Exercises and control questions to Chapter 1:

1.

Pattern	textural	structural
idiomorphic	X	
fine grained	X	
pegmatitic	X	
pumiceous	X	
fluidal		X
brecchiated	X	

2. A. "homogeneous" and "anisotropic". B. "heterogeneous" and "anisotropic". C. "isotropic" and "heterogeneous (see Fig. S1.1).

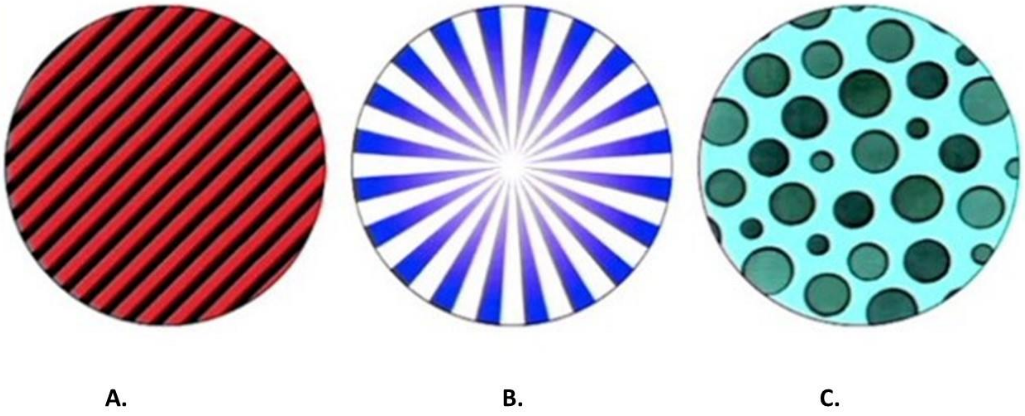


Fig. S1.1 Pattern examples for the Exercise 2 (http://www.soloentendidos.com/principio-cosmologico-isotropico-y-homogeneo-1390).

3. (a) erosion, reworking, and transportation of rock components,
   (b) deposition and sedimentation of the material, and
   (c) compaction and diagenetic processes.
4. (a) gravimetry, geodata, (b) seismic, acoustics of rocks, (c) magnetic field measurements.
5. II: median= 95 , sorting=65; III: median= 70 , sorting=100; IV: median= 35 , sorting=60; V: median= 5 , sorting=40.
6. (a) angular velocity is a vector, tensor rank 1, (b) moment of inertia is the tensor rank 2, (c) surface is a vector, tensor rank 1, (d) density is scalar, tensor rank 0, (e)

piezoelectric coefficient is the proportionality coefficient between the electric polarization (vector) and stress (tensor rank 2), so it is a tensor rank 3?

7. Using the data set in Fig. 1.9 of Chapter1one plots the density distribution function of grain sizes shown in Fig S1.2.

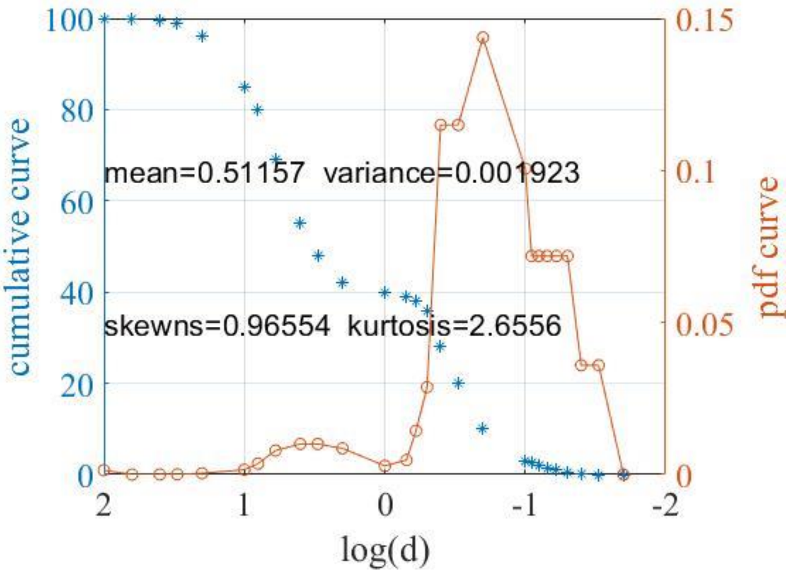


Fig. S1.2 Problem 7: MATLAB plots of cumulative and pdf curves of sample C.

The MATLAB code is shown below:

```
d=[100 63 40 30 20 10 8 6 4 3 2 1 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0.09 0.08
0.07 0.06 0.05 0.04 0.03 0.02];
CUM=[100 100 99.5 99 96 85 80 69 55 48 42 40 39 38 36 28 20 10 3 2.5 2 1.5
1 0.5 0.25 0 0];%
PDF=diff([0 CUM]) ./ diff([0 d]);
PDF=PDF./sum(PDF);
figure
yyaxis left
plot(log10(d),CUM,'*')
ylabel('cumulative curve')
yyaxis right
plot(log10(d),PDF,'-o')
ylabel('pdf curve')
xlabel('log(d)')
set(gca, 'XDir','reverse')
set(gca,'FontSize',20)
a = get(gca,'XTickLabel');
set(gca,'XTickLabel',a,'FontName','Times','fontsize',18)
grid
hold on
M=sum(PDF.*d); S=var(PDF);
skewns = @(x) (sum((x-mean(x)).^3)./length(x)) ./ (var(x,1).^1.5);
kurtss = @(x) (sum((x-mean(x)).^4)./length(x)) ./ (var(x,1).^2);
SK=skewns(PDF); KTS=kurtss(PDF);
txt1 = strcat('mean=',num2str(M),'  variance=',num2str(S));
txt2=strcat('skewns=',num2str(SK),'  kurtosis=',num2str(KTS));
text(2,0.1,txt1,'Fontsize',16);
text(2,0.05,txt2,'Fontsize',16)
```

8. The specific surface area per grain is:

$$A_{sp}\left[\frac{m^2}{g}\right] = \frac{\overbrace{\pi d^2}^{surface\ of\ a\ spherical\ grain}}{\underbrace{d^3 \cdot \rho \cdot \left(1 - \frac{\pi}{6}\right)}_{mass\ of\ a\ unit\ cell}} = \frac{\pi}{d \cdot \rho \cdot \left(1 - \frac{\pi}{6}\right)} = \frac{3.14}{30 \cdot 10^{-6} \cdot 3 \cdot 10^{6} \cdot 0.4764} \approx 0.15 \frac{m^2}{g}$$

In the case of parallel cylindrical tubes: $A_{sp}[\frac{m^2}{g}] = \frac{\overbrace{\pi d \cdot l}^{surface\ of\ a\ scylindrical\ grain}}{\underbrace{d^2 \cdot l \cdot \rho \cdot \left(1 - \frac{\pi}{4}\right)}_{mass\ of\ a\ cylindrical\ grain}} =$

$\frac{\pi}{d \cdot \rho \left(1 - \frac{\pi}{4}\right) \cdot} = \frac{3.144}{30 \cdot 10^{-6} \cdot 3 \cdot 10^{6} \cdot 0.2146} \approx 0.16 \frac{m^2}{g}$. In the case of cubic cells having edge length 3d the unit cell volume 27d³. The density is $\rho^* = (1 - \frac{\pi \cdot d^3}{6 \cdot 27 d^3}) \cdot \rho \approx 0.981 \cdot \rho$. The specific area per unit mass

$$A_{sp}[\frac{m^2}{g}] = \frac{\overbrace{\pi d^2}^{surface\ of\ a\ spherical\ grain}}{\underbrace{27 d^3 \cdot \rho \cdot (1 - \frac{\pi}{6 \cdot 27})}_{mass\ of\ a\ unit\ cell}} = \frac{\pi}{27 \cdot d \cdot \rho^*} = \frac{\pi}{27 \cdot 30 \cdot 10^{-6} \cdot 3 \cdot 10^{6} \cdot 0.981} \approx 1.3 \cdot 10^{-3} \frac{m^2}{g}.$$

9. The grain size analysis of a clastic sediment gives the results as follows:

Particle size, mm	Mass, g	Component	%
0-0.02	1.2	Colloidal+silt+clay+fine sand	≈1
0.02-0.04	3.1	Medium size sand	64.7
0.04-0.063	3.7	Medium size sand	
0.063-0.1	6.1	Medium size sand	
0.1.-0.2	7.4	Medium size sand	
0.2-0.4	35.6	Medium size sand	
0.4-0.63	25.2	Medium size sand	
0.63-1.0	17.1	Coarse size sand	13.6
1.0-2.0	9.5	Fine gravel	12.8
2.0-6.3	6.5	Fine gravel	
6.3-10	5.3	Medium gravel	4.2
10-20	4.7	Coarse gravel	3.7
Sum:	125.4 g		

The histogram of grain size distribution is shown in Fig. S1.3.

