Chapter 11

```
> with(plots):
Warning, the name changecoords has been redefined

> with(DEtools):
> with(linalg):
Warning, the name adjoint has been redefined

Warning, the protected names norm and trace have been redefined and
unprotected

> with(plottools):
Warning, the name translate has been redefined
```

Question 1

(i)

```
> sol:=dsolve({diff(y(t),t)=beta*f(u)-beta*(1-xi)*y(t),y(0)=y0}
  ,y(t));
```

$$sol := y(t) = -\frac{f(u)}{-1+\xi} + \frac{e^{(\beta(-1+\xi)t)}(f(u) - y0 + y0\,\xi)}{-1+\xi}$$

```
> solve(0=beta*f(u)-beta*(1-xi)*ye,ye);
```

$$-\frac{f(u)}{-1+\xi}$$

```
> sol2:=collect(sol,y0);
```

$$sol2 := y(t) = e^{(\beta(-1+\xi)t)}\,y0 - \frac{f(u)}{-1+\xi} + \frac{e^{(\beta(-1+\xi)t)}\,f(u)}{-1+\xi}$$

```
> collect(sol2,exp(beta*(-1+xi)*t));
```

$$y(t) = \left(y0 + \frac{f(u)}{-1+\xi}\right)e^{(\beta(-1+\xi)t)} - \frac{f(u)}{-1+\xi}$$

(ii)

Note that this expression is,

$$y(t) = ye + e^{(-\beta(1-\xi))}(y0 - ye)$$

where

$$ye = \frac{f(u)}{1-\xi}$$

and is asymptotically stable so long as $\beta(1-\xi) < 1$.

Question 2

For the system

$$y_{t-1} = 9 + .2\,(5 - p_{t-1})$$

$$\pi_t = \frac{\alpha\,(y_{t-1} - 6)}{6}$$

we can solve for p_t.

```
> expand(solve((p-p1)/p1=alpha*((9+(1/5)*(5-p1)-6)/6),p));
```

$$p1+\frac{2}{3}\alpha\, p1-\frac{1}{30}\alpha\, p1^2$$

We can solve for p as follows,

```
> solve(p=p+(2/3)*alpha*p-(1/30)*alpha*p^2,p);
```

$$0, 20$$

which is independent of α.

The difference equation in terms of p and α is given by,

$$p_t = \mathrm{f}(p_{t-1}) = \left(1+\frac{2\,\alpha}{3}\right)p_{t-1}-\frac{\alpha\, {p_{t-1}}^2}{30}$$

whose stability we can investigate by considering,

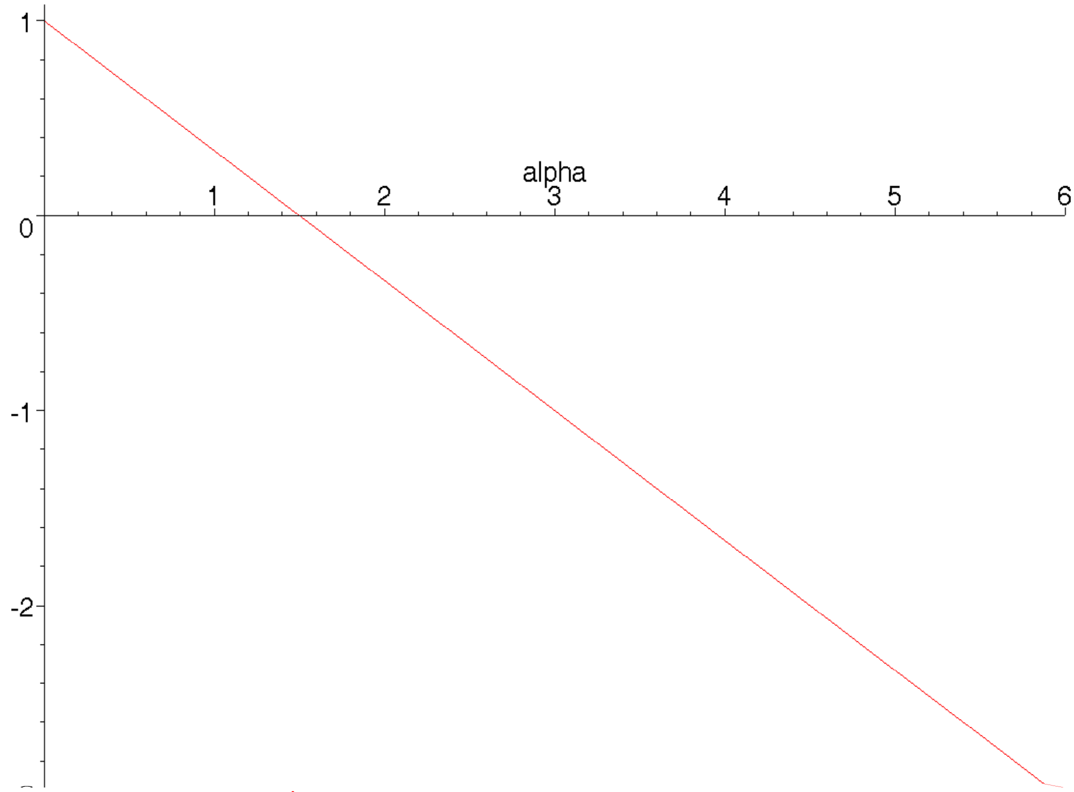
```
> slope:=diff(p1+2/3*alpha*p1-1/30*alpha*p1^2,p1);
```

$$slope := 1+\frac{2}{3}\alpha-\frac{1}{15}\alpha\, p1$$

```
> eqslope:=subs(p1=20,slope);
```

$$eqslope := 1-\frac{2}{3}\alpha$$

```
> plot(1-(2*alpha/3),alpha=0..6);
```




```
> solve(1-(2*alpha/3)=0,alpha);
```

$$\frac{3}{2}$$

```
> solve(1-(2*alpha/3)=-1,alpha);
```

$$3$$

```
> solve(1-(2*alpha/3)=1,alpha);
```

$$0$$

Thus, the system is asymptotically stable if $0 < \alpha < 3$. The system converges steadily on $pe = 20$, if $0 < \alpha < 3/2$ an oscillates to equilibrium if $3/2 < \alpha < 3$. Beyond $\alpha = 3$ the system gives rise to an explosive oscillation. To see these alternatives let

$\alpha = \{1.2, 2, 3.5\}$

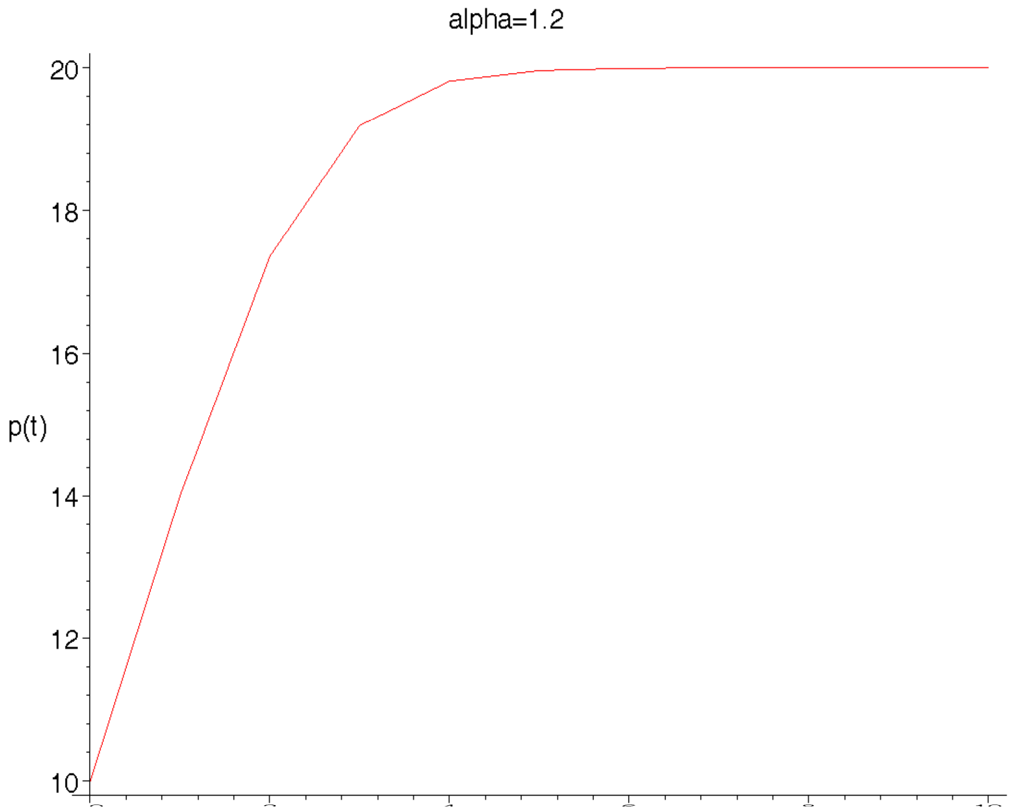
```
> f:='f':
> f:=p->p+(2*p*alpha/3)-(alpha*p^2/30);
```

$$f := p \to p + \frac{2}{3} p\,\alpha - \frac{1}{30}\alpha\,p^2$$

```
> f1:=p->subs(alpha=1.2,f(p));
```

$$f1 := p \to \text{subs}(\alpha = 1.2, \text{f}(p))$$

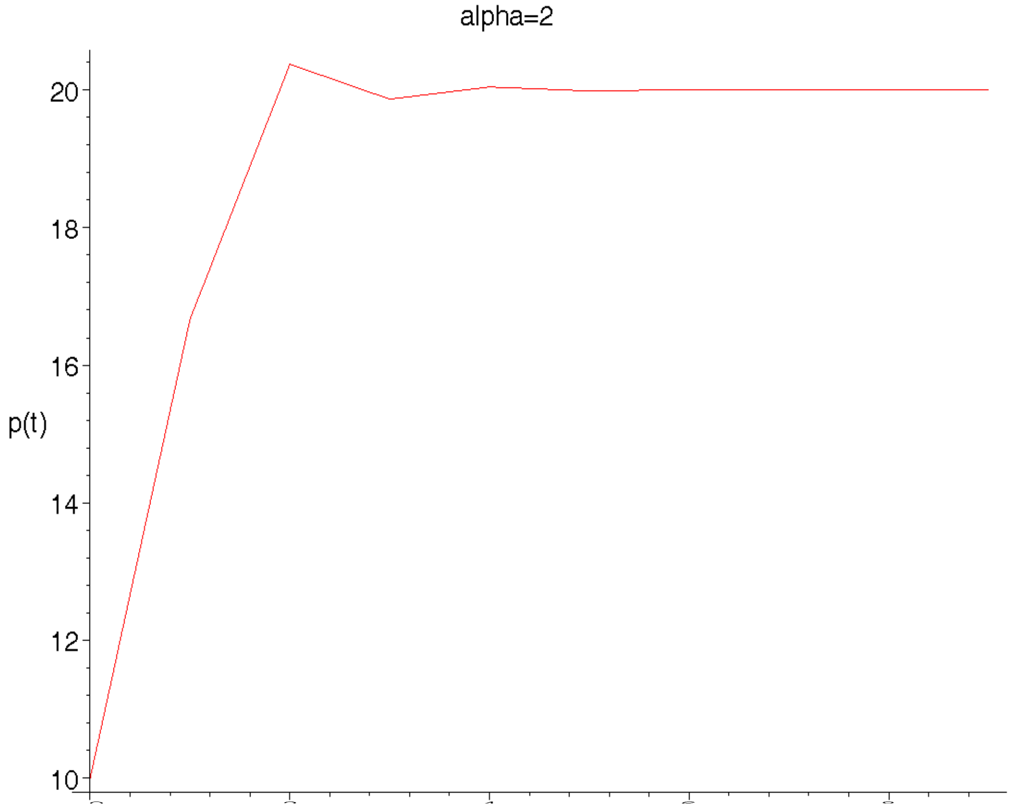
```
> path1:=seq([k,(f1@@k)(10)],k=0..10):
> plot([path1],title="alpha=1.2",labels=["t","p(t)"]);
```




```
> f2:=p->subs(alpha=2,f(p));
```

