**Problem Solutions**

Lidar Engineering

Introduction to Basic Principles

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**Chapter 2**

2.6.1 The Poisson distribution given in Eq. (2.5) is often said to closely resemble the Gauss distribution given in Eq. (2.7) for mean values of 6 or greater. Is this correct? Plot the two distributions for mean = 6 on the same graph and compare them.

Solution:

The distributions can be calculated using Eqs. (2.5) and (2.7) or by using built-in spreadsheet functions. In either case, the variance is equal to the mean in Poisson statistics, so the standard deviation in the Gauss distribution is equal to the square root of 6. The two distributions are plotted below. They are quite similar even for a mean value as small as 6.



Poisson and Gauss distributions for mean = 6.

Triangles – Poisson; circles – Gauss.

* + 1. A lidar researcher makes a nighttime measurement at a certain range only once, and he collects 100lidar photoelectrons. What is his best estimate of the average value he would find if he repeated the measurement many times? What is his best estimate of the standard deviation of each set of measurements?

Solution:

According to statistical theory, having only the one measurement, his best estimate of the mean is simply the number he collected (100), and his best estimate of the standard deviation is the square root of that number (10). Averaging many such measurements together would improve the estimates, of course, but from a single measurement, those numbers are the best available.

* + 1. In Figure 2.3, SNR increases with  by a factor of ten every decade on the right vertical axis (background limited), but only with every two decades on the left axis (signal limited). Does this fact mean that the common lidar technique of multi-pulse averaging causes a linear increase in SNR in background-limited conditions? Using the general formula in Eq. (2.8), show that the increase in SNR obtained by summing the photon detections from  pulses is always proportional to .

Solution:

Equation (2.8) for one pulse is

.

For  pulses,

,

so SNR is always increased by , because multi-pulse averaging increases and  by the same factor.

* + 1. Estimated errors are normally expressed as fractions of the mean value, for example ±10 %. How is the fractional uncertainty related to the SNR?

Solution:

Photon SNR is defined as the mean divided by the standard deviation, so 1/SNR is the standard deviation divided by the mean, and (1/SNR) x 100% is the fractional uncertainty in percent that corresponds to ± one standard deviation. For example, SNR = 100 corresponds to a fractional uncertainty of ±1 %.

* + 1. The Standard Atmosphere air density profile plotted in Fig. 1.3 is almost a straight line on the semi-logarithmic graph, which means that the decrease with altitude is very nearly exponential. Using the data shown in Fig. 1.3, find the *scale height * of the atmospheric density, in the relation  , with  and **in km. Assume that  is 1 kg/m3 and  is 1 x 10-6 kg/m3. How much of the mass of the atmosphere is in the troposphere? Assume the tropopause height is 15 km as shown in Figure 1.2 and use the scale height found above.

Solution:

The ratio of densities at the two altitudes is

 .

Taking the natural logarithm of both sides and solving for the scale height yields  = 7.2 km. In practice, different scale heights are used for different altitude regions because the profile shown in Figure 1.3 is not quite a straight line.

Integrating the scale height equation  from the surface  to any reference altitude  yields the atmospheric mass in a square-meter column from the surface to as

  (kg/m2).

The fraction of mass in the troposphere is the value of the integral at 15 km divided by the value at infinity (which is ), or . As a rule of thumb, the troposphere is often said to contain about 90% of the atmosphere.

* + 1. Did EARL meet the design goal of 10-minute measurements of clear air backscatter at 4 km altitude during both day and night with an SNR of at least 100? Values of signal and background photons are shown in the table below for both receiver channels during both day and night, for the conditions of Figure 2.9. Fill in the missing entries, recalling that the quantum efficiency is 0.12. Find the SNR values using Eq. (2.8). The dark count rate is effectively zero.

Solution:

The completed table is shown below. Yes, the daytime design goal was met. During nighttime, even the short-range receiver channel achieves SNR > 100.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  | *NS* | *nS* | *NB* | *nB* | SNR |
| Short Range | Day | 3.33 x 105 | 4.00 x 104 | 2.94 x 108 | 3.53 x 107 | 7 |
| Night | 3.33 x 105 | 4.00 x 104 | 2.94 x 102 | 3.53 x 101 | 200 |
| Long Range | Day | 2.15 x 106 | 2.58 x 105 | 4.14 x 107 | 4.97 x 106 | 113 |
| Night | 2.15 x 106 | 2.58 x 105 | 4.14 x 101 | 4.97 x 100 | 508 |

**Chapter 3**

3.6.1. The Weather Surveillance Radar 1988 Doppler (WSR-88D) detects radio waves scattered by raindrops. The radar has an operating wavelength of 3 cm and raindrops are on the order of 1 mm in diameter. Which of the three types of scattering does the radar detect?

Solution:

The scattering regime is determined by the dimensionless scattering parameter . Putting in the relevant parameters,  = (2 x 3.14 x 0.5 x 10-3) / (3 x 10-2) = 0.1, which is less than 0.5, so the scattering is in the Rayleigh regime as defined in Table 3.2.

3.6.2. Is extinction due to molecular scattering an important effect in tropospheric lidar? Consider a 10 km horizontal path with the backscatter coefficient at STP given by Eq. (3.11) and the extinction coefficient calculated using Eq. (3.13.): what is the one-way OD at 355, 532, and 1064 nm? How do these OD values compare to the total atmospheric (zenith) ODs in Table 3.3?