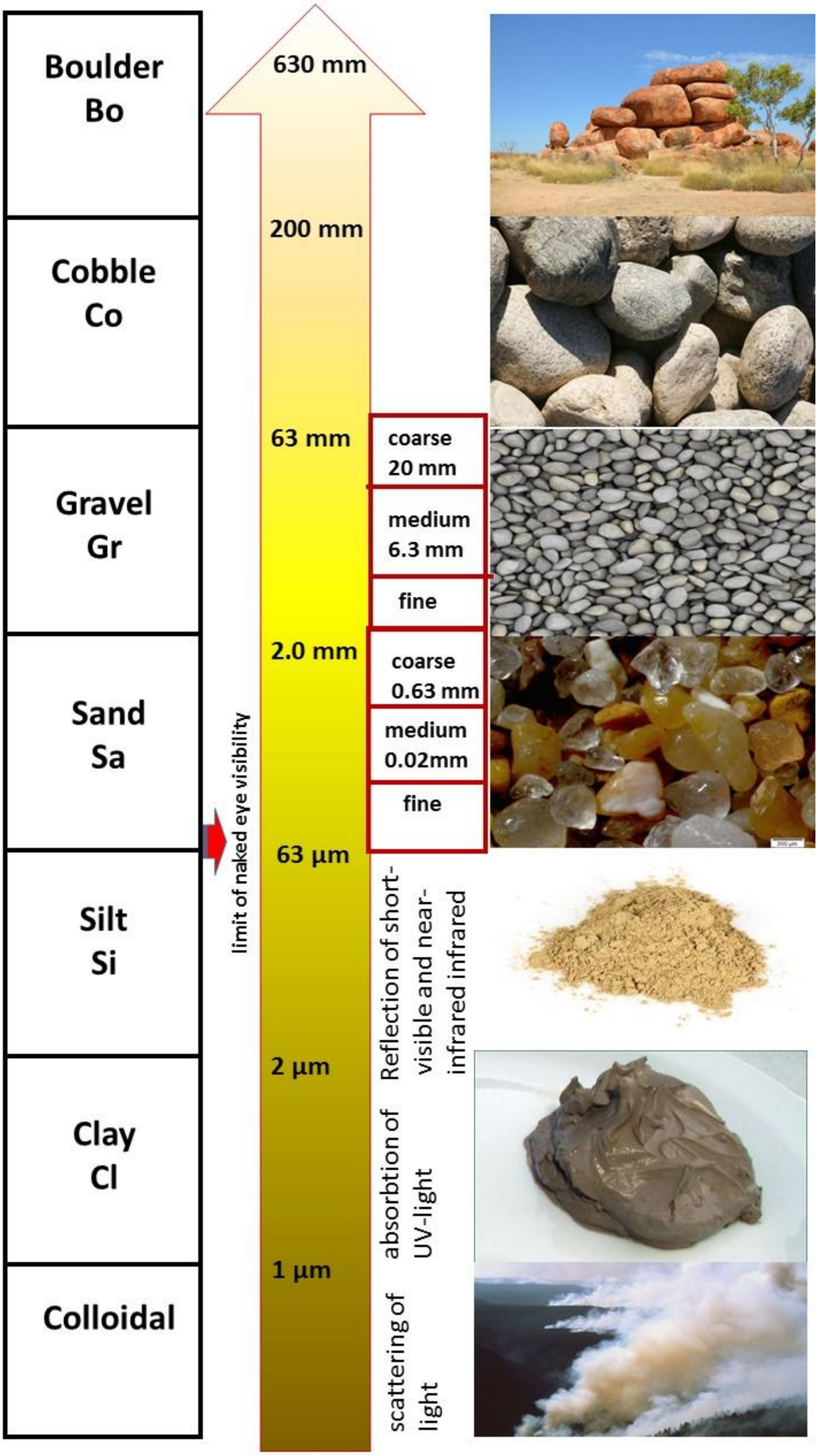


Fig. S1.3a The grain size scale in clastic sediments (adapted from Sebastian, 2009 after ISO 14688-1).

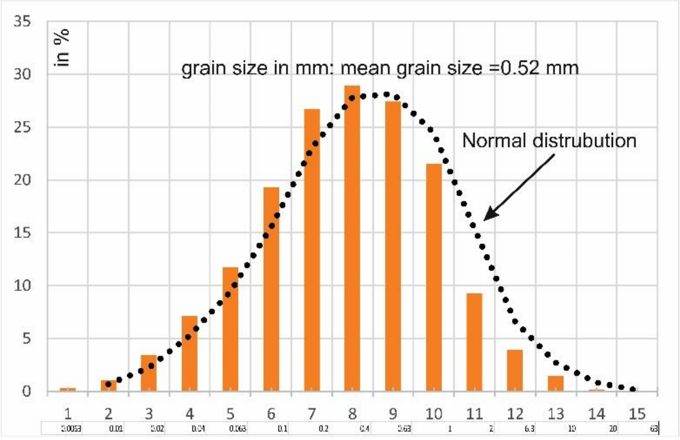


Fig. S1.3b Problem 9: Histogram of grain size distribution.

10. From cumulative curve of grain sizes shown in Fig. S1.4 the probability distribution function of grain sizes PDF(d_i) may be reconstructed by the method described in Exercise 7. For a particular grain size interval the specific grain surface erea per unit mass may be calculated as: $A_{sp}(d_i) \approx \frac{6}{\rho \cdot d_i}$. The total grain surface area is the sum: $A_{sp}^{total} = \sum_i A_{sp}(d_i) \cdot PDF(d_i)$. The results is A_{sp}=0.26 m²/g.

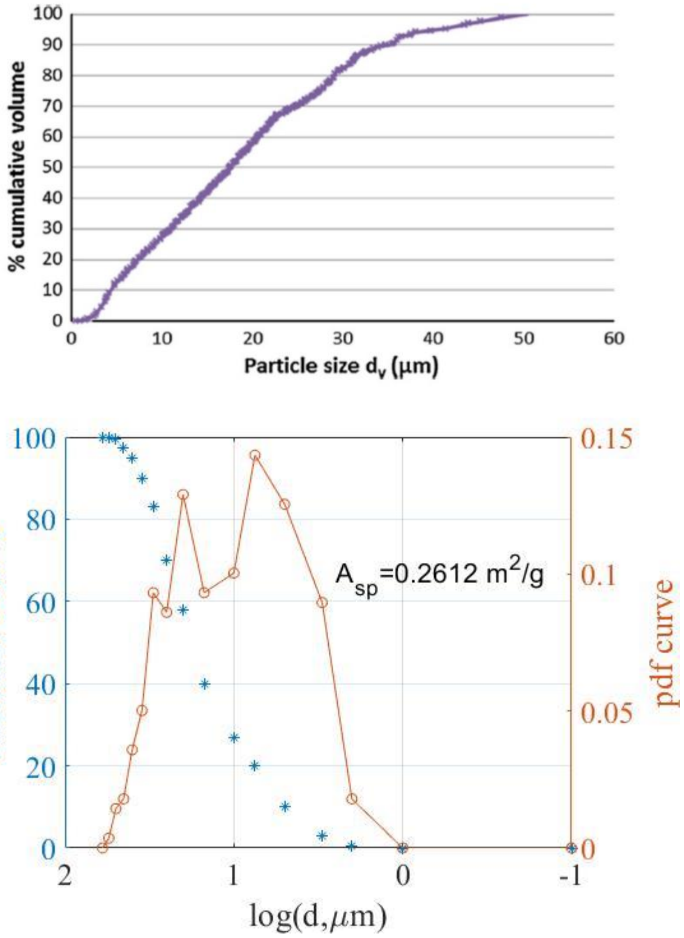


Fig. S1.4 Problem 10: (upper panel) Particle size distribution of a blast furnace slag sample as determined by image analysis (Arvaniti et al., 2015). (lower panel) Particle size distribution function.

Solutions to Chapter 2:

1. (a) Consider 3 spheres in the layer B (Fig. S2.1 upper left panel) B1, B2 and B3. The apper vertex sphere of the tetrahedra B1B2B3A1 the point A1 is in the centre of the hexagonal of layer A. The side length of tetrahedra is $a=b=1$. The height B1H=1/2. The angle OB1H=30°, dann OH=BH/tg30°=$1/2\sqrt{3}$. The angle A1B1H=60°, then A1H=$\sqrt{3}/2$. From the triangular ΔOHA1 one obtains that OA1= $\sqrt{2}/\sqrt{3}$. The distance between two adjacent layers B is $c=2\sqrt{2}/\sqrt{3}=1.633$

   (b) In the elementary cell in the layer A there are 7 spheres (see Fig. S2.1 left lower panel). But only a half of the central sphere belongs to the elementary cell. From 6 spheres forming a hexagon only 1/6th of each belongs to the cell. Totally, it makes 1.5 spheres per layer A. In the layer B 3 spheres belong to the cell, in the next layer A consisting of 7 only 1.5 spheres count to the cell. Totally, per unit cell there are 6 spheres having the total volume: $6\cdot\pi/6\cdot 1^3=\pi$. The volume of the unit cell is the product of height × area of hexagon which is formed in the layer A. The height is: $2\sqrt{2}/\sqrt{3}$. The area of hexagon is: $6\cdot ½\cdot\sqrt{3}/2\cdot 1$. Thus, the ratio of the spheres volume to the unit cell is: $\pi/3\sqrt{2}=0.74$. The porosity of hexagonal dense packing is given by: 1-0.74=0,26 (Gauss, 1831).

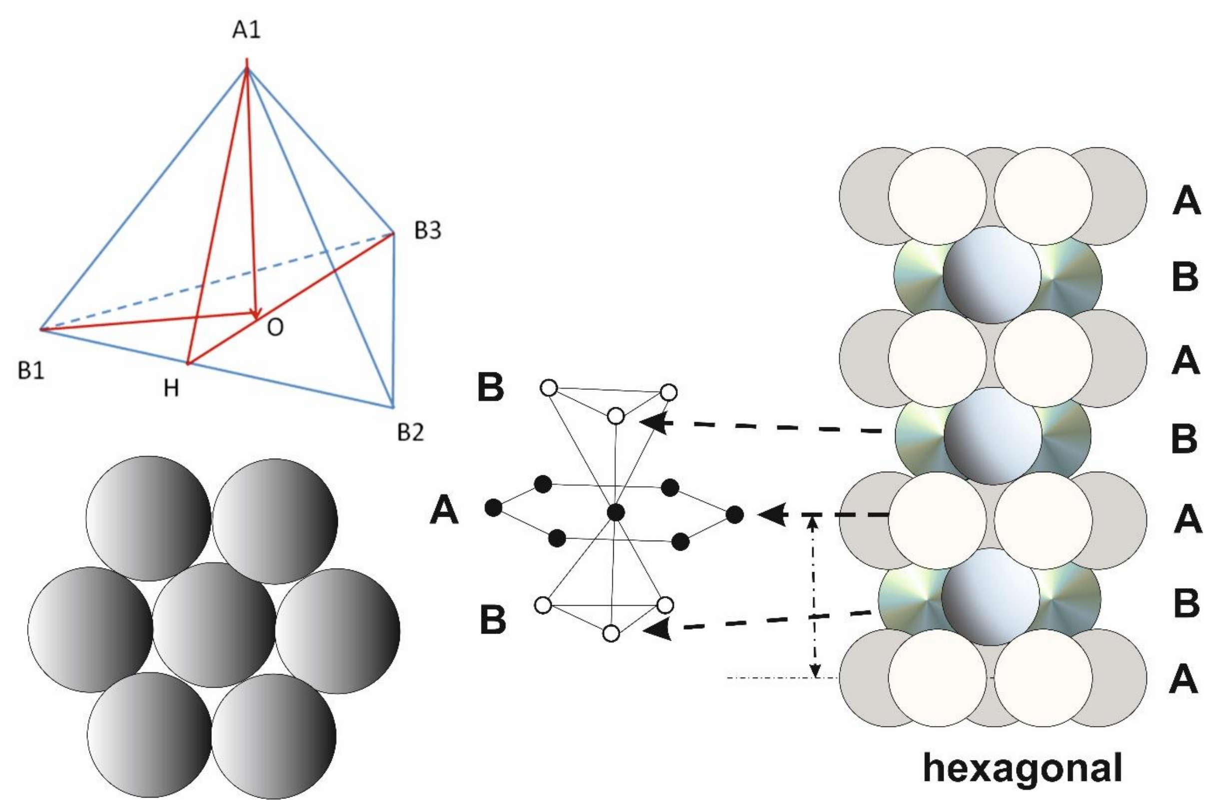


Fig. S2.1 Problem 1.