$$f2 := p \to \text{subs}(\alpha = 2, \text{f}(p))$$

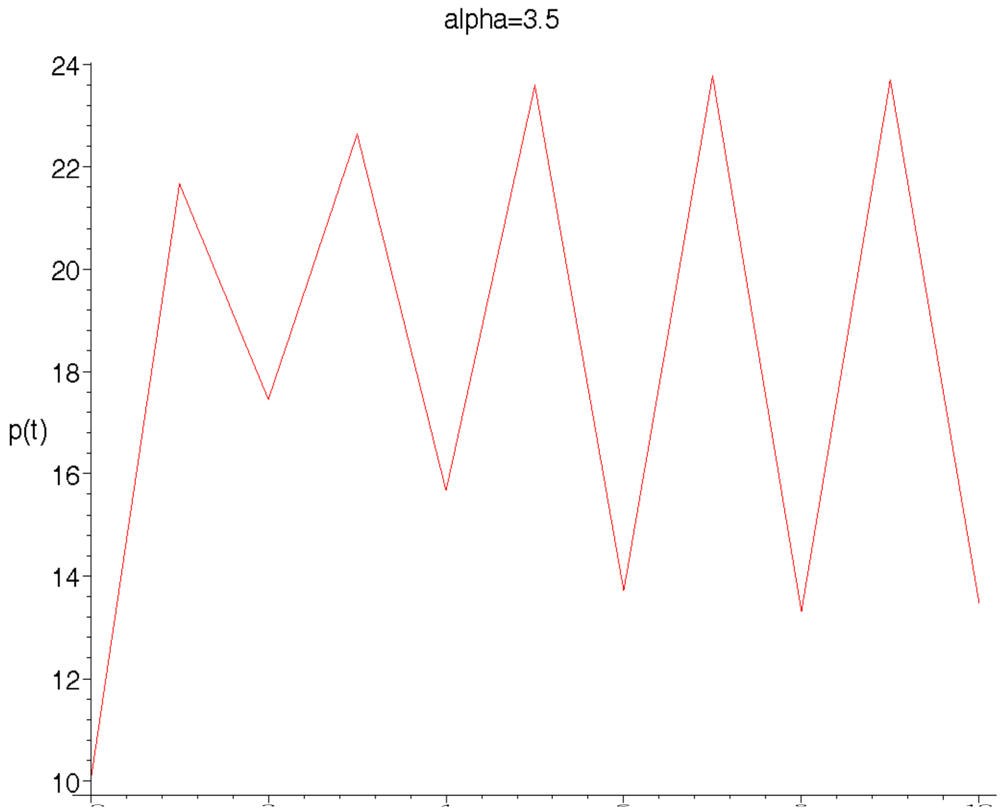
```
> path2:=seq([k,(f2@@k)(10)],k=0..10):
> plot([path2],title="alpha=2",labels=["t","p(t)"]);
```




```
> f3:=p->subs(alpha=3.5,f(p));
```

$$f3 := p \rightarrow \text{subs}(\alpha = 3.5, \text{f}(p))$$

```
> path3:=seq([k,(f3@@k)(10)],k=0..10):
>
> plot([path3],title="alpha=3.5",labels=["t","p(t)"]);
```




```
>
```

Question 3

Let

```
> A:=matrix([[-1.85, -10], [0.3, 0]]);
>
```

$$A := \begin{bmatrix} -1.85 & -10 \\ .3 & 0 \end{bmatrix}$$

```
> sol:=eigenvalues(A);
```

$$sol := -.9250000000 + 1.464368465\, I, -.9250000000 - 1.464368465\, I$$

```
> Re(sol[1]);
```

$$-.9250000000$$

```
> Im(sol[1]);
```

$$1.464368465$$

```
> Re(sol[2]);
```

$$-.9250000000$$

```
> Im(sol[2]);
```

$$-1.464368465$$

Clear that the real part of the complex conjugate roots is $\alpha = -.925$.

Question 4

```
> pe1:=solve(pe(t+1)=pe(t)+(1-lambda)*(p(t)-pe(t)),pe(t+1));
```

$$pe1 := \mathrm{p}(t) - \lambda\, \mathrm{p}(t) + \lambda\, \mathrm{pe}(t)$$

```
> collect(pe1,p(t));
```

$$(1 - \lambda)\, \mathrm{p}(t) + \lambda\, \mathrm{pe}(t)$$

```
> RHS:=array(0..3);
```

$$RHS := \mathrm{array}(0 .. 3, [\ ])$$

```
> for i from 0 to 3 do RHS[i]:=(1-lambda)*p(t-i)+lambda*pe(t-i)
  od;
>
```

$$RHS_0 := (1 - \lambda)\, \mathrm{p}(t) + \lambda\, \mathrm{pe}(t)$$

$$RHS_1 := (1 - \lambda)\, \mathrm{p}(t-1) + \lambda\, \mathrm{pe}(t-1)$$

$$RHS_2 := (1 - \lambda)\, \mathrm{p}(t-2) + \lambda\, \mathrm{pe}(t-2)$$

$$RHS_3 := (1 - \lambda)\, \mathrm{p}(t-3) + \lambda\, \mathrm{pe}(t-3)$$

```
> array([seq([RHS[i]],i=0..3)]);
```

$$\begin{bmatrix} (1 - \lambda)\, \mathrm{p}(t) + \lambda\, \mathrm{pe}(t) \\ (1 - \lambda)\, \mathrm{p}(t-1) + \lambda\, \mathrm{pe}(t-1) \\ (1 - \lambda)\, \mathrm{p}(t-2) + \lambda\, \mathrm{pe}(t-2) \\ (1 - \lambda)\, \mathrm{p}(t-3) + \lambda\, \mathrm{pe}(t-3) \end{bmatrix}$$

```
> RHS[0];
```

$$(1 - \lambda)\, \mathrm{p}(t) + \lambda\, \mathrm{pe}(t)$$

```
> RHS[1];
```

$$(1 - \lambda)\, \mathrm{p}(t-1) + \lambda\, \mathrm{pe}(t-1)$$

```
> simplify((1-lambda)*p(t)+lambda*((1-lambda)*p(t-1)+lambda*pe(t-1)));
```

$$p(t) - \lambda\, p(t) + p(t-1)\,\lambda - p(t-1)\,\lambda^2 + \lambda^2\, pe(t-1)$$

```
> collect(p(t)-lambda*p(t)+p(t-1)*lambda-p(t-1)*lambda^2+lambda^2*pe(t-1),{p(t),p(t-1)});
```

$$(1-\lambda)\, p(t) + (\lambda-\lambda^2)\, p(t-1) + \lambda^2\, pe(t-1)$$

```
> collect((1-lambda)*p(t)+(lambda-lambda^2)*p(t-1)+lambda^2*((1-lambda)*p(t-2)+lambda*pe(t-2)),{p(t),p(t-1),p(t-2)});
```

$$(1-\lambda)\, p(t) + (\lambda-\lambda^2)\, p(t-1) + \lambda^2 (1-\lambda)\, p(t-2) + \lambda^3\, pe(t-2)$$

```
> collect((1-lambda)*p(t)+(lambda-lambda^2)*p(t-1)+lambda^2*(1-lambda)*p(t-2)+lambda^3*((1-lambda)*p(t-3)+lambda*pe(t-3)),{p(t),p(t-1),p(t-2),p(t-3)});
```

$$(1-\lambda)\, p(t) + (\lambda-\lambda^2)\, p(t-1) + \lambda^2 (1-\lambda)\, p(t-2) + \lambda^3 (1-\lambda)\, p(t-3) + \lambda^4\, pe(t-3)$$

This is the same as the expression

```
> (1-lambda)*sum(((lambda^k)*p(t-k), k=0..3))+lambda^4*pe(t-3);
```

$$(1-\lambda)\,(p(t) + p(t-1)\,\lambda + \lambda^2\, p(t-2) + \lambda^3\, p(t-3)) + \lambda^4\, pe(t-3)$$

Taking the series to the limit, we have

$$E_t P_{t+1} = (1-\lambda)\left(\sum_{k=0}^{\infty} \lambda^k p_{t-k}\right)$$

```
> solve(b*((ep/p)-1)=a-M/p,p);
```

$$\frac{b\, ep + M}{b+a}$$

```
> subs(ep=(1-lambda)*sum(lambda^k*p[t-k],k = 0 .. infinity),(b*ep+M)/(b+a));
```

$$\frac{b\,(1-\lambda)\left(\sum_{k=0}^{\infty} \lambda^k p_{t-k}\right) + M}{b+a}$$

Therefore,

$$P_t = \frac{M_t}{a+b} + \frac{b\,(1-\lambda)\left(\sum_{k=0}^{\infty} \lambda^k p_{t-k}\right)}{a+b}$$

Question 5

(i)

Setting $y_t^d = y_t^s = y_t$ and solving for y_t and p_t under the assumption of fixed expectations. In what follows we let ep denote the expression $E_{t-1} p_t$.

```
> solpy:=solve({y=a0+a1*(m-p),y=yn+b1*(p-ep)}, {y,p});
```

$$solpy := \left\{p = \frac{a0 + a1\, m - yn + b1\, ep}{a1 + b1},\ y = \frac{a0\, b1 + a1\, m\, b1 + a1\, yn - a1\, b1\, ep}{a1 + b1}\right\}$$

Next we impose the money supply following a systematic component $m = \mu 0$.

```
> subs(m=mu0,solpy);
```

$$\{p = \frac{a0 + a1\,\mu 0 - yn + b1\,ep}{a1 + b1}, y = \frac{a0\,b1 + a1\,\mu 0\,b1 + a1\,yn - a1\,b1\,ep}{a1 + b1}\}$$

Since

$$p_t = \frac{a_0 - y_n}{a_1 + b_1} + \frac{a_1\,\mu 0}{a_1 + b_1} + \frac{b_1\,E_{t-1}\,p_t}{a_1 + b_1}$$

then

$$E_{t-1}\,p_t = \frac{a_0 - y_n}{a_1 + b_1} + \frac{a_1\,\mu 0}{a_1 + b_1} + \frac{b_1\,E_{t-1}\,p_t}{a_1 + b_1}$$

```
> solve(ep=(a0+b1*ep-yn+a1*mu0)/(a1+b1),ep);
```

$$\frac{a0 + a1\,\mu 0 - yn}{a1}$$

or

$$E_{t-1}\,p_t = \frac{a_0 - y_n}{a_1} + \mu 0$$

Substituting this into the solution for y, we obtain

```
> solve({y=(a0*b1+a1*mu0*b1+a1*yn-a1*b1*ep)/(a1+b1),ep=(a0+a
1*mu0-yn)/a1},{y,ep});
```

$$\{ep = \frac{a0 + a1\,\mu 0 - yn}{a1}, y = yn\}$$

(ii)

```
> subs(m=mu0+z,solpy);
```

$$\{p = \frac{a0 + a1\,(\mu 0 + z) - yn + b1\,ep}{a1 + b1}, y = \frac{a0\,b1 + a1\,(\mu 0 + z)\,b1 + a1\,yn - a1\,b1\,ep}{a1 + b1}\}$$

$E_{t-1}\,p_t$ is as before since $E_{t-1}\,z_t = 0$.

```
> solve({y=(a0*b1+a1*(mu0+z)*b1+a1*yn-a1*b1*ep)/(a1+b1),ep=(
a0+a1*mu0-yn)/a1},{y,ep});
```

$$\{y = \frac{b1\,yn + a1\,b1\,z + a1\,yn}{a1 + b1}, ep = \frac{a0 + a1\,\mu 0 - yn}{a1}\}$$

Or,

$$y_t = y_n + \frac{a_1\,b_1\,z_t}{a_1 + b_1}$$

Income deviates from its natural level only in so far as changes in the money supply are unexpected.

Question 6

(i)

Since

$$r\,q = max_q\left[\mathrm{x}(e) - a\,\lambda - w - s\,q + \left(\frac{\partial}{\partial t}q\right)\right]$$

and

$$w = b + \frac{(r+s+\mathrm{h}(q,e))\,b}{\lambda}$$

then

$$r\,q = \mathrm{x}(e) - b - s\,q + \left(\frac{\partial}{\partial t}q\right) - min_q\left[a\,\lambda + \frac{(r+s+\mathrm{h}(q,e))\,b}{\lambda}\right]$$

Consequently the value of λ which maximises $r\,q$ is that which minimises the expression in square brackets. Let this be denoted z. Then,

```
> z:=a*lambda+(r+s+h(q,e))*(b/lambda);
```

$$z := a\,\lambda + \frac{(r+s+\mathrm{h}(q,e))\,b}{\lambda}$$

```
> diff(z,lambda);
```

$$a - \frac{(r+s+\mathrm{h}(q,e))\,b}{\lambda^2}$$

```
> diff(z,lambda$2);
```

$$2\,\frac{(r+s+\mathrm{h}(q,e))\,b}{\lambda^3}$$

This last expression is positive. So the result of setting the first derivative to zero gives a minimum.

```
> solve(diff(z,lambda)=0,lambda);
```

$$\frac{\sqrt{a\,(r+s+\mathrm{h}(q,e))\,b}}{a}, -\frac{\sqrt{a\,(r+s+\mathrm{h}(q,e))\,b}}{a}$$

