover, at a time such that $\Omega_0 t \approx R_m^{1/2}$, the second term in the expression for B_ϕ in Eq.(6) dominates, which implies a value of

$$B_\phi \approx R_m^{1/2} B_0,$$

as required.

(b) *Steady State*

As $t \to \infty$, the solution of (3) is likely to tend to a steady state $[f(R)]$ satisfying

$$i\Omega(R) f = \eta \left(\frac{1}{R} \frac{d}{dR} R \frac{df}{dR} - \frac{1}{R^2} \right) f, \tag{8}$$

subject to the condition at large distances that

$$f(R) \to R \quad \text{as} \quad R \to \infty,$$

so that the field there is unaffected.

For the particular angular velocity profile

$$\Omega(R) = k_0^2/\eta \quad for \quad R < R_0,$$

$$\Omega(R) = 0 \quad for \quad R > R_0,$$

where $k_0^2 = \eta\Omega_0$, the required solution to (8) has the form

$$f(R) = D J_1(pR) \quad for \quad R < R_0$$

and

$$f(R) = R + C/R \quad for \quad R > R_0,$$

where $J_1(pR)$ is a Bessel function and $p = (1 - ik_0)/\sqrt{2}$.

The constants C and D are determined by the conditions that B_R and B_ϕ be continuous at $R = R_0$, which imply

$$D = \frac{2}{pJ_0(pR_0)}$$

and

$$C = \frac{R_0[2J_1(pR_0) - pR_0 J_0(pR_0)]}{pJ_0(pR_0)}.$$

In the limit as $R_m \to \infty$, we have $k_0 \to \infty$ and $p \to \infty$, so that $D \to 0$ and $C \to -R_0^2$. In other words $f(R)$ becomes

$$f(R) = 0 \qquad for \qquad R < R_0$$

and

$$f(R) = R - R_0^2/R \qquad for \qquad R > R_0,$$

with flux function

$$A = 0 \qquad for \qquad R < R_0$$

and

$$A = B_0 \left(R - R_0^2/R\right) \sin\phi \qquad for \qquad R > R_0.$$

In other words, the magnetic field is totally expelled from the rotating region, as required, and there is a current sheet on the boundary $R = R_0$.