Solution:

Equation (3.11) is at STP. From Eq. (3.13), , so . Using these equations with the given wavelengths, along with the 10-km path, yields the results in the following table:

|  |  |  |  |
| --- | --- | --- | --- |
|  (µm) |  (m-1sr-1) |  (m-1) | OD |
| 0.355 | 8.01 x 10-6 | 6.71 x 10-5 | 0.671 |
| 0.532 | 1.59 x 10-6 | 1.33 x 10-5 | 0.133 |
| 1.064 | 9.92 x 10-8 | 8.32 x 10-7 | 0.00832 |

These horizontal-path numbers are larger than the total zenith atmospheric ODs in Table 3.3, which is consistent with the fact that the troposphere contains most of the atmosphere’s molecules. At 0.355 µm, molecular scattering causes a loss of lidar signal at 5 km horizontal range (a two-way pathlength of 10 km) of approximately a factor of 2.

3.6.3. Consider the molecules in nitrogen gas at temperature  = 300 K. The molecules have an average kinetic energy of , where  is Boltzmann’s constant (1.38 x 10-23 J/K), and is the gas temperature in Kelvins. The kinetic energy is  , where  is the particle mass and is its speed.

1. What is the average speed of the molecules? The mass of the most common isotope of nitrogen, 14N, is 14 AMU, so assume that the N2 molecular mass is 28 AMU. One AMU = 1.66 x 10-27 kg.

1. What is the Doppler shift of backscattered 532-nm laser light (in nm) caused by this speed, for molecules moving directly away from the laser? The Doppler shift can be found from the relation  . What is this Doppler shift expressed in GHz?

Solutions:

1. From , we find that . Inserting the values listed above,

v = ((3 x 1.38 x 10-23) x 300) / (28 x (1.66 x 10-27))1/2 = 517 m/s.

1. From , using the speed from a) and the speed of light c = 3 x 108 m/s, we find that  = 3.45 x 10-6. At 532 nm, the shift is therefore 532 x (3.45 x 10-6) = 1.84 x 10-3 nm, or about 0.002 nm. Because this shift is so small, it is usually expressed in frequency units. From the relation  , we find  and , so  . But for the Doppler shift, , and by substituting that relation we find , which is (-2 x 517) / (532 x 10-9) = -1.94 x 109 Hz for molecules moving away from the lidar, so the width of the Doppler-broadened lidar signal is on the order of ±2 GHz, which is consistent with Figure 3.10.

3.6.4. In atmospheric molecules, many rotational energy levels are populated but almost all the molecules are in the lowest vibrational state. The Boltzmann distribution shows that the ratio of the populations in two states is , where  is the energy difference of the two states. Calculate that ratio for the two lowest rotational and vibrational levels in N2, for which = 1.99 cm-1 (when the molecule is in its lowest vibrational state) and  = 2331 cm-1. Note that the value of  at 300 K is 207 cm-1, as shown in Figure 3.24, and ignore centrifugal stretching and anharmonicity.

Solution:

From Eq. (3.15), the energy difference in wavenumbers between the  and  levels is , and from Eq. (3.16) the energy difference between the  and levels is . The two Boltzmann factors are therefore

Rotational:  = 0.981 and

Vibrational:  = 1.29 x 10-5.

These factors are consistent with the statement that many rotational energy levels are populated but almost all the molecules are in the lowest vibrational state, at atmospheric temperatures.

**Chapter 4**

4.8.1 Equation (4.2) gives a value for the scattering cross section  of a single air molecule. Show that it is consistent with the volume angular scattering coefficient for air given in Eq. (3.9).

Solution:

Equation (4.2) is

 (m2/sr).

To convert this to the volume angular scattering coefficient, multiply by the number of scatterers per unit volume  and replace  with the value 3, which is an accurate approximation for air. The result is then

 (1/m-sr),

which is Eq. (3.9).

* + 1. A radiation fog forms when the air temperature drops below the dew point and water vapor rapidly condenses on haze particles that were already present. The number density does not change appreciably, but the optical properties do. Consider scattering from water droplets with the radii shown in Figure 4.4 at the wavelength 0.532 µm. The refractive index of water is 1.33 at VIS wavelengths. Use an online calculator such as the Oregon Medical Laser Center, Mie Scattering Calculator at <https://omlc.org/calc/mie_calc.html>. to find the extinction and backscatter cross sections. Discuss their changes as functions of droplet radius.

Solution:

The radii in Figure 4.4 are 0.05, 0.10, 0.25, and 0.5 µm, so they span one order of magnitude. Using a Mie calculator yields the values in the table below.

|  |  |  |
| --- | --- | --- |
| Radius (µm) | ExtinctionCross section (µm2) | BackscatterCross section (µm2) |
| 0.05 | 1.02 x 10-4 | 1.31 x 10-4 |
| 0.10 | 5.12 x 10-3 | 3.36 x 10-3 |
| 0.25 | 3.32 x 10-1 | 1.89 x 10-2 |
| 0.50 | 3.07 x 100 | 3.59 x 10-1 |

The extinction cross section grows by a factor of 3.01 x 104 and the backscatter cross section grows by a factor of 2.74 x 103 as the droplet radius increases from 0.05 to 0.5 µm. The increases are dramatic because the scattering efficiency grows as  in the Rayleigh regime while the area grows as , as shown by Eqs. (4.6) and (4.7). This calculation explains why an unnoticeable haze can quickly become a serious visibility issue as a fog forms, and why backscattered light can become blinding when driving at night.