Ignoring the negative value of λ, then

$$\lambda = \sqrt{\frac{b}{a}}\,\sqrt{r+s+\mathrm{h}(q,e)}$$

(ii)

```
> solve({w=b+(r+s+h(q,e))*(b/lambda),lambda=sqrt(b/a)*sqrt(r+s+h(q,e))},{w,lambda});
```

$$\left\{\lambda = \sqrt{\frac{b}{a}}\,\sqrt{r+s+\mathrm{h}(q,e)},\ w = \frac{b\left(\sqrt{\frac{b}{a}}\,\sqrt{r+s+\mathrm{h}(q,e)} + r+s+\mathrm{h}(q,e)\right)}{\sqrt{\frac{b}{a}}\,\sqrt{r+s+\mathrm{h}(q,e)}}\right\}$$

where w can be expressed,

$$w = b + \sqrt{a\,b}\,\sqrt{r+s+\mathrm{h}(q,e)}$$

To solve for profit $\pi(q,e)$ we note that,

$$\pi(q,e) = MRP_L - w = \mathrm{x}(e) - a\,\lambda - b - \sqrt{a\,b}\,\sqrt{r+s+\mathrm{h}(q,e)}$$

```
> simplify(x(e)-a*sqrt(b/a)*sqrt(r+s+h(q,e))-b-sqrt(a*b)*sqrt(r+s+h(q,e)));
```

$$x(e) - a\sqrt{\frac{b}{a}}\sqrt{r+s+h(q,e)} - b - \sqrt{a\,b}\sqrt{r+s+h(q,e)}$$

i.e.,

$$\pi(q,e) = x(e) - b - 2\sqrt{a\,b}\sqrt{r+r+h(q,e)}$$

Question 7

(i)

```
> sol7:=solve({m=(u*v)^(1/4),(m*q/v)=c},{m,v});
```

$$sol7 := \left\{v = \frac{\mathrm{RootOf}(c\,\_Z^3 - u\,q, label = \_L3)\,q}{c},\; m = \left(\frac{u\,\mathrm{RootOf}(c\,\_Z^3 - u\,q, label = \_L3)\,q}{c}\right)^{(1/4)}\right\}$$

```
> ### WARNING: allvalues now returns a list of symbolic values instead of a sequence of lists of numeric values
allvalues(sol7);
```

$$\left\{v = \frac{\left(\frac{u\,q}{c}\right)^{(1/3)} q}{c},\; m = \left(\frac{u\left(\frac{u\,q}{c}\right)^{(1/3)} q}{c}\right)^{(1/4)}\right\},$$

$$\left\{v = \frac{\left(\frac{u\,q}{c}\right)^{(1/3)} (-1)^{(2/3)}\, q}{c},\; m = \left(\frac{u\left(\frac{u\,q}{c}\right)^{(1/3)} (-1)^{(2/3)}\, q}{c}\right)^{(1/4)}\right\},$$

$$\left\{v = -\frac{\left(\frac{u\,q}{c}\right)^{(1/3)} (-1)^{(1/3)}\, q}{c},\; m = \left(-\frac{u\left(\frac{u\,q}{c}\right)^{(1/3)} (-1)^{(1/3)}\, q}{c}\right)^{(1/4)}\right\}$$

Taking only the positive real value for *v*, we have

$$v = \frac{(q\,u\,c^2)^{\left(\frac{1}{3}\right)} q}{c^2}$$

or

$$v = \left(\frac{q}{c}\right)^{\left(\frac{4}{3}\right)} u^{\left(\frac{1}{3}\right)}$$

(ii)

Since,

$$h(q,e) = \frac{m(u,v)}{u} = \frac{u^{\left(\frac{1}{4}\right)} v^{\left(\frac{1}{4}\right)}}{u} = u^{\left(-\frac{3}{4}\right)} v^{\left(\frac{1}{4}\right)} \qquad v = \left(\frac{q}{c}\right)^{\left(\frac{4}{3}\right)} u^{\left(\frac{1}{3}\right)} \qquad u = 1 - e$$

then $h(q, e)$ is eqal to

```
> h:=simplify((1-e)^(-3/4)*(q/c)^(1/3)*(1-e)^(1/12));
```

$$h := \frac{\left(\frac{q}{c}\right)^{(1/3)}}{(1-e)^{(2/3)}}$$

(iii)

```
> diff(h,q);
```

$$\frac{1}{3}\,\frac{1}{(1-e)^{(2/3)}\left(\frac{q}{c}\right)^{(2/3)}c}$$

```
> diff(h,e);
```

$$\frac{2}{3}\,\frac{\left(\frac{q}{c}\right)^{(1/3)}}{(1-e)^{(5/3)}}$$

which are both positive.

(iv)

```
> e:='e': s:='s': q:='q':
```

Since

$$\frac{\partial}{\partial t}e = (1-e)\,\mathrm{h}(q, e) - s\,e$$

then

```
> edot:=(1-e)*h-s*e;
```

$$edot := (1-e)^{(1/3)}\left(\frac{q}{c}\right)^{(1/3)} - s\,e$$

```
> solve(edot=0,q);
```

$$-\frac{s^3\,e^3\,c}{-1+e}$$

or

$$q = \frac{c\,s^3\,e^3}{1-e}$$

Question 8

```
> m:='m': u:='u': v:='v': e:='e': h:='h': c:='c': q:='q':
```

(i)

Given

$$m = \sqrt{u\,v} \text{ and } \frac{m\,q}{u} = c$$

and noting that $u = 1 - e$, then

```
> solve({(sqrt(u*v)/u)*q=c,u=1-e},{v,u});
```

$$\{u = 1 - e, v = -\frac{(-1+e)\,c^2}{q^2}\}$$

Hence,

$$v = (1-e)\left(\frac{c}{q}\right)^2$$

(ii)

```
> solve({h=sqrt(v/(1-e)),v=(1-e)*(c/q)^2},{h,v});
```

$$\{v = -\frac{(-1+e)\,c^2}{q^2}, h = \sqrt{\frac{c^2}{q^2}}\}$$

Thus, h(q, e) = $\frac{c}{q}$ and is independent of e.

Question 9

```
> a:='a': s:='s': beta:='beta': lambda:='lambda':
  delta:='delta': n:='n': x:='x':
> kdot:=s*a*k^beta-(n+delta)*k;
```

$$kdot := s\,a\,k^{\beta} - (n+\delta)\,k$$

```
> para0:={a=2, s=0.2,beta=0.25,lambda=0.05, delta=0.03,n=0.02};
```

$$para0 := \{a = 2, s = .2, \beta = .25, \lambda = .05, \delta = .03, n = .02\}$$

```
> kdot0:=subs(para0,kdot);
```

$$kdot0 := .4\,k^{.25} - .05\,k$$

```
> solve(kdot0=0,k);
```

$$16., 0.$$

Since

$$x = y\,r^{\left(-\frac{1}{4}\right)}, \quad y = \mathrm{f}(k) = \alpha\,k^{\beta} \quad \frac{\partial}{\partial k} f = \beta\,a\,k^{(\beta-1)}$$

```
> solve(x=a*k^beta*r^(-1/4),r);
```

$$\frac{a^4\,(k^{\beta})^4}{x^4}$$

```
> xdot:=(beta*a*k^(beta-1)+lambda-delta-n-a^4*(k^beta)^4/(x^4))
  *x;
```

$$xdot := \left(\beta\,a\,k^{(\beta-1)} + \lambda - \delta - n - \frac{a^4\,(k^{\beta})^4}{x^4}\right)x$$

```
> xdot0:=subs(para0,xdot);
```

$$xdot0 := \left(.50\,\frac{1}{k^{.75}} - \frac{16\,k^{1.00}}{x^4}\right)x$$

```
> solve(xdot0=0,x);
```

$$2\,2^{(1/4)}\,(k^{(7/4)})^{(1/4)}, 2\,I\,2^{(1/4)}\,(k^{(7/4)})^{(1/4)}, -2\,2^{(1/4)}\,(k^{(7/4)})^{(1/4)},$$

$$-2\, I\, 2^{(1/4)} \left(k^{(7/4)}\right)^{(1/4)}$$

The only positive real soltuion for x is

$$x = 2.\ 2^{\left(\frac{1}{4}\right)} \left(k^{\left(\frac{7}{4}\right)}\right)^{\left(\frac{1}{4}\right)}$$

or

```
> evalf(2.*2^(1/4)*(k^(7./16)));
```

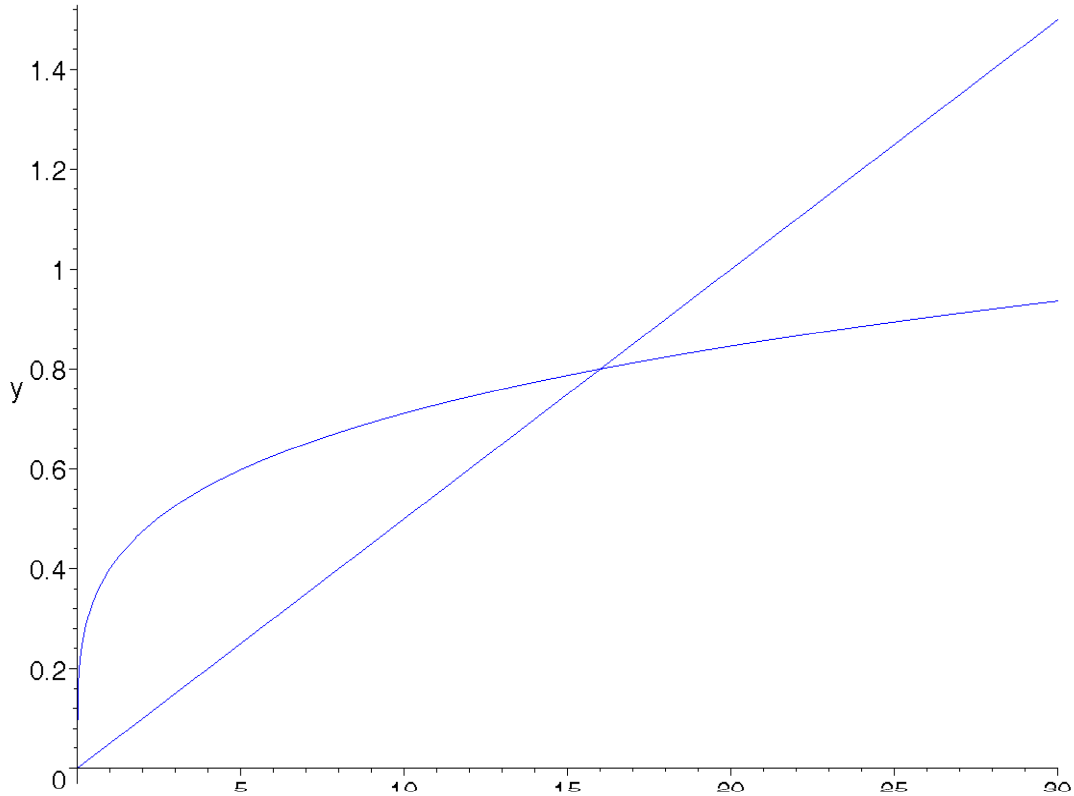
$$2.378414230\, k^{.4375000000}$$

```
> solve({x=2.378414230*k^.4375000000,k=16},{k,x});
```

$$\{k = 16.,\ x = 8.000000000\}$$

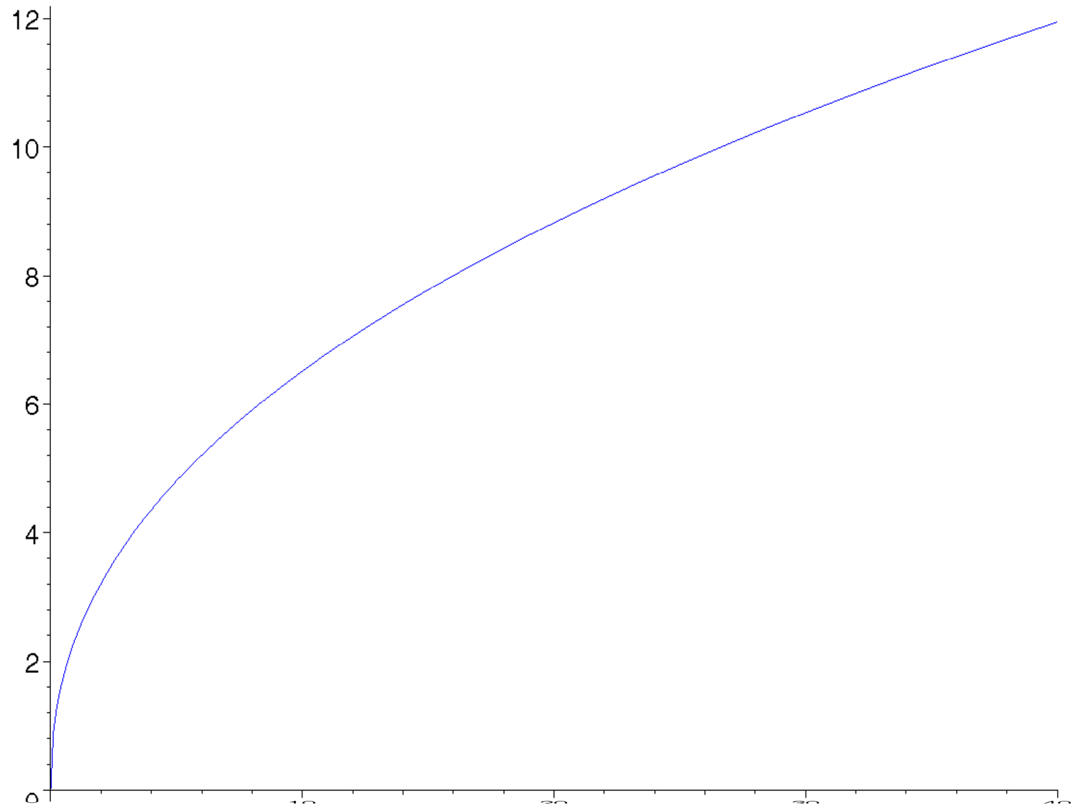
Equilibrium k is shown by the intersection of $s\ f(k)$ and $(n + \delta)\ k$.

```
> initdiag:=plot({0.05*k,0.4*k^0.25},k=0..30,colour=blue):
> display(initdiag,labels=["k","y"]);
```