2. Solution: density =0.4 · 2.65 + 0.5 · (0.5·2.57 +0.5·2.59) +0.06 · 2.85 +0.04 · 3.05=2.64 g/cm³.

3. The solution is presented in Fig. S2a&b.

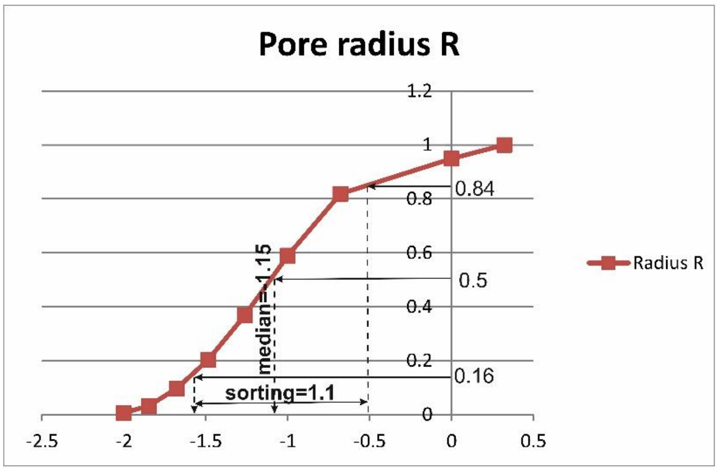


Fig. S2.2a Problem 3: Cumulative curve of pore radius as a function of ϕ.

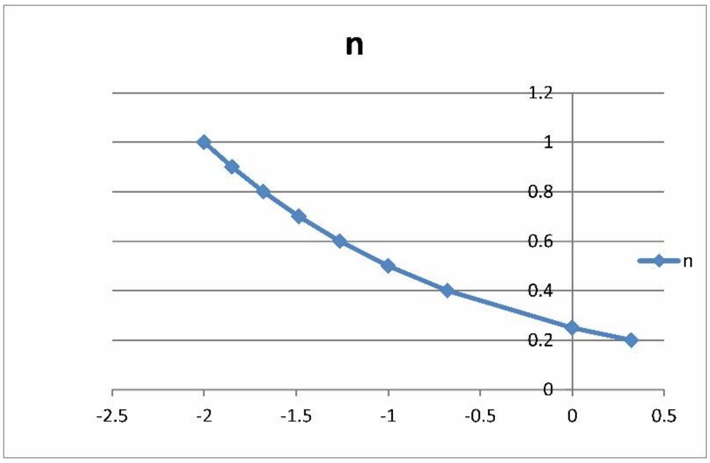


Fig. S2.2b Problem 3: Number of particles as a function of ϕ.

4. (a) For bulk density ρ_B one uses the relationship:

$$\rho_B = \frac{607\,g}{413\,cm^3} \approx 1{,}47\,g/cm^3$$

(b) In order to calculate porosity the bulk density may be presented as the arithmetic mean:: $\rho_B = (1-\phi)\cdot\rho_S \Rightarrow 1{,}47 = (1-\phi)\cdot 2{,}65 \Rightarrow \phi = \frac{2{,}65-1{,}47}{2{,}65} \approx 0{,}445$ from which it follows that: ϕ=44.5%

(c) $e = V_{Pores}/V_{mineral\ grains}$ $e = \frac{V\cdot\phi}{V\cdot(1-\phi)} = \frac{0{,}445}{0{,}555} \approx 0{,}8$

(d) for the degree of saturation S: $\underbrace{\frac{727\,g}{413\,cm^3}}_{1{,}76\,g/cm^3} = \underbrace{(1-\phi)\cdot\rho_S}_{1{,}47\,g/cm^3} + \underbrace{\phi\cdot\rho_W}_{0{,}445\,g/cm^3}\cdot S \Rightarrow S \approx 0{,}65$ or S=65%,

(e) the water content in the rocks is given by: $w = \frac{727-607}{607} = 0{,}198$ or S=19.8%.

5. $P = 2.65\cdot 10^3\cdot 9.81\cdot(1-\varphi) = 20000Pa$, i.e. $\varphi = 1-0.77 = 0.23$.

6. In order to calculate the total amount of oil one has to integrate the prosity as function of depth z:

$$V = A\cdot\beta\cdot\int_0^H \phi_0\cdot e^{-\alpha\cdot z}\cdot dz = \frac{A\cdot\beta\cdot\phi_0}{\alpha}\cdot\int_H^0 \cdot e^{-\alpha\cdot z}\cdot d\underbrace{(-\alpha\cdot z)}_{u} = \frac{A\cdot\beta\cdot\phi_0}{\alpha}\cdot\int_H^0 \cdot e^u\cdot du = \frac{A\cdot\beta\cdot\phi_0}{\alpha}\cdot(1-e^{-\alpha\cdot H})$$

$$V[km^3] = 0{,}01\cdot A[km^2]\cdot\phi_0\cdot(1-e^{-\cdot H[km]})$$

7. The equation may be rewritten in a dimensionless form for quartz exsolution and deposition on the surface of pores in a sandstone as follows:

$\frac{\omega\cdot(\varphi_0-\varphi_c)}{S_0\cdot A\cdot\overline{v_Q}\cdot\left(\frac{C-C_{eq}}{C_{eq}}\right)}\cdot\frac{d\xi}{\xi^n\cdot dz}=-e^{-\frac{E}{R\cdot T_0\cdot[1+\frac{1}{T_0}\left(\frac{dT}{dz}\right)\cdot z]}}\approx -e^{-\frac{E}{R\cdot T_0}\cdot\left[1-\frac{1}{T_0}\cdot\left(\frac{dT}{dz}\right)\cdot z\right]}$, where ξ is dimensionless variable of porosity: ξ=$\frac{\varphi-\varphi_c}{\varphi_0-\varphi_c}$. Here the expansion of $1/(1+\frac{1}{T_0}\left(\frac{dT}{dz}\right)\cdot z)$ in the Taylor series has been done because $\frac{1}{T_0}\left(\frac{dT}{dz}\right)\sim$ 10^{-1} 1/km and after the integration of this differential equation from z=0 to z assuming the boundary conditions at *z=0* $\phi=\phi_0$ and $T=T_0$ one gets:

$\frac{\frac{E}{R\cdot T^2{}_0}\cdot\left[\left(\frac{dT}{dz}\right)\right]\cdot\omega\cdot(\varphi_0-\varphi_c)\cdot e^{\frac{E}{R\cdot T_0}}}{(1-n)\cdot S_0\cdot A\cdot\overline{v_Q}\cdot\left(\frac{C-C_{eq}}{C_{eq}}\right)}\cdot[\left(\frac{\varphi-\varphi_c}{\varphi_0-\varphi_c}\right)^{1-n}]\approx 1-e^{\frac{E}{R\cdot T_0}\cdot\left[\frac{1}{T_0}\cdot\left(\frac{dT}{dz}\right)\cdot z\right]}$,

and in the case when n=1

$\frac{\frac{E}{R\cdot T^2{}_0}\cdot\left[\left(\frac{dT}{dz}\right)\right]\cdot\omega\cdot(\varphi_0-\varphi_c)\cdot e^{\frac{E}{R\cdot T_0}}}{S_0\cdot A\cdot\overline{v_Q}\cdot\left(\frac{C-C_{eq}}{C_{eq}}\right)}\cdot[ln\left(\frac{\varphi-\varphi_c}{\varphi_0-\varphi_c}\right)\ ]\approx 1-e^{\frac{E}{R\cdot T_0}\cdot\left[\frac{1}{T_0}\cdot\left(\frac{dT}{dz}\right)\cdot z\right]}$.

After the substitution of typical parameters for quartz this relationship for z in km and burial rate ω in km/Myr is then

$\left(\frac{\varphi-\varphi_c}{\varphi_0-\varphi_c}\right)^{1-n}\approx\frac{(1-e^{3.6\cdot z})\cdot(1-n)}{1.86\cdot 10^4\ \omega\cdot(\varphi_0-\varphi_c)}$, and for n=1

$ln\left(\frac{\varphi-\varphi_c}{\varphi_0-\varphi_c}\right)\ \approx\frac{1-e^{3.6\cdot z}}{1.86\cdot 10^4\ \omega\cdot(\varphi_0-\varphi_c)}$.

The characteristic scale decrease of porosity by factor 2 corresponds to about z=3.4 km with the burial rate 50 km/Myr, ϕ_0=0.3 and ϕ_c=0.05.

8. Surface tension and capillary forces:

a) The contact angle for water on clean quartz surface is approximately zero. What is the surface tension of water at 20 ° C when a column of water in a quartz capillary tube with an inside diameter of 0.6 mm rises to 4.96 cm high. The density of water at 20 ° C is 998.2 kg / m^3.

b) What is the maximum diameter of the guiding capillary in a 30 m high rock column in order to raise water to the upper surface of a column by capillary forces? Set the calculation for different wetting angles of 0°, 10°, 20°?

a) The surface tension force is balanced by the weight of the water column in the capillary tube: $2\pi\cdot r\cdot\sigma\cdot cos\theta=\rho\cdot g\cdot h\cdot\pi r^2$. When the contact angle θ=0, then the surface tension of water is : $\sigma=\frac{\rho\cdot g\cdot h\cdot r}{2}$=0.073 N/$m^2$.

b) One rearranged the previous identity and obtains: $d_{max} = \frac{4\cdot\sigma\cdot cos\theta}{\rho\cdot g\cdot h} \approx\ 10^{-6}m\ for\ \theta = 0,$ $0.985\cdot 10^{-6}m\ for\ \theta = 10^{0}, 0.94\cdot 10^{-6}m\ for\ \theta = 20^{0}.$

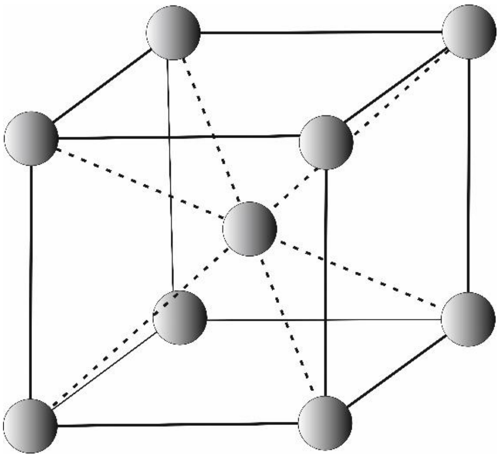


Fig. S2.3 Problem 9**:** *bcc* lattice.

