Solutions to exercises

Exercise 1.1 (a) As the mass and mass accretion rate are the same in both cases, the ratio of the accretion luminosities is simply the inverse ratio of the radii:

$$\frac{L_{\mathrm{acc,\,WD}}}{L_{\mathrm{acc,\,Sun}}} = 1\,\frac{\mathrm{R}_\odot}{R_{\mathrm{WD}}} = 100.$$

The luminosity $L_{\mathrm{acc,\,WD}} = 100\,\mathrm{L}_\odot$ much exceeds the white dwarf's intrinsic luminosity except for the very youngest, hottest white dwarfs.

(b) For the neutron star the accretion luminosity is also larger than the Sun's by a factor

$$\frac{M_{\mathrm{NS}}/R_{\mathrm{NS}}}{1\,\mathrm{M}_\odot/\mathrm{R}_\odot} = \frac{1.4}{1} \times \frac{6.96 \times 10^8\,\mathrm{m}}{20 \times 10^3\,\mathrm{m}} \approx 4.9 \times 10^4.$$

So the accretion luminosity is a few times $10^4\,\mathrm{L}_\odot$; this is not much below the luminosity of the brightest, most massive stars.

Exercise 1.2 The accretion efficiency η_{acc} is defined by $L_{\mathrm{acc}} = \eta_{\mathrm{acc}}\dot{M}c^2$.

Equating this to Equation 1.3 and solving for η_{acc} gives

$$\eta_{\mathrm{acc}} = \frac{GM}{Rc^2}.$$

For a neutron star with mass $1\,\mathrm{M}_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$ and radius $10\,\mathrm{km} = 10^4\,\mathrm{m}$, this is

$$\eta_{\mathrm{acc}} = \frac{6.673 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} \times 1.99 \times 10^{30}\,\mathrm{kg}}{1 \times 10^4\,\mathrm{m} \times (2.998 \times 10^8\,\mathrm{m\,s^{-1}})^2} \approx 0.15.$$

Exercise 1.3 (a) The mass defect $\Delta m = 4.40 \times 10^{-29}\,\mathrm{kg}$ involved in the fusion of four protons into one helium nucleus translates into an energy gain of $\Delta E = \Delta mc^2$ per four protons. The energy input is the mass energy of the four protons, $4m_{\mathrm{p}}c^2$, so the efficiency = gain/input is

$$\eta_{\mathrm{H}} = \frac{\Delta mc^2}{4m_{\mathrm{p}}c^2} = \frac{4.40 \times 10^{-29}\,\mathrm{kg}}{4 \times 1.673 \times 10^{-27}\,\mathrm{kg}} \approx 0.0066.$$

(b) From part (a), the efficiency of hydrogen burning, the most common nuclear fusion reaction in the Universe, is only $\eta_{\mathrm{H}} \approx 0.7\%$. In other words, if one kilogram of hydrogen accretes onto a neutron star, it liberates about 20 times more energy (in the form of heat and radiation, say) than if this kilogram of hydrogen undergoes nuclear fusion into helium.

There is no other process in the Universe that could persistently sustain the conversion of such a large fraction of mass energy ($\gtrsim 10\%$) into energy for a macroscopic amount of mass. There are processes with 100% efficiency such as the annihilation of electron–positron pairs (see Chapters 7 and 8), but these involve antimatter, which is not abundant in the known Universe.

Exercise 1.4 The accretion disc luminosity is

$$L_{\mathrm{disc}} = \frac{1}{2}\frac{GM\dot{M}}{R}, \qquad \text{(Eqn 1.8)}$$

where R is the inner disc radius. Equating this with $L_{\rm acc} = \eta_{\rm acc}\dot{M}c^2$ and solving for $\eta_{\rm acc}$ gives

$$\eta_{\rm acc} = \frac{1}{2}\frac{GM}{Rc^2}.$$

We set $R = 3R_{\rm S}$ and use Equation 1.11 to obtain

$$\eta_{\rm acc} = \frac{GM}{6R_{\rm S}c^2} = \frac{GM}{12GMc^2/c^2} = \frac{1}{12} \approx 0.083 = 8.3\%.$$

Exercise 1.5 On the right-hand side, the first term is the gravitational potential of the primary star. The denominator is the magnitude of the vector pointing from the primary to the point of reference.

The second term is the corresponding gravitational potential of the secondary.

The third term describes the effect of the centrifugal force. The quantity $(\boldsymbol{\omega} \times (\boldsymbol{r} - \boldsymbol{r}_{\rm c}))^2$ is the scalar product of the vector $\boldsymbol{\omega} \times (\boldsymbol{r} - \boldsymbol{r}_{\rm c})$ with itself. The vector $\boldsymbol{\omega} \times (\boldsymbol{r} - \boldsymbol{r}_{\rm c})$ has the magnitude $\omega r_\perp$, where $r_\perp$ is the distance of the point of reference from the rotational axis. The vector $\boldsymbol{\omega}$ is parallel to the rotational axis and has magnitude ω, the orbital angular speed.

Exercise 1.6 If the two stars with masses M_1 and M_2 are at $x = 0$ and $x = a$, respectively, and the centre of mass is at $x = x_{\rm c}$, then $M_1 x_{\rm c} = M_2(a - x_{\rm c})$, so that $(M_1 + M_2)x_{\rm c} = M_2 a$, and hence

$$x_{\rm c} = \frac{M_2}{M_1 + M_2}a = \frac{M_2}{M}a,$$

where $M = M_1 + M_2$ is the total binary mass. This is also the distance a_1 of the primary from the centre of mass. The distance of the secondary from the centre of mass is $a_2 = a - x_{\rm c} = (M_1/M)a$.

Equation 1.16 describing the Roche potential contains the following vectors: $\boldsymbol{r} = (x, 0, 0)$, $\boldsymbol{r}_1 = (0, 0, 0)$, $\boldsymbol{r}_2 = (a, 0, 0)$, $\boldsymbol{r}_{\rm c} = (x_{\rm c}, 0, 0)$ and $\boldsymbol{\omega} = (0, 0, \omega)$. So we have $|\boldsymbol{r} - \boldsymbol{r}_1| = x$, $|\boldsymbol{r} - \boldsymbol{r}_2| = a - x$, and

$$(\boldsymbol{\omega} \times (\boldsymbol{r} - \boldsymbol{r}_{\rm c}))^2 = \omega^2(x - x_{\rm c})^2 = \omega^2\left(x - \frac{M_2}{M}a\right)^2.$$

Therefore, for $0 < x < a$, the Roche potential as a function of the coordinate x is

$$\Phi_{\rm R}(x) = -\frac{GM_1}{x} - \frac{GM_2}{a - x} - \tfrac{1}{2}\omega^2\left(x - \frac{M_2}{M}a\right)^2.$$

Now in the x-direction, $\boldsymbol{\nabla}\Phi_{\rm R} = {\rm d}\Phi_{\rm R}(x)/{\rm d}x$, so

$$\frac{{\rm d}\Phi_{\rm R}(x)}{{\rm d}x} = +\frac{GM_1}{x^2} - \frac{GM_2}{(a - x)^2} - \omega^2\left(x - a\frac{M_2}{M}\right).$$

The Roche potential at the centre of mass, i.e. at $x = x_{\rm c} = aM_2/M$, is (note that $a - x_{\rm c} = aM_1/M$)

$$\Phi_{\rm R}(x_{\rm c}) = \frac{GM_1M}{M_2a} - \frac{GM_2M}{M_1a} - 0 = \frac{GM}{a}\left(\frac{M_2^2 - M_1^2}{M_1M_2}\right).$$

The gradient at $x = x_{\rm c}$ is

$$\frac{{\rm d}\Phi_{\rm R}(x)}{{\rm d}x} = \frac{GM_1M^2}{M_2^2a^2} - \frac{GM_2M^2}{M_1^2a^2} - 0 = \frac{GM^2}{a^2}\left(\frac{M_1}{M_2^2} - \frac{M_2}{M_1^2}\right).$$

So the force $\boldsymbol{F} = -m\,\mathrm{d}\Phi_\mathrm{R}(x)/\mathrm{d}x$ has magnitude

$$F = \frac{GmM^2}{a^2}\left|\frac{M_1}{M_2^2} - \frac{M_2}{M_1^2}\right|$$

and it is in the $-x$-direction for $M_1 > M_2$ ($+x$-direction for $M_1 < M_2$).

Exercise 1.7 We have $1\,\mathrm{pc} = 3.086 \times 10^{16}\,\mathrm{m}$ and $1\,\mathrm{AU} = 1.496 \times 10^{11}\,\mathrm{m}$, so $1\,\mathrm{pc} = 2.063 \times 10^5\,\mathrm{AU}$. Therefore $l = 2 \times 10^3\,\mathrm{AU} \approx 10^{-2}\,\mathrm{pc}$.

Exercise 1.8 The size of the emitting region is

$$r \approx 30\ \text{light-days} = 30 \times 86\,400 \times 3 \times 10^8\,\mathrm{m} \approx 10^{15}\,\mathrm{m}.$$

From Equation 1.19, the mass is

$$M \simeq \frac{\langle v^2\rangle r}{G} = \frac{(6\times 10^6)^2\,\mathrm{m^2\,s^{-2}} \times 10^{15}\,\mathrm{m}}{6.673\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}} = 5.4\times 10^{38}\,\mathrm{kg} \simeq 3\times 10^8\,\mathrm{M_\odot}.$$

Exercise 1.9 (a) We recall that $1\,\mathrm{M_\odot\,yr^{-1}} = 6.31 \times 10^{22}\,\mathrm{kg\,s^{-1}}$ (Equation 1.4). From Equation 1.21 we find

$$(2\times T_\mathrm{peak})^4 \simeq \frac{3\times 6.673\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}}{8\pi\times 5.671\times 10^{-8}\,\mathrm{J\,m^{-2}\,K^{-4}\,s^{-1}}}\ \frac{0.6\times 1.99\times 10^{30}\,\mathrm{kg}\times 10^{-9}\times 6.31\times 10^{22}\,\mathrm{kg\,s^{-1}}}{(8.7\times 10^6\,\mathrm{m})^3},$$

which gives $T_\mathrm{peak} \approx 3.2 \times 10^4\,\mathrm{K}$.

(b) For the neutron star we have instead

$$T_\mathrm{peak} \simeq 0.5\times\left(\frac{3\times 6.673\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}\times 1.4\times 1.99\times 10^{30}\,\mathrm{kg}\times 10^{-8}\times 6.31\times 10^{22}\,\mathrm{kg\,s^{-1}}}{8\pi\times 5.671\times 10^{-8}\,\mathrm{J\,m^{-2}\,K^{-4}\,s^{-1}}\times(10^4\,\mathrm{m})^3}\right)^{1/4}$$

or $T_\mathrm{peak} \approx 1.1 \times 10^7\,\mathrm{K}$.

Exercise 1.10 (a) Using Equations 1.21 and 1.11, we find

$$\begin{aligned}(2\times T_\mathrm{peak})^4 &\simeq \frac{3GM\dot{M}}{8\pi\sigma(3R_\mathrm{S})^3} = \frac{3GM\dot{M}}{8\pi\sigma\times 3^3\times(2GM/c^2)^3} = \frac{c^6\dot{M}}{576\,\pi G^2\sigma M^2}\\
&= \frac{c^6\times\mathrm{M_\odot\,yr^{-1}}}{576\,\pi G^2\sigma\,\mathrm{M_\odot^2}}\left(\frac{M}{\mathrm{M_\odot}}\right)^{-2}\left(\frac{\dot{M}}{\mathrm{M_\odot\,yr^{-1}}}\right)\\
&= \frac{(2.998\times 10^8\,\mathrm{m})^6\times 6.31\times 10^{22}\,\mathrm{kg\,s^{-1}}}{576\,\pi\times(6.673\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}})^2\times 5.671\times 10^{-8}\,\mathrm{J\,m^{-2}\,K^{-4}\,s^{-1}}\times(1.99\times 10^{30}\,\mathrm{kg})^2}\\
&\quad\times\left(\frac{M}{\mathrm{M_\odot}}\right)^{-2}\left(\frac{\dot{M}}{\mathrm{M_\odot\,yr^{-1}}}\right).\end{aligned}$$

This gives

$$T_\mathrm{peak} \simeq 1.1\times 10^9\,\mathrm{K}\left(\frac{M}{\mathrm{M_\odot}}\right)^{-1/2}\left(\frac{\dot{M}}{\mathrm{M_\odot\,yr^{-1}}}\right)^{1/4}. \qquad (1.17)$$

Note that the actual peak temperature is slightly different from this value because general relativistic corrections have to be applied.