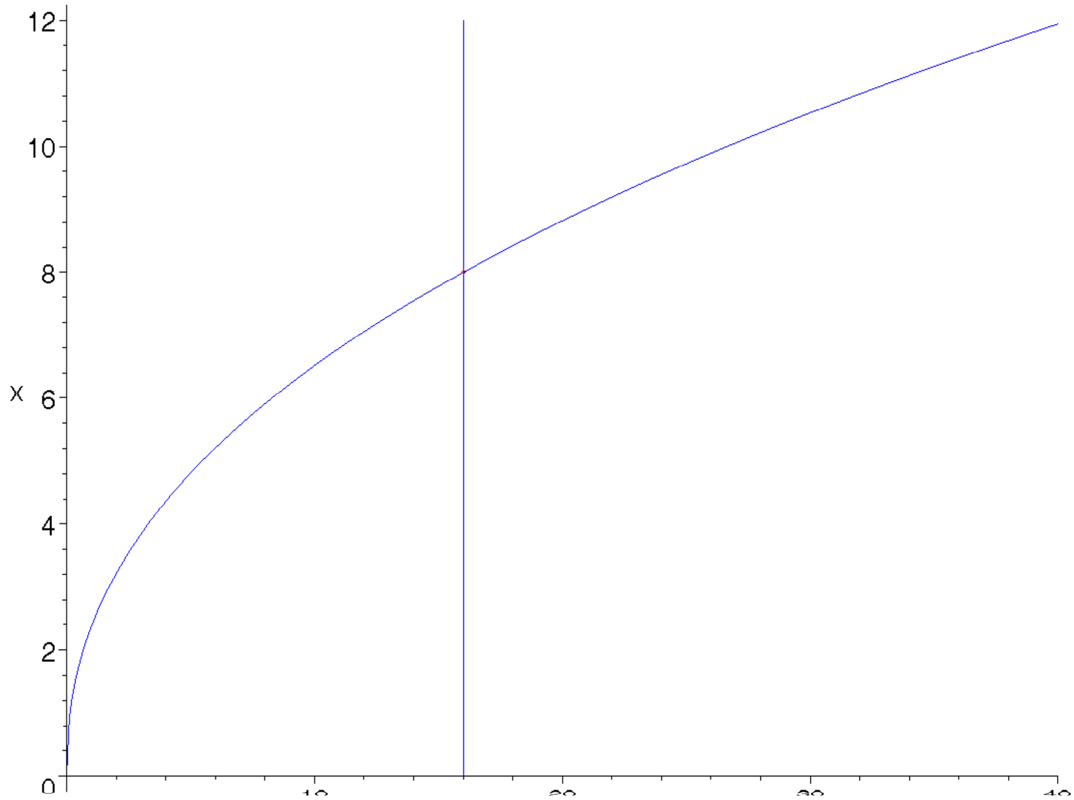


The dynamics are shown by the direction-field diagram. In plotting this we first derive the isoclines and plot these along with the equilibrium point. These are then superimposed on the direction-field diagram.

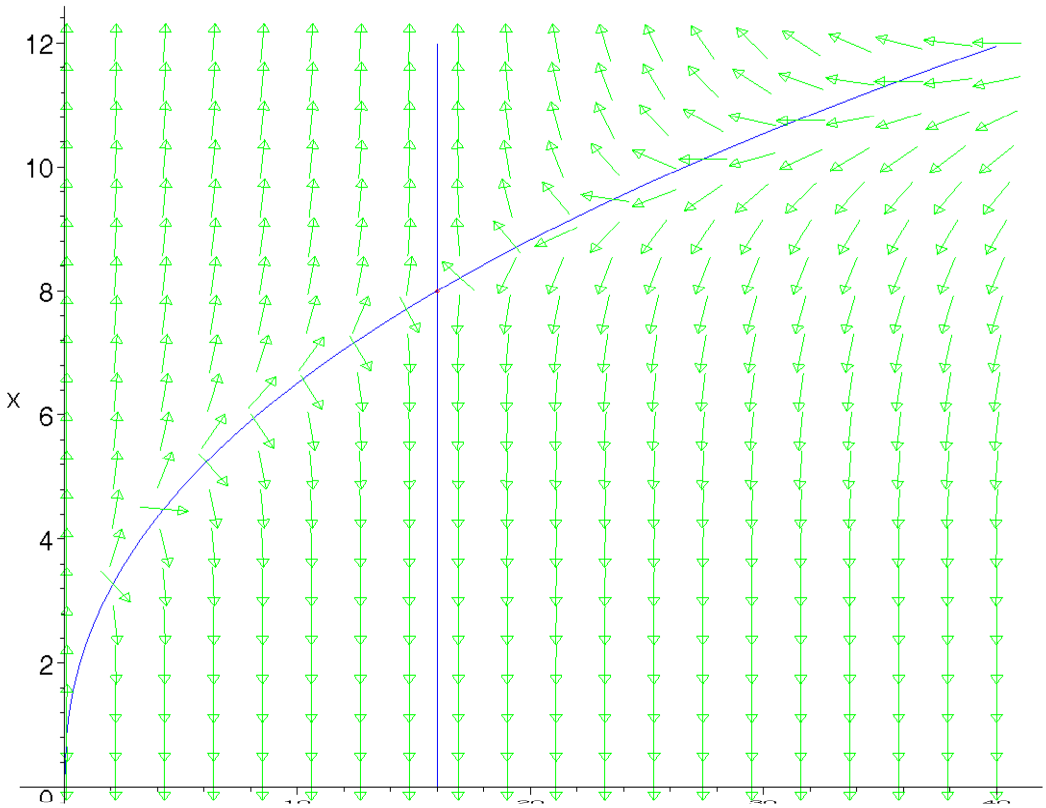
```
> xcline0:=plot(2.378414230*k^.4375000000,k=0..40,colour=blue):
> display(xcline0);
```




```
> linept:=[line([16,0],[16,12],colour=blue),point([16,8],colour
  =red)]:
> display(xcline0,linept,labels=["k","x"]);
```




```
> field0:=dfieldplot([diff(k(t),t)=0.4*k(t)^0.25-0.05*k(t),diff
  (x(t),t)=((0.5/(k(t)^0.75))-(16*k(t)/(x(t)^4)))*x(t)],[k(t),x
  (t)],t=0..1,k=0.1..40,x=0.1..12,arrows=SLIM,colour=green):
> display(field0,xcline0,linept,labels=["k","x"]);
```



(i) Rise in s from 0.2 to 0.3

```
> para1:={a=2, s=0.3,beta=0.25,lambda=0.05,
  delta=0.03,n=0.02};
```

$$para1 := \{s = .3, a = 2, \beta = .25, \lambda = .05, \delta = .03, n = .02\}$$

```
> kdot1:=subs(para1,kdot);
```

$$kdot1 := .6\,k^{.25} - .05\,k$$

```
> solve(kdot1=0,k);
```

$$0., 27.47314182$$

```
> xdot1:=subs(para1,xdot);
```

$$xdot1 := \left(.50\,\frac{1}{k^{.75}} - \frac{16\,k^{1.00}}{x^4}\right)x$$

```
> solve(xdot1=0,x);
```

$$2\,2^{(1/4)}\,(k^{(7/4)})^{(1/4)}, 2\,I\,2^{(1/4)}\,(k^{(7/4)})^{(1/4)}, -2\,2^{(1/4)}\,(k^{(7/4)})^{(1/4)},$$
$$-2\,I\,2^{(1/4)}\,(k^{(7/4)})^{(1/4)}$$

```
> evalf(2.*2.0^(1/4)*(k^(7.0/16)));
```

$$2.378414230\,k^{.4375000000}$$

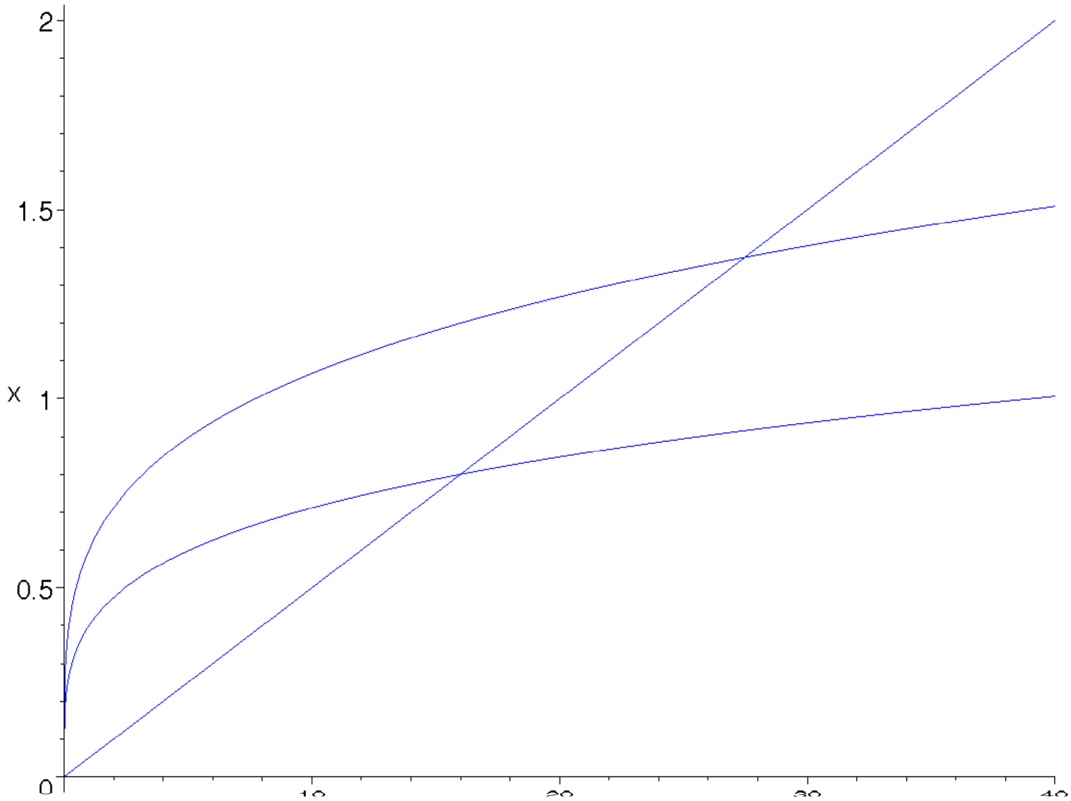
which is the same isocline as for $\frac{dx}{dt} = 0$.

```
> solve({x=2.378414230*k^.4375000000,k=27.47314182},{k,x});
```

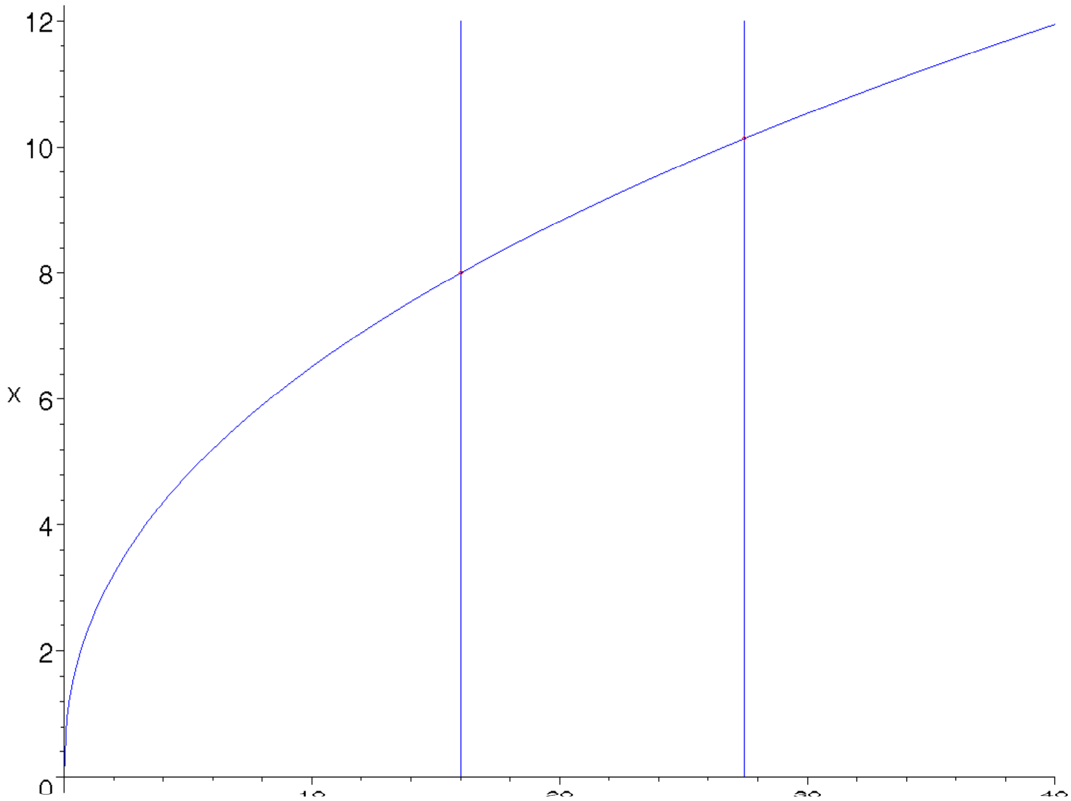
$$\{k = 27.47314182, x = 10.13467644\}$$

```
> diag1:=plot({0.05*k,0.4*k^0.25,0.6*k^0.25},k=0..40,labels=
  ["k","x"],colour=blue):
```

```
> display(diag1);
```