9. Per unit cell there are 2 spheres: one in the center and 8 spheres are shared between 8 adjacent unit cells $8\cdot\frac{1}{8} = 1$, the volume of two spheres =$\frac{8\cdot\pi\cdot r^3}{3}$ If the side length of unit cell is a, then the largest diagonal is $\sqrt{3}\cdot a$. This diagonal equals 2 dimeters of spheres, because spheres are touching each other: $\sqrt{3}\cdot a = 4\cdot r$. The unit cell volume is a^3=$\frac{64\cdot r^3}{3\sqrt{3}}$. The filling factor of bcc is: $\frac{8\cdot\pi\cdot r^3}{3}\cdot\frac{3\sqrt{3}}{64\cdot r^3}$=$\frac{\pi\cdot\sqrt{3}}{8}$=68%.

10. The first step is to covert the pressure into Pa: 1 Torr= 133.322 Pa, and to convert adsorbed volume from standard conditions to 77K and corresponding pressure using the ideal gas law:

p [Torr]	**30**	**50**	100	150	200	250
V [cm^3]	29.1	32	36.3	40	44	48.5
p[Pa]	4000	6666.66667	13333.3333	20000	26666.6667	33333.3333
V[m^3]	0.00019119	0.00012614	7.1547E-05	5.256E-05	4.3362E-05	3.8237E-05
Asp.[m^2/kg]	166.484219	183.07543	207.676191	228.844288	251.728717	277.473699
p[bar]	0.004	0.00666667	0.01333333	0.02	0.02666667	0.03333333

$\Delta\gamma$ = --0.01391345= -13.9 mJ/m².

The results of the experiment are presented in Fig. S2.4a.

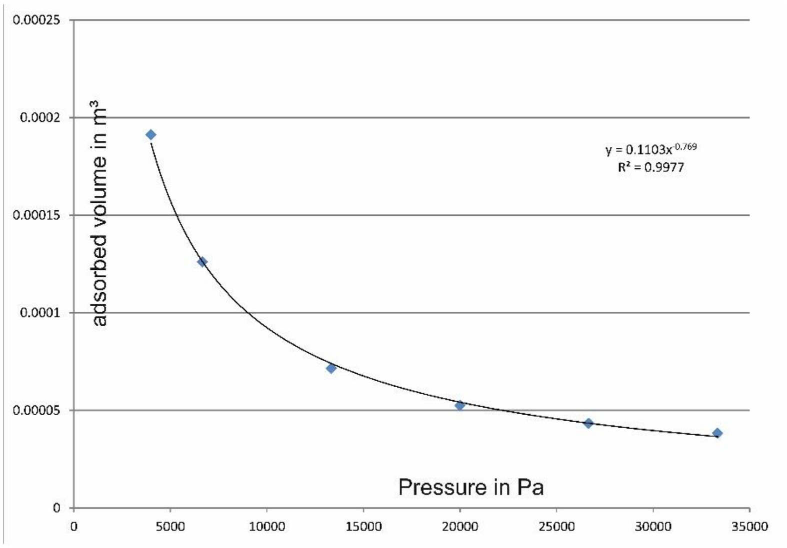


Fig. S2.4a Adsorbed volume vs. pressure.

The specific area of inner open surface per unit mass is calculated as follows: the number of moles of adsorbed gas is $\mu = \frac{P\cdot V}{R\cdot T}$, the number of gas adsorbed molecules is $\frac{P\cdot V}{R\cdot T}\cdot N_A$, the corresponding inner surface is $\frac{P\cdot V\cdot a\cdot N_A}{T\cdot R}$, and $A_{sp.}\left[\frac{m^2}{kg}\right] = \frac{P\cdot V\cdot a\cdot N_A}{T\cdot R}\cdot\frac{1}{m}$. Fig. S2.4b presents the results of specific area calculations as a function of pressure. The surface energy change Δγ of the rock sample (in mJ/m²) as the pressure increases during the experiment from 30 to 250 Torr: $\Delta\gamma = -R\cdot T\cdot\frac{ln\left(\frac{250}{30}\right)}{a\cdot N_A} = -13.9\ mJ/m^2$.

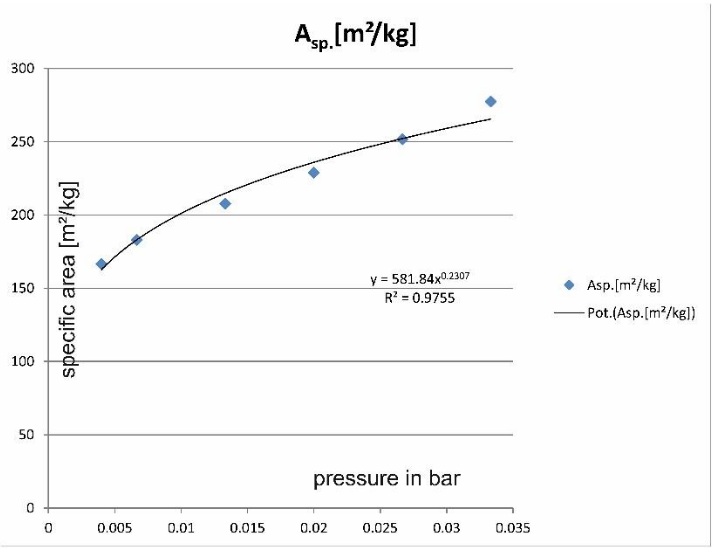


Fig. S2.4b Specific area A_{sp} as a function of pressure.

11. KCl: $r_K = 0.133 \cdot 10^{-9} m, r_{Cl} = 0.182 \cdot 10^{-9} m, \frac{r_K}{r_{Cl}} = 0.735$, the coordination number is 8, the dense packing corresponds to *bcc* lattice, and the atom in unit cell are touching along the main diagonal: $a = \frac{2}{\sqrt{3}} \cdot (r_K + r_{Cl}) = 0.365 \cdot 10^{-9} m$. There are 2 atoms per unit cell, 1 of K and 1 of Cl. The theoretical density of the dense packing is: $\rho = \frac{(39.0983+35.453) g/mol}{(0.365 \cdot 10^{-9})^3 \cdot 6.023 \cdot 10^{23} 1/mol} = 2.546 \cdot 10^6 \frac{g}{m^3}$ =2.546 g/cm^3.

MgO: $r_{Mg} = 0.066 \cdot 10^{-9} m, r_O = 0.132 \cdot 10^{-9} m, \frac{r_K}{r_{Cl}} = 0.5$, the coordination number is 6, the dense packing corresponds to *fcc* lattice, and the atom in unit cell are touching along the edge of cube: $a = 2 \cdot (r_K + r_{Cl}) = 0.396 \cdot 10^{-9} m$. There are 8 atoms per unit cell (see Fig. S2.5), 4 of Mg and 4 of O: 8 atoms in vertexes shared between 8 adjacent unit cells =1 atom, 12 atoms in the middle of edges shared between four adjacent unit cells =3 atoms, 6 atomes in the middle of faces shared between 2 adjecent unit cells =3 atoms and 1 atom in the center of the unit cell. The theoretical density of the dense packing is: $\rho = \frac{4 \cdot (24.312+16) g/mol}{(0.395 \cdot 10^{-9})^3 \cdot 6.023 \cdot 10^{23} 1/mol} = 4.313 \cdot 10^6 \frac{g}{m^3}$ =4.313 g/cm^3.

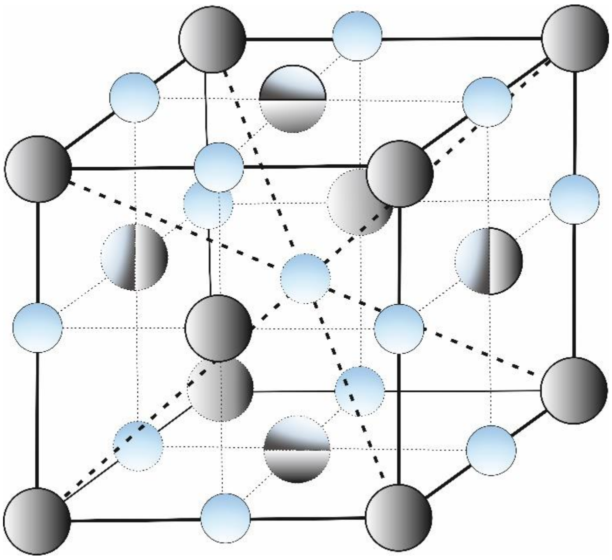


Fig. S2.5 Problem 11: *fcc* lattice.

Exercises and control questions to Chapter 3:

1. The parameter λ of the Poisson's distribution is the number of cracks per 5 m: $\lambda=\frac{12100}{1000}\cdot 5=60.5$.

(a) The probability to get exactly 8 cracks is: $e^{-60.5}\cdot\frac{(60.5)^8}{8!}\approx 1.6\cdot 10^{-17}$

(b) The cumulative probability of 0 to 10 cracks is: $e^{-60.5}\cdot\frac{(60.5)^{10}}{10!}+..e^{-60.5}\cdot\frac{(60.5)^{1}}{1!}+e^{-60.5}\approx 10^{-15}$

(c) $e^{-60.5}\cdot\frac{(60.5)^{9}}{9!}+..e^{-60.5}\cdot\frac{(60.5)^{15}}{15!}\approx 2.7\cdot 10^{12}$

(d) $1-\overbrace{\left(e^{-60.5}\cdot\frac{(60.5)^{10}}{10!}+..e^{-60.5}\cdot\frac{(60.5)^{1}}{1!}+e^{-60.5}\right)}^{10^{-15}}$.

2. There are 42 fractures on average in a 10 m long borehole core. If the fraction between-length (l) follows an exponential distribution how would you define the average spacing between? How would you estimate the number of fractures (n) per 1m of the core? How are the exponential distribution and the Poisson distribution look like?

The parameter λ of the exponential distribution is the number of cracks per meter, $\lambda=4.2\ m^{-1}$. The average spacing is $=\frac{1}{\lambda}=0.24\ m$. The exponential cumulative distribution of value x is given by: $F(x)=\int_0^x\lambda\cdot e^{-\lambda\cdot t}\cdot dt$=1-$e^{-\lambda\cdot x}$. The median value $\tilde{x}$ corresponds to the cumulative probability: $\frac{1}{2}$=1-$e^{-\lambda\cdot\tilde{x}}\rightarrow\tilde{x}=\frac{\ln(2)}{\lambda}$= 0.165.