The extinction values in the table above can also be calculated from the scattering efficiency factors listed in Appendix J of E.J. McCartney, *Optics of the Atmosphere*. New York: Wiley, 1976, along with Eq. (4.7), although some interpolation is required.

* + 1. The quotient  appears occasionally in the pre-2007 linear depolarization lidar literature with no explanation of what it represents. Show that it is the polarized fraction of the received lidar signal.

Solution:

The polarized fraction of the signal is given by Eq. (4.17) as

 .

Multiplying the right-hand side of the equation by  yields

 .

* + 1. At 22:47:38 UTC (the last data points) on 23 Sep 2014 the AERONET Georgia\_Tech station reported the data in the table below. What value of the Angstrom parameter describes the wavelength dependence in this table? Consider the data from 380 to 1020 nm only. Is the Angstrom parameter consistent with Figure 4.24?

|  |  |
| --- | --- |
| λ (nm) | AOD |
| 340 | 0.2237 |
| 380 | 0.2089 |
| 440 | 0.1618 |
| 500 | 0.1352 |
| 675 | 0.0799 |
| 870 | 0.0508 |
| 1020 | 0.0380 |
| 1640 | 0.0178 |

Solution:

Making an Angstrom plot like Figure 4.23, which is a plot of ln() vs. ln(  ), results in the figure shown below. The straight line fitted to the data points using the method of least squares has a slope of about -1.72, so the Angstrom parameter is 1.72, in agreement with the last data point in Figure 4.24.



Angstrom plot of the data in the table above, with linear fit.

**Chapter 5**

5.6.1 Derive Eq. (5.3).

Solution:

Equation (5.3) gives the total power in a Gaussian beam in terms of the on-axis irradiance  and the 1/e2 beam radius . The total power is the integral of the irradiance  and the differential of area . Because a Gaussian beam has axial symmetry, the differential of area  is chosen as an annular ring with area . Using Eq. (5.2), the integral is therefore

 .

Recalling that, if  then  , the equation for power can be written in an integrable form as

,

which is evaluated as

,

 which is Eq. (5.3). Equation (5.4) for encircled power is derived in a similar way.

* + 1. Is EARL eye safe? As shown in Figure 5.21, the ANSI MPE is 2 x 10-7 J/cm2 for 10 ns pulses at 523.5 nm (it is the same for EARL’s 2.5 ns pulses). However, there is a correction factor Cp = n-0.25 for multiple-pulse exposures, where n is the number of pulses. Recall that an 0.25 s exposure is assumed for visible light and use the transmitter parameters listed in Table 5.6.

Solution:

The transmitted fluence is (15 x 10-6) J / [π(202)/4] cm2 = 0.48 x 10-7 J/cm2, so it is safe for a single pulse, but the MPE must corrected for multiple pulses. The factor is Cp = n-0.25, where n is the number of pulses in 0.25 s, which is 2500/4 = 625. Cp = 1/625.25 = 1/5, so the MPE is 0.40 x 10-7 J/cm2. EARL’s fluence is slightly above the MPE. EARL met the ANSI eye safety standard at the time it was designed and built, which was before a 2014 revision to the MPEs.

* + 1. What is the 21 CFR 1040 laser class of EARL’s transmitter? Table 5.5 is for CW lasers, for pulsed lasers, see <http://www.accessdata.fda.gov/scripts/cdrh/cfdocs/cfcfr/CFRSearch.cfm?FR=1040.10> (this is [13] in Chapter 5). The parameters k1 and k2 are defined in a table at the end.

Solution:

Finding the class of a laser is done by starting at Table I and proceeding to higher-numbered tables until the laser meets the requirements. Note that the exposure levels in the various 21 CFR 1040.10 tables are in different units. Table 1 is in radiant energy. The limit is 2 x 10-7 J, which is much less than EARL’s pulse energy, so EARL is not Class I.

Table II-A is in radiant power with a limit of 3.9 x 10-6 W, which is much lower than EARL’s power, so EARL is not Class IIa.

Table II has a maximum radiant power of 1 mW, and EARL is higher and so not Class II.

Table III-A, for exposures > 3.8 x 10-4 s, has a limit of 5 mW, so again EARL is higher and is not Class IIIa.

Table III-B is in units of radiant exposure and its upper limit is 10 J/cm2. EARL’s fluence is lower than this limit, so EARL’s transmitter is Class IIIb.

* + 1. An industrial laser produces a beam with wavelength 0.532 µm and BPP 2.00. If the laser beam is focused into an f/2 cone of light, find the spot diameter  at the beam waist.

Solution:

This problem is misstated: it should say that  is 2.00, not the BPP.

Equation (5.6) is  , so rearranging, . The diameter  is twice  , so

 .

All the parameters are given in the problem statement except the half angle, . Recall that in an f/2 cone of light, the distance from the lens to the beam waist is twice the beam diameter at the lens. The half angle is found from the beam radius at the lens, which is ¼ the beam waist distance. The angle is therefore given by  radians. Using all the numbers in the equation for  yields a spot diameter of 2.76 µm.