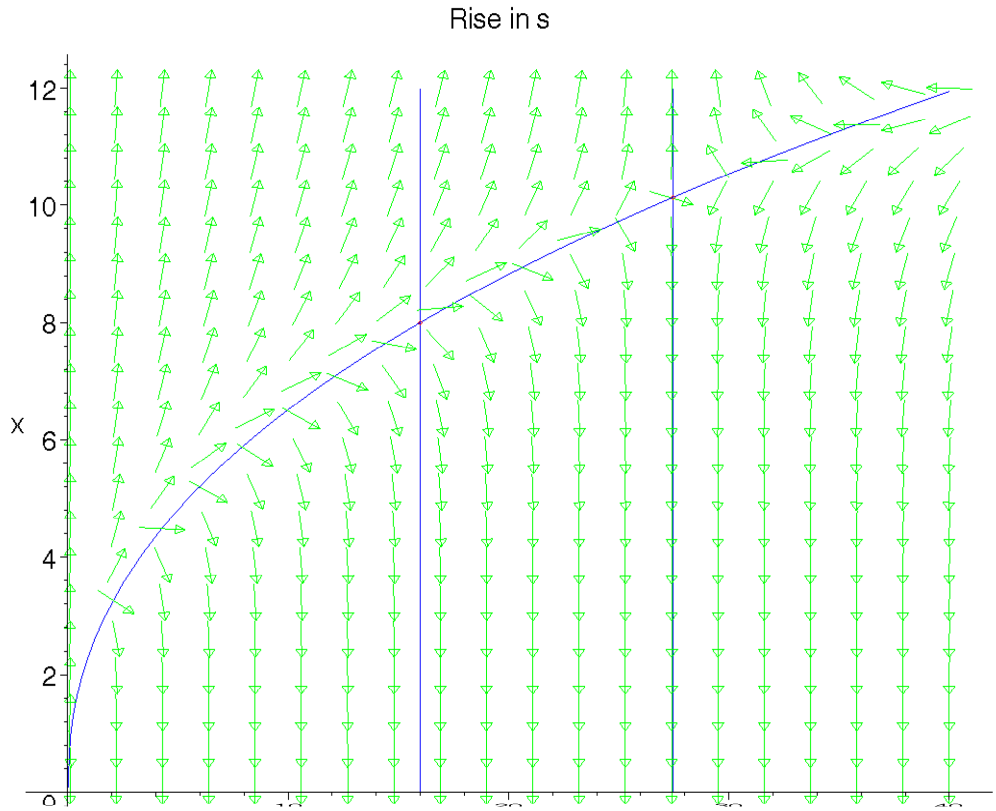



```
> linept1:=[line([16,0],[16,12],colour=blue),line([27.47314182,0],[27.47314182,12],colour=blue),point([16,8],colour=red),point([27.47314182,10.13467644],colour=red)]:
> display(linept,linept1,xcline0,labels=["k","x"]);
```




```
> field1:=dfieldplot([diff(k(t),t)=0.6*k(t)^0.25-0.05*k(t),diff(x(t),t)=((0.5/(k(t)^0.75))-(16*k(t)/(x(t)^4)))*x(t)],[
```

```
k(t),x(t)],t=0..1,k=0.1..40,x=0.1..12,arrows=SLIM,colour=green):
> display(field1,xcline0,linept,linept1,labels=["k","x"],title="Rise in s");
```



Note that a rise in s shifts only $s\ f(k)$ in (k, y)-space and only the $\frac{dk}{dt} = 0$ isocline in (k, x)-space.

(ii) A rise in *n* from 0.02 to 0.03

```
> para2:={a=2,s=0.2,beta=0.25,lambda=0.05,delta=0.03,n=0.03};
```

$$para2 := \{n = .03,\ a = 2,\ s = .2,\ \beta = .25,\ \lambda = .05,\ \delta = .03\}$$

```
> kdot2:=subs(para2,kdot);
```

$$kdot2 := .4\ k^{.25} - .06\ k$$

```
> solve(kdot2=0,k);
```

$$12.54714705,\ 0.$$

```
> xdot2:=subs(para2,xdot);
```

$$xdot2 := \left(.50\ \frac{1}{k^{.75}} - .01 - \frac{16\ k^{1.00}}{x^4} \right) x$$

```
> solve(xdot2=0,x);
```

$$2.\ \frac{\left(-100\ (k^4 + 125000\ k^{(7/4)} + 2500\ k^{(5/2)} + 50\ k^{(13/4)})\ (-6250000 + k^3)^3\right)^{(1/4)}}{-6250000 + k^3},$$

$$\frac{2.\, I\, \left(-100\, \left(k^4 + 125000\, k^{(7/4)} + 2500\, k^{(5/2)} + 50\, k^{(13/4)}\right) \left(-6250000 + k^3\right)^3\right)^{(1/4)}}{-6250000 + k^3},$$

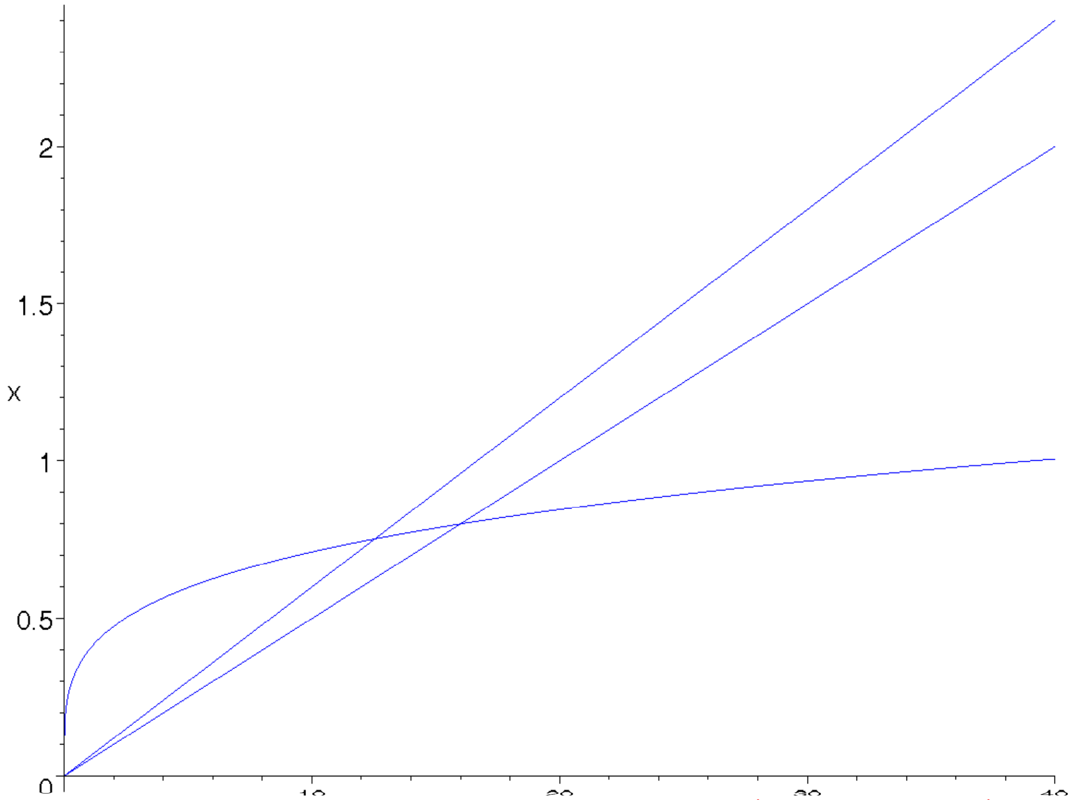
$$-2.\, \frac{\left(-100\, \left(k^4 + 125000\, k^{(7/4)} + 2500\, k^{(5/2)} + 50\, k^{(13/4)}\right) \left(-6250000 + k^3\right)^3\right)^{(1/4)}}{-6250000 + k^3},$$

$$\frac{-2.\, I\, \left(-100\, \left(k^4 + 125000\, k^{(7/4)} + 2500\, k^{(5/2)} + 50\, k^{(13/4)}\right) \left(-6250000 + k^3\right)^3\right)^{(1/4)}}{-6250000 + k^3}$$

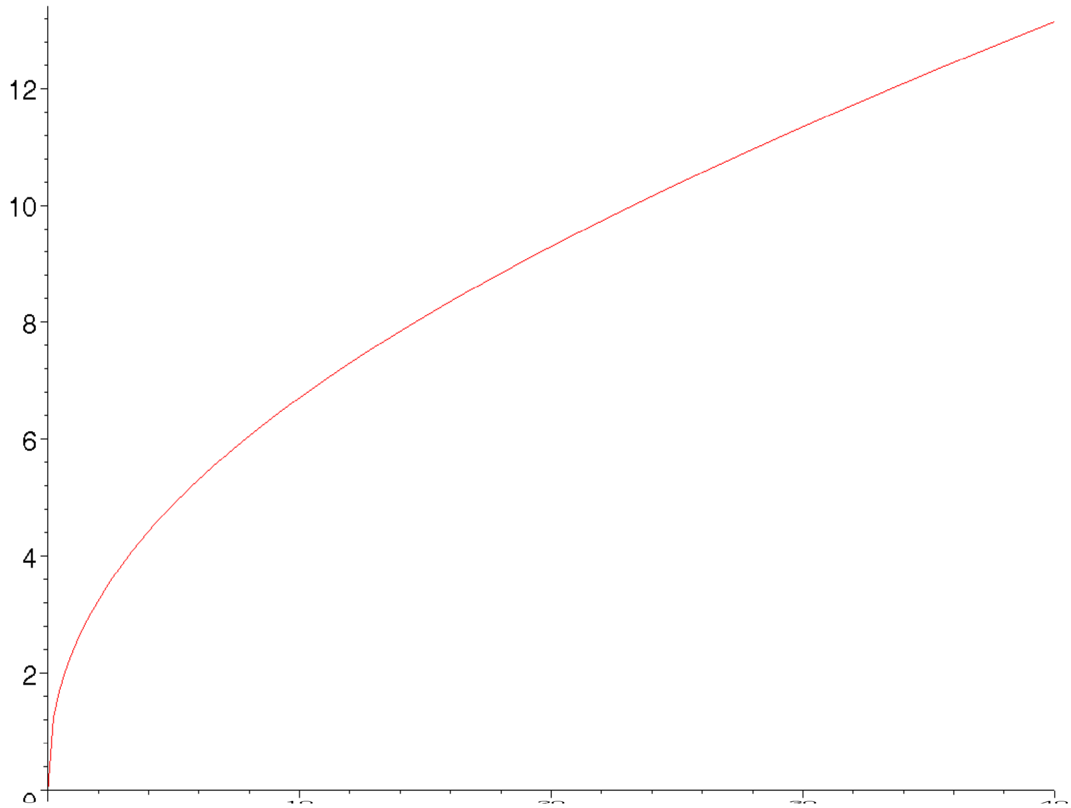
```
> solve({x=-2.*sqrt(10)*(-(2500*k^(5/2)+50*k^(13/4)+k^4+1250
  00*k^(7/4))*(-6250000+k^3)^3)^(1/4)/(-6250000+k^3),k=12.54
  714705},{k,x});
```

$$\{k = 12.54714705,\, x = 7.454832733\}$$

```
> diag2:=plot({0.05*k,0.06*k,0.4*k^0.25},k=0..40,labels=["k"
  ,"x"],colour=blue):
> display(diag2);
```