Fig. S3.1 illustrate the difference between the Poisson's and exponential distributions.

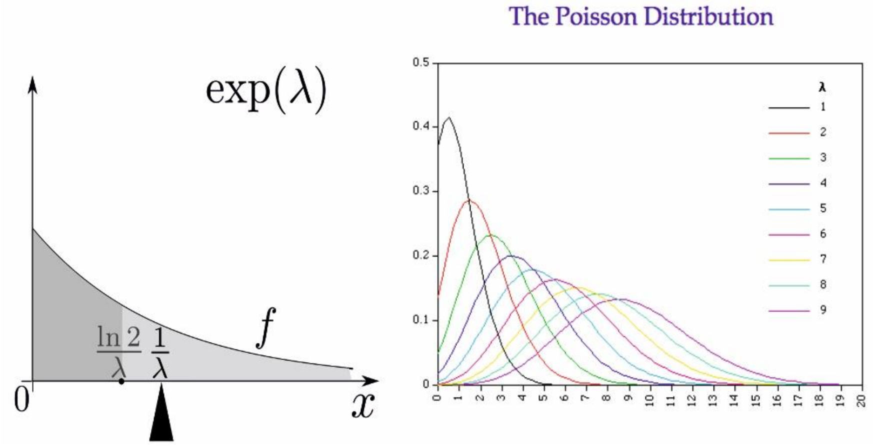


Fig. S3.1. Exponential and Poisson distributions.

3. The Reuss average is:

$$E_R=\left(\frac{0,2}{96,4}+\frac{0,1}{63,6}+\frac{0,2}{74,9}+\frac{0,25}{78,5}+\frac{0,25}{128,8}\right)^{-1}\cdot 10^9\ [N/m^2]=87,4\cdot 10^9\ [N/m^2]$$

The Voigt average is:

$$E_V=(96,4\cdot 0,2+63,6\cdot 0,1+74,9\cdot 0,2+78,5\cdot 0,25+128,8\cdot 0,25)\cdot 10^9\ [N/m^2]=92,4\cdot 10^9\ [N/m^2]$$

4. (a) Fig. S3.2 illustrate the application of the force acting on the surface

$$\sigma_{ZZ} = \frac{F \cdot cos\psi}{A} = \frac{10^3 \cdot cos(45°)}{0{,}001[m^2]} = 707100Pa$$

$$\tau_Y = \frac{F \cdot sin\psi \cdot sin\varphi}{A} = \frac{10^3 \cdot 0{,}707 \cdot 0{,}5}{0{,}001} = 353500Pa$$

$$\tau_X = \frac{F \cdot sin\psi \cdot cos\varphi}{A} = \frac{10^3 \cdot 0{,}707 \cdot 0{,}87}{0{,}001} = 612300Pa$$

(b)

$$\Delta l_Z = \sqrt{A} \cdot \frac{\sigma_Z}{E} = 2{,}8 \cdot 10^{-7}\, m$$

$$\varepsilon_{ZZ} = \frac{\sigma_Z}{E} = 0{,}0009\%$$

(c)

$$\gamma_X = \frac{\tau_X}{2\mu} = \frac{612300}{2 \cdot 30 \cdot 10^9} = 10^{-5}\, rad$$

$$\gamma_Y = \frac{\tau_Y}{2\mu} = \frac{353500}{2 \cdot 30 \cdot 10^9} = 0{,}6 \cdot 10^{-5}\, rad$$

The angle deformation is :$\gamma_X + \gamma_Y = 1.6 \cdot 10^{-5}\ rad.$ The shear strain is given by:

$$\varepsilon_{XY} = \frac{\gamma_X + \gamma_Y}{2} = 0{,}8 \cdot 10^{-5}\, rad$$

The twist angle between the base and the surface plane is: $\omega_{XY} = (\gamma_X - \gamma_Y) = 0.4 \cdot 10^{-5}$ rad.

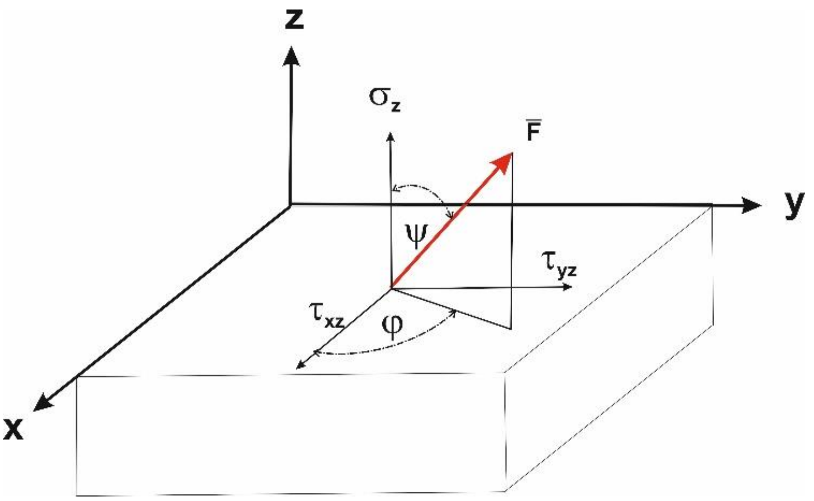


Fig. S3.2 Problem 4.

5. To the side surface of the block of rock with the edge lengths *a*, *b*, and *c* the force F is applied which is evenly distributed over the surface (Fig. E3.2).

Given: F = 10 kN, *a* = 3 cm, *b* = 4 cm, *c* = 2 cm.
How big are tangential and normal stresses to the grey, diagonal cut surface?

<u>Lösung:</u>

a. The forces parallel and orthogonal to the cut plane are:

$$F_t = F \cdot cos\alpha = 10 \cdot \frac{4}{5} = 8kN$$

$$F_N = F \cdot sin\alpha = 10 \cdot \frac{3}{5} = 6kN$$

b. he tangential and normal stressest o the cut plane are::

$$\sigma_t = \frac{F_t}{5 \cdot 10^{-2} \cdot 2 \cdot 10^{-2}} = 8 \cdot 10^6 Pa = 8MPa$$

$$\sigma_N = \frac{F_N}{5 \cdot 10^{-2} \cdot 2 \cdot 10^{-2}} = 6 \cdot 10^6 Pa = 6MPa$$

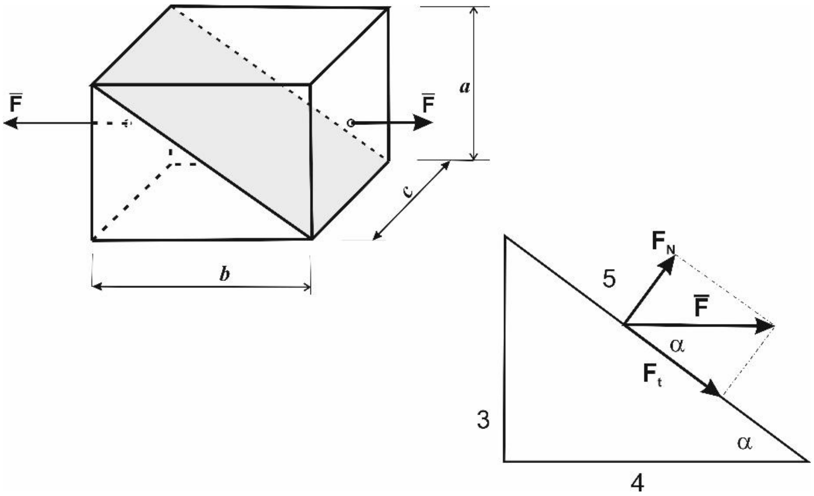


Fig. S3.3 Problem 5.

6. a) Axial stress along z-axis is: $\sigma_z = \frac{25000 \cdot 4}{\pi d^2} \approx \frac{10^5}{3{,}14 \cdot 4 \cdot 10^{-4}} \approx 79{,}6MPa$. Radial strain is: $\boldsymbol{\varepsilon_r = -\frac{0{,}003}{20} \cdot 100\% = -0{,}015\%}$, axial strain is: $\varepsilon_z = +\frac{0{,}164}{200} \cdot 100\% = +0{,}082\%$. The Young's modulus is: $E = \frac{\sigma_z}{\varepsilon_z} = \frac{79{,}6MPa}{0{,}00082} = 97{,}07GPa$.

b) The Poisson's ratio is: $\nu = -\frac{\varepsilon_r}{\varepsilon_z} = 0.183$; $\boldsymbol{K = \frac{E}{3 \cdot (1 - 2 \cdot \nu)} = 76{,}3GPa}$ and

$$\boldsymbol{\mu = \frac{E}{2 \cdot (1 + \nu)} = 41GPa}$$

c) The strength of granite sample is given by:

$$\sigma_F = \frac{4 \cdot F_{max}}{\pi d^2} = \frac{4 \cdot 125000}{3{,}14 \cdot 10^{-6}(20 - 0{,}033)^2} \approx 399{,}4MPa$$

7. In Fig. S3.4 the construction of the Mohr's circle and the principle stress ellipse is illucidated. The stress vector σ(45°) corresponds to the point on the stress ellipse having two components [$\sigma_1 \cdot \cos(45°)$, $\sigma_2 \cdot \sin(45°)$]. The length of the vector is:

$|\sigma(45^o)| = \sqrt{\sigma_1^2 \cdot cos^2(45^o) + \sigma_2^2 \cdot sin^2(45^o)} = \sqrt{\frac{50^2+25^2}{2}}$= 39.53 MPa, where σ_1 and σ_2 are two given principal stresses. The vector σ(45°) builds the angle β with the σ_1 axis:

$$\boldsymbol{tan(\beta) = \frac{\sigma_2 \cdot sin(45°)}{\sigma_1 \cdot cos(45°)} = 0{,}5}$$

$$\boldsymbol{\beta = 26{,}56°}$$

Then, from the right triangle it follows:

$$\sigma_n(45°) = \sigma(45°) \cdot \cos(45° - \beta) = 37{,}5MPa$$
$$\tau(45°) = \sigma(45°) \cdot \sin(45° - \beta) = 12{,}5MPa$$