**Chapter 6**

* + 1. The paraxial approximation.
			1. How many milliradians are in one degree?
			2. Fill in the table below, to eight digits beyond the decimal point. How good is the paraxial approximation for these angles?

Solutions:

1. Angle is defined as arc length over radius. The circumference of a circle is equal to 2π times its radius, therefore a circle contains 2π radians. A circle also contains 360 degrees, by definition, so 1 degree = 2π/360 = 17.45 mrad.

1. Even for angles as large as 10 degrees, the values of sin(θ) and tan(θ) are within one percent of θ. For one degree, the values agree to 0.01 percent. For smaller angles, the agreement is even better.

|  |  |  |  |
| --- | --- | --- | --- |
| Angle  | (radians) | sin() | tan() |
| 10 degrees | 0.17453293 | 0.17364818 | 0.17632698 |
| 1 degree | 0.01745329 | 0.01745241 | 0.01745506 |
| 10 mrad | 0.01000000 | 0.00999983 | 0.01000033 |
| 1 mrad | 0.00100000 | 0.00100000 | 0.00100000 |
| 100 µrad | 0.00010000 | 0.00010000 | 0.00010000 |

* + 1. Lidar receiver parameters.
1. EARL has an *f*/4.21 receiver based on a parabolic mirror with a diameter of 0.61 m. What is the defocus distance for backscattered light from a range of 500 m?
2. What is the receiver cone angle ?
3. If the field stop diameter is 3.0 mm, what is the FOV?

Solutions:

1. Equation (6.14) gives the defocus distance as . Recalling that the receiver f-number is the focal length divided by the mirror diameter,  is equal to 4.21 x 0.61 = 2.57 m. The defocus distance is therefore

= 0.013 m = 13 mm.

1. Equation (6.17) gives the approximate receiver cone angle as . The value is therefore  = 0.237 radians
2. As mentioned in Section 6.1.1, . The FOV angle is therefore

= 0.00117 radians = 1.17 mrad.

* + 1. Show that Eq. (6.23) is consistent with conservation of the product of image size and cone angle, as discussed in Section 6.3.2, within the limits of the approximations used to derive the equation.

Solution:

Referring to a diagram like Figure 6.22 with an object and an image, the conservation law is stated as . In lidar, the object is the scattering volume. Its diameter at the range of full crossover is given by , and its corresponding plane angle is .

The image has a plane angle of . At the crossover range, the receiver light cone just fits through the field stop, and the image size must be calculated using the defocus distance, as shown in Figure 6.27. The image diameter is the field stop diameter minus the product of the cone angle and the defocus distance, or . The conservation law is then

.

Remembering that , after some algebra this reduces to

 ,

which is Eq. (6.23), so this equation for the crossover range is consistent with the conserved optical quantity, within the limits of the approximations that were used in Chapter 6.

* + 1. Derive Equation (6.30) for biaxial crossover plots.

Solution:

Referring to Figure 6.25, increasing the value of results in an angular shift of the cone of light in the receiver of , whereas increasing the distance  results in an angular shift to the right of . One-half of these values is simply added to the terms on the right-hand side of Eq. (6.28), resulting in Eq. (6.30) (remember that the plus and minus values of  each describe one-half of the cone of received light).

* + 1. Derive Equation (6.32) for the incorrect crossover range.

Solution:

In Figure 6.35, crossover is said to be complete at the range where the cone representing the transmitted beam intersects the FOV cone. Imagine a horizontal line emerging from the top edge of the transmitter. The top edge of the transmitted beam is at a distance equal to above that line. Let distance be negative above the line and positive below. Assuming that the optical axes of the transmitter and receiver are parallel, the top edge of the FOV cone intersects the transmitted cone at the range where

,

which reduces to the result

.

This result is incorrect, as can be seen by comparing it to Eq. 6.31 with  and  set to zero. The problem with Figure 6.35 is that it is not an optical diagram – the FOV cannot be represented by a cone extending from the receiver. Note also that for a co-axial lidar, a drawing like Figure 6.35 would imply a crossover range of zero.

**Chapter 7**

7.5.1 Consider a thin BK7 mirror with a diameter of 0.61 m and a thickness of 25 mm. If it is supported only on its edge, as in Figure 7.7, how much does the mirror sag in the middle, and how many wavelengths of light at 532 nm is the sag? Assume Poisson’s ratio ν is 0.2.

Solution:

The sag in a mirror mounted as in Figure 7.7 is calculated according to Eq. (7.7), which is

.

From Table 1, the density of BK7 glass is 2530 kg/m3 and the elastic modulus is 82 GPa.

First find the weight, which is density times volume times acceleration due to gravity:

= (2530 kg/m3) x (π(0.61m2) / 4)x (0.025) x (9.8 ms-2) = 181 N. The parameter m = 1/ν = 5.

Putting numbers into the formula,

ΔY = (-3 x 181 x 4 x 26 x 0.612) / [64 x 25 x π x 8.2 x 1010 x (0.025)3] = -3.3 x 10-6 m = -3.3 µm.

The number of wavelengths of 532 nm light is 3300/532 = 6.2.

* + 1. If a mirror has a focal length of 2.00 m, how much will the focal length change with a 10 °C change in temperature, when
1. the mirror is made of BK7, and
2. the mirror is ULE?