(b) For $M_1 = 10\,\mathrm{M}_\odot$ and $\dot{M} = 10^{-7}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$, Equation 1.17 becomes

$$T_{\text{peak}} \simeq 1.1 \times 10^9\,\mathrm{K} \times 10^{-7/4} \times 10^{-1/2} = 6 \times 10^6\,\mathrm{K}.$$

(c) For $M_1 = 10^7\,\mathrm{M}_\odot$ and $\dot{M} = 1\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$, Equation 1.17 becomes

$$T_{\text{peak}} \simeq 1.1 \times 10^9\,\mathrm{K} \times 10^{-7/2} = 3 \times 10^5\,\mathrm{K}.$$

Exercise 1.11 We have $1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$ and $T \simeq E_{\text{ph}}/k$. So for $E_{\text{ph}} = 1\,\mathrm{eV}$ the temperature is

$$T \simeq \frac{1\,\mathrm{eV}}{1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}} = \frac{1.602 \times 10^{-19}\,\mathrm{J}}{1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}} = 1.160 \times 10^4\,\mathrm{K},$$

which is of order $10^4\,\mathrm{K}$.

Exercise 1.12 (a) We make use of Equation 1.25 (rather than the rule of thumb) to work out the typical photon energy:

$$\frac{E_{\text{ph}}}{\mathrm{eV}} = 2.70 \times \frac{1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}}{1.602 \times 10^{-19}\,\mathrm{J}} \times T = 2.3 \times 10^{-4}\,\mathrm{K^{-1}} \times T.$$

For $T = 3.2 \times 10^4\,\mathrm{K}$ (white dwarf) this gives $E_{\text{ph}} = 7.4\,\mathrm{eV}$, while for $T = 1.1 \times 10^7\,\mathrm{K}$ (neutron star) we obtain $E_{\text{ph}} = 2.5\,\mathrm{keV}$ (these temperatures were found in Exercise 1.9).

Also, for $T = 6 \times 10^6\,\mathrm{K}$ (stellar mass black hole) this gives $E_{\text{ph}} = 1.4\,\mathrm{keV}$, while for $T = 3 \times 10^5\,\mathrm{K}$ (AGN) we obtain $E_{\text{ph}} = 69\,\mathrm{eV}$ (these temperatures were found in Exercise 1.10).

(b) Photon energy and wavelength λ are related as

$$E_{\text{ph}} = h\frac{c}{\lambda}.$$

With Equation 1.25 this gives

$$\lambda = \frac{hc}{2.7kT} = \frac{6.626 \times 10^{-34}\,\mathrm{J\,s} \times 2.998 \times 10^8\,\mathrm{m\,s^{-1}}}{2.7 \times 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}} \times \frac{1}{T} = 5.3275 \times 10^{-3}\,\mathrm{mK} \times \frac{1}{T}.$$

For $T = 3.2 \times 10^4\,\mathrm{K}$ (white dwarf) this gives $\lambda = 1.7 \times 10^{-7}\,\mathrm{m}$. For $T = 1.1 \times 10^7\,\mathrm{K}$ (neutron star) we obtain $\lambda = 4.8 \times 10^{-10}\,\mathrm{m} = 0.48\,\mathrm{nm}$. These wavelengths are much shorter than the wavelengths of visible light (≈ 400–$800\,\mathrm{nm}$). The first is in the classical X-ray range, the second in the ultraviolet range.

For the accreting stellar mass black hole we find $\lambda = 0.9\,\mathrm{nm}$ (soft X-rays), while for the AGN we obtain $\lambda = 18\,\mathrm{nm}$ (near the ultraviolet/X-ray boundary).

Exercise 1.13 For $h\nu \gg kT$ we have $\exp(h\nu/kT) \gg 1$ and hence $\exp(h\nu/kT) - 1 \simeq \exp(h\nu/kT)$, so that the Planck function becomes the Wien tail (Equation 1.27).

For the case $h\nu \ll kT$ we introduce the quantity $x = h\nu/kT$. As $x \ll 1$ we can use the first-order expansion $\exp(x) \simeq 1 + x$ to obtain for the denominator in Equation 1.23 $\exp(h\nu/kT) - 1 \simeq h\nu/kT$. Hence

$$B_\nu(T) \simeq \frac{2h\nu^3}{c^2} \times \frac{kT}{h\nu} = 2kT\nu^2/c^2,$$

confirming Equation 1.28 for the Rayleigh–Jeans tail.

Exercise 2.1 The results are given in the following table.

q	$f(q)$ Eggleton	$f(q)$ Paczyński	$\Delta f/f$ in %
0.5	0.3208	0.3203	0.14
1.0	0.3789	0.3667	3.2
2.0	0.4400	0.4036	8.3

The last column denotes the difference between the f values calculated according to Equation 2.7 (column 2) and Equation 2.8 (column 3), divided by the value in column 2, expressed in %.

Given that Eggleton's approximation is accurate to within 1%, it is clear that Paczyński's relation is at most 4% off for $q < 1$, and even for $q = 2$ it is good to within 9%.

Exercise 2.2 (a) We solve Kepler's law for the period,

$$P_{\text{orb}}^2 = a^3 \frac{4\pi^2}{GM},$$

and multiply both the numerator and denominator on the right-hand side by $(R_{\text{L},2}/a)^3$. Hence

$$P_{\text{orb}}^2 = \frac{a^3 4\pi^2}{GM} \frac{(R_{\text{L},2}/a)^3}{(R_{\text{L},2}/a)^3} = \frac{R_{\text{L},2}^3\, 4\pi^2}{GM} \frac{1}{(R_{\text{L},2}/a)^3}.$$

Inserting Paczyński's approximation for $R_{\text{L},2}/a$ gives

$$P_{\text{orb}}^2 \simeq \frac{R_{\text{L},2}^3\, 4\pi^2}{GM} \frac{M}{0.462^3 M_2} = \frac{4\pi^2}{0.462^3\, G} \frac{R_{\text{L},2}^3}{M_2}.$$

Taking the square root and noting that

$$\overline{\rho} = \frac{M_2}{(4\pi/3) \times R_{\text{L},2}^3}$$

(the stellar radius R_2 equals the Roche-lobe radius $R_{\text{L},2}$), we have

$$P_{\text{orb}} \approx \left(\frac{4\pi^2}{0.462^3\, G}\right)^{1/2} \left(\frac{4\pi}{3}\overline{\rho}\right)^{-1/2} = \left(\frac{3\pi}{0.462^3\, G}\right)^{1/2} \overline{\rho}^{\,-1/2}.$$

So

$$\frac{P_{\text{orb}}}{\text{h}} = \frac{P_{\text{orb}}}{3600\,\text{s}} = \frac{1}{3600\,\text{s}} \left(\frac{3\pi}{0.462^3 \times 6.673 \times 10^{-11}\,\text{N}\,\text{m}^2\,\text{kg}^{-2}}\right)^{1/2} \times \left(\frac{\overline{\rho}}{10^3\,\text{kg}\,\text{m}^{-3}} \times 10^3\,\text{kg}\,\text{m}^{-3}\right)^{-1/2}$$

and hence

$$\frac{P_{\text{orb}}}{\text{h}} \cong 10.5 \left(\frac{\overline{\rho}}{10^3\,\text{kg}\,\text{m}^{-3}}\right)^{-1/2},$$

as required.

(b) We used Paczyński's approximation for $R_{\text{L},2}/a$, so Equation 2.9 is valid only in the range of mass ratios q where this approximation is good, i.e. for $q \lesssim 0.8$.

Exercise 2.3 In the previous exercise we obtained the expression

$$P_{\text{orb}}^2 \simeq \frac{4\pi^2}{0.462^3\, G} \frac{R_{\text{L},2}^3}{M_2}$$

with R_2 instead of $R_{\text{L},2}$, so

$$P_{\text{orb}} \simeq \frac{2\pi}{0.462^{3/2}\, G^{1/2}} \frac{R_2^{3/2}}{M_2^{1/2}}$$

or

$$\frac{P_{\text{orb}}}{\text{h}} \simeq \frac{2\pi \mathrm{R}_\odot^{3/2}}{3600\,\text{s} \times 0.462^{3/2}\, (G\,\mathrm{M}_\odot)^{1/2}} \left(\frac{R_2}{\mathrm{R}_\odot}\right)^{3/2} \left(\frac{M_2}{\mathrm{M}_\odot}\right)^{-1/2}.$$

With the dimensionless constant

$$k_2 = \frac{2\pi \mathrm{R}_\odot^{3/2}}{3600\,\text{s} \times 0.462^{3/2}\, (G\,\mathrm{M}_\odot)^{1/2}} \approx 8.856$$

this becomes

$$\frac{P_{\text{orb}}}{\text{h}} \simeq k_2 \left(\frac{R_2}{\mathrm{R}_\odot}\right)^{3/2} \left(\frac{M_2}{\mathrm{M}_\odot}\right)^{-1/2}.$$

Taking logs in this equation reproduces Equation 2.10, as $\log_{10} k_2 = 0.9472$.

Exercise 2.4 Consider two point masses M_1 and M_2 on circular orbits around the common centre of mass, with separation a. The masses M_1 and M_2 execute circular orbits with radii a_1 and a_2, respectively, and angular speed ω about the common centre of mass. The total orbital angular momentum in the system is then $J = M_1 a_1^2 \omega + M_2 a_2^2 \omega$. From $a = a_1 + a_2$ and $a_1 M_1 = a_2 M_2$ we find $a_1 = (M_2/M)a$ and $a_2 = (M_1/M)a$. Using Kepler's law we have $\omega^2 = 4\pi^2/P^2 = GM/a^3$, so

$$J = M_1 \left(\frac{M_2}{M}\right)^2 a^2 \frac{(GM)^{1/2}}{a^{3/2}} + M_2 \left(\frac{M_1}{M}\right)^2 a^2 \frac{(GM)^{1/2}}{a^{3/2}} = \frac{M_1 M_2 (Ga)^{1/2}}{M^{3/2}} (M_2 + M_1)$$

$$= M_1 M_2 \left(\frac{Ga}{M}\right)^{1/2},$$

which reproduces Equation 2.17.

Exercise 2.5 Paczyński's approximation for the Roche-lobe radius is

$$R_{\text{L},2} \approx 0.462 \left(\frac{M_2}{M}\right)^{1/3} a. \qquad \text{(Eqn 2.8)}$$

Taking the logarithmic derivative gives

$$\frac{\dot{R}_{\text{L},2}}{R_{\text{L},2}} = \frac{1}{3}\frac{\dot{M}_2}{M_2} - \frac{1}{3}\frac{\dot{M}}{M} + \frac{\dot{a}}{a} = \frac{1}{3}\frac{\dot{M}_2}{M_2} + \frac{\dot{a}}{a}, \qquad (2.18)$$

as $\dot{M} = 0$ for conservative mass transfer. To find the logarithmic derivative of a, we solve the expression for the orbital angular momentum (Equation 2.17) for a,

$$a = \frac{J^2 M}{G M_1^2 M_2^2},$$

and take its logarithmic derivative:

$$\frac{\dot{a}}{a} = 2\frac{\dot{J}}{J} + \frac{\dot{M}}{M} - 2\frac{\dot{M}_1}{M_1} - 2\frac{\dot{M}_2}{M_2} = 2\frac{\dot{J}}{J} + 2\frac{\dot{M}_2}{M_1} - 2\frac{\dot{M}_2}{M_2} \tag{2.19}$$

($\dot{M} = 0$, $\dot{M}_1 = -\dot{M}_2$). Substituting from Equation 2.19 into Equation 2.18 gives

$$\frac{\dot{R}_{\mathrm{L},2}}{R_{\mathrm{L},2}} = \left(\tfrac{1}{3} - 2\right)\frac{\dot{M}_2}{M_2} + 2\frac{M_2}{M_1}\frac{\dot{M}_2}{M_2} + 2\frac{\dot{J}}{J}.$$

Collecting terms gives

$$\frac{\dot{R}_{\mathrm{L},2}}{R_{\mathrm{L},2}} = 2\frac{\dot{J}}{J} + \left(2q - \tfrac{5}{3}\right)\frac{\dot{M}_2}{M_2}.$$

Comparing this with Equation 2.18 shows that $\zeta_{\mathrm{L}} = 2q - 5/3$.