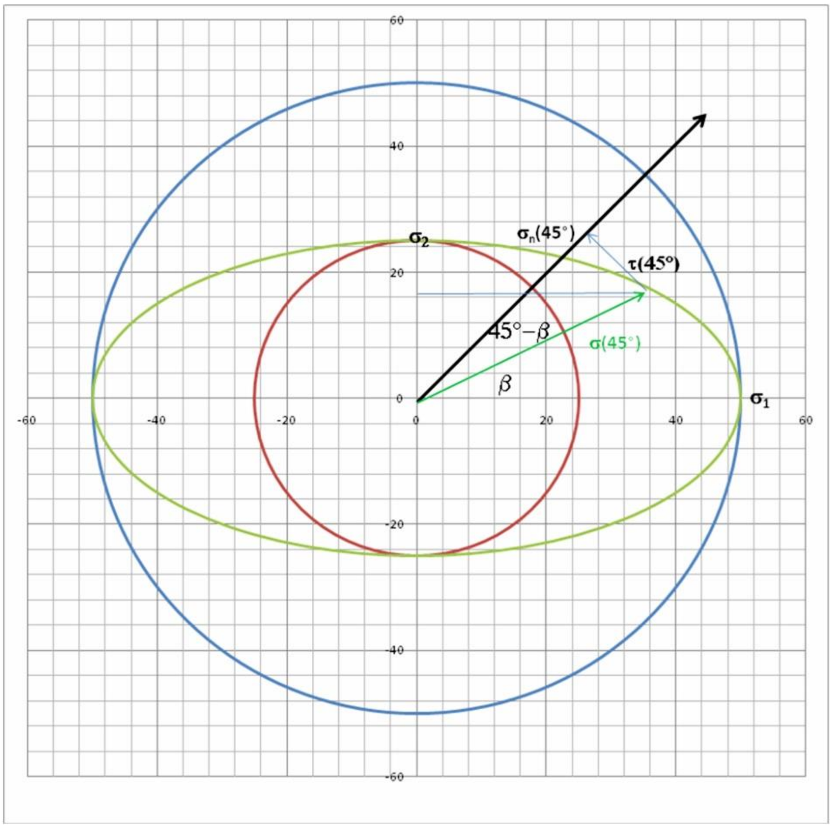


Fig. S3.4 Problem 7.

8. (a) Analytical solution. In ordert o calculate the principal stresses one needs to find eigenvalues oft he matrix equation:

$$\boldsymbol{det\begin{vmatrix} 60-\lambda & 15 \\ 15 & 20-\lambda \end{vmatrix} = 0}$$

The root oft he quadratic equation $\lambda_{1,2}$ are:

$$\boldsymbol{\lambda^2 - 80 \cdot \lambda + 975 = 0}$$
$$\boldsymbol{\lambda_{1,2} = 40 \pm 25}$$

So two principle stresses are: σ_1=65 and σ_2=15 MPa: $\boldsymbol{S^* = \begin{vmatrix} 65 & 0 \\ 0 & 15 \end{vmatrix}}$

The new coordinate systen in which the stress matrix is diagonal are defined by the rotation angle of axis θ:

$$\theta = \frac{1}{2} arctan\left(\frac{2\tau}{\sigma_x - \sigma_y} \right) = 0{,}5 \cdot arctan(\frac{2 \cdot 15}{40}) = 18{,}44°$$

(b) Graphic solution. One plots in Cartesian coordinate system two points: A(σ_x,+τ) und B(σ_y, -τ) and connects A and B with a straight line (red line in Fig. S3.5). Then, one estimates the radius of the Mohr's circle as a half distance between points A and B. The ordinates of intersection points of the circle with the horizontal axis S_1 und S_2 define the principal stresses σ_1 and σ_2. The direction of the principal stress is defined by the half angle between the line AB and the *x*-axis =1/2 · *angle*(AOS_1) or equals the angle(AS_2O).

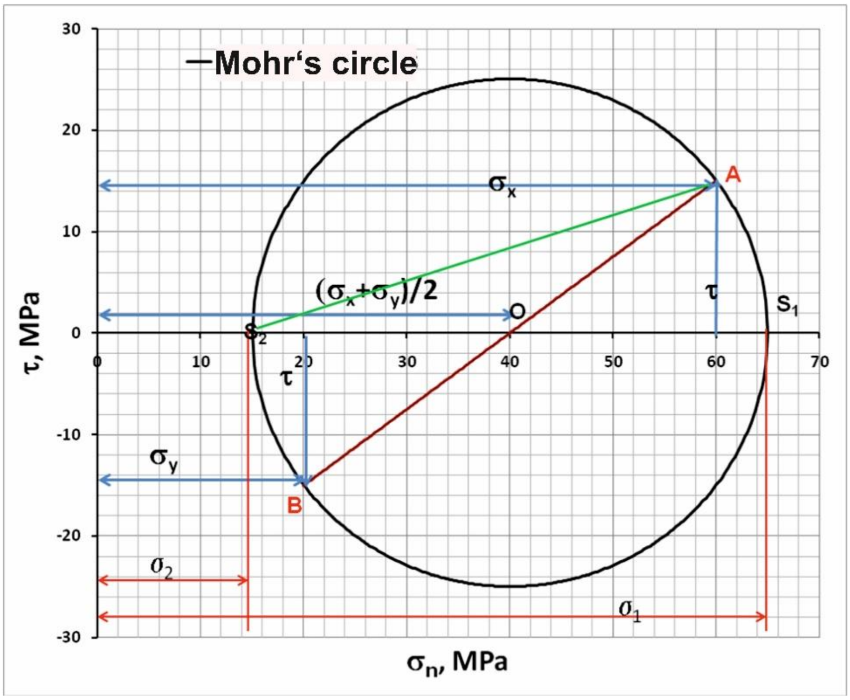


Fig. S3.5 Problem 8.

9. (a) The division of the stress state into hydrostatic and deviatorial part can be given in a tensor form. How does the equation look like? Derive the hydrostatic and the deviatoric stress components explicitly trough principal stresses. What type of deformation do the hydrostatic and the deviatoric parts of stress tensor correspond to?

(b) The invariants of the deviatoric part of stress tensor are:

$$J_1 = s_{11} + s_{22} + s_{33}$$

$$J_2 = s_{11} \cdot s_{22} + s_{11} \cdot s_{33} + s_{33} \cdot s_{22}$$

$$J_3 = s_{11} \cdot s_{22} \cdot s_{33}$$

Express the deviatoric invariants for the uniaxial tensile test through principal stresses. The stress tensor invariants are the coefficients in the cubic equation which roots define tree principal stresses. In the matrix form this equation is:

$\begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} - \lambda \cdot \overbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}^{I} = 0 \ \rightarrow \lambda^3 - I_1 \cdot \lambda^2 - I_2 \cdot \lambda - I_3 = 0$, where $I_{1,2,3}$ are the stress matrix invariants. The hydrostatic stress is the mean value of diagonal stress matrix elements: $\sigma_m = \frac{\overbrace{trace(\sigma_{ij})}^{I_1}}{3}$. So the matrix $\sigma_m \cdot I$ represents hydrostatic part of the stress matrix. The deviatoric stress matrix is given by: $\begin{vmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{vmatrix} -$ $\sigma_m \cdot I = \begin{vmatrix} \sigma_{xx} - \sigma_m & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - \sigma_m & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - \sigma_m \end{vmatrix} = S$. If $\lambda_{1,2,3}$ are the eigenvalues of the stress matrix, then, the eigenvalues of the deviatoric matrix are $s_{1,2,3} = \lambda_{1,2,3} - \sigma_m$, or in terms of three principal stresses $\sigma_{1,2,3}$:

$$s_1 = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3}$$

$$s_2 = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{3}$$

$$s_3 = \frac{2\sigma_3 - \sigma_2 - \sigma_1}{3}$$

(b) The invariants of deviatoric stress matrix are:

$J_1 = s_1 + s_2 + s_3 \equiv 0$, The trace of deviatoric matrix is trivial zero, since the sum of three roots of deviatoric matrix eigenvalues are corrected to the mean value of the roots.

$J_2 = \sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2 - (\sigma_{xx} - \sigma_m) \cdot (\sigma_{yy} - \sigma_m) - (\sigma_{xx} - \sigma_m) \cdot (\sigma_{zz} - \sigma_m) - (\sigma_{yy} - \sigma_m) \cdot (\sigma_{zz} - \sigma_m) = \sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yz}^2 - (\sigma_{xx} \cdot \sigma_{yy} + \sigma_{xx} \cdot \sigma_{zz} + \sigma_{yy} \cdot \sigma_{zz}) + 3\sigma_m^2 = -(s_1 \cdot s_2 + s_1 \cdot s_2 + s_2 \cdot s_3)$, since the trace of deviatoric matrix is zero. Or $J_2 - (s_1 \cdot s_2 + s_1 \cdot s_2 + s_2 \cdot s_3) + \frac{(s_1 + s_2 + s_3)^2}{2} = \frac{s_1^2 + s_2^2 + s_3^2}{2}$. The third invariant is the matrix determinant andfor the diagonal matrix: $J_3 = s_1 \cdot s_2 \cdot s_3$.

10.

a) The principal stresses are 210 and 100 MPa: $(155 - \lambda)^2 = 55^2$, $\lambda_1 = 155 + 55 = 210, \lambda_2 = 155 - 55 = 100$.

b) According to Tresca yield criterion is yielding starts when the maximum shear stress in the material equals the yielding stress limit: $\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = 55\ MPa < k_f$. The material deforms elastically.

c) According to von Mises in 2D case the yielding criterion is: $\sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \cdot \sigma_2}$ = = 181.9 MPa $< k_f$. The material responded also elastically.

d) In Fig. S3.6 the difference between Tresca and von Mises yield criterion is depicted. Depending on the ratio between two principle stresses $\alpha = \frac{\sigma_2}{\sigma_1}$ the difference between predicted yielding criteria may be significant only at $\alpha = 0, 1\ and\ \infty$ two criteria predicted the same yielding stress.

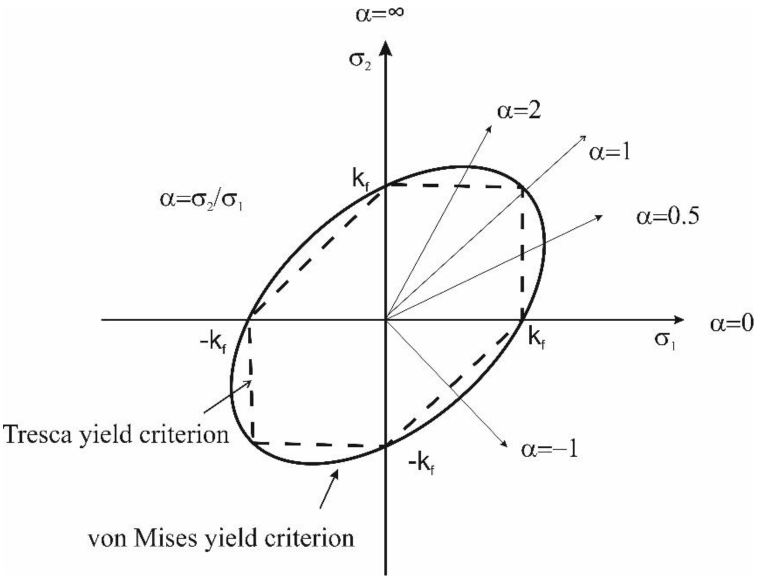


Fig. S3.6 Problem 10: comparison between Tresca and von Mises yielding criteria.

