CN

Chapter 2

CT

Supplement: Energy and Power

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Energy

In “CBGL” and in the preceding section of the supplement, we have seen how to find the electric field of the generated wave; however, in practice, we are often more interested in knowing the power or energy produced by the nonlinear interaction. In this section, we will see how to calculate these quantities. In some cases, it is more convenient to perform this calculation in the time-domain, while in others, the frequency-domain is more natural; hence, we will examine both.

Calculation of the power of an electromagnetic wave is based upon the concept of the Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ which describes the power per unit area being carried by the electromagnetic wave. The Poynting vector can then be integrated over a particular surface of interest to determine the total power P passing through that surface: $\mathrm{P} =\oint_{\Sigma} \mathbf{S}\cdot\widehat{\mathbf{n}}\ d\sigma$, where $\widehat{\mathbf{n}}$ is a unit vector perpendicular to the surface of interest Σ and $d\sigma$ is a differential element of area on that surface.

When we evaluate the Poynting vector for a plane wave, we find that the power per unit area (or intensity) is given in the time-domain by the expression $\mathrm{I}\,(x,t) = \dfrac{[E(x,t)]^2}{\eta}$, where $\eta=\sqrt{\dfrac{\mu}{\epsilon}}$ is the wave impedance. We can obtain the energy per unit area by integrating with respect to time; thus:

$$\mathrm{U}\,(x) = \int \mathrm{I}\,(x,t)\ dt = \frac{1}{\eta}\int [E(x,t)]^2\,dt \tag{2.1}$$

Usually, we measure the energy at some fixed location, such as $x = 0$ for the input waves or $x = l$ for the output wave.

Equation 2.1 tells us how to calculate the energy in the wave from its time-domain representation. We can also calculate the energy from the frequency-domain representation, using a theorem of Fourier analysis that is sometimes called Parseval’s

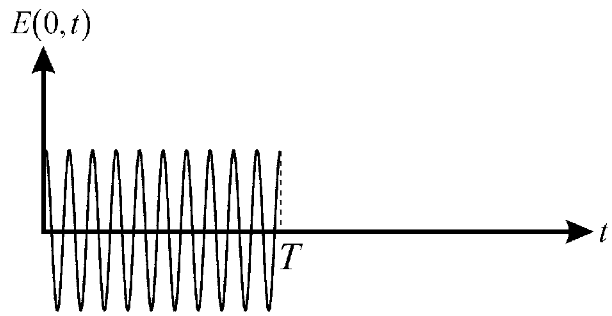

Figure 2.1: Figure S-2-1: Sinusoidal wave of limited time duration

theorem:

$$\mathrm{U}\left(x\right) = \frac{1}{\eta}\int_{-\infty}^{\infty}\left[E(x,t)\right]^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\left|\mathcal{E}(x,\omega)\right|^2}{\eta}d\omega \tag{2.2}$$

Since the integral over frequency gives the total energy, we can call $\left|\mathcal{E}(x,\omega)\right|^2/\eta$ the *energy spectral density* corresponding to $E(x,t)$.

Suppose we have a monochromatic plane wave, $E(x,t) = A_o\cos(\omega_o t - k_o x)$, for which we desire to determine the power at $x = 0$. If we insert $E(0,t) = A_o\cos(\omega_o t)$ into Eq. 2.1 or 2.2, and evaluate the integral from $-\infty$ to $+\infty$ as indicated, we find that this wave contains infinite energy. However, such a never-ending wave is a mathematical fiction; real light waves do not persist for an infinite time but instead have a finite duration. Thus, we will consider a time-limited sinusoidal wave, $E(0,t) = A_o u(t)\cos(\omega_o t)$ (Fig.S-1).