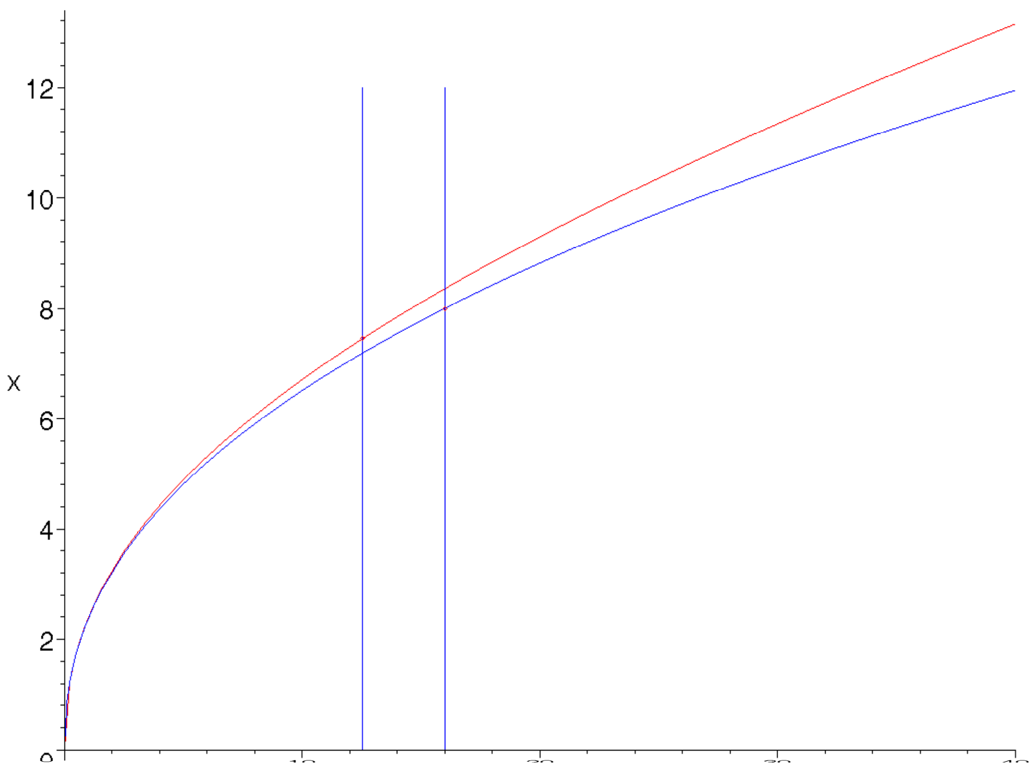



```
> xcline2:=plot(-2.*sqrt(10)*(-(2500*k^(5/2)+50*k^(13/4)+k^4
  +125000*k^(7/4))*(-6250000+k^3)^3)^(1/4)/(-6250000+k^3),k=
  0..40):
> display(xcline2);
```




```
> linept2:=[line([16,0],[16,12],colour=blue),line([12.54714705,0],[12.54714705,12],colour=blue),point([16,8],colour=red),point([12.54714705,7.454832733],colour=red)]:
> display(linept,linept2,xcline0,xcline2,labels=["k","x"],title="Rise in n");
```

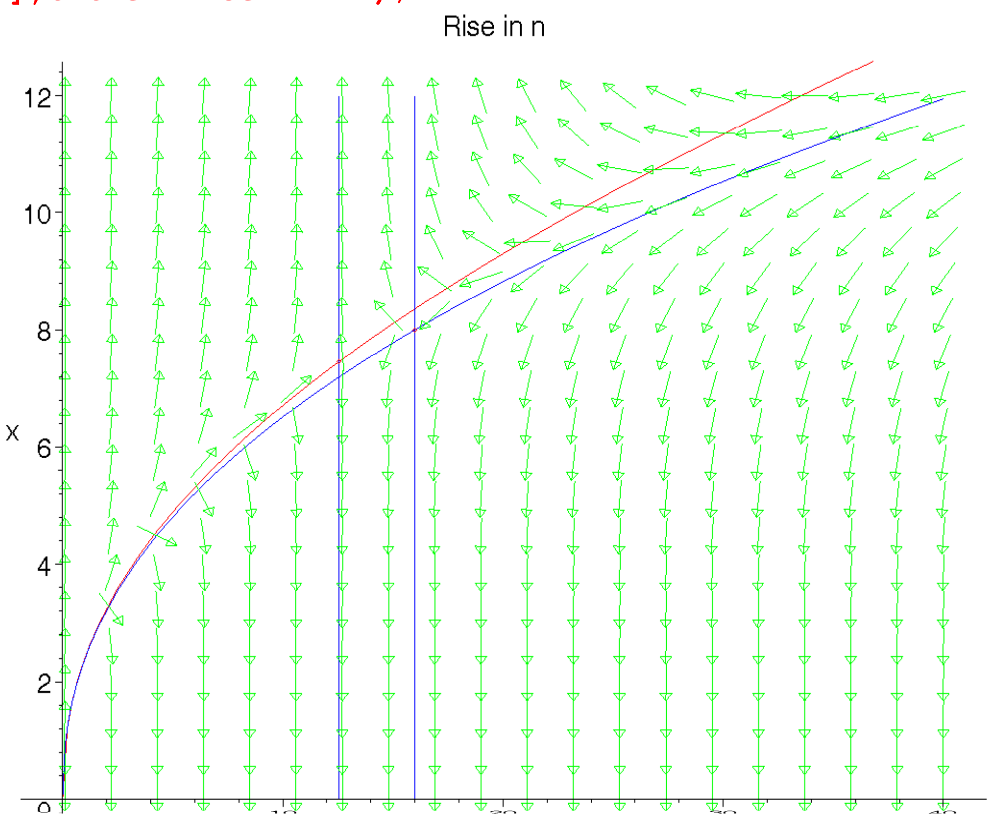
Rise in n




```
> field2:=dfieldplot([diff(k(t),t)=0.4*k(t)^0.25-0.06*k(t),diff(x(t),t)=(-0.01+(0.5/(k(t)^0.75))-(16*k(t)/(x(t)^4)))*x
```

```
(t)],[k(t),x(t)],t=0..1,k=0.1..40,x=0.1..12,arrows=SLIM,co
lour=green):
```

```
> display(field2,linept,linept2,xcline0,xcline2,labels=["k",
"x"],title="Rise in n");
```



Note that a rise in n shifts only $(n+\delta)\,k$ in (k, y)-space but shifts both isoclines in $(k, x$ $)$-space.

(iii) A rise in technology (a rise in *a* from 2 to 5)

```
> para3:={a=5,s=0.2,beta=0.25,lambda=0.05,
delta=0.03,n=0.02};
```

$$para3 := \{s = .2, \beta = .25, \lambda = .05, \delta = .03, n = .02, a = 5\}$$

```
> kdot3:=subs(para3,kdot);
```

$$kdot3 := 1.0\, k^{.25} - .05\, k$$

```
> solve(kdot3=0,k);
```

$$0., 54.28835233$$

```
> xdot3:=subs(para3,xdot);
```

$$xdot3 := \left(1.25\,\frac{1}{k^{.75}} - \frac{625\,k^{1.00}}{x^4}\right) x$$

```
> solve(xdot3=0,x);
```

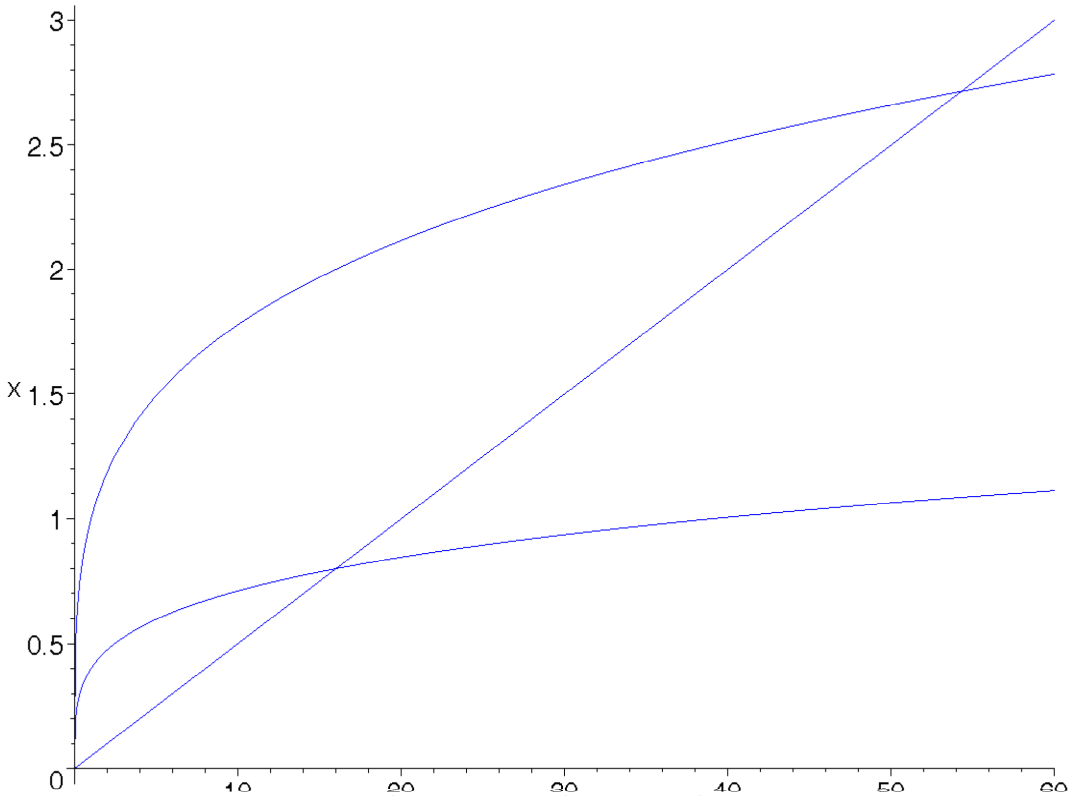
$$500^{(1/4)}\,(k^{(7/4)})^{(1/4)},\ I\,500^{(1/4)}\,(k^{(7/4)})^{(1/4)},\ -500^{(1/4)}\,(k^{(7/4)})^{(1/4)},$$
$$-I\,500^{(1/4)}\,(k^{(7/4)})^{(1/4)}$$

```
> solve({x=(500^(1/4))*(k^(7/16)),k=54.28835233},{k,x});
```

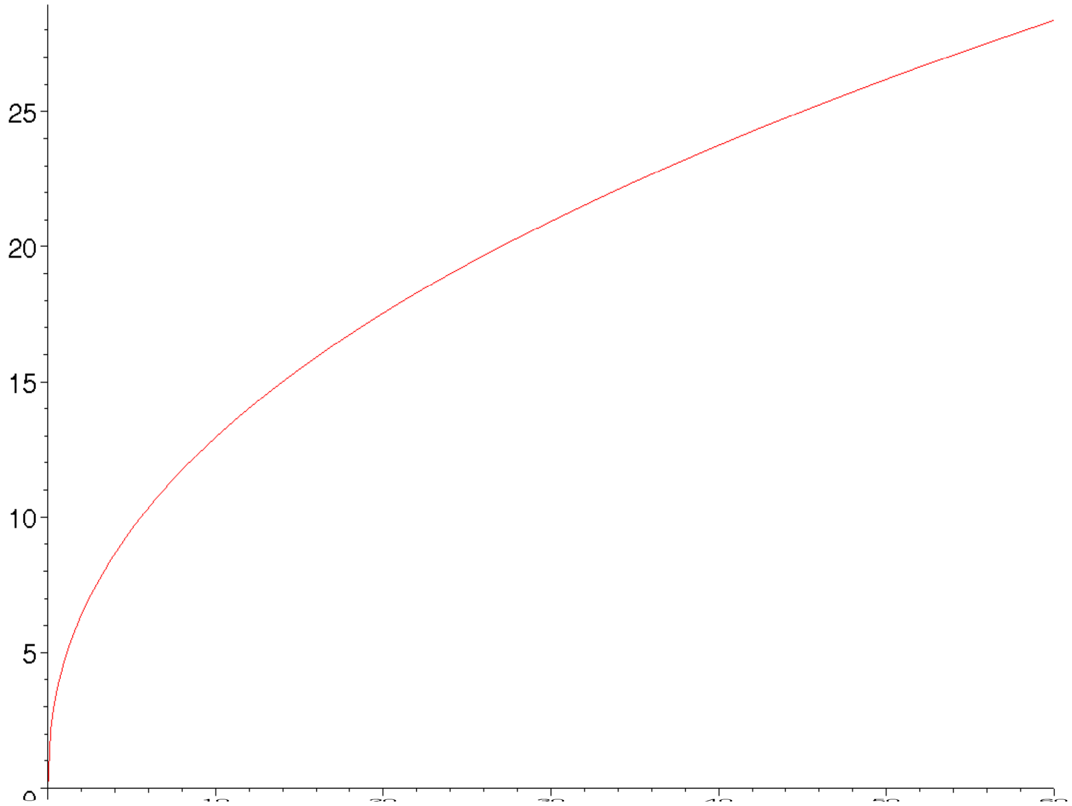
$$\{x = 27.14417617, k = 54.28835233\}$$

```
> diag3:=plot({0.05*k,0.4*k^0.25,k^0.25},k=0..60,labels=["k"
```

```
,"x"],colour=blue):
> display(diag3);
```