Exercise 2.6 The table below gives a representative radius of the $5\,\mathrm{M}_\odot$ star at the beginning of the corresponding mass transfer case. The orbital period P_{orb} was calculated using Equation 2.10. The mass ratio is just larger than 1, so it is still acceptable to use Paczyński's approximation (Equation 2.8), and Equation 2.10 does indeed use this approximation.

Case	$\log_{10} R/\mathrm{R}_\odot$	$R/\mathrm{R}_\odot$	P_{orb}
A	0.5	3	22.3 h
B	1.0	10	5.2 d
C	2.0	100	165 d

If both stars formed at the same time, and this is the first time the system experiences mass transfer, then a mass ratio $q < 1$ is unphysical because the more massive binary component evolves faster and fills its Roche lobe first. So at the start of a case A, B or C mass transfer, the mass ratio is > 1. (There are exceptions, however, such as systems where very strong wind losses have reduced the mass of the primary so much that, at the point of first contact with its Roche lobe, it is less massive than the less evolved secondary star.)

Exercise 2.7 For the Sun we have

$$t_{\mathrm{KH}} = \frac{6.673 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} \times (1.99 \times 10^{30}\,\mathrm{kg})^2}{6.96 \times 10^8\,\mathrm{m} \times 3.83 \times 10^{26}\,\mathrm{J\,s^{-1}}} = 9.91 \times 10^{14}\,\mathrm{s} = 3.1 \times 10^7\,\mathrm{yr}.$$

With $R \propto M$ and $L \propto M^4$, we also have

$$t_{\mathrm{KH}} \propto \frac{M^2}{RL} \propto \frac{M^2}{MM^4} \propto M^{-3},$$

so

$$t_{\mathrm{KH}} \simeq 3.1 \times 10^7\,\mathrm{yr}\left(\frac{M_2}{\mathrm{M}_\odot}\right)^{-3}.$$

Therefore the Kelvin–Helmholtz time for a $0.5\,\mathrm{M}_\odot$ main-sequence star is about 2.5×10^8 yr, while for a $5\,\mathrm{M}_\odot$ main-sequence star it is about 2.5×10^5 yr.

Exercise 2.8 The mass transfer rate is

$$\frac{\dot{M}_2}{M_2} = \frac{2\dot{J}_{\mathrm{sys}}/J - (\dot{R}_2/R_2)_{\mathrm{nuc}}}{\zeta - \zeta_{\mathrm{L}}}. \qquad \text{(Eqn 2.22)}$$

By assumption we have $\zeta - \zeta_L = 0 - \zeta_L \simeq 1$, $\dot{J}_{sys}/J = 0$, $(\dot{R}_2/R_2) = 1/t_{th}$, and therefore $-\dot{M}_2 = M_2/t_{th}$. The thermal time t_{th} is just the Kelvin–Helmholtz time,

$$t_{KH} \simeq 3.1 \times 10^7\,\mathrm{yr}\left(\frac{M_2}{\mathrm{M}_\odot}\right)^{-3}$$

(see Exercise 2.7). So, Equation 2.22 becomes

$$-\dot{M}_2(\text{case B}) \simeq 3 \times 10^{-8}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1} \times \left(\frac{M_2}{\mathrm{M}_\odot}\right)^4. \tag{2.20}$$

(Hence the case B transfer rate is $2 \times 10^{10}/3.1 \times 10^7 \approx 6 \times 10^2$ times larger than the case A rate; see Worked Example 2.2.) For $M_2 = 0.5\,\mathrm{M}_\odot$, $1\,\mathrm{M}_\odot$, $5\,\mathrm{M}_\odot$ this is 2×10^{-9}, 3×10^{-8}, $2 \times 10^{-5}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$, respectively.

Exercise 2.9 For conservative mass transfer and $\zeta = 1$, Equation 2.22 becomes

$$\frac{-\dot{M}_2}{M_2} = \frac{-\dot{J}_{GR}/J}{4/3 - M_2/M_1}.$$

According to Equation 2.11 we also have $M_2/\mathrm{M}_\odot \simeq P_{orb}/8.8\,\mathrm{h} = 0.23$. Hence Equation 2.25 becomes

$$\frac{\dot{J}_{GR}}{J} = -1.27 \times 10^{-8}\,\mathrm{yr}^{-1} \times \frac{1 \times 0.23}{1.23^{1/3}} \times 2^{-8/3} = -4.29 \times 10^{-10}\,\mathrm{yr}^{-1},$$

so putting this value into Equation 2.22

$$\frac{-\dot{M}_2}{0.23\,\mathrm{M}_\odot} = \frac{4.29 \times 10^{-10}\,\mathrm{yr}^{-1}}{4/3 - 0.23/1}.$$

This gives $-\dot{M}_2 = 8.9 \times 10^{-11}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$.

Exercise 2.10 Assuming conservative mass transfer, the stability criterion requires $q \lesssim 1$, or $M_2 < M_1 = 1.4\,\mathrm{M}_\odot$. The longest orbital period is realized for the most massive donor that still allows stable mass transfer, so $P_{orb} \simeq 8.8\,\mathrm{h} \times 1.4 = 12\,\mathrm{h}$ (Equation 2.11). A note of caution: the actual radius of a $1.4\,\mathrm{M}_\odot$ main-sequence star can be up to a factor of 2 larger than what was assumed in Equation 2.11, so the period could be up to $2^{1.5} \approx 3$ times longer than the value that we have just calculated.

Exercise 2.11 The orbital speed v of the companion is given by Equation 2.1, but applied to the star with mass M_1 instead of M_2. As by assumption $M_2 \ll M_1$, we can set $M_1/(M_1 + M_2) \simeq 1$, so

$$v = \left(\frac{GM_1}{a}\right)^{1/2},$$

where a is the orbital separation of the circular pre-supernova orbit. The escape speed of the companion from the binary is

$$v_{esc} = \left(\frac{2GM_1}{a}\right)^{1/2}. \tag{Eqn 1.10}$$

Immediately after a prompt supernova explosion, the orbital speed of the companion is still v as calculated above, but the escape speed has changed. The binary will remain bound if v is smaller than the new escape speed, i.e.

$$\left(\frac{GM_1}{a}\right)^{1/2} < \left(\frac{2G(M_1 - \Delta M)}{a}\right)^{1/2}.$$

Here ΔM is the mass ejected in the supernova explosion, so the primary has a post-supernova mass $M_1 - \Delta M$. Therefore $M_1 < 2(M_1 - \Delta M)$ or $\Delta M < M_1/2$.

Exercise 3.1 A rather famous differentially rotating body is the Sun. This can be seen when groups of sunspots move across the disc of the Sun. Sunspots at higher latitudes lag behind sunspots that are closer to the equatorial region. Hence the angular velocity in equatorial regions is larger than in polar regions.

An indirect example of differential rotation can be seen when sprinters in separate lanes follow the curve of a stadium (e.g. in a 400 m heat). Even if the athletes in the inner and outer lanes run at the same speed, if they start at the same point, the one in the inner lane will be ahead of the one in the outer lane as the inner lane is closer to the centre of the circle that defines the bend. The angular velocity of the inner sprinter is larger than that of the outer sprinter, so the group of sprinters 'rotates' differentially. (Of course, to compensate for this, the lanes are staggered so that the sprinter in the inner lane starts further back than the one in the outer lane.)

Exercise 3.2 The product rule gives

$$\frac{\partial(r\omega)}{\partial r} = r\frac{\partial\omega}{\partial r} + \omega\frac{\partial r}{\partial r} = r\frac{\partial\omega}{\partial r} + \omega,$$

and this is non-zero (i.e. equal to ω) even in the *absence* of shear. But viscous stresses exist only in the *presence* of shearing motion. If $\omega =$ constant, there is no shear, hence no stress, so we *must* have $\sigma_\mathrm{s} = 0$ in this case. Therefore only the first term, $r\,\partial\omega/\partial r$, can contribute to the shear stress σ_s.

Exercise 3.3 Equation 3.8 reads

$$G_\mathrm{vis} = 2\pi r\,\nu_\mathrm{vis}\,\Sigma r^2\,\frac{\partial\omega}{\partial r}.$$

The unit of the right-hand side is

$$\mathrm{m} \times (\mathrm{m} \times \mathrm{m\,s^{-1}}) \times (\mathrm{kg\,m^{-2}}) \times \mathrm{m}^2 \times (\mathrm{s^{-1}\,m^{-1}}).$$

Collecting terms, this is $\mathrm{m^{5-3}\,s^{-2}\,kg} = \mathrm{m^2\,s^{-2}\,kg}$.

With torque $=$ force $\times$ distance, the corresponding unit is

$$\mathrm{N} \times \mathrm{m} = (\mathrm{kg\,m\,s^{-2}}) \times \mathrm{m} = \mathrm{kg\,m^2\,s^{-2}},$$

as above.

Exercise 3.4 In the case of Keplerian motion the angular speed is (Equation 3.3)

$$\omega = \left(\frac{GM}{r^3}\right)^{1/2}.$$

So the radius derivative is

$$\frac{\mathrm{d}\omega}{\mathrm{d}r} = (GM)^{1/2}\left(-\tfrac{3}{2}r^{-5/2}\right).$$

Inserting this into Equation 3.11 gives

$$D(r) = \tfrac{1}{2}\nu_\mathrm{vis}\,\Sigma r^2 GM\left(-\tfrac{3}{2}\right)^2 r^{-5}.$$

Hence

$$D(r) = \tfrac{9}{8}\nu_{\text{vis}}\,\Sigma\frac{GM}{r^3},$$

as required.

Exercise 3.5 Equation 3.17 describes the conservation of angular momentum in the disc:

$$r\frac{\partial}{\partial t}(\Sigma r^2\omega) + \frac{\partial}{\partial r}(rv_r\Sigma r^2\omega) = \frac{1}{2\pi}\frac{\partial G_{\text{vis}}}{\partial r}.$$

The two terms on the left-hand side describe the angular momentum balance when $\partial G_{\text{vis}}/\partial r = 0$, i.e. in the absence of the so-called source term on the right-hand side. In this case the angular momentum J of a disc ring between radii r and $r + \Delta r$ changes only if there is an imbalance between the angular momentum that flows into the ring via the mass that flows into the ring, and the angular momentum leaving the ring via the mass flowing out of the ring. We find the flow rate of angular momentum at radius r by multiplying the mass flow rate $\mathrm{d}M/\mathrm{d}t$ with the specific angular momentum that this mass has. The specific angular momentum is just $r^2\omega$, and $\mathrm{d}M/\mathrm{d}t$ is given by Equation 3.15 as

$$\dot{M}(r,t) = -2\pi r v_r\Sigma.$$

So the local flow rate of angular momentum is just

$$\dot{J} = -2\pi r v_r\Sigma r^2\omega.$$

To work out the net change ΔJ in the angular momentum J of the disc ring due to this mass flow in a small time interval Δt, we take the difference between the local flow rates at $r + \Delta r$ and r, and multiply it by Δt. This can be written as

$$\Delta J = \left[\dot{J}(r+\Delta r,t) - \dot{J}(r,t)\right] \times \Delta t \approx \Delta r\,\frac{\partial\dot{J}}{\partial r} \times \Delta t.$$

Hence

$$\frac{\Delta J}{\Delta t} = \Delta r\,\frac{\partial\dot{J}}{\partial r} = -\Delta r\,\frac{\partial(2\pi r v_r\Sigma r^2\omega)}{\partial r}.$$

This becomes

$$\frac{\Delta J}{\Delta t} = -2\pi\,\Delta r\,\frac{\partial(v_r r\Sigma r^2\omega)}{\partial r}. \qquad (3.21)$$

On the other hand, the total angular momentum J in the disc ring is

$$J = \text{mass in the ring} \times \text{specific angular momentum} = 2\pi r\,\Delta r \times \Sigma \times r^2\omega.$$

Hence the time derivative of J can be written as

$$\frac{\partial J}{\partial t} = -2\pi r\,\Delta r\,\frac{\partial}{\partial t}(\Sigma r^2\omega). \qquad (3.22)$$

Note that r and Δr are not affected by the partial derivative with respect to t, as by definition this has to be taken for fixed r. For small time intervals Δt, the expression $\Delta J/\Delta t$ in Equation 3.21 becomes the derivative $\partial J/\partial t$. Equating the right-hand side of Equation 3.22 with the right-hand side of Equation 3.21, and dividing by $2\pi\,\Delta r$, finally reproduces the first two terms in Equation 3.17.