Here, the wave is turned on at time $t = 0$ and off at $t = T$, and this modulation is accounted for through the function $u(t)$. If we calculate power in the time-domain, we obtain:

$$\begin{aligned}\mathrm{U}\left(0\right) = \frac{1}{\eta}\int_{-\infty}^{\infty}\left[E(0,t)\right]^2 dt &= \frac{A_o^2}{\eta}\int_0^T\left[\cos(\omega_o t)\right]^2 dt = \frac{A_o^2}{\eta}\int_0^T\left[\frac{1}{2}+\frac{1}{2}\cos(2\omega_o t)\right]\,dt \\ &= \frac{A_o^2 T}{2\eta}\left[1+\frac{\sin(2\omega_o T)}{2\omega_o T}\right] = \frac{A_o^2 T}{2\eta}\left[1+\mathrm{sinc}(2\omega_o T)\right]\end{aligned} \tag{2.3}$$

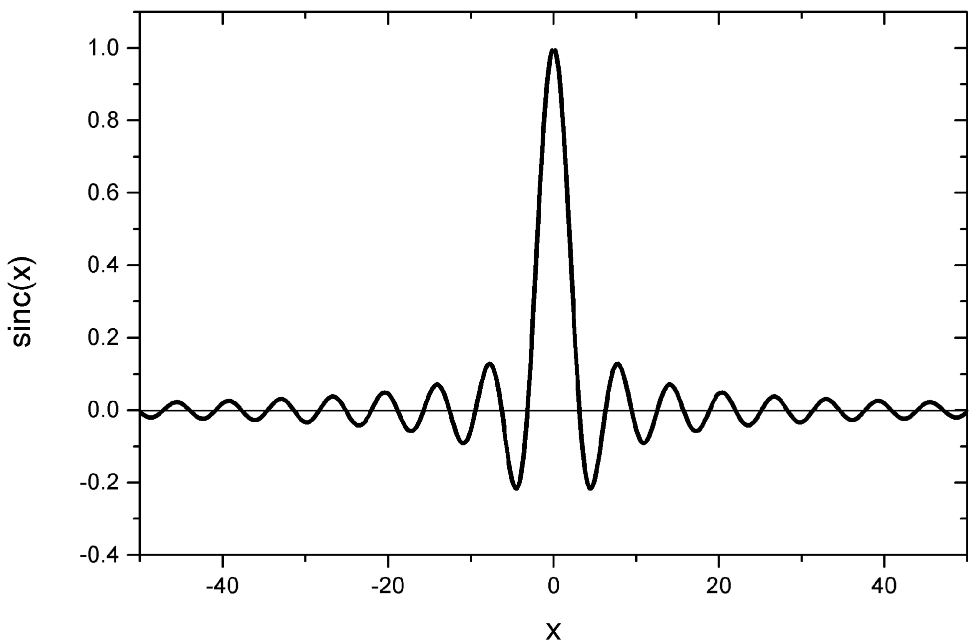

Figure 2.2: Figure S-2-2:The sinc function

If, as is usually the case, there are a large number of cycles in the wave, such that $\omega_o T \gg 1$, the sinc term above approaches zero (see Fig. S-2-2), and we have $\mathrm{U}(0) = \frac{A_o^2 T}{2\eta}$.

Alternatively, we can find the energy from the frequency domain. To find the Fourier transform corresponding to $E(0,t)$ we use the following relationship:

$$\begin{aligned}\mathcal{E}(0,\omega) &= \mathfrak{F}\{A_o u(t)\cos(\omega_o t)\} = \frac{A_o}{2\pi}\mathfrak{F}\{u(t)\} * \mathfrak{F}\{\cos(\omega_o t)\} \\ &= \frac{A_o}{2}\mathcal{U}(\omega) * [\delta(\omega-\omega_o) + \delta(\omega+\omega_o)] = \pi A_o \left\{\frac{1}{2\pi}[\mathcal{U}(\omega-\omega_o) + \mathcal{U}(\omega+\omega_o)]\right\}\end{aligned} \tag{2.4}$$

where $\mathcal{U}(\omega)$ is the Fourier transform of $u(t)$:

$$\mathcal{U}(\omega) = \int_{-\infty}^{\infty} u(t)e^{-j\omega t}dt = T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)e^{\frac{-j\omega T}{2}} \tag{2.5}$$