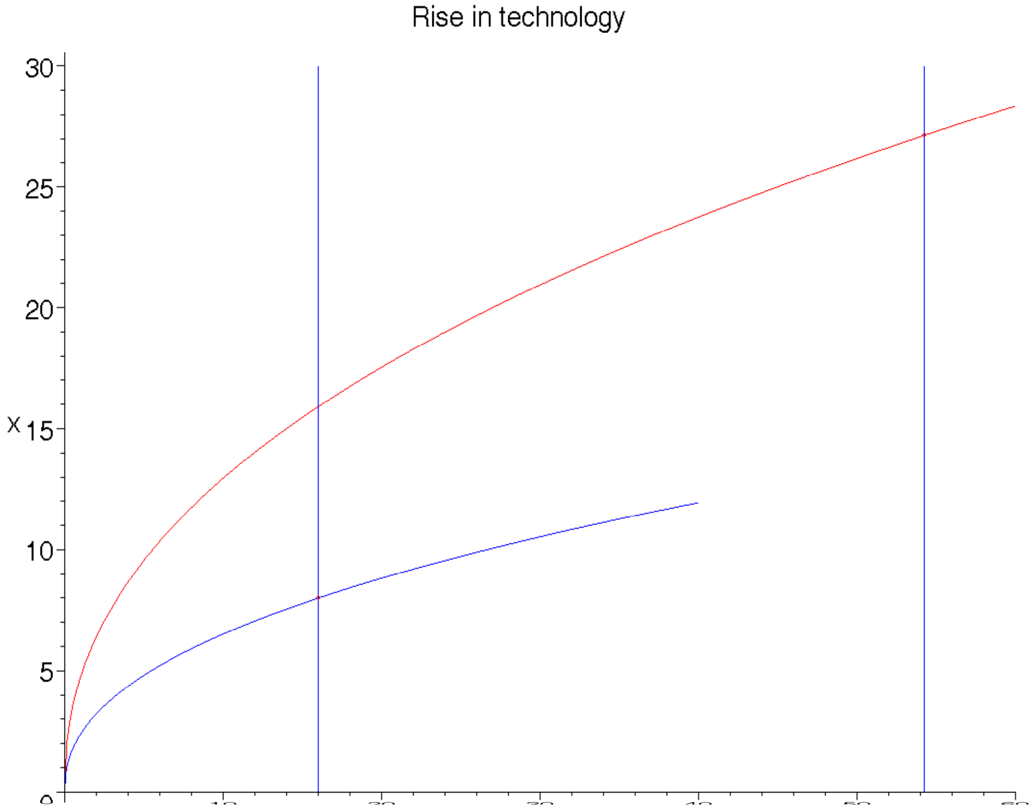



```
> xcline3:=plot(500^(1/4)*(k^(7/4))^(1/4),k=0..60):
> display(xcline3);
```

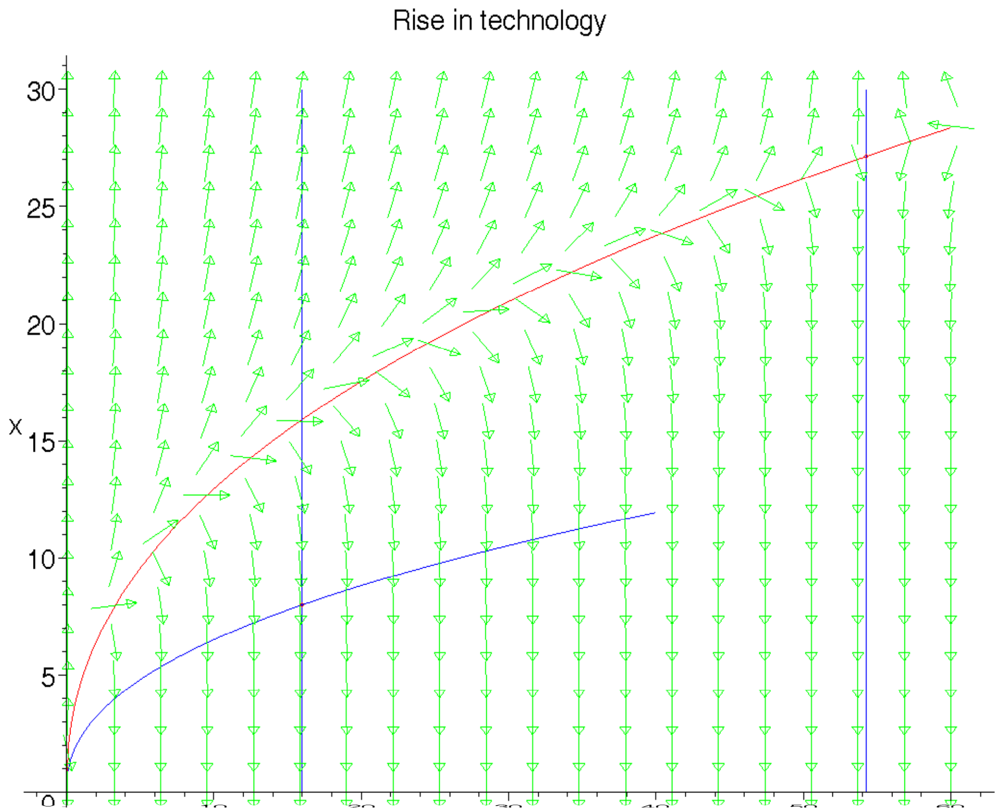



```
> linept3:=[line([16,0],[16,30],colour=blue),line([54.288352
33,0],[54.28835233,30],colour=blue),point([16,8],colour=re
d),point([54.28835233,27.14417617],colour=red)]:
```

```
> display(linept,linept3,xcline0,xcline3,labels=["k","x"],title="Rise in technology");
```




```
> field3:=dfieldplot([diff(k(t),t)=k(t)^0.25-0.05*k(t),diff(x(t),t)=((1.25/(k(t)^0.75))-(625*k(t)/(x(t)^4)))*x(t)],[k(t),x(t)],t=0..1,k=0.1..60,x=0.1..30,arrows=SLIM,colour=green):
> display(field3,linept,linept3,xcline0,xcline3,labels=["k","x"],title="Rise in technology");
```



Note that a rise in technology shifts only s f(k) in (k, y)-space but shifts both isoclines in (k, x)-space.

Question 10

This is a continuation of the previous question and we shall identify all plots with the number 4.

```
> para4:={a=2,s=0.2,beta=0.25,lambda=0.06, delta=0.03,n=0.02};
```

$$para4 := \{\lambda = .06, a = 2, s = .2, \beta = .25, \delta = .03, n = .02\}$$

```
> kdot4:=subs(para4,kdot);
```

$$kdot4 := .4\,k^{.25} - .05\,k$$

```
> solve(kdot4=0,k);
```

$$16., 0.$$

```
> xdot4:=subs(para4,xdot);
```

$$xdot4 := \left(.50\,\frac{1}{k^{.75}} + .01 - \frac{16\,k^{1.00}}{x^4}\right)x$$

```
> solve(xdot4=0,x);
```

$$2.\,\frac{100^{(1/4)}\left(\left(2500\,k^{(5/2)} - 50\,k^{(13/4)} + k^4 - 125000\,k^{(7/4)}\right)\left(-6250000 + k^3\right)^3\right)^{(1/4)}}{-6250000 + k^3},$$

$$\frac{2.\,I\,100^{(1/4)}\left(\left(2500\,k^{(5/2)} - 50\,k^{(13/4)} + k^4 - 125000\,k^{(7/4)}\right)\left(-6250000 + k^3\right)^3\right)^{(1/4)}}{-6250000 + k^3},$$

$$-2.\,\frac{100^{(1/4)}\left(\left(2500\,k^{(5/2)}-50\,k^{(13/4)}+k^4-125000\,k^{(7/4)}\right)\left(-6250000+k^3\right)^3\right)^{(1/4)}}{-6250000+k^3},$$

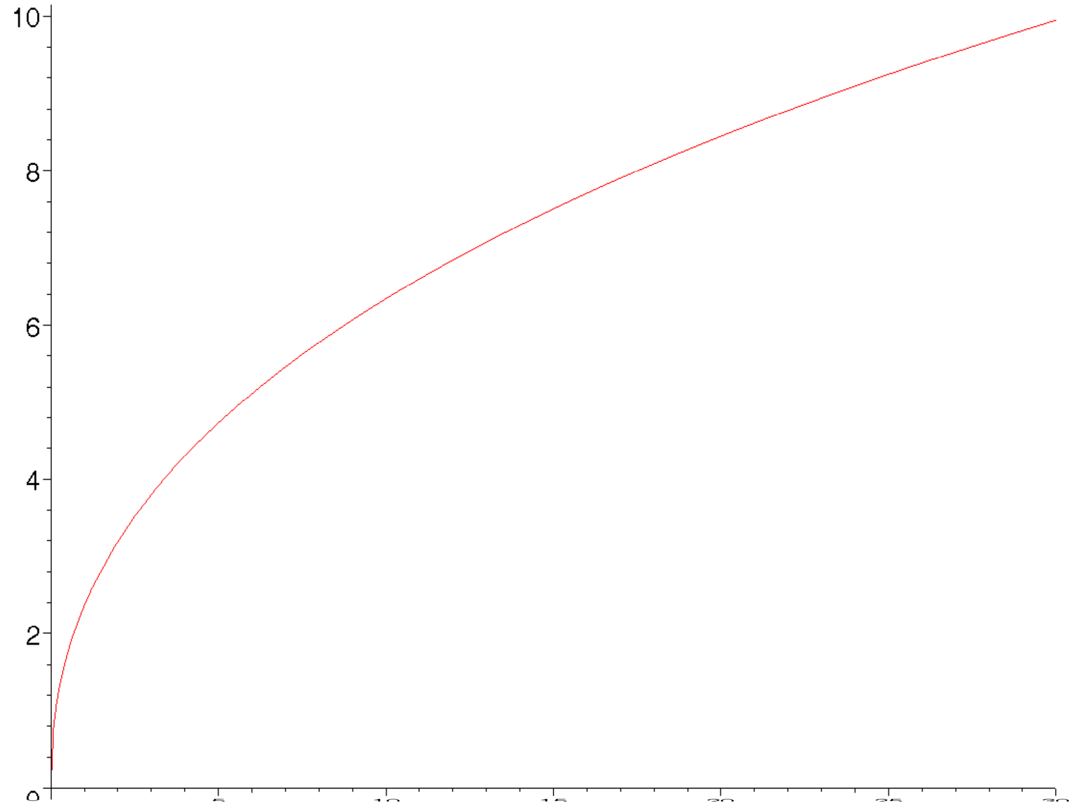
$$\frac{-2.\,I\,100^{(1/4)}\left(\left(2500\,k^{(5/2)}-50\,k^{(13/4)}+k^4-125000\,k^{(7/4)}\right)\left(-6250000+k^3\right)^3\right)^{(1/4)}}{-6250000+k^3}$$

```
> solve({x=-2.*sqrt(10)*((2500*k^(5/2)-50*k^(13/4)+k^4-125000*k^(7/4))*(-6250000+k^3)^3)^(1/4)/(-6250000+k^3),k=16},{k,x});
```

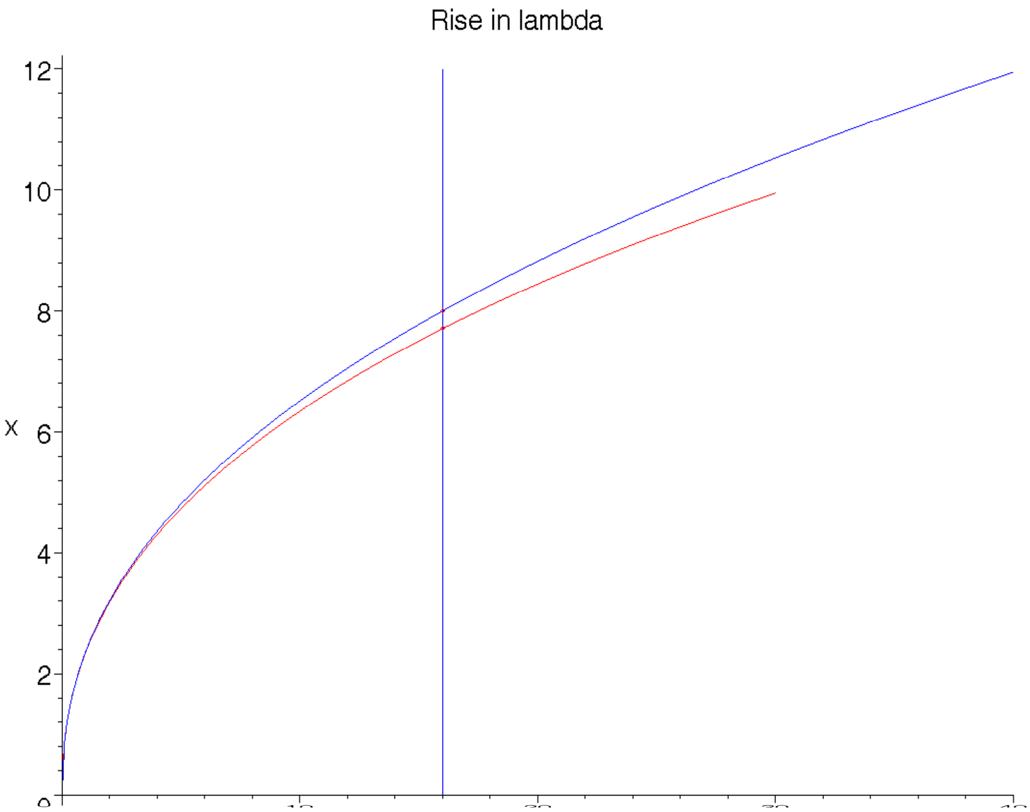
$$\{x = 7.708599627,\ k = 16.\}$$

Since a change in λ affects neither $(n+\delta)\,k$ nor $s\,f(k)$, then there is no change in equilibrium k, and the diagram in $(k, y$ $\frac{dx}{dt}=0$ iscoline shifts.