Exercise 3.6 Equation 3.27 describes the luminosity of a disc ring with inner radius r_1 and outer radius r_2:

$$L(r_1, r_2) = \frac{3GM\dot{M}}{2}\left\{\frac{1}{r_1}\left[1 - \frac{2}{3}\left(\frac{R_1}{r_1}\right)^{1/2}\right] - \frac{1}{r_2}\left[1 - \frac{2}{3}\left(\frac{R_1}{r_2}\right)^{1/2}\right]\right\}.$$

We obtain the luminosity of the whole disc if we set r_1 equal to the radius of the accreting object (or inner rim of the accretion disc, if this is different), and r_2 to infinity. This is appropriate for an idealized, infinitely extended disc. A real disc in, for example, a binary system is limited by the size of the Roche lobe of the accreting star. But even in that case the choice $r_2 = \infty$ is usually a rather good approximation, as $r_2 \gg r_1$. So, with $r_1 = R_1$ and $r_2 = \infty$ we have

$$L_{\text{disc}} = L(R_1, \infty) = \frac{3GM\dot{M}}{2}\left\{\frac{1}{R_1}\left[1 - \frac{2}{3}\right] - \frac{1}{\infty}\left[1 - \frac{2}{3}\left(\frac{R_1}{\infty}\right)^{1/2}\right]\right\}.$$

Clearly the second term in curly brackets is identical to 0 (division by ∞). So

$$L_{\text{disc}} = \frac{3GM\dot{M}}{2}\left\{\frac{1}{3R_1} - 0\right\} = \frac{GM\dot{M}}{2R_1}.$$

Exercise 3.7 (a) Introducing T_* into Equation 3.28 gives

$$T_{\text{eff}}^4(r) = T_*^4\left(\frac{R_1}{r}\right)^3\left[1 - \left(\frac{R_1}{r}\right)^{1/2}\right]$$

or

$$\frac{T_{\text{eff}}^4(r)}{T_*^4} = \left(\frac{R_1}{r}\right)^3\left[1 - \left(\frac{R_1}{r}\right)^{1/2}\right].$$

(b) Hence with $y = (T_{\text{eff}}/T_*)^4$ and $x = r/R_1$ we have

$$y(x) = x^{-3}(1 - x^{-1/2}) = x^{-3} - x^{-3.5}.$$

(c) The maximum value of y is reached at a point x_0 where $\mathrm{d}y/\mathrm{d}x = 0$. As

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^{-4} - (-3.5)x^{-4.5},$$

we have at the maximum

$$0 = -3x_0^{-4} - (-3.5)x_0^{-4.5}.$$

Solving for x_0, this becomes

$$3x_0^{-4} = 3.5x_0^{-4.5} \quad \text{or} \quad x_0^{1/2} = 3.5/3,$$

hence $x_0 = (7/6)^2$.

(d) The function $(T_{\text{eff}}/T_*)^4$ attains a maximum value at the same radius as T_{eff} itself does. This radius is $r_0 = R_1 x_0$. Hence

$$r_0 = R_1 \times \left(\tfrac{7}{6}\right)^2 = \tfrac{49}{36}R_1,$$

as requested.

(e) Inserting the value for x_0 in the expression for y gives

$$y(x_0) = x_0^{-3} - x_0^{-3.5} = \left(\tfrac{6}{7}\right)^6 - \left(\tfrac{6}{7}\right)^7 = \left(\tfrac{6}{7}\right)^6 \left(1 - \tfrac{6}{7}\right) = \left(\tfrac{6}{7}\right)^6 \times \tfrac{1}{7}.$$

The maximum temperature $T_{\text{eff}} = y(x_0)^{1/4} T_*$ is therefore

$$T_{\text{eff}} = \frac{(6/7)^{3/2}}{7^{1/4}} T_* \simeq 0.488 T_*.$$

Exercise 3.8 The prime observational quantity is the energy flux through the surface area (Equation 3.11). In a steady-state disc this is *independent* of viscosity (Equation 3.24) because in a steady state, the viscosity *must* adjust itself to obey the equilibrium condition for the surface density and mass accretion rate expressed in Equation 3.23. So no matter what mechanism is causing the viscosity, the value of $\nu_{\text{vis}}\,\Sigma$ is always the same.

As a further consequence the surface temperature of a steady-state disc, which is in principle accessible via the emitted spectrum, is also *independent* of the viscosity (Equation 3.28).

Exercise 3.9 We have

$$\frac{H}{r} \simeq \frac{c_{\text{s}}}{v_{\text{K}}} \qquad \text{(Eqn 3.35)}$$

so we need to estimate the sound speed and the Keplerian speed at r_{o}. From Equation 3.32 with $T = 10^4\,\text{K}$ we have $c_{\text{s}} \simeq 10^4\,\text{m s}^{-1}$, while from Equation 1.5 we obtain

$$\begin{aligned} v_{\text{K}} &= \left(\frac{GM}{r}\right)^{1/2} \\ &= \left(\frac{6.673 \times 10^{-11}\,\text{N m}^2\,\text{kg}^{-2} \times 1.99 \times 10^{30}\,\text{kg}}{0.5 \times 6.96 \times 10^8\,\text{m}}\right)^{1/2} \\ &= 6.18 \times 10^5\,\text{m s}^{-1}. \end{aligned}$$

So we have

$$\tan\delta = \frac{H}{r} \simeq \frac{10^4}{6.18 \times 10^5} = 0.016, \qquad (3.23)$$

so $\delta \simeq 0.92°$. The disc is indeed rather flat!

Exercise 4.1 From Equations 4.7 and 3.6 we find

$$t_{\text{th}} \simeq \frac{c_{\text{s}}^2}{\nu_{\text{vis}}\,GM/r^3} = \frac{c_{\text{s}}^2}{\alpha c_{\text{s}} HGM/r^3} = \frac{c_{\text{s}} r^3}{\alpha HGM}.$$

Noting Equation 3.35, we also have $H/r = c_{\text{s}}/v_{\text{K}}$, hence

$$t_{\text{th}} \simeq \frac{c_{\text{s}} r^2}{\alpha GM}\,\frac{r}{H} = \frac{c_{\text{s}} r^2}{\alpha GM}\,\frac{v_{\text{K}}}{c_{\text{s}}} = \frac{r^2}{\alpha GM}\left(\frac{GM}{r}\right)^{1/2} = \frac{1}{\alpha}\left(\frac{r^3}{GM}\right)^{1/2} = \frac{1}{\alpha}\,\frac{1}{\omega_{\text{K}}} = \frac{1}{\alpha} t_{\text{dyn}},$$

as required. We have used the identity for the Keplerian angular speed, $\omega_{\text{K}} = (GM/r^3)^{1/2}$.

Exercise 4.2 As $\nu_{\text{vis}} =$ constant, we can move it to the front, and Equation 4.1 becomes

$$\frac{\partial\Sigma}{\partial t} = \frac{3\nu_{\text{vis}}}{r}\,\frac{\partial}{\partial r}\left\{r^{1/2}\frac{\partial}{\partial r}(\Sigma r^{1/2})\right\}.$$

Using the product rule on the inner derivative gives

$$\frac{\partial \Sigma}{\partial t} = \frac{3\nu_{\text{vis}}}{r}\frac{\partial}{\partial r}\left\{r^{1/2}\left[\Sigma\frac{\partial}{\partial r}(r^{1/2}) + \frac{\partial \Sigma}{\partial r}r^{1/2}\right]\right\} = \frac{3\nu_{\text{vis}}}{r}\frac{\partial}{\partial r}\left\{r^{1/2}\left[\Sigma\frac{1}{2r^{1/2}} + \frac{\partial \Sigma}{\partial r}r^{1/2}\right]\right\}.$$

Factoring in $r^{1/2}$ gives

$$\frac{\partial \Sigma}{\partial t} = \frac{3\nu_{\text{vis}}}{r}\frac{\partial}{\partial r}\left\{\frac{\Sigma}{2} + \frac{\partial \Sigma}{\partial r}r\right\}.$$

Using the sum rule, this becomes

$$\frac{\partial \Sigma}{\partial t} = \frac{3\nu_{\text{vis}}}{r}\left\{\frac{\partial}{\partial r}\left(\frac{\Sigma}{2}\right) + \frac{\partial}{\partial r}\left(\frac{\partial \Sigma}{\partial r}r\right)\right\}.$$

Now using the product rule again gives

$$\frac{\partial \Sigma}{\partial t} = \frac{3\nu_{\text{vis}}}{r}\left\{\frac{1}{2}\frac{\partial \Sigma}{\partial r} + \frac{\partial^2 \Sigma}{\partial r^2}r + \frac{\partial \Sigma}{\partial r}\right\} = \frac{3\nu_{\text{vis}}}{r}\left\{\frac{3}{2}\frac{\partial \Sigma}{\partial r} + \frac{\partial^2 \Sigma}{\partial r^2}r\right\}.$$

Thus we finally obtain Equation 4.12:

$$\frac{\partial \Sigma}{\partial t} = \frac{9\nu_{\text{vis}}}{2r}\frac{\partial \Sigma}{\partial r} + 3\nu_{\text{vis}}\frac{\partial^2 \Sigma}{\partial r^2}.$$

Exercise 4.3 The viscous time $t_{\text{vis}} = r_{\text{c}}^2/\nu_{\text{vis}}$ is an appropriate estimate for the time it takes the torus at the circularization radius to spread into a disc-like structure. With $\nu_{\text{vis}} = \alpha H c_{\text{s}}$ and $H \simeq c_{\text{s}}/\omega_{\text{K}}$ (Equation 3.34) we obtain

$$\nu_{\text{vis}} = \alpha\frac{c_{\text{s}}^2}{(GM/r_{\text{c}}^3)^{1/2}},$$

so

$$t_{\text{vis}} \simeq \frac{r_{\text{c}}^2}{\alpha c_{\text{s}}^2} \times \left(GM/r_{\text{c}}^3\right)^{1/2} = \frac{(GMr_{\text{c}})^{1/2}}{\alpha c_{\text{s}}^2}.$$

Then using Equation 3.32 for the sound speed, we have

$$t_{\text{vis}} \simeq \frac{\left(6.673 \times 10^{-11}\,\text{N}\,\text{m}^2\,\text{kg}^{-2} \times 0.6 \times 1.99 \times 10^{30}\,\text{kg} \times 0.2 \times 6.96 \times 10^{8}\,\text{m}\right)^{1/2}}{0.3 \times (10^4\,\text{m}\,\text{s}^{-1})^2}$$

$$= 3.5 \times 10^6\,\text{s}.$$

This is about 40 days, i.e. a little over a month.