Solution:

As pointed out in Section 6.3.3, if a drawing of a lidar optical system is scaled to a larger size, all the distances will be increased by a common scale factor and all the angles will be unchanged. A cross section of a solid object expands like a photographic enlargement as it is heated, so the focal length scales like a dimension of the mirror. From Table 7.1, the CTE = 7.1 x 10-6/°C for BK7 and the CTE = 0.03 x 10-6/°C for ULE. Therefore, the focal length changes are

1. BK7: 7.1 x 10-6/°C x 10 °C x 2 m = 142 x 10-6 m = 142 µm.
2. ULE: 0.03 x 10-6/°C x 10 °C x 2 m = 0.6 x 10-6 m = 0.6 µm = 600 nm.

* + 1. A lidar engineer has a pendulum clock that keeps accurate time during winter but loses about one minute per month during summer. He suggests that the reason is that his home is warmer in the summer, so the pendulum is longer due to thermal expansion. Is this reasonable? The pendulum is steel, and the indoor temperatures are 20 °C in winter and 24 °C in summer. The period of a pendulum is given by the relation , where  is the length and  is the acceleration due to gravity.

Solution:

Thermal expansion will increase  and therefore the period, making the clock run slower. Quantitatively, Eq. (7.6) describes a fractional change. There are 60 x 24 x 30 = 43,200 minutes in a 30-day month, so the fractional change in the period is 1/(43,200) = 2.3 x 10-5. Table 7.1 lists the CTE for steel as 12.1 x 10-6, and the temperature difference ΔT is 4 °C.

Recalling that if , then , so the fractional change in the period due to thermal expansion is ½(4 x 1.21 x 10-5) = 2.4 x 10-5, which is about the same as the observed clock rate error, so the suggested explanation is reasonable.

7.5.4 What temperature difference between two opposite sides of the EARL structure illustrated in Figure 7.22 would cause misalignment of the long-range receiver channel by changing the transmitted beam direction? Assume that the steel framework is 2.5 m tall and 0.7 m wide. As stated in Chapter 6, EARL’s misalignment tolerance is 137 µrad.

Solution:

The tilt angle is equal to the difference in the side lengths divided by the width, or /0.7, so if > 0.7 m x 137 µrad = 96 µm, EARL will be misaligned. From Equation (7.5),  or = (96 x 10-6 m) / [(12.1 x 10-6 / °C) x 2.5 m] = 3.2 °C. EARL is actually about twice this sensitive to temperature differences, because the secondary mirror is also tilted by expansion of one side of the structure, moving the received cone of light in the same direction as the transmitter tilt.

* + 1. If a 1-cm thick mirror is athermalized by the mounting method shown in Fig. 7.18 and the steel tube is 2 cm long, what is the length of the aluminum tube?

Solution:

From Table, 1, the CTEs are 7.1 ppm for glass, 12.1 ppm for steel, and 23.0 ppm for aluminum. Letting  be the distance from the attachment point to the mirror surface and using Eq. (7.6),

 = [(12.1 x 2) – (7.1 x 1) – (23.0 x L)] where L is the aluminum tube length. Setting equal to zero and solving for L yields the length of the aluminum tube as 0.74 cm.

**Chapter 8**

8.9.1 What is the maximum value of for a detector with unity gain? Derive an equation for  for a detector with as a function of wavelength in nm.

Solution:

From Eq. (8.9), . Using  and , we have

 

Recalling  ≅ 2.0 x 10-25 and  = 1.602 x 10-19 C and using 1 nm = 1 x 10-9 m yields

.

* + 1. Using Eqs. (8.10) and (8.12), show that the signal-limited power SNR is just the mean number of photoelectrons, as expected from Poisson statistics.

Solution:

In signal-limited detection, the only noise source is shot noise associated with the signal current, so from Eq. (8.12), . Equation (8.10) states that  , so

 ,

where the definition of  in Table 8.1 has been used. Equation (8.5) implies that the quantity  is the mean number of signal photoelectrons  generated in a time . Power  is therefore the square of the photon described in Chapter 2, as expected.

* + 1. Show that the SNR in Eq. (8.18) is the square of Eq. (2.8).

Solution:

Equation (8.18) is

.

Recalling that  where  is the sampling interval and that the conversion from power to photoelectrons per sampling interval is , multiply the right side of the equation by the identity  to obtain the equation

,

which is the square of Eq. (2.8) when the dark count is zero.

* + 1. A certain lidar has a wavelength of 532 nm, a bandwidth of 10 MHz, a detector QE of 0.5, and a daytime background power on the detector of 2.6 x 10-10 W. Considering only statistical noise, is the NEP signal-limited or background-limited?

Solution:

Remembering that  and using the parameters listed above in Eqs. (8.19), (8.20), and (8.21), we find these values:

 NEP = 7.1 x 10-11 W

 NEPSL = 1.5 x 10-11 W

 NEPBL = 6.3 x 10-11 W

The NEP is therefore almost background-limited, because NEPBL is close to NEP.

* + 1. Derive Eq. (8.25).

Solution:

For power SNR, equation (8.11) is

 .

The signal current is given by Eq. (8.15) as

 ,

and the noise currents are given by



because the conversion factor from power to current is  , as shown in Eq. (8.22). Making these substitutions and after some algebra,

 .

The first term in the denominator is shown in Eq. (8.20) to be  and first part of the second term is shown in Eq. (8.21) to be  . The remaining noise power terms follow in the same way as the derivation of Eq. (8.21), yielding

 ,

which is Eq. (8.25).