Note that $\lim_{T\to\infty}\frac{\mathcal{U}(\omega)}{2\pi} = \delta(\omega)$, so that as $T\to\infty$, $\mathcal{E}(0,\omega) \to \pi A_o[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]$,

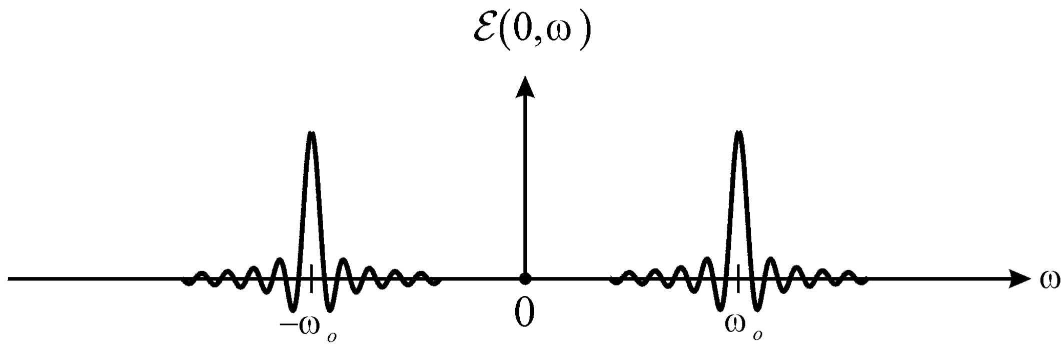

Figure 2.3: Figure S-2-3: Fourier transform of the signal in Figure S-2-1.

as might be expected. The frequency spectrum of this signal is shown in Fig.S-2-3.

If we again assume that $\omega_o T \gg 1$, so that the peaks are well-separated, the integral of Eq. 2.2 becomes:

$$\mathrm{U} = \frac{1}{2\pi\eta} \int_{-\infty}^{\infty} |\mathcal{E}(0,\omega)|^2 \, d\omega \tag{2.6}$$

$$= \frac{A_o^2 T^2}{8\pi\eta} \int_{-\infty}^{\infty} \left\{ \left[\frac{\sin\left(\frac{(\omega-\omega_o)\,T}{2}\right)}{\frac{(\omega-\omega_o)\,T}{2}} \right]^2 + \left[\frac{\sin\left(\frac{(\omega-\omega_o)\,T}{2}\right)}{\frac{(\omega-\omega_o)\,T}{2}} \right]^2 \right\} d\omega \tag{2.7}$$

$$= \frac{A_o^2 T^2}{8\pi\eta} \left\{ \frac{2\pi}{T} + \frac{2\pi}{T} \right\} = \frac{A_o^2 T}{2\eta} \tag{2.8}$$

as before.

C Power

When dealing with pulsed waveforms, it is often the total energy of the pulse that is of interest. However, with continuous-wave (cw) signals, we are usually interested in measuring the average power, rather than the energy. For our truncated sinusoid, the average intensity is:

$$\mathrm{I}_{av}\left(0\right)=\frac{1}{T}\left\{\frac{1}{\eta}\int\limits_{\tau}^{\tau+T}\left[E(0,t)\right]^{2}dt\right\}=\frac{A_{o}^{2}}{2\eta} \tag{2.9}$$