Exercise 4.4 A stability analysis studies the reaction of a physical system (e.g. an accretion disc) to perturbations. Initially the system is assumed to be in equilibrium. Then a perturbation is applied to the system, and the reaction of the system is calculated. The stability analysis is said to be linear if the initial perturbations are sufficiently small, so that the resulting change of other quantities can be described by the first (linear) term in the corresponding Taylor expansion with respect to the perturbing quantity. The stability analysis is local if the reaction of the system is studied only in the immediate vicinity of a given point in the system. Therefore the reaction at this point is assumed to be determined by its immediate vicinity only, not by events far away from this point.

Exercise 4.5 With Kramers' opacity $\kappa_R \propto \rho T^{-3.5}$, the denominator of Equation 3.41 scales as

$$\kappa_R \rho H \propto \rho^2 T^{-3.5} H,$$

so that with $\rho = \Sigma/H$,

$$\kappa_R \rho H \propto \Sigma^2 \frac{1}{H^2} T^{-3.5} H \propto \Sigma^2 H^{-1} T^{-3.5} \propto \Sigma^2 T^{-4},$$

where for the last step we have used $H \propto T^{1/2}$ (Equation 3.36). With this,

$$F(H) \propto \frac{T^4}{\Sigma^2 T^{-4}} \propto T^8 \Sigma^{-2},$$

as required.

Exercise 4.6 (a) For $r \gg R_1$ the temperature profile of a steady-state disc is

$$T_{\text{eff}}^4(r) = \frac{3GM_1\dot{M}}{8\pi\sigma r^3} \qquad \text{(Eqn 3.28)}$$

(with M_1 as the mass of the central accretor, the white dwarf). Setting $r = r_D$, $T_{\text{eff}}(r_D) = T_H$ and solving for $\dot{M}$ gives

$$\dot{M} = \frac{8\pi\sigma T_H^4 r_D^3}{3GM_1}.$$

To determine r_D we note that

$$R_{L,1} \simeq 0.462 \left(\frac{M_1}{M_1 + M_2}\right)^{1/3} a \qquad \text{(Eqn 2.8)}$$

can be used here as $1/q = M_1/M_2 \simeq 1$ (but note that in general, short-period CVs would have $1/q \gg 1$ in which case Equation 2.8 is *not* a good approximation). Therefore

$$r_D^3 = (0.5R_{L,1})^3 \simeq 0.231^3 \frac{M_1}{M} a^3.$$

With Kepler's law $a^3 = G(M_1 + M_2)P_{\text{orb}}^2/4\pi^2$, this becomes

$$r_D^3 \simeq 0.231^3 \frac{GM_1}{4\pi^2} P_{\text{orb}}^2.$$

Inserting into the above expression for $\dot{M}$, we have

$$\dot{M} = \frac{0.231^3 \times 8\pi\sigma \times T_H^4 GM_1}{3 \times 4\pi^2 GM_1} P_{\text{orb}}^2$$

or

$$\dot{M} = \frac{0.231^3 \times 2\sigma \times T_H^4}{3\pi} P_{\text{orb}}^2.$$

This is a lower limit for the mass transfer rate if the disc is meant to be stable. Note that it scales as $\dot{M} \propto P_{\text{orb}}^2$.

(b) We have

$$\frac{\dot{M}}{\text{M}_\odot\,\text{yr}^{-1}} = \frac{0.231^3 \times 2 \times 5.671 \times 10^{-8}\,\text{J}\,\text{m}^{-2}\,\text{K}^{-4}\,\text{s}^{-1} \times (6 \times 10^3\,\text{K})^4 \times (3600\,\text{s})^2}{3\pi \times 6.31 \times 10^{22}\,\text{kg}\,\text{s}^{-1}} \left(\frac{P_{\text{orb}}}{\text{h}}\right)^2$$

$$= 4 \times 10^{-11} \left(\frac{P_{\text{orb}}}{\text{h}}\right)^2,$$

so $\dot{M} \simeq 4 \times 10^{-10}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$ at $P_{\mathrm{orb}} = 3\,\mathrm{h}$. In nova-like systems with periods longer than 3 h, the observationally estimated mass accretion rate is a few times $10^{-9}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$, while for shorter periods the rate is thought to be as low as a few times $10^{-11}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$ — and the vast majority of short-period cataclysmic variables ($P_{\mathrm{orb}} \lesssim 2\,\mathrm{h}$) are indeed dwarf novae.

Exercise 5.1 Cataclysmic variables are ideal laboratories for the study of accretion phenomena for the following reasons:

- The mass donor is faint and does not swamp the optical and ultraviolet radiation emitted by the accretion flow itself.
- The irradiation of the accretion disc by the hot accreting white dwarf is negligible.
- The size of the orbit is compact enough so that orbital changes can be observed within hours — a convenient timescale for human observers.
- Eclipses and radial velocity studies allow one to map the accretion flow.
- Major brightness variations of the disc due to thermal and viscous evolution occur on a convenient timescale of weeks to months.

Exercise 5.2 (a) The accretion luminosity is given by Equation 1.3:

$$L_{\mathrm{acc}} = \frac{GM\dot{M}}{R}.$$

We set $\dot{M} = 10^{-9}\,\mathrm{M}_\odot\,\mathrm{yr}^{-1}$, $M = 1\,\mathrm{M}_\odot$ and $R = 8.7 \times 10^6$ m. Then

$$\begin{aligned} L_{\mathrm{acc}} &= (6.673 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}) \times (1.99 \times 10^{30}\,\mathrm{kg}) \times 10^{-9} \times 1.99 \times 10^{30}\,\mathrm{kg} \\ &\quad \times (365.25 \times 24 \times 3600\,\mathrm{s})^{-1}/(8.7 \times 10^6\,\mathrm{m}) \\ &= 9.6 \times 10^{26}\,\mathrm{N\,m\,s^{-1}} \\ &\approx 10^{27}\,\mathrm{J\,s^{-1}}. \end{aligned}$$

The solar luminosity is $\mathrm{L}_\odot \approx 4 \times 10^{26}\,\mathrm{J\,s^{-1}}$. Hence, using the definition for astronomical magnitudes, for the difference between the absolute magnitude M_{CV} of the CV and the absolute magnitude M_{Sun} (not to be confused with the solar mass!) we have

$$M_{\mathrm{CV}} - M_{\mathrm{Sun}} = -2.5\log_{10}\left(\frac{L_{\mathrm{acc}}}{\mathrm{L}_\odot}\right).$$

Hence

$$M_{\mathrm{CV}} - 4.83 = -2.5\log_{10}\left(\frac{10^{27}}{4 \times 10^{26}}\right),$$

which gives

$$M_{\mathrm{CV}} = 4.83 - 2.5\log_{10}(2.5) = 3.84.$$

Using the distance modulus (with zero extinction)

$$m = M_{\mathrm{CV}} - 5 + 5\log_{10}(d/\mathrm{pc}),$$

we find the apparent magnitude of the CV when it is located at a distance $d = 1000$ pc:

$$m = 3.84 - 5 + 5\log_{10}(1000) = 3.84 - 5 + 15.$$

Hence $m = 13.84$.

(b) With a transfer rate $\dot{M} = 10^{-9}\,\mathrm{M_\odot\,yr^{-1}}$, this CV is one of the brighter ones anyway, and still its apparent magnitude, at a distance of $1000\,\mathrm{pc}$, is only about 14. (Note: In reality a bolometric correction should also be applied as the calculation here leads to the bolometric magnitude not the visual magnitude. This correction of about 2 magnitudes would make the CV even fainter than calculated in the V band.) This CV would only just be observable with a 12-inch telescope. Larger telescopes and deeper surveys of the sky would of course easily detect such a system, and also much fainter CVs, but the problem then is to distinguish the CVs in the survey field from the much more numerous ordinary stars.

Exercise 5.3 For the compact binary disc, the inner disc radius is close to the compact star's radius, i.e. $r_\mathrm{in} \simeq 10^{-2}\mathrm{R_\odot}$, which is 10^9 m for a white dwarf and 10^4 m for a neutron star. For the supermassive black hole accretor, we assume that the inner disc radius is close to the last stable circular orbit, i.e.

$$r_\mathrm{in} \simeq 3\frac{2GM}{c^2} \approx \frac{6 \times 7 \times 10^{-11}\,\mathrm{N\,kg^{-2}\,m^2} \times 10^8 \times 2 \times 10^{30}\,\mathrm{kg}}{(3 \times 10^8\,\mathrm{m\,s^{-1}})^2} \approx 10^{12}\,\mathrm{m}.$$

The outer radius for the discs in binaries is a fraction of the Roche-lobe radius of the accretor, i.e. of order the orbital separation. For short-period systems this is $\lesssim 1\,\mathrm{R_\odot} \simeq 10^9$ m. For AGN, this is perhaps $\lessapprox 10^{-2}\,\mathrm{pc} \simeq 10^{14}$ m.

Hence we have $r_\mathrm{out}/r_\mathrm{in} \simeq 10^2$ for cataclysmic variables and AGN, and $r_\mathrm{out}/r_\mathrm{in} \simeq 10^5$ for LMXBs.

Exercise 5.4 The surface pattern arises from lines that connect points in the disc with constant magnitude of the radial velocity, i.e. constant magnitude of the y-component of the orbital velocity v.

Consider now two points, A and B, in the accretion disc that are mirror-symmetric with respect to the y-axis. If point A has coordinates (x_0, y_0), then point B must have coordinates $(-x_0, y_0)$. The symmetry with respect to the y-axis arises because v_y at A has the same magnitude but opposite sign to v_y at B. Therefore the only difference between a point 'to the left' (B) and 'to the right' (A) of the y-axis is that the plasma to the left is approaching, while the plasma to the right is receding from the observer (if the orbital motion is anticlockwise).

The situation is similar if we consider two points, A and C, in the accretion disc that are mirror-symmetric with respect to the x-axis. As point A has coordinates (x_0, y_0), point C must have coordinates $(x_0, -y_0)$. The symmetry with respect to the x-axis arises because v_y at A has the same magnitude *and* sign as v_y at C! The velocities at A and C differ only in the sign of the x-component of v.

Exercise 5.5 (a) (i) Disc plasma at a distance r from the accretor with mass M has the Keplerian speed $v_\mathrm{K} = (GM/r)^{1/2}$ (Equation 1.5). For an edge-on system ($i = 90°$), the line-of sight velocity $v_\parallel$ varies with azimuth ϕ as $v_\parallel = v_\mathrm{K}\cos\phi$. So the lines of constant line-of-sight velocity in Figure 5.3 are defined by the relation

$$\frac{\cos\phi}{r^{1/2}} = \frac{v_\parallel}{(GM)^{1/2}}.$$

(ii) Converting the polar coordinates (r, ϕ) into Cartesian coordinates (x, y) (see Figure S5.1), we have $\cos\phi = x/r$ and $r^2 = x^2 + y^2$, so

$$\frac{v_\parallel}{(GM)^{1/2}} = \frac{x}{r^{3/2}} = \frac{x}{(x^2+y^2)^{3/4}}.$$

Solving for y,

$$x^2 + y^2 = \left(x \times \frac{(GM)^{1/2}}{v_\parallel}\right)^{4/3} = x^{4/3} \times \left(\frac{GM}{v_\parallel^2}\right)^{2/3}$$

or

$$y = \left[x^{4/3} \times \left(\frac{GM}{v_\parallel^2}\right)^{2/3} - x^2\right]^{1/2}.$$

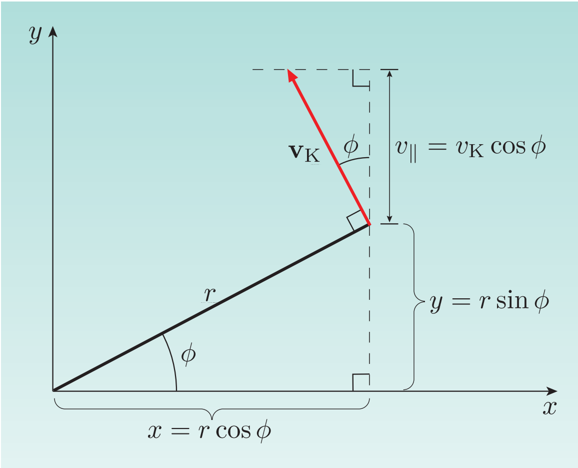

Figure S5.1 Cartesian and polar coordinates for Exercise 5.5.