11. Elastic modulus E=90 · 10^9 N/m^2, Poisson number ν= 0.28.
   (a) The axial stress is σ_z =600 N/m².
   (b) The axial strain is $\varepsilon_z = \frac{\sigma_z}{E} = 6.7\cdot 10^{-7}$. The lateral strain is $\varepsilon_l = -\nu \cdot \varepsilon_z = -1.86\cdot 10^{-7}$. Volume strain is: $\varepsilon_v = (1-2\nu)\cdot\frac{\sigma_z}{E} = 2.9\cdot 10^{-7}$.
   (c) The width is changed by: 0.1· ε_l =-1.87· 10^{-8}, the length is elongated by: 0.2 · ε_z=1.3· 10^{-7} m.
   12. The graphical solution of the problem is shown in Fig. S3.7.

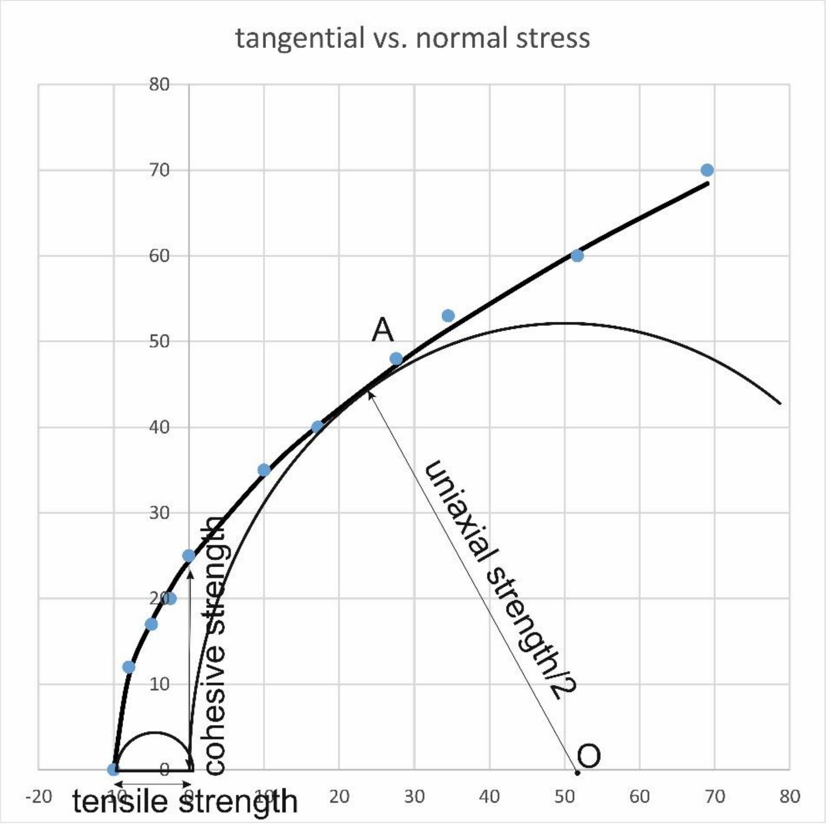


Fig. S3.7 Problem 12.

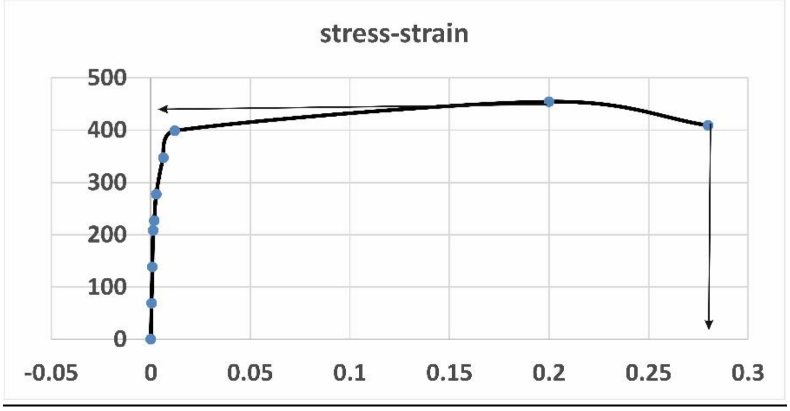
$\tau = 7.7 \cdot \sqrt{\sigma_n + 10}$ from this relationship the tensile strength is defined by $7.7 \cdot \sqrt{\sigma_n + 10}$=0 $\tau_{tesile\ strength} = 10\ MPa$. The cohesive strength is $\tau_{cohesive} = 7.7 \cdot \sqrt{0 + 10}$ =24.3 MPa. The uniaxial compressive strength is defined by the Mohr's diameter D of a circle with centre O($\frac{D}{2}, 0$). From the graph it is about 108 MPa. The analytical solution is as follows: The equation of tangential line to the Mohr's circle at point A(σ_0, τ_0) having diameter D and passing through the point (0,0) is: $\tau = \overbrace{\frac{7.7}{2 \cdot \sqrt{\sigma_0 + 10}}}^{tangent\ slope} \cdot (\sigma_n - \sigma_0) + \overbrace{\tau_0}^{7.7 \cdot \sqrt{\sigma_0 + 10}}$. The line which is orthogonal to the tangent passing through the point A and O is: $\tau = \overbrace{-\frac{2 \cdot \sqrt{\sigma_0 + 10}}{7.7}}^{orthogonal\ to\ tangent\ line\ slope} \cdot (\sigma_n - \sigma_0) + 7.7 \cdot \sqrt{\sigma_n + 10}$. The point O($\frac{D}{2}, 0$) satisfied this straight line equation: 0=$-\frac{2 \cdot \sqrt{\sigma_0 + 10}}{7.7} \cdot \left(\frac{D}{2} - \sigma\right) + 7.7 \cdot \sqrt{\sigma_n + 10}$. From this relationship one obtains the identity: $D = 2\sigma_0 + (7.7)^2$. From other side, the point A(σ_0, τ_0)belongs to the Mohr's circle: $\frac{D^2}{4} = \left(\sigma_0 - \frac{D}{2}\right)^2 + \tau_0^2$. After plugging $\tau_0 = 7.7 \cdot \sqrt{\sigma_0 + 10}$, one obtain the second identity: D=σ_0 +(7.7)^2$\cdot$ $(1 + \frac{10}{\sigma_0})$. Solving two equations relative to σ_0 and D, one obtains: $= 7.7 \cdot \sqrt{10} = \tau_{cohesive} = 24.3\ MPa$, and D=$2 \cdot 24.3 + (7.7)^2$ =108 MPa.

The stress state corresponds to the largest Mohr's circle radius $\frac{D}{2}$=(200-20)/2=90 MPa at this normal stress the maximum possible tangential stress is =7.7$\cdot$ $\sqrt{90 + 10} = 77$ MPa. So the stress state corresponds to failure of rock.

Exercises and control questions to Chapter 4:

1. Table S4.1

Tensile Force, kN	Measured length, mm
0	50.80
8.9	50.82
17.8	50.84
26.8	50.86
29.2	50.89
35.7	50.94
44.6	51.12
51.3	51.41
53.1 (max)	60.96
52.6 (failure)	65.02



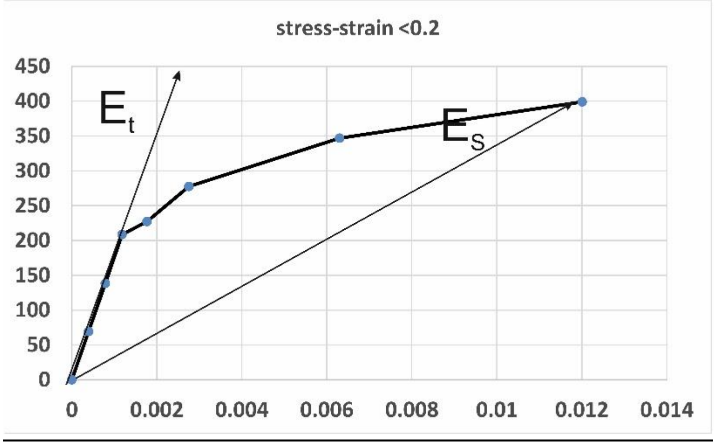


Fig. S4.1 Problem 1.

a. Fig. S4.1 represents the data on the stress-strain curve σ as function of ε:

Tangential Young's modulus: E_t=210 MPa/0.0012=175 GPa; mean Young's modulus: E_M= 190 MPa/0.001= 190 GPa; secant Young's modulus: E_S=400 MPa/0.012=33.3 GPa.

b. ε_{zz}=(60.96-50.80)/50.80=0.2; $\varepsilon_{xx}=\varepsilon_{yy}$=(12.2-12.8)/12.8=-0.047; Poisson's ratio ν=0.234.

c. Fron Fig. S5.1 it follows:: ε_{zz}=2%=0.02, σ=400 MPa.

d. $\sigma_{strength}=53.1\cdot10^3$ N/(3.14·$(12.2\cdot10^{-3})^2$/4)=454 GPa

e. $\varepsilon_{failure}$=(65.02-50.8)/50.8=0.28

2. From Fig. S4.2 it follows: K=500 MPa/0.1=5000 MPa= 5GPa; B=500 MPa; R= 300 MPa.

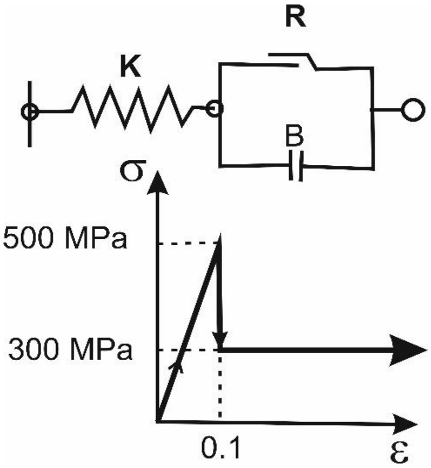


Fig. S4.2 Elasto-brittle model with the rest strength. Problem 2.

