Project B.5: Stability boundaries for double-diffusive instabilities

(A)

1. Both results follow from (B.35): the product of three numbers is zero if and only if at least one of the numbers is zero.
2. Suppose that $\alpha_0 = 0$, and assume without loss of generality that $s_1 = 0$. In that case, (B.33) and (B.34) become

$$\begin{aligned} s_2 + s_3 &= -1 \quad (3)\\ s_2 s_3 &= \alpha_1. \quad (4) \end{aligned}$$

   Because $\alpha_1 > 0$, (4) tells us that s_2 and s_3 have the same sign, and (3) tells us that they must be negative.
3. Consider s as a function defined on the $\alpha_0 - \alpha_1$ plane (as in figure 1). Suppose we start at some point on the line $\alpha_0 = 0$, where $s = 0$, and move by some small distance $\Delta\alpha_0$, with α_1 held constant. What is the corresponding change in s? Differentiating (B.32), we have

$$3s^2\Delta s + 2s\Delta s + \alpha_1 \Delta s + \Delta\alpha_0 = 0. \quad (5)$$

   Evaluating this on $\alpha_0 = 0$ (where $s = 0$) and taking the limit $\Delta\alpha_0 \to 0$ we find

$$\left(\frac{\partial s}{\partial \alpha_0}\right)_{\alpha_0=0} = -\frac{1}{\alpha_1}, \quad (6)$$

   which is negative since $\alpha_1 > 0$. Therefore, s is positive if $\alpha_0 < 0$ and negative if $\alpha_0 > 0$.

(B)

Let the single real root be s_1 and the complex conjugate pair be $s_{2,3} = s_r \pm s_i$. We then have

$$\begin{aligned} s_1 + 2s_r &= -1 \quad (7)\\ 2s_1 s_r + s_r^2 + s_i^2 &= \alpha_1 \quad (8)\\ s_1(s_r^2 + s_i^2) &= -\alpha_0. \quad (9) \end{aligned}$$

1. Assuming that the complex root is not zero, (9) tells us that $s_1 = 0$ if and only if $\alpha_0 = 0$.
2. If $s_r = 0$, then from (7) and (8) we have $s_1 = -1$ and $s_i^2 = \alpha_1$. Substituting these results into (9), we obtain $\alpha_1 = \alpha_0$. Is the converse true? Assuming that $\alpha_1 = \alpha_0$, we can add (8) and (9) to get

$$s_r[(s_r + 1)^2 + s_i^2] = 0. \quad (10)$$

   This tells us that s_r must be zero. (The term in square brackets cannot be zero because $s_i \neq 0$ for a complex root.) Therefore, $s_r = 0$ if and only if $\alpha_1 = \alpha_0$.
3. The real and imaginary parts of (B.32) are

$$\begin{aligned} s_r^3 - 9s_r s_i^2 + s_r^2 - s_i^2 + \alpha_1 s_r + \alpha_0 &= 0, \quad (11)\\ s_i(3s_r^2 - s_i^2 - 2s_i + \alpha_1) &= 0. \quad (12) \end{aligned}$$

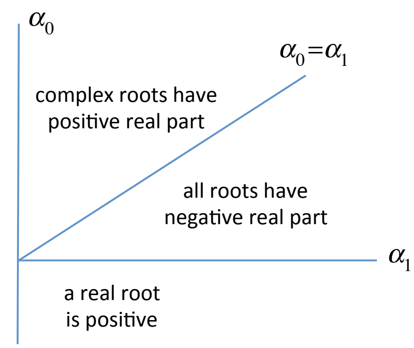

Figure 1: Regime sketch of the $\alpha_0 - \alpha_1$ plane for part C.

4. One possible solution of (12) is $s_i = 0$, i.e. the real root. For that root, (11) reduces to (B.32), so the argument is the same as in the earlier problem **A**(iii), hence the sign of the root is opposite to the sign of α_0.
5. Solving (12) for s_i gives
$$s_i^2 = 3s_r^2 - 2s_i + \alpha_1. \tag{13}$$
Substituting this into (11) gives (B.36).
6. As was done in **A**(iii), differentiate (B.36) along a line $\alpha_1 = constant$:
$$78s_r^2\Delta s_r - 32s_r\Delta s_r - 2(1-4\alpha_1)\Delta s_r - \Delta\alpha_0 = 0. \tag{14}$$
Evaluating at $\alpha_0 = \alpha_1$, $s_r = 0$ gives the desired result. Therefore, in the case of complex conjugate roots, the real part is positive for $\alpha_0 > \alpha_1$ and negative for $\alpha_0 < \alpha_1$.

(C)

See figure 1.

(D)

The stability boundaries are found by substituting the definitions of A_0, A_1 and A_2 from (9.8) into $A_0 < 0$ and $A_0 > A_1A_2$. The first condition, $A_0 > 0$, pertains to $R_\rho > 1$ and therefore to the case of warm, salty water overlying cool fresh water. The second condition, $A_0 > A_1A_2$, works when $R_\rho < 1$ and the opposite stratification exists.