* + 1. Derive coherent detection Eq. (8.32) from Eq. (8.30).

Solution:

Equation (8.30) is

 .

Squaring the sum of two terms yields

.

Using the trigonometric identity  ,

.

Using the trigonometric identity  and dropping all optical frequency terms, we have

 ,

which is Eq. (8.32).

* + 1. Show that two rays of light at 1 µm wavelength travelling through two different eddies at 300 K and 299 K and a pressure of 1000 mbar will be completely out of phase after propagating only ½ m.

Solution:

Using Eq. (8.41), the refractive indices of air at the two temperatures are = 1 + 263 ppm and = 1 + 264 ppm. The number of waves  in a path length  is , so = (0.5 x 106) x [1/(1.000263) – 1/(1.000264)].

Remembering that (1+x)-1 ~ 1-x for small x, the difference in number of waves is 

(0.5 x 106) x [(264-263) x 10-6] = 0.5 wave, which is completely out of phase.

8.9.8 Show that a 1 % change in the 800 V bias voltage on a Hamamatsu R7400U PMT will cause a gain change of about 7 %. The R7400U has 8 dynodes and model parameters are a = 0.1 and α = 0.888.

Solution:

Equation (8.44) is

.

Using the parameters listed above in Eq. (8.44) with  as both 800 and 808 V results in gains of 6.99 x 105 and 7.50 x 105, respectively. Their ratio is 1.073, meaning that the gain increased by 7.3 %.

* + 1. Derive Eq. (8.59) for APDs.

Solution:

By analogy with the way Eq. (8.21) was derived from Eq. (8.18),

 ,

where the spectral NEP must be multiplied by the square root of bandwidth to be in watts. Squaring both sides, cancelling terms, and re-arranging yields

.

Multiplying both sides by ,

.

The term on the right is the optical power on the detector that corresponds to  multiplied by the factor that coverts optical power on the detector to photoelectrons per second, as shown in Eq. (8.5). Multiplying by the sampling interval (the time during which the signal and background photons were detected) and dividing by  yields the number of equivalent received photons , which is multiplied by the factor as it is used in the denominator in Eq. (8.58). Dividing that factor out yields the correct definition

 ,

which is Eq. (8.59). The reason for dividing out  is that  includes all detector and amplifier noise and it is not subject to the excess noise factor. The excess noise factor  only pertains to quantities that experience the detector gain mechanism (the signal and background powers).

As an alternative solution, Eq. (8.59) can be derived from Eq. (8.25), which is for power SNR, by

substituting Eq. (8.20) and Eq. (8.21) along with  for the NEP terms in the denominator. Remember that the right side of Eq, (8.25) is multiplied by for an APD, that , and that the number of photoelectrons is the power multiplied by , as in the derivation given above.

**Chapter 9**

9.5.1 The DIAL technique requires very high SNR. An ozone DIAL developer requires a voltage SNR of 10,000 and plans to use a 10-bit digitizer. If the digitizer is ideal, will it provide sufficient SNR for a single laser shot? If not, how many pulses must be averaged to provide it?

Solution:

The required SNR in dB = 20 log (104) = 80 dB.

Table 9.1 shows that an ideal 10-bit digitizer has 1,024 levels and it provides an SNR of 62 dB, so no, it will not provide sufficient SNR with a single laser shot.

For multi-pulse averaging, Eq. (9.5) is

.

Solving this equation for the closest integer  yields 64 laser shots. For non-ideal digitizers, the improvement by a factor of in the argument of the logarithm should be applied to the actual SNR, which is generally taken to be the specified SNDR.

* + 1. Derive the code probabilities for the code density test, as given by Eq. (9.13).

Solution:

Equation (9.12), which is based on Figure 9.9, is

,

Where is the probability that a sample lies in the interval . In discrete variables, the voltage interval is one quantum voltage , so the code probabilities become

,

where the range of the code index  is from 1 to 2N (the code values are -1).

Using , as defined after Eq. (9.1), this becomes

,

but we must shift the index by half of full scale when zero volts is at the center of the range, as shown in Figure 9.9. One-half of  is . Making this adjustment yields Eq. (9.13):

.

The value of  is zero for voltages greater than  and less than .

* + 1. If the histogram test is employed on a 12-bit digitizer, how many samples are required for an accuracy of 0.1 LSB with a confidence of 99 %?

Solution:

Equation (9.14) states that

,

where  is the number of samples,  is the number of bits,  is the confidence level expressed as the number of standard deviations from the mean, and  is the desired accuracy. For a confidence of 99%,  is 2.58, as mentioned in Chapter 2. Putting in the numbers shows that  is about 4.3 x 106. This is an example of the requirement for large samples in statistical tests.  is especially large because a large sample is required to determine each code probability, and in this case, there are 4,096 codes.

* + 1. Is Johnson noise a consistent explanation for EARL’s digitizer noise measurements?
1. Calculate EARL’s noise floor in dB due to Johnson noise. The bandwidth is 5 MHz. Assume that room temperature is 290 K, and compare the noise power to the maximum signal power, assuming a sine wave input voltage range of ± 1 V. Recall that power is  ; in this case  is the r.m.s. signal voltage and  is the input impedance of 50 ohms.