3. As it follows from Fig. S4.3 from 0 to 0.03 strain the elastic spring K_1 is only acting. Thus, the slope on the stress-strain curve defines the elastic constant K_1: K_1=500/0,03=16.7 GPa. If the stress on the element R eqials 500 MPa, then, it breaks and the stress exerted on the spring K_2 is also 500 MPa, and two springs connected in sequence K2 und K1 are at the stress R= 500 MPa. The slope on the stress-strain curve is $(1/K_1+ 1/K_2)^{-1}$=(800-500)/(0.35-0.03)=0.938 GPa. From this identity, it follows that K_2= $(0.937^{-1}\text{-}16.7^{-1})^{-1}$=0.994 GPa. At stress 800 GPa breaks the element B, so B=800 MPa.

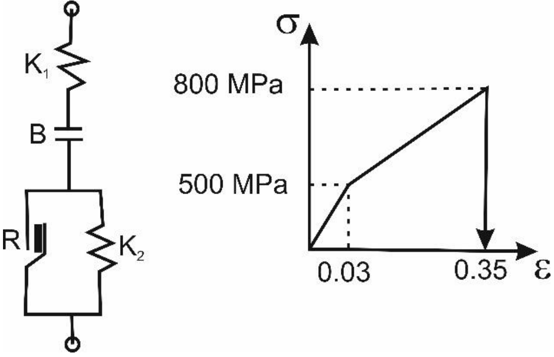


Fig. S4.3 Brittle model with the Saint-Venant element. Problem 3.

4. When atoms are at equilibrium positions, then, the sum of attraction and repulsion forces is zero:.

$$U_0 = -\frac{A}{r_0^n} + \frac{B}{r_0^m}; F = 0 = -\left(\frac{dU}{dr}\right)_{r=r_0} \Rightarrow \frac{n \cdot A}{r_0^n} = \frac{m \cdot B}{r_0^m}$$

From these two equations one can eliminated consants A and B:

$$U=\frac{U_0}{(n-m)}\cdot\left[-\frac{m}{\left(\frac{r}{r_0}\right)^n}+\frac{n}{\left(\frac{r}{r_0}\right)^m}\right];F=\frac{n\cdot m\cdot U_0}{(n-m)\cdot r_0}\cdot\left[\frac{1}{\left(\frac{r}{r_0}\right)^{n+1}}-\frac{1}{\left(\frac{r}{r_0}\right)^{m+1}}\right]$$

The Young's modulus can be understand in this situation as a force gradient per unit distance between atoms:

$$E=-\frac{1}{\pi\cdot r_0}\cdot\left(\frac{dF}{dr}\right)_{r=r_0}=\frac{U_0}{\pi\cdot r_0^3}\cdot\frac{m\cdot n}{n-m}\cdot\left[\frac{n+1}{\left(\frac{r}{r_0}\right)^{n+2}}-\frac{m+1}{\left(\frac{r}{r_0}\right)^{m+2}}\right]_{r=r_0}=\frac{n\cdot m\cdot U_0}{\pi\cdot r_0^{\ 3}}$$

5. The strain is: ε=(0.455-0.452)/0.452=0.00664; the Young's modulus is: E=1000 MPa/0.00664=150.7 GPa.

6. The Arrhenius dependence of viscosity on temperature is given by: η ∝ $e^{\frac{Ea}{RT}}$. For the temperature T_1=1150+273=1423K: $\frac{\sigma_{failure}}{\eta(T_1)}$=$\frac{0.01}{3.2\cdot 10^6 a}$, and for temperature T_2=1050+273=1323K: $\frac{\sigma_{failure}}{\eta(T_2)}$=$\frac{0.01}{8.5\cdot 10^6 a}$, Thus, $\frac{\eta(T_2)}{\eta(T_1)}$=$\frac{8.5\cdot 10^6 a}{3.2\cdot 10^6 a}$ =2.656. From the Arrhenius equation it follows: $\frac{\eta(T_2)}{\eta(T_1)}= e^{\frac{Ea}{R}\cdot\left(\frac{1}{T_2}-\frac{1}{T_1}\right)}$, or $\frac{Ea}{R}=\frac{ln\left(\frac{\eta(T_2)}{\eta(T_1)}\right)}{\left(\frac{1}{T_2}-\frac{1}{T_1}\right)}$=$\frac{0.9769}{4.9445\cdot 10^{-4}}=19757$. At T_3=1050+273=1393K the viscosity $\frac{\eta(T_3)}{\eta(T_1)}=e^{18757\cdot\left(\frac{1}{1393}-\frac{1}{1423}\right)}$=1.3485. At 1393K the break may be achieved in $4.3\cdot 10^6 a$. From 0.9% to 1% of strain the elapsed time is 0.43· $10^6 a$.

7. (a) In deformed state the coordinate of points in Fig. S4.4 are: A'(2,0.5) $\cdot 10^{-3}\,mm$, B'(1+5· 10^{-3} ,2.5· 10^{-3}), C'(1+10· 10^{-3} , 1 +5.5· 10^{-3}) and D'(7· 10^{-3} , 1 + 3.5· 10^{-3}) .
(b) The components of strain tensor are: $\varepsilon_{ij}=\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)\\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2}\end{vmatrix}=\begin{vmatrix}3 & 3.5\\ 3.5 & 3\end{vmatrix}\cdot$ 10^{-3} for all points the same.
(c) Angle deformation is $\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)$ =7· 10^{-3} . The angle DAB 90° is deformed by c. 0.4° in D'A'B'.

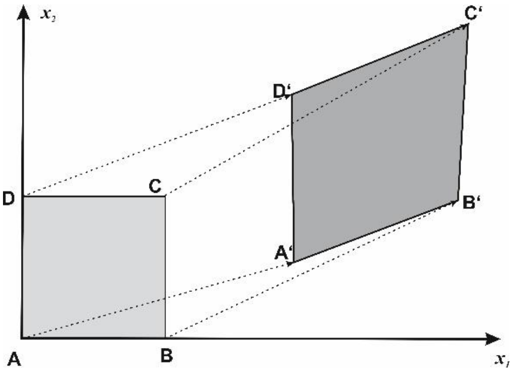


Fig. S4.4 Problem 7.

8. (a) The length of cuboid diagonal OA is chanced by $\Delta d = \sqrt{(a+\Delta a)^2 + (b+\Delta b)^2 + (c+\Delta c)^2}$-$\sqrt{a^2+b^2+c^2} \approx 4 \cdot 10^{-3}\ m$. The shape of the cuboid in the deformed state (see Fig. S4.5) implies that there is a linear spatial distortion of cuboid sides, and because orthogonal sides remain orthogonal even after deformation, the shear distortions in the cuboid are zero. The components of strain tensor are: $\varepsilon_{ij} = \begin{vmatrix} \frac{\Delta a}{a} & 0 & 0 \\ 0 & \frac{\Delta b}{b} & 0 \\ 0 & 0 & \frac{\Delta c}{c} \end{vmatrix}$ $= \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1.67 & 0 \\ 0 & 0 & -0.5 \end{vmatrix} \cdot 10^{-3}$.

(b) The volumetric strain is the sum of the diagonal elements: $\varepsilon_V = 3.17 \cdot 10^{-3}$. The diagonal in the non-deformed state is characterized by three cosines: $cos\alpha = \frac{1}{\sqrt{14}}$, $cos\beta = \frac{3}{\sqrt{14}}$, $cos\gamma = \frac{2}{\sqrt{14}}$. The strain along the direction $\vec{n} = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ is given by: $\varepsilon'_d = \varepsilon_{11} \cdot cos^2\alpha + \varepsilon_{22} \cdot cos^2\beta + \varepsilon_{33} \cdot cos^2\gamma = \frac{2\cdot1+1.67\cdot9-0.5\cdot4}{14} \cdot 10^{-3} \approx 1.07 \cdot 10^{-3}$.

(c) The mean normal strain is the arithmetic mean of diagonal strain tensor elements: $\varepsilon_m = \frac{2+1.67-0.5}{3} \cdot 10^{-3} = 1.06 \cdot 10^{-3}$

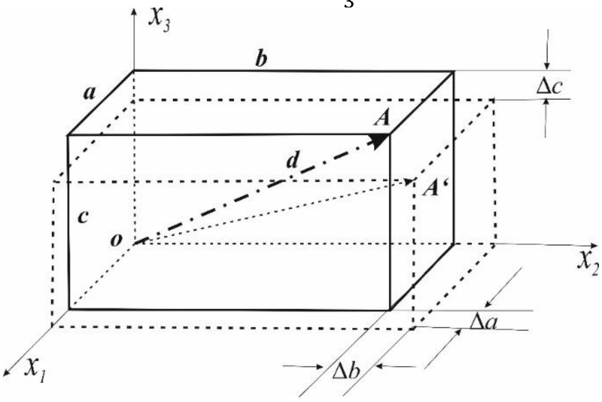


Fig. S4.5 Problem 8.

9. Formulate the equations and graphically explain the behavior and subsequent relief of Maxwell and Perzyna bodies shown in Fig. E4.5, under the constant tension stress σ_0 acting during the time period t_1.
   (a) Maxwell body: one denotes the viscosity of dashpot by h and the elastic modulus of spring as K (see Fig. S4.6A). Then the stress on the both elements is the same $\sigma = \sigma_D = \sigma_S = \eta \cdot \frac{d\varepsilon}{dt} = K \cdot \varepsilon$, because they connected in series, and the total strain is the

sum of strains of dashpot and spring, $\varepsilon = \varepsilon_D + \varepsilon_S = \frac{\sigma}{\eta} \cdot t + \frac{\sigma}{K}$. If a Maxwell body is suddenly subjected to a stress σ_0, then the elastic element would suddenly deform by $\frac{\sigma_0}{K}$, and the viscous element would deform with a constant rate, $\frac{\sigma_0}{\eta}$ (see Fig. S4.6 C). If at some time t_1 one releases the stress to 0, then the deformation of the elastic spring releases by back deformation $\frac{\sigma_0}{K}$, and the deformation of the viscous element would not change and stays at the level of $\frac{\sigma_0}{\eta} \cdot t_1$.

(b) Perzyna body: the difference with the Maxwell fluid body is the Saint-Venant element connected in parallel with the viscous dashpot (see Fig. Fig. S4.6B). For Perzyna body, the rate of viscous strain is a function of the initial yield stress σ_y and viscosity. The sliding element represents a constant yielding stress when the elastic limit is exceeded irrespective of the strain. Depending of the level of applied stress σ_0 the reaction of the body to a sudden applied or released stress σ_0 is as follows: at $|\sigma_0|<\sigma_y$ $\varepsilon = \frac{\sigma_0}{K}$, and at $\sigma_0>\sigma_y$ $\varepsilon = \frac{\sigma_0}{K} + \frac{(\sigma_0 - \sigma_y)}{\eta} \cdot t$.