(b) From the last equation we see that $y(x_0) = 0$ if $x_0^{4/3} \times (GM/v_\parallel^2)^{2/3} = x_0^2$, or $x_0^{2/3} = (GM/v_\parallel^2)^{2/3}$, hence $x_0 = GM/v_\parallel^2$.

(c) Alternatively, for $y = 0$ we always have $x = r$. Therefore $v_\parallel = v_\mathrm{K}(r)$, i.e. the line-of-sight velocity is just the Kepler speed as the disc plasma moves straight towards (or directly away from) the observer. This gives $r = GM/v_\mathrm{K}(r)^2$, which is equivalent to the expression for x_0 that we have just found.

Exercise 5.6 (a) The Keplerian speed at distance r from the white dwarf with mass M is (Equation 1.5)

$$v_\mathrm{K} = \left(\frac{GM}{r}\right)^{1/2}.$$

For $M = 0.8\,\mathrm{M}_\odot$ ($1\,\mathrm{M}_\odot = 1.99 \times 10^{30}$ kg) and $R = R_\mathrm{out} = 3.0 \times 10^8$ m, we therefore have the velocity

$$v_\mathrm{K,\,out} = \left(\frac{6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} \times 0.8 \times 1.99 \times 10^{30}\,\mathrm{kg}}{3.0 \times 10^8\,\mathrm{m}}\right)^{1/2}$$
$$= 5.95 \times 10^5\,\mathrm{m\,s^{-1}} = 595\,\mathrm{km\,s^{-1}}.$$

For the inner edge of the accretion disc we set $R = R_\mathrm{in} = 7 \times 10^6$ m (comparable to the white dwarf radius). As

$$v_\mathrm{K,\,in} = v_\mathrm{K,\,out}\left(\frac{R_\mathrm{out}}{R_\mathrm{in}}\right)^{1/2},$$

we have

$$v_{\mathrm{K,\,in}} = 5.95 \times 10^5\,\mathrm{m\,s^{-1}} \times \left(\frac{3.0 \times 10^8}{7.0 \times 10^6}\right)^{1/2}$$
$$= 3.90 \times 10^6\,\mathrm{m\,s^{-1}} = 3900\,\mathrm{km\,s^{-1}}.$$

(b) The Doppler shift is given by Equation 5.1

$$\frac{\Delta\lambda}{\lambda_{\mathrm{em}}} = \frac{v_{\parallel}}{c}.$$

Hence at the outer edge of the disc the Doppler shift is

$$\Delta\lambda_{\mathrm{out}} = \frac{5.95 \times 10^5}{3.0 \times 10^8} \times 656\,\mathrm{nm} = 1.3\,\mathrm{nm},$$

while at the inner edge of the disc the Doppler shift is

$$\Delta\lambda_{\mathrm{in}} = \frac{3.90 \times 10^6}{3.0 \times 10^8} \times 656\,\mathrm{nm} = 8.5\,\mathrm{nm}.$$

Exercise 5.7 The hot spot appears brightest when it faces the observer, i.e. immediately before phase 0 (in Figure 5.4 this is at phase 0.875). The binary is at phase 0 when the secondary star is closest to the observer. At the opposite phase, close to phase 0.5, the hot spot is facing away from the observer. Hardly any light from the spot reaches the observer as the disc is in the way. This variable contribution from the hot spot gives rise to an orbital 'hump' in the optical light curve. The hump is most pronounced when the system is seen nearly edge-on. If we see the system face-on, there is no such hump. In this case the hot spot always contributes roughly the same (small) amount to the total light.

Exercise 5.8 The 'shadow' in the figure indicates those regions on the accretion disc from where the Earth (i.e. the telescope that collects photons emitted from the disc) cannot be seen because it is obscured by the donor star. The shadow is long if the inclination is high. In a system seen edge-on (inclination 90°), the shadow formally has an infinite length, while in a system seen face-on (inclination 0°), there is no shadow.

Exercise 5.9 The Keplerian speed for accretion disc material is given by Equation 1.5: $v_{\mathrm{K}} = (GM/r)^{1/2}$, i.e. the speed increases with decreasing r. In a velocity map, the surface brightness of the accretion disc is plotted as a function of the x- and y-components of the velocity v of the emitting material. Hence the rapidly moving material from the inner regions of the accretion disc will appear at large values of v_x and v_y, while the slowly moving material from the outer regions of the accretion disc will appear at small values of v_x and v_y.

Exercise 5.10 (a) The shift between the continuum and line flux is about 40 days.

(b) The emission line light curve needs to be shifted about 40 days earlier in order to correlate with the continuum light curve. The emission line light curve therefore lags about 40 days after the continuum light curve.

(c) This indicates that in this case, the emission line flux could be caused by reprocessing or reflection of the continuum flux after an interval of about 40 days.

This may represent the light travel time from the site of emission of the continuum to the site of emission of the emission line light.

Exercise 6.1 The energy of the photon is given by

$$E = h\nu = \frac{hc}{\lambda}.$$

Hence the frequency ν for a 1 keV photon is given by

$$\nu = \frac{E}{h} = \frac{10^3\,\mathrm{eV} \times 1.602 \times 10^{-19}\,\mathrm{J\,eV^{-1}}}{6.626 \times 10^{-34}\,\mathrm{J\,s}} = 2.4 \times 10^{17}\,\mathrm{Hz}.$$

Also,

$$\lambda = \frac{hc}{E} = \frac{(6.626 \times 10^{-34}\,\mathrm{J\,s}) \times (2.998 \times 10^8\,\mathrm{m\,s^{-1}})}{10^3\,\mathrm{eV} \times 1.602 \times 10^{-19}\,\mathrm{J\,eV^{-1}}}$$
$$= 1.2 \times 10^{-9}\,\mathrm{m} = 1.2\,\mathrm{nm}.$$

Exercise 6.2 (a) Using Equation 6.9, the Eddington limit becomes

$$L_{\mathrm{Edd}} = \frac{4\pi \times 6.673 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} \times 1.673 \times 10^{-27}\,\mathrm{kg} \times 2.998 \times 10^8\,\mathrm{m\,s^{-1}}}{6.652 \times 10^{-29}\,\mathrm{m^2}} \times M$$
$$= 6.322 \times (1.99 \times 10^{30}) \times \left(\frac{M}{\mathrm{M_\odot}}\right)\,\mathrm{W}$$
$$= 1.26 \times 10^{31}\left(\frac{M}{\mathrm{M_\odot}}\right)\,\mathrm{W}.$$

(b) In this case, $M = 1.4\,\mathrm{M_\odot}$, and $L_{\mathrm{Edd}} = 1.8 \times 10^{31}\,\mathrm{W}$.

(c) For a $10\,\mathrm{M_\odot}$ black hole, $L_{\mathrm{Edd}} = 1.3 \times 10^{32}\,\mathrm{W}$.

Exercise 6.3 (a) First we need to calculate the cross-section at 1 keV:

$$\sigma(E) = (c_0 + c_1 \times E + c_2 \times E^2)E^{-3} \times 10^{-24}\,\mathrm{cm^2},$$

where E is measured in keV. Hence

$$\sigma(1\,\mathrm{keV}) = \big(120.6 + (169.3 \times 1) + (-47.7 \times 1^2)\big) \times 1^{-3} \times 10^{-24}\,\mathrm{cm^{-2}},$$

giving $\sigma(1\,\mathrm{keV}) = 2.42 \times 10^{-22}\,\mathrm{cm^{-2}}$.

The fraction of radiation transmitted through the absorber, $f_{\mathrm{trans}}(E)$, is given by Equation 6.15. For $N_{\mathrm{H}} = 1.5 \times 10^{22}$ atom $\mathrm{cm^{-2}}$,

$$f_{\mathrm{trans}}(1\,\mathrm{keV}) = \exp\big[-(1.5 \times 10^{22}) \times (2.42 \times 10^{-22})\big] = 0.0265.$$

The fraction of energy absorbed (f_{abs}) is $1 - f_{\mathrm{trans}}$, i.e. $f_{\mathrm{abs}} \simeq 97\%$.

(b) For 5 keV photons,

$$\sigma(5\,\mathrm{keV}) = \big(433 - (2.4 \times 5) + (0.75 \times 5^2)\big) \times 5^{-3} \times 10^{-24}\,\mathrm{cm^{-2}},$$

hence $\sigma(5\,\mathrm{keV}) = 3.5 \times 10^{-24}\,\mathrm{cm^2}$. Therefore

$$f_{\mathrm{abs}} = 1 - \exp\big[-(1.5 \times 10^{22}) \times (3.5 \times 10^{-24})\big] = 5\%.$$

Exercise 6.4 (a) The lowest frequency is given by

$$f_{\mathrm{min}} = \frac{1}{N\,\Delta T} = \tfrac{1}{4}\,\mathrm{Hz}.$$

The highest frequency is given by

$$f_{\max} = \frac{1}{2\,\Delta T} = \tfrac{1}{2}\,\text{Hz}.$$

There are $N/2$ frequencies in the PDS, i.e. 2.

(b) Equation 6.19 tells us that the power for frequency k, $P(f_k)$, is given by

$$P(f_k) = C\left(\left[\sum_{j=1}^{N} I_j \cos(2\pi f_k t_j)\right]^2 + \left[\sum_{j=1}^{N} I_j \sin(2\pi f_k t_j)\right]^2\right),$$

where C is a constant, $k = 1, 2$, $j = 1, 2, 3, 4$.

First, we shall tackle $k = 1$, where $f_k = \frac{1}{4}$:

$$P(\tfrac{1}{4}) = C\left(\left[\sum_{j=1}^{4} I_j \cos(2\pi \times \tfrac{1}{4} \times t_j)\right]^2 + \left[\sum_{j=1}^{4} I_j \sin(2\pi \times \tfrac{1}{4} \times t_j)\right]^2\right).$$

Summing the cosine terms:

$$\sum_{j=1}^{4} I_j \cos(2\pi \times \tfrac{1}{4} \times t_j) = (3 \times 0) + (1 \times -1) + (3 \times 0) + (1 \times 1) = 0.$$

Now summing the sine terms:

$$\sum_{j=1}^{4} I_j \sin(2\pi \times \tfrac{1}{4} \times t_j) = (3 \times 1) + (1 \times 0) + (3 \times -1) + (1 \times 0) = 0.$$

Since both the sine and cosine terms are zero, $P(\frac{1}{4}) = 0$.

Now we shall tackle $k = 2$, where $f_k = \frac{1}{2}$:

$$P(\tfrac{1}{2}) = C\left(\left[\sum_{j=1}^{4} I_j \cos(2\pi \times \tfrac{1}{2} \times t_j)\right]^2 + \left[\sum_{j=1}^{4} I_j \sin(2\pi \times \tfrac{1}{2} \times t_j)\right]^2\right).$$

Summing the cosine terms:

$$\sum_{j=1}^{4} I_j \cos(2\pi \times \tfrac{1}{2} \times t_j) = (3 \times -1) + (1 \times 1) + (3 \times -1) + (1 \times 1) = -4.$$

Summing the sine terms:

$$\sum_{j=1}^{4} I_j \sin(2\pi \times \tfrac{1}{2} \times t_j) = (3 \times 0) + (1 \times 0) + (3 \times 0) + (1 \times 0) = 0.$$

Hence

$$P(\tfrac{1}{2}) = C\left[(-4)^2 + (0)\right]^2 = C \times 16.$$

This tells us that the light curve is produced by a cosine with frequency $\frac{1}{2}$.

Exercise 6.5 (a) The detected flux F is given by

$$F = \frac{L}{4\pi d^2}, \qquad \text{(Eqn 6.20)}$$

so

$$L = F \times 4\pi d^2.$$

The distance d is estimated to be 8 kpc, i.e. $8000 \times 3.086 \times 10^{16}$ m. Hence

$$L = (4.5 \times 10^{-11}\,\mathrm{W\,m^{-2}}) \times 4\pi \times (8000 \times 3.086 \times 10^{16}\,\mathrm{m})^2$$
$$= 3.4 \times 10^{31}\,\mathrm{W}.$$

This luminosity is higher than expected for a $1.4\,\mathrm{M_\odot}$ neutron star accreting hydrogen, by about a factor of 2 (see Exercise 6.2), but is in line with observations of LMXBs in globular clusters.