How can we determine the average power using the frequency domain representation? From Eqs. 2.10 and 2.2, we can write

$$\begin{aligned}\mathrm{I}_{av}\left(0\right) &= \frac{1}{T}\int\limits_{-\infty}^{\infty}\frac{\left|E(0,t)\right|^{2}}{\eta}dt=\frac{1}{2\pi T}\int\limits_{-\infty}^{\infty}\frac{\left|\mathcal{E}(0,\omega)\right|^{2}}{\eta}d\omega & (2.10)\\ &= \int\limits_{-\infty}^{\infty}\left[\frac{\left|\mathcal{E}(0,\omega)\right|^{2}}{T\eta}\right]\ d\omega & (2.11)\end{aligned}$$

We can define the *power spectral density* corresponding to $E(z,t)$ as:

$$\mathcal{G}(0,\omega)=\left[\frac{\left|\mathcal{E}(0,\omega)\right|^{2}}{T\eta}\right] \tag{2.12}$$

so that

$$\mathrm{I}_{av}\left(0\right)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\mathcal{G}(0,\omega)\ d\omega \tag{2.13}$$

Performing this integration for our truncated cosine wave, we find again that $\mathrm{I}_{av}\left(0\right)=\frac{A_{o}^{2}}{2\eta}$.

Although real signals have a definite start and stop, for mathematical convenience it is often expedient to adopt the fiction of an unending cosinusoidal wave. If we let $T\rightarrow\infty$ in $\mathrm{U}\left(0\right)=\frac{A_{o}^{2}T}{2\eta}$ and in $\mathrm{I}_{av}\left(0\right)=\frac{A_{o}^{2}}{2\eta}$, we find that such a sinusoid has an infinite energy (as mentioned before), but a finite power. To be rigorously correct for such waves, we must define the spectral densities in terms of limits:

$$\mathrm{I}_{av}\left(0\right) = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\frac{\left|E(0,t)\right|^2}{\eta}dt = \lim_{T\to\infty}\frac{1}{2\pi T}\int_{-\infty}^{\infty}\frac{\left|\mathcal{E}(0,\omega)\right|^2}{\eta}d\omega \tag{2.14}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{T\to\infty}\left[\frac{\left|\mathcal{E}(0,\omega)\right|^2}{T\eta}\right]\ d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{G}\left(0,\omega\right)\ d\omega \tag{2.15}$$

where the power spectral density is now defined as:

$$\mathcal{G}(0,\omega) = \lim_{T\to\infty}\left[\frac{\left|\mathcal{E}(0,\omega)\right|^2}{T\eta}\right] \tag{2.16}$$

If we use the expression for $\mathcal{E}(0,\omega)$ given for the truncated sinusoid in Eqs. 2.4 and 2.6, and take the limit as $T\to\infty$, we obtain

$$\mathcal{G}(0,\omega) = \frac{\pi A_o^2}{2\eta}\left[\delta\left(\omega-\omega_o\right)+\delta\left(\omega+\omega_o\right)\right] \tag{2.17}$$

When this expression is integrated over frequency according to Eq. 2.15, we obtain the now-familiar result that $\mathrm{I}_{av}\left(0\right) = A_o^2/2\eta$.

To summarize, an electric field which has the time domain representation $E(0,t) = A_o\cos(\omega_o t)$ has the spectrum $\mathcal{E}\left(0,\omega\right) = \pi A_o\left[\delta\left(\omega-\omega_o\right)+\delta\left(\omega+\omega_o\right)\right]$ and power spectral density $\mathcal{G}(0,\omega) = \frac{\pi A_o^2}{2\eta}\left[\delta\left(\omega-\omega_o\right)+\delta\left(\omega+\omega_o\right)\right]$. In general, if we have a spectrum consisting of delta functions with the form $\mathcal{E}\left(0,\omega\right) = \widetilde{E}\delta\left(\omega-\omega_o\right)+\widetilde{E}^*\delta\left(\omega+\omega_o\right)$, the power spectral density will be $\mathcal{G}(0,\omega) = \frac{\left|\widetilde{E}\right|^2}{2\pi\eta}\left[\delta\left(\omega-\omega_o\right)+\delta\left(\omega+\omega_o\right)\right]$