1. What noise floor (in dB) is represented by EARL’s measured standard deviation of 0.17 digitizer levels? The digitizer has 16 bits.

Solution:

a) The noise power is  , where Boltzmann’s constant is 1.38 x 10-23 J/K,  = 290 K, and  = 5 x 106 s-1. Using these numbers, the calculated noise power is 2.00 x 10-14 W.

The r.m.s. value of a sine wave is  times the peak voltage (1 volt in this case), so the signal power is 0.5/50 = 0.01 W.

The noise floor is 10log(2.00 x 10-14/0.01) = -117 dB, which is consistent with the typical value illustrated in Figure 9.7.

b) The measured EARL SNR with inputs terminated is given by

 = -112 dB, which is roughly consistent with the value found in part a).

* + 1. A lidar data system starts acquiring data when a laser pulse is transmitted, and it must stop data acquisition before the next pulse is transmitted. The lidar’s maximum range is therefore limited by its PRF.

a) Derive an expression for the maximum lidar range.

b) The GTRI lidar at the Starfire Optical Range had a PRF of 5 kHz. What was its maximum range? The digitization rate was 10 MHz. What was the maximum number of range bins?

Solutions:

1. Lidar range is  and the time  is 1/PRF, so the expression is .
2. The maximum range was (3 x 108) / (2 x 5 x 103) = 30 km. A 10 MHz digitization rate corresponds to 15-m range bins, so the maximum number of bins was 30,000/15 = 2000.

**Chapter 10**

10.5.1 The lidar data shown in Figure 9.14 were recorded at a sampling frequency of 40 MHz and smoothed with a 41-bin rectangular filter.

1. By what factor was the r.m.s. variation in the data reduced?

* + - 1. What is the vertical resolution of the filtered data, by the NDACC definitions? Hint – for the second definition, implement Eq. (10.6) in a spreadsheet or math application to find the cutoff frequency.

Solutions:

1. All 41 of the filter weights have the value unity, so the normalizing factor is 1/41 and the coefficients all have the value 1/41. Using Eq. (10.3),



The r.m.s factor is the square root of the variance factor, or 0.156.

1. The range bin width is  = (3 x 108) / [(40 x 106) x (2)] = 3.75 m. Using the definition that resolution is the FWHM of the filter,  = 41 bins so the resolution is 154 m. For the second definition, using Eq. (10.6) with trial and error, the cutoff frequency  is determined to be 0.0147, so  34 bins, or 128 m.

* + 1. By inspecting Figures 10.4 and 10.6, find the resolution of the Hann and Hamming 25-point filters according to the first NDACC definition of a FIR filter’s resolution. What rectangular filter width would yield the same resolution?

Solution

The half-maximum points are at ± 6 bins for both Hann and Hamming, so their resolution is 13 bins. For the rectangular filter, the FWHM is the full filter width, so a 13-bin rectangular filter would have the same resolution, by the first NDACC definition. As noted before Figure 10.9, a 9-point rectangular filter’s resolution is closer to the Hann and Hamming resolutions by the second definition. This fact can be verified by using the same spreadsheet or math application used in the previous problem.

* + 1. The atmospheric and aerosol depolarization ratios are not the same, because the atmospheric value depends on the aerosol concentration.

1. Show that Eq. (10.15) yields the aerosol depolarization ratio.
2. Show that when  and , .

Solution:

1. Equation (10.15) is



and the scattering ratios are defined in Eqs. (13) and (14) as

 and .

Putting the scattering ratio definitions into Eq. (10.15) and expanding the definition of  in Eq. (10.12) into the two atmospheric components yields

,

which, after some algebra, reduces to

 ,

which is the aerosol depolarization ratio.

1. Starting from

,

when  and , terms of the type  become . The factor in square brackets then becomes unity and .

**Chapter 11**

11.8.1 A classic example of a transcendental equation is . Solve this equation with an error of ± 1% or less by using the Newton-Raphson method, with an initial guess that  = 0.6 radian.

Solution:

In this case, the equation to be solved is , so  and the algorithm in Eq. (11.1) is

 .

Implementing this algorithm in a spreadsheet results in the table below, which demonstrates rapid convergence and hence computational efficiency, even when the initial guess is not very close to the correct solution. The error is much less than 1% after three iterations.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Iteration |  |  |  -  | % difference |
| 1 | 0.6 | 0.825336 | -0.22534 | -37.5559 |
| 2 | 0.744017 | 0.735754 | 0.008264 | 1.110665 |
| 3 | 0.73909 | 0.739082 | 8.94 x 10-06 | 0.00121 |

11.8.2. One of the transmittance peaks in Figure 11.4 is at 500 nm. Where are the others?

Solution:

Figure 11.4 is a plot of Eq. (11.4), which is

 .

The transmittance is unity when the sine function is zero, which occurs when the argument is 0, π, 2π, …, so peaks occur when the argument  is equal to  where  is integer, or when . The wavelength is therefore found from . As stated in the text,  is 1.5 and  is 100 µm, so  is 300 µm. When the peak occurred at 500 nm (0.5 µm),  was therefore 600, so that . The other peaks in Figure 11.4 occur at higher  values as shown in the table below.

|  |  |
| --- | --- |
| *m* | λ (nm) |
| 600 | 500.0 |
| 601 | 499.2 |
| 602 | 498.3 |
| 603 | 497.5 |
| 604 | 496.7 |
| 605 | 495.9 |

11.8.3 Two differential absorption techniques are used in laser remote sensing, DIAL and IPDA. Elucidate their basic data analysis relations by

1. deriving Eq. (11.16) from Eq. (11.15); and
2. deriving Eq. (11.18) from Eq. (11.17).