(b) The X-ray peak of a radius-expanding burst corresponds to the photosphere shrinking down to normal size, so the black body radius gives us the radius of the neutron star, R_{NS}. For black body radiation,

$$L = 4\pi R_{\mathrm{NS}}^2\,\sigma T^4.$$

First, we need to work out the temperature corresponding to $\mathrm{k}T = 2.1\,\mathrm{keV}$:

$$T = \frac{2.1 \times 1000 \times (1.602 \times 10^{-19})\,\mathrm{J}}{1.381 \times 10^{23}\,\mathrm{J\,K^{-1}}} = 2.4 \times 10^7\,\mathrm{K}.$$

We then obtain R_{NS} using the Stefan–Boltzmann law:

$$R_{\mathrm{NS}} = \left(\frac{3.4 \times 10^{31}\,\mathrm{W}}{4 \times \pi \times 5.671 \times 10^{-8}\,\mathrm{W\,m^{-2}\,K^{-4}} \times (2.4 \times 10^7\,\mathrm{K})^4}\right)^{1/2} \approx 1.2 \times 10^4\,\mathrm{m} = 12\,\mathrm{km}.$$

Exercise 6.6 We estimate the donor mass from the approximate relation

$$P_{\mathrm{orb}} \simeq 8.8\,\mathrm{h}\,\frac{M_2}{\mathrm{M_\odot}}. \qquad \text{(Eqn 2.11)}$$

This gives

$$\frac{M_2}{\mathrm{M_\odot}} \simeq \frac{P_{\mathrm{orb}}}{8.8\,\mathrm{h}} = \frac{4.4}{8.8} = 0.5.$$

So $M_2 = 0.5\,\mathrm{M_\odot}$. With $R_2/\mathrm{R_\odot} \simeq M_2/\mathrm{M_\odot}$, the radius of the donor star is $0.5\,\mathrm{R_\odot}$, and as the donor is Roche-lobe filling, this is also equal to the Roche-lobe radius, $R_{\mathrm{L,2}} \approx 3.5 \times 10^8$ m.

Since the LMXB exhibits X-ray bursts, it must contain a neutron star primary, so we may assume a mass of $M_1 = 1.4\,\mathrm{M_\odot}$. Therefore the mass ratio of the system is $q = M_2/M_1 \approx 0.5/1.4 = 0.36$.

We find the neutron star's Roche-lobe radius by noting that

$$\frac{R_{\mathrm{L,1}}}{R_{\mathrm{L,2}}} = \frac{f_1(1/q)}{f_2(q)},$$

where $f(q)$ is given by Eggleton's approximation

$$f(q) \simeq \frac{0.49q^{2/3}}{0.6q^{2/3} + \log_e(1 + q^{1/3})}. \qquad \text{(Eqn 2.7)}$$

For $q = 0.36$ we have $1/q = 2.8$ and so $f(2.8)/f(0.36) \approx 1.6$. The Roche-lobe radius of the neutron star is therefore $R_{\mathrm{L,1}} \simeq 1.6 \times R_{\mathrm{L,2}} = 5.6 \times 10^8$ m.

To estimate the size of the corona, we rearrange Equation 6.21 to get

$$D_{\mathrm{ADC}} = \frac{2\pi R_{\mathrm{disc}}\,\Delta T_{\mathrm{ing}}}{P_{\mathrm{orb}}}.$$

R_{disc} is estimated to be 30–50% of the Roche-lobe radius. So the smallest corona size is obtained by taking $R_{\mathrm{disc}} = 0.3 \times R_{\mathrm{L},1}$. Then

$$\begin{aligned} D_{\mathrm{ADC}} &= \frac{2\pi \times 0.3 \times (5.6 \times 10^8)\,\mathrm{m} \times 2000\,\mathrm{s}}{4.4 \times 3600\,\mathrm{s}} \\ &= 1.3 \times 10^8\,\mathrm{m}. \end{aligned}$$

For $R_{\mathrm{disc}} = 0.5 \times R_{\mathrm{L},1}$, the corona is larger by a factor $0.5/0.3 = 1.7$. Hence the corona has a diameter of $\sim$130 000–230 000 km.

Exercise 6.7 We have the relation $H/r = \tan\delta$ (see Figures 6.22 and 6.23). Hence $\delta = \tan^{-1}(0.2) = 11.3°$.

Exercise 6.8 We have to calculate the radius in the disc where the orbital frequency equals the QPO frequency f. The angular speed ω at distance r from the accretor with mass M is

$$\omega = \left(\frac{GM}{r^3}\right)^{1/2}. \qquad \text{(Eqn 3.3)}$$

Rearranging this to solve for r yields

$$r = \left(\frac{GM}{\omega^2}\right)^{1/3}.$$

Since $\omega = 2\pi f$,

$$r = \left[\frac{(6.673 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}) \times (1.4 \times 1.99 \times 10^{30}\,\mathrm{kg})}{(2 \times \pi \times 6.0\,\mathrm{s^{-1}})^2}\right]^{1/3}.$$

Hence $r = 5.2 \times 10^5$ m, or $\sim$520 km. This is only about 30 times the neutron star radius.

Exercise 6.9 Observations of Galactic LMXBs with known neutron star primaries suggest that transitions from the low state to the high state occur at $0.1L_{\mathrm{Edd}}$ or less. For a neutron star with a mass of $2.1\,\mathrm{M}_\odot$,

$$L_{\mathrm{Edd}} \simeq 1.26 \times 10^{31} \times \frac{2.1\,\mathrm{M}_\odot}{1\,\mathrm{M}_\odot} = 2.7 \times 10^{31}\,\mathrm{W}.$$

Since the transition occurs at $\sim 0.1L_{\mathrm{Edd}}$ or less, we do not expect to see neutron star LMXBs exhibit the low state at 0.01–1000 keV luminosities higher than $\sim 2.7 \times 10^{30}$ W.

Exercise 6.10 (a) The Eddington luminosity is given by

$$L_{\mathrm{Edd}} = 1.26 \times 10^{31} \left(\frac{M}{\mathrm{M}_\odot}\right) \mathrm{W}, \qquad \text{(Eqn 6.10)}$$

while the accretion luminosity for a Schwarschild black hole can be written as $L_{\mathrm{acc}} = \eta_{\mathrm{acc}}\, c^2 \dot{M}$ (Equation 1.9), with $\eta_{\mathrm{acc}} = 0.057$ (see Subsection 1.2.1). Equating these luminosities and solving for $\dot{M}$ gives

$$\dot{M}_{\mathrm{Edd}} = \frac{1.26 \times 10^{31}\,\mathrm{W}}{0.057 \times c^2} \left(\frac{M}{\mathrm{M}_\odot}\right)$$

or

$$\dot{M}_{\text{Edd}} = 3.9 \times 10^{-8}\,\text{M}_\odot\,\text{yr}^{-1}\left(\frac{M}{\text{M}_\odot}\right). \tag{6.24}$$

(b) Inserting Equation 6.24 into Equation 1.17 (in the solution to Exercise 1.10) gives

$$T_{\text{peak}} = 1.5 \times 10^{7}\,\text{K}\left(\frac{M}{\text{M}_\odot}\right)^{-1/4}. \tag{6.25}$$

(c) For a $100\,\text{M}_\odot$ black hole accreting at the Eddington limit, this is $T_{\text{peak}} \simeq 5 \times 10^6$ K, a factor of 2–4 lower than the observed value.

Exercise 6.11 (a) If the source is unbeamed, its luminosity would be

$$\begin{aligned} L &= F \times 4\pi d^2 \\ &= (3.5 \times 10^{-15}\,\text{W}) \times 4\pi \times (3.26 \times 10^{6}\,\text{pc} \times 3.059 \times 10^{16}\,\text{m}\,\text{pc}^{-1})^2. \end{aligned}$$

So $L = 4.5 \times 10^{32}$ W.

(b) From Equation 6.10,

$$L_{\text{Edd}} = 1.26 \times 10^{31}\left(\frac{10\,\text{M}_\odot}{\text{M}_\odot}\right) = 1.26 \times 10^{32}\,\text{W}.$$

The source therefore exceeds its Eddington luminosity unless it is beamed by at least the minimum beaming factor

$$b_{\text{min}} = \frac{4.5 \times 10^{32}\,\text{W}}{1.26 \times 10^{32}\,\text{W}} = 3.6.$$

(c) From Equation 6.25, b_{min} is also defined as $b_{\text{min}} = 4\pi/\Delta\Omega$. Hence

$$\Omega = \frac{4\pi}{b_{\text{min}}} = \frac{4 \times 3.141}{3.6} = 3.5\,\text{sr}.$$

This is a fraction $3.5/4\pi = 28\%$ of the whole sphere. If the beaming factor is larger than b_{min}, then $\Delta\Omega$ decreases, hence this is a maximum solid angle.

Exercise 7.1 (a) At a mass loss rate of $10^{-5}\,\text{M}_\odot\,\text{yr}^{-1}$, the Wolf–Rayet star will lose $\Delta M = \dot{M}\,\Delta t_{\text{wind}} \simeq 10^{-5} \times 0.1 \times 5 \times 10^{6}\,\text{M}_\odot = 5\,\text{M}_\odot$, which is a large fraction of its initial mass!

(b) At a mass loss rate of $10^{-14}\,\text{M}_\odot\,\text{yr}^{-1}$, the Sun will lose just $10^{-4}\,\text{M}_\odot$ over its lifetime, which is a tiny fraction of its mass.

Exercise 7.2 Taking 6000 Å as a typical optical wavelength, the application of Equation 5.1 suggests a speed of $c/150$ or $2000\,\text{km}\,\text{s}^{-1}$. The component along the line of sight coincides with the actual velocity as the outflow is spherically symmetric (a wind).

Exercise 7.3 In 27 days from 3rd April to 30th April, the two jets appear in the sky to have moved apart by about 700 milliarcsec, which corresponds to 0.2×0.7 ly $= 0.14$ ly. Thus the speed at which the two bright spots appear to move apart is $v = (0.14\,\text{ly})/(27\,\text{days})$. We may express this directly in terms of the speed of light by noting that a light-year is the distance that light travels in

1 year. Thus $v = (0.14 \times 365c)/27 \simeq 1.9c$. The left bright spot has moved further from the centre than the right spot. It is roughly at twice the distance from the cross than the right spot. Thus the left spot has moved from the centre at about $1.3c$, and the right at about $0.6c$. The left spot appears to be moving at a speed faster than light!

Exercise 7.4 Solving Equation 7.1 for V/c gives

$$\frac{V}{c} = \left(1 - \frac{1}{\gamma^2}\right)^{1/2}.$$

Using the first-order expansion $(1+x)^{1/2} \simeq 1 + x/2$ $(x \ll 1)$ with $x = -1/\gamma^2$, this becomes

$$\frac{V}{c} \simeq 1 - \frac{1}{2\gamma^2},$$

as required.

Exercise 7.5 (a) Equation 7.13 gives the maximum value of α for observing a star that is at right angles to the ecliptic ($\theta = \pi/2$). Thus $\alpha_{\text{max}} = v/c$. Converting the angle to radians, we get $\alpha = 21\pi/(180 \times 60 \times 60)\,\text{rad} = 1.0 \times 10^{-4}\,\text{rad}$. So $v/c \approx 10^{-4}$ and $v \approx 30\,\text{km}\,\text{s}^{-1}$.