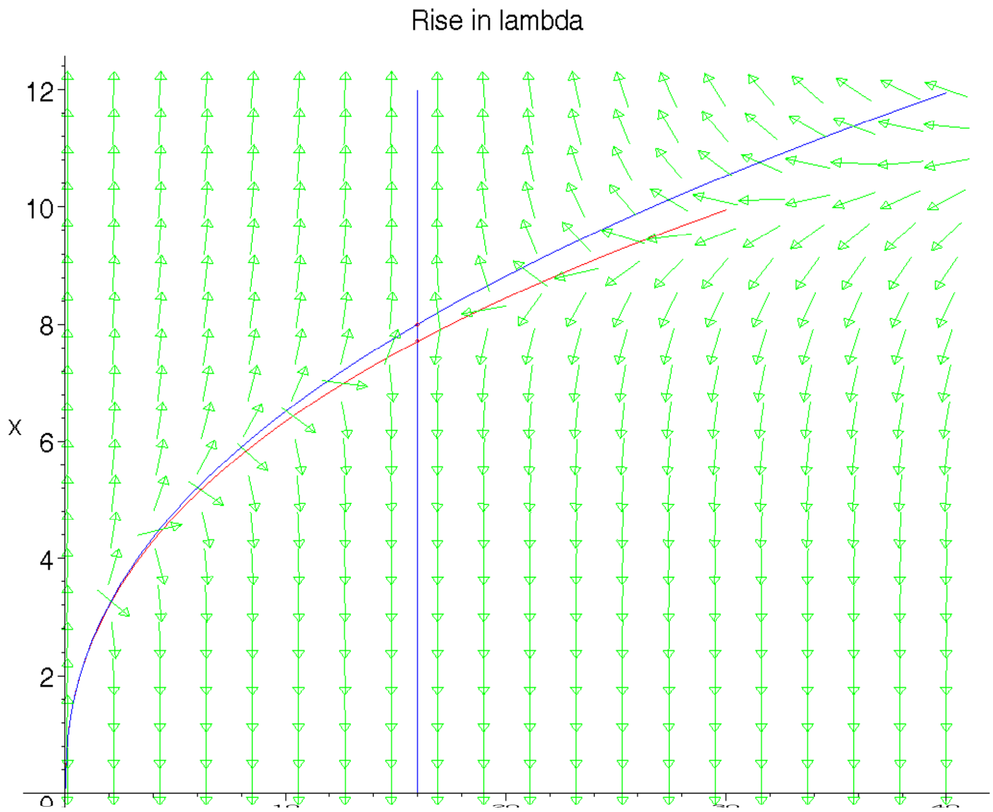
```
> xcline4:=plot(-2.*sqrt(10)*((2500*k^(5/2)-50*k^(13/4)+k^4-125000*k^(7/4))*(-6250000+k^3)^3)^(1/4)/(-6250000+k^3),k=0..30):
> display(xcline4);
```




```
> linept4:=[line([16,0],[16,12],colour=blue),point([16,8],colour=red),point([16,7.708599627],colour=red)]:
> display(linept,linept4,xcline0,xcline4,labels=["k","x"],title="Rise in lambda");
```




```
> field4:=dfieldplot([diff(k(t),t)=0.4*k(t)^0.25-0.05*k(t),diff
  (x(t),t)=((0.01+0.5/(k(t)^0.75))-(16*k(t)/(x(t)^4)))*x(t)],[k
  (t),x(t)],t=0..1,k=0.1..40,x=0.1..12,arrows=SLIM,colour=green
  ):
> display(field4,linept,linept4,xcline0,xcline4,labels=["k","x"
  ],title="Rise in lambda");
```




```
>
```

The rise in λ shifts the $\frac{dx}{dt} = 0$ isocline down, equilibrium k remains at 16 while equilibrium s falls from 8 to 7.7. The economy's trajectory, therefore, is vertically down from one equilibrium to the other.

Question 11

```
> A:='A':B:='B':C:='C':DD:='D':EE:='EE':FF:='FF':G:='G':H:='H':
  J:='J':
```

(i) and (ii)

```
> A:=(-alpha*(a+i0+g)/(1-b*(1-t)+(k*h/u))+alpha*yn;
```

$$A := -\frac{\alpha\,(a + i0 + g)}{1 - b\,(1 - t) + \frac{k\,h}{u}} + \alpha\, yn$$

```
> B:=-alpha*(h/u)/(1-b*(1-t)+(k*h/u));
```

$$B := -\frac{\alpha\, h}{u\left(1 - b\,(1 - t) + \frac{k\,h}{u}\right)}$$

```
> C:=-((alpha*h/(1-b*(1-t)+(k*h/u)))+1);
```

$$C := -\frac{\alpha\, h}{1 - b\,(1 - t) + \frac{k\,h}{u}} - 1$$

```
> subs({a=100,b=0.8,i0=600,yn=3000,g=525,t=0.25,k=0.25,h=2.5,u=
  5,alpha=0.2,beta=0.05},A);
```

$$133.3333333$$

```
> subs({a=100,b=0.8,i0=600,yn=3000,g=525,t=0.25,k=0.25,h=2.5,u=
  5,alpha=0.2,beta=0.05},B);
```

$$-.1904761905$$

```
> subs({a=100,b=0.8,i0=600,yn=3000,g=525,t=0.25,k=0.25,h=2.5,u=
  5,alpha=0.2,beta=0.05},C);
```

$$-1.952380952$$

```
> solve(0=133.3333333-.1904761905*ms-1.952380952*pie,pie);
```

$$68.29268292 - .09756097564\, ms$$

Since D and E are protected, we call the coefficients DD, EE and FF respectively.

```
> DD:=(alpha*beta*(a+i0+g)/(1-b*(1-t)+(k*h/u)))-alpha*beta*yn;
```

$$DD := \frac{\alpha\,\beta\,(a + i0 + g)}{1 - b\,(1 - t) + \frac{k\,h}{u}} - \alpha\,\beta\, yn$$

```
> EE:=alpha*beta*(h/u)/(1-b*(1-t)+(k*h/u));
```

$$EE := \frac{\alpha\,\beta\, h}{u\left(1 - b\,(1 - t) + \frac{k\,h}{u}\right)}$$

```
> FF:=alpha*beta*h/(1-b*(1-t)+(k*h/u));
```

$$FF := \frac{\alpha\,\beta\,h}{1 - b\,(1-t) + \frac{k\,h}{u}}$$

```
> subs({a=100,b=0.8,i0=600,yn=3000,g=525,t=0.25,k=0.25,h=2.5,u=5,alpha=0.2,beta=0.05},DD);
```

$$-6.66666667$$

```
> subs({a=100,b=0.8,i0=600,yn=3000,g=525,t=0.25,k=0.25,h=2.5,u=5,alpha=0.2,beta=0.05},EE);
```

$$.009523809524$$

```
> subs({a=100,b=0.8,i0=600,yn=3000,g=525,t=0.25,k=0.25,h=2.5,u=5,alpha=0.2,beta=0.05},FF);
```

$$.04761904762$$

```
> solve(0=-6.66666667+.9523809524e-2*ms+.4761904762e-1*pie,pie);
```

$$140.0000001 - .2000000000\,ms$$

```
> solve({0=133.3333333-.1904761905*ms-1.952380952*pie,0=-6.66666667+.9523809524e-2*ms+.4761904762e-1*pie},{ms,pie});
```

$$\{ms = 700.0000009,\ pie = -.1140000001\ 10^{-6}\}$$

```
> mA:=matrix([[-.1904761905,-1.952380952],[.9523809524e-2,.4761904762e-1]]);
```

$$mA := \begin{bmatrix} -.1904761905 & -1.952380952 \\ .009523809524 & .04761904762 \end{bmatrix}$$

```
> trace(mA);
```

$$-.1428571429$$

```
> det(mA);
```

$$.009523809524$$

```
> eigenvals(mA);
```

$$-.07142857144 + .06649638111\,I,\ -.07142857144 - .06649638111\,I$$

```
> eigenvects(mA);
```

$$[-.07142857148 + .06649638110\,I,\ 1,\ \{[-6.982120019 - 12.50000000\,I,\ 0. + 1.\,I]\}],$$
$$[-.07142857148 - .06649638110\,I,\ 1,\ \{[-6.982120019 + 12.50000000\,I,\ 0. - 1.\,I]\}]$$

(iii)

```
> G:=(k/u)*(a+i0+g)/(1-b*(1-t)+(k*h/u));
```

$$G := \frac{k\,(a + i0 + g)}{u\left(1 - b\,(1-t) + \frac{k\,h}{u}\right)}$$

```
> H:=((h/u)*(k/u)/(1-b*(1-t)+(k*h/u)))-(1/u);
```

$$H := \frac{h\,k}{u^2\left(1 - b\,(1-t) + \frac{k\,h}{u}\right)} - \frac{1}{u}$$

```
> J:=(k*h/u)/(1-b*(1-t)+(k*h/u));
```

$$J := \frac{k\,h}{u\left(1 - b\,(1-t) + \frac{k\,h}{u}\right)}$$

```
> subs({a=100,b=0.8,i0=600,yn=3000,g=525,t=0.25,k=0.25,h=2.5,u=
5,alpha=0.2,beta=0.05},G);
```

116.6666667

```
> subs({a=100,b=0.8,i0=600,yn=3000,g=525,t=0.25,k=0.25,h=2.5,u=
5,alpha=0.2,beta=0.05},H);
```

-.1523809524

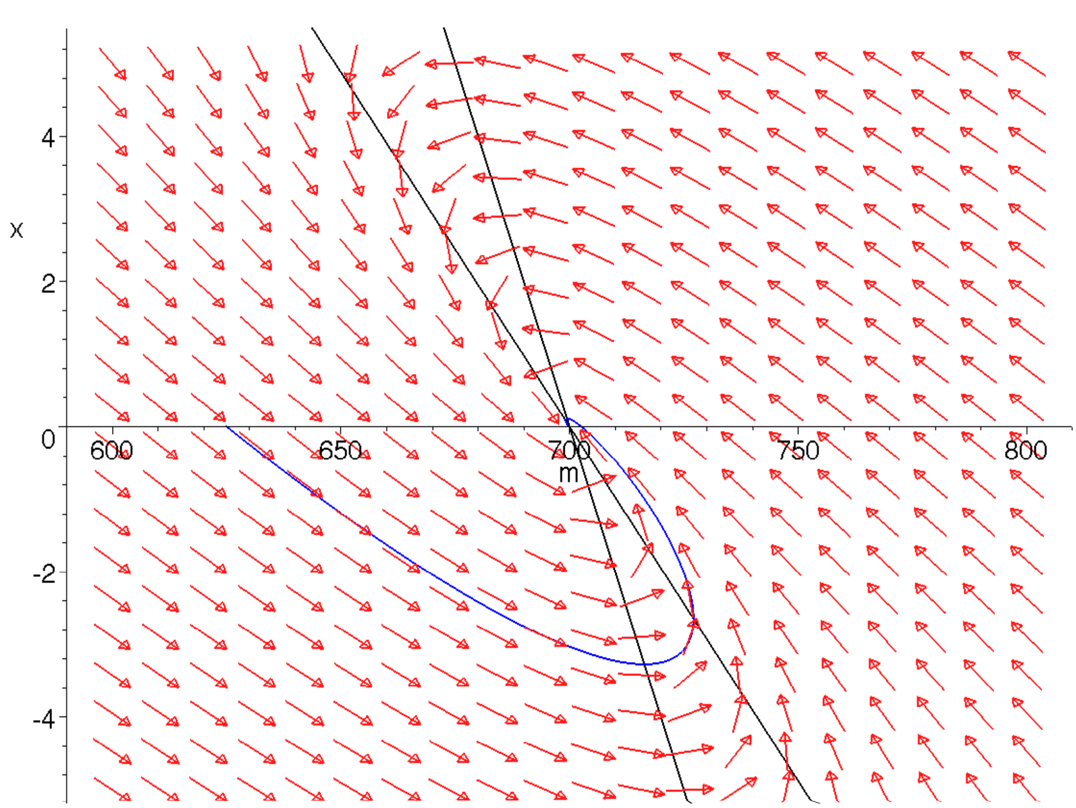
```
> subs({a=100,b=0.8,i0=600,yn=3000,g=525,t=0.25,k=0.25,h=2.5,u=
5,alpha=0.2,beta=0.05},J);
```

.2380952381

```
> solve(0=116.6666667-.1523809524*ms+.2380952381*pie,pie);
```

$-490.0000001 + .6400000001\, ms$

```
> traj:=phaseportrait(
[D(m)(t)=133.3333333-.1904761905*m(t)-1.952380952*x(t),D(x)(t
)=-6.66666667+.9523809524e-2*m(t)+.4761904762e-1*x(t)],
[m(t),x(t)],t=0..100,
[[m(0)=625,x(0)=0]],
m=600..800,x=-5..5,
stepsize=.05,
linecolour=blue,
arrows=SLIM,
thickness=2):
> lines:=plot({68.29268292-.9756097564e-1*m,140-0.2*m},m=600..8
00,colour=black, thickness=2):
> display(traj,lines);
```



Question 12

(i)

Since

$$S = \frac{\Delta M}{P} = \frac{\Delta M}{M} \text{ x } \frac{M}{P} = \frac{\lambda M}{P} \text{ then}$$

$$\ln(S) = \ln(\lambda) + m - p$$

$$= \ln(\lambda) - \alpha \pi \qquad \text{since } \pi^e = \pi$$

Hence,

$$\ln(S) = \ln(\lambda) - \alpha \lambda$$

(ii)

```
> solve(diff(ln(lambda)-alpha*lambda,lambda)=0,lambda);
```

$$\frac{1}{\alpha}$$