Solutions:

1. Equation (11.15) is

.

The numerical derivative with respect to  of a function  is . Therefore,

,

and by using properties of the logarithm function and rearranging,

,

which is Eq. (11.16).

1. Equation (11.17) is

.

Writing this equation for the “off” and “on” wavelengths and finding their ratio yields

 ,

where all the constants were assumed to be the same in both signals. Rearranging after finding the logarithm yields Eq. (11.18), which is

,

where .

11.8.4 Tropospheric ozone lidars often operate at 288.9 nm and 299.1 nm because these

wavelengths can be generated in SRS cells with D2 and H2 pumped by the fourth harmonic of the Nd:YAG laser at 266 nm. GTRI’s ozone lidar known as NEXLASER was designed for a vertical resolution of 300 m and an ozone uncertainty of 10 ppbv. What was the maximum relative error for the term in brackets in Eq. (11.16)? The ozone absorption cross sections are 1.59 x 10-18 cm2 at 289 nm and 4.55 x 10-19 cm2 at 299 nm, so is 1.14 x 10-18 cm2.

Solution:

Recalling that, if  , then ,

,

and solving for the desired quantity yields

,

where the ozone uncertainty is expressed as number density and the term in square brackets is the total fractional uncertainty from the four lidar signal measurements that go into it.

After converting to m2, the multiplier  is (2) x (1.14 x 10-22) x (300) = 6.84 x 10-20 m3/molecule. The ozone uncertainty of 10 ppbv is 10-8 times the number density of air, or (10-8) x (2.55 x 1025) = 2.55 x 1017 molecules/m3.

The relative uncertainty  is then (6.84 x 10-20) x (2.55 x 1017) = 1.74 x 10-2. The SNR is the inverse of the relative uncertainty, or 57.3, but it represents four separate measurements. If the errors are uncorrelated, the required SNR of each measurement is twice that value, or 115. Such SNRs were reached at altitudes up to 3 km with a few minutes of averaging at 10 Hz PRF and pulse energies of a few mJ, by filtering data in 15-m range bins to 300 m resolution.

11.8.5 What is the bandwidth of MERLIN’s detector-TIA combination? What is its NEP in watts? The detector’s impulse response time is 111 ns and the spectral NEP is 43 fWHz-1/2.

Solution:

Using the definition of bandwidth in Table 8.1 with 111 ns as  , B = 1 / [2 x (111 x 10-9)] = 4.51 MHz. Multiplying the spectral NEP by the square root of B then yields the NEP in watts as (43 x 10-15) x (4.51 x 106)1/2 = 9.13 x 10-11 W.

11.8.6 Derive Eq. (11.19) for water vapor Raman lidar.

Solution:

The Raman lidar equations for the two gases are

 and

.

Finding their ratio and lumping all constant terms into one big constant  yields

  .

Solving for the desired mixing ratio,

 ,

which is Eq. (11.19).

11.8.7 The lidars known as PollyXT have photon counting data systems with paralyzable counters. The maximum photon count rate is 60 Mcps, which is said to result in correction factors less than 1.3. Assume the dead time  is 4 ns.

a) Are these numbers consistent with Eq. (9.19)?

1. Are such large count rates consistent with the design parameters? Use Eq. (2.14) with the parameters for the 532 nm Near receiver listed in the table below, which were taken from the literature. Assume a sky radiance of 40 W/m2-µm-sr, a receiver efficiency kR of 0.3 and a detector quantum efficiency η of 0.2 to calculate the daytime background count rate.

PollyXT 532 Near receiver design parameters

|  |  |  |
| --- | --- | --- |
| Parameter | Value | Units |
| Diameter | 0.05 | m |
| FOV | 2.2 | mrad |
| Optical bandwidth | 1.0 | nm |

Solutions:

1. Equation (9.19) is

  .

Using the given numbers yields an observed count rate of 47.2 Mcps, so the correction factor is 1.27, which is less than 1.3, and the stated numbers are consistent with the equation.

1. Equation (2.14) is . The first two terms were given. The receiver area is calculated from the diameter as πD2/4 = 1.96 x 10-3 m2, the receiver solid angle is (π/4) x (FOV2) = 3.80 x 10-6 sr, the optical bandwidth is 10-3 µm, and  is 2.66 x 1018 photons/W. Note that  is 1 s because we are calculating cps. Multiplying these terms together with  yields the number of background photocounts per second as 47.6 Mcps. This high count rate is consistent with the stated count rates.

11.8.8 Derive Eq. (11.21) from Eq. (11.20) for the HSRL.

Solution:

Equation (11.20) is

.

In the Rayleigh channel, there is no aerosol backscatter but there is aerosol extinction, and Rayleigh backscatter is proportional to the molecular number density, so Eq. (11.20) becomes

,

where  is the Rayleigh backscatter cross section (m2/sr) per molecule and  is the number density of the molecules. Solving for the aerosol extinction profile yields

,

which can be expanded as

,

but the range derivative of  is zero. Rearranging gives the final result as

,

which is Eq. (11.21) for the HSRL.