(b) The Earth's rotation period is $P_\oplus = 24\,\text{h}$. The rotational velocity of a point at the surface of the Earth's equator (of radius $R_\oplus$) is

$$v = 2\pi R_\oplus/P_\oplus = 2\pi \times 6.38 \times 10^6/(24 \times 60 \times 60)\,\text{m}\,\text{s}^{-1} \approx 464\,\text{m}\,\text{s}^{-1}.$$

We may use this in Equation 7.13 to calculate α_{max}. Alternatively, we may exploit the result of part (a): the ratio of the two speeds is equal to the ratio of the aberration angles in the two situations. Thus $\alpha_{\oplus,\text{max}}/21'' = 464/30\,000$, giving $\alpha_{\oplus,\text{max}} \approx 0.32''$.

Exercise 7.6 The transformation equation for x is Equation 7.7

$$x = \gamma(x' + Vt'),$$

while for t it is Equation 7.10: $t = \gamma(t' + Vx'/c^2)$. Hence we have

$$\mathrm{d}x = \gamma(\mathrm{d}x' + V\,\mathrm{d}t') \qquad \text{and} \qquad \mathrm{d}t = \gamma\left(\mathrm{d}t' + \frac{V\,\mathrm{d}x'}{c^2}\right).$$

So we obtain for the velocity

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x' + V\,\mathrm{d}t'}{\mathrm{d}t' + (V/c^2)\,\mathrm{d}x'} = \frac{\mathrm{d}x'/\mathrm{d}t' + V}{1 + (V/c^2)(\mathrm{d}x'/\mathrm{d}t')} = \frac{v'_x + V}{1 + Vv'_x/c^2}.$$

(The corresponding transformations for v_y and v_z can be obtained in a similar way.)

Exercise 7.7 (a) v_{ap} attains a maximum for $\mathrm{d}v_{\text{app}}/\mathrm{d}\theta = 0$. From Equation 7.24, we get

$$\frac{\mathrm{d}v_{\text{ap}}}{\mathrm{d}\theta} = \frac{V\cos\theta}{1 - (V/c)\cos\theta} - \frac{V\sin\theta}{(1 - (V/c)\cos\theta)^2}\,\frac{V}{c}\sin\theta,$$

and applying the condition for a maximum

$$0 = \left(1 - \frac{V}{c}\cos\theta_{\text{max}}\right)V\cos\theta_{\text{max}} - \frac{V^2}{c}\sin^2\theta_{\text{max}}.$$

Manipulating this further gives

$$\cos\theta_{\max}-\frac{V}{c}\cos^2\theta_{\max}-\frac{V}{c}\sin^2\theta_{\max}=0 \qquad \text{or} \qquad \cos\theta_{\max}-\frac{V}{c}(\cos^2\theta_{\max}+\sin^2\theta_{\max})=0,$$

thus

$$\cos\theta_{\max}=\frac{V}{c} \qquad \text{and so} \qquad \theta_{\max}=\cos^{-1}(V/c).$$

(b) From $\cos\theta_{\max}=(V/c)$, we get

$$\sin\theta_{\max}=\sqrt{1-(V/c)^2}=\frac{1}{\gamma}.$$

Equation 7.24 thus gives

$$v_{\text{ap}}=\frac{V\sin\theta_{\max}}{1-(V/c)\cos\theta_{\max}}=\frac{V}{\gamma[1-(V^2/c^2)]}=\frac{V\gamma^2}{\gamma}=\gamma V.$$

Exercise 7.8 (a) For $\theta=\pi/2$, Equation 7.28 gives $\mathcal{D}=1/\gamma$.

(b) For $\theta=0$, Equation 7.28 becomes $\mathcal{D}=[\gamma(1-V/c)]^{-1}$.

From the definition of γ (Equation 7.1), we have

$$\gamma^2=[(1+V/c)(1-V/c)]^{-1}.$$

So $\gamma(1-V/c)=[\gamma(1+V/c)]^{-1}$.

Thus $\mathcal{D}=\gamma(1+V/c)$, which for highly relativistic speeds becomes $\mathcal{D}\simeq 2\gamma$.

(c) For a source moving at right angles to the observer, the received frequency is redshifted: $\nu_{\text{rec}}=\nu'_{\text{em}}/\gamma$.

For a source moving towards the observer, the received frequency is blueshifted: $\nu_{\text{rec}}\simeq 2\gamma\nu'_{\text{em}}$.

Exercise 7.9 For the given values of the parameters, $\gamma_1=1$, $m_1\simeq m_2/\gamma_2$, $\gamma_{\text{b}}=\gamma_2/2$ and $E_{\text{th}}\simeq\gamma_2 m_1 c^2/4$, Equation 7.39 gives

$$\varepsilon=\frac{\gamma_2 m_1 c^2/4}{(m_1+\gamma_2 m_2)c^2}=\frac{\gamma_2/4}{1/\gamma_2+\gamma_2}=\frac{1}{4}\left(\frac{\gamma_2}{1/\gamma_2+\gamma_2}\right).$$

As $\gamma_2\gg 1$, we therefore have $\varepsilon\approx\frac{1}{4}=25\%$.

Exercise 7.10 If the electrons execute a circular motion with speed v and radius r, we have $|\boldsymbol{v}\times\boldsymbol{B}|=vB_\perp$, so

$$evB_\perp=\frac{m_{\text{e}}v^2}{r}. \tag{7.26}$$

Hence the circular speed is $v=eB_\perp r/m_{\text{e}}$. The frequency is given by $\nu_{\text{cy}}=v/(2\pi r)$, so

$$\nu_{\text{cy}}=\frac{eB_\perp r}{m_{\text{e}}2\pi r}=\frac{eB_\perp}{2\pi m_{\text{e}}}. \tag{7.27}$$

Exercise 7.11 A comparison between the power emitted by a single electron by each mechanism (Equations 7.53 and 7.59) shows that $P_{\text{sy}}/P_{\text{ic}}=U_B/U_{\text{rad}}$. So it is the relative intensity of the respective underlying fields that determines which component dominates.

Exercise 8.1 Solving Equation 7.49 for E, we get $E = 4\pi d^2 S$. The distance has to be converted to m, using $1\,\mathrm{pc} = 3 \times 10^{16}\,\mathrm{m}$.

(a) For a Galactic halo source,

$$d \approx 50\,\mathrm{kpc} = 50 \times 10^3\,\mathrm{pc} \times 3 \times 10^{16}\,\mathrm{m\,pc^{-1}} = 1.5 \times 10^{21}\,\mathrm{m}$$

and

$$E \approx 4\pi \times (1.5 \times 10^{21}\,\mathrm{m})^2 \times 10^{-9}\,\mathrm{J\,m^{-2}} \approx 3 \times 10^{34}\,\mathrm{J}.$$

(b) Similarly, for the cosmological source,

$$d \approx 9 \times 10^{25}\,\mathrm{m} \approx 10^{26}\,\mathrm{m}$$

and

$$E \approx 4\pi \times (10^{26}\,\mathrm{m})^2 \times 10^{-9}\,\mathrm{J\,m^{-2}} \approx 10^{44}\,\mathrm{J}.$$

We have

$$\mathrm{M_\odot}\, c^2 \approx 2 \times 10^{30}\,\mathrm{kg} \times (2.998 \times 10^8\,\mathrm{m\,s^{-1}})^2 \approx 2 \times 10^{47}\,\mathrm{J}.$$

The energy implied for a Galactic halo source is about 10 times higher than for an X-ray burst, and much less compared to a supernova or the mass energy of a solar mass. The energy in a source at a cosmological distance is comparable to the energy in a supernova, and $1/1000$ of the mass energy available in 1 solar mass.

Exercise 8.2 The shorter timescale, $\Delta t_{\mathrm{var}} \simeq 1\,\mathrm{ms}$, provides the more stringent constraint, $\Delta r \lesssim c\,\Delta t_{\mathrm{var}} \approx 300\,\mathrm{km}$. This suggests a compact stellar object (neutron star or black hole).

Exercise 8.3 We need to estimate n_γ for use in Equation 8.3. From Equation 7.50, $n_\gamma \simeq L_\gamma/(4\pi(\Delta r)^2 ch\nu)$, and using $\Delta r = c\,\Delta t_{\mathrm{var}}$, we get $\tau_{\gamma\gamma} \simeq \sigma_{\mathrm{T}} L_\gamma/(4\pi c^2 \Delta t_{\mathrm{var}}\, h\nu)$. Substituting values in and converting MeV to SI units by $1\,\mathrm{eV} = 1.602 \times 10^{-19}\,\mathrm{J}$, we obtain

$$\begin{aligned}\tau_{\gamma\gamma} &\approx \frac{0.665 \times 10^{-28}\,\mathrm{m^{-2}} \times 10^{44}\,\mathrm{J\,s^{-1}}}{4\pi \times (2.998 \times 10^8\,\mathrm{m\,s^{-1}})^2 \times 10 \times 10^{-3}\,\mathrm{s} \times 0.5 \times 10^6\,\mathrm{eV} \times 1.602 \times 10^{-19}\,\mathrm{J\,eV^{-1}}} \\ &\approx 7 \times 10^{12}.\end{aligned}$$

Exercise 8.4 The source reached the apparent size R_{def} in 4 weeks' time. 4 weeks corresponds to $\Delta t = 4 \times 7 \times 24 \times 60^2\,\mathrm{s} = 2.4 \times 10^6\,\mathrm{s}$. An estimate of the speed V of expansion is obtained from $R_{\mathrm{def}} \simeq V\,\Delta t$, thus

$$V \approx \frac{10^{15}\,\mathrm{m}}{2.4 \times 10^6\,\mathrm{s}} \approx 4.13 \times 10^8\,\mathrm{m\,s^{-1}} = \frac{4.13 \times 10^8}{3 \times 10^8}\, c \approx 1.4c,$$

an apparent superluminal expansion as in AGN jets and microquasars.

Exercise 8.5 Using Equation 7.36 we obtain

$$r_{\mathrm{dis}} \simeq 2\gamma^2 c\,\Delta t_{\mathrm{var}} = 2 \times 300^2 \times (2.998 \times 10^8\,\mathrm{m\,s^{-1}}) \times (10 \times 10^{-3}\,\mathrm{s}) = 5.4 \times 10^{11}\,\mathrm{m} \approx 5 \times 10^{11}\,\mathrm{m}.$$

Exercise 8.6 (a) $1\,\mathrm{cm} = 10^{-2}\,\mathrm{m}$. Therefore $1\,\mathrm{cm}^3 = 10^{-6}\,\mathrm{m}^3$, which gives $1\,\mathrm{cm}^{-3} = 10^6\,\mathrm{m}^{-3}$.

(b) Substituting values in Equation 8.12, we obtain

$$r_{\text{dec}} \approx \left(\frac{3}{4\pi \times 1.67 \times 10^{-27}\,\text{kg} \times (2.998 \times 10^{8}\,\text{m}\,\text{s}^{-1})}\right)^{1/3} \times \left(\frac{10^{44}\,\text{J}}{10^{6}\,\text{m}^{-3} \times 300^{2}}\right)^{1/3}$$
$$\approx 1.2 \times 10^{14}\,\text{m},$$

which is about $0.004\,\text{pc}$ or $800\,\text{AU}$.

Exercise 8.7 We may use the result of Exercise 8.6, $r_{\text{dec}} \approx 1.2 \times 10^{14}\,\text{m}$, in Equation 8.14:

$$t_{\text{dec}} \approx \frac{1.2 \times 10^{14}\,\text{m}}{(2 \times 300^{2}) \times (2.998 \times 10^{8}\,\text{m}\,\text{s}^{-1})} \approx 2\,\text{s}.$$

Exercise 8.8 For $\Theta_{\text{J}} = 10° \approx 0.17\,\text{rad}$, the subtended solid angle of one jet is $\Delta\Omega = 2\pi(1 - \cos(\Theta_{\text{J}}/2))$ (Equation 7.20). For two jets the solid angle is

$$\Delta\Omega = 4\pi(1 - \cos 5°) = 4.8 \times 10^{-2}.$$

(a) The required energy is $4\pi/\Delta\Omega \approx 260$ times less than estimated in Exercise 8.1.

(b) As the emission is confined to a narrow jet, on average only 1 out of a few hundred GRBs will be detected by an observer on Earth. Hence the required source rate is a few hundred times higher than estimated in Worked Example 8.2, where it was assumed that GRBs emit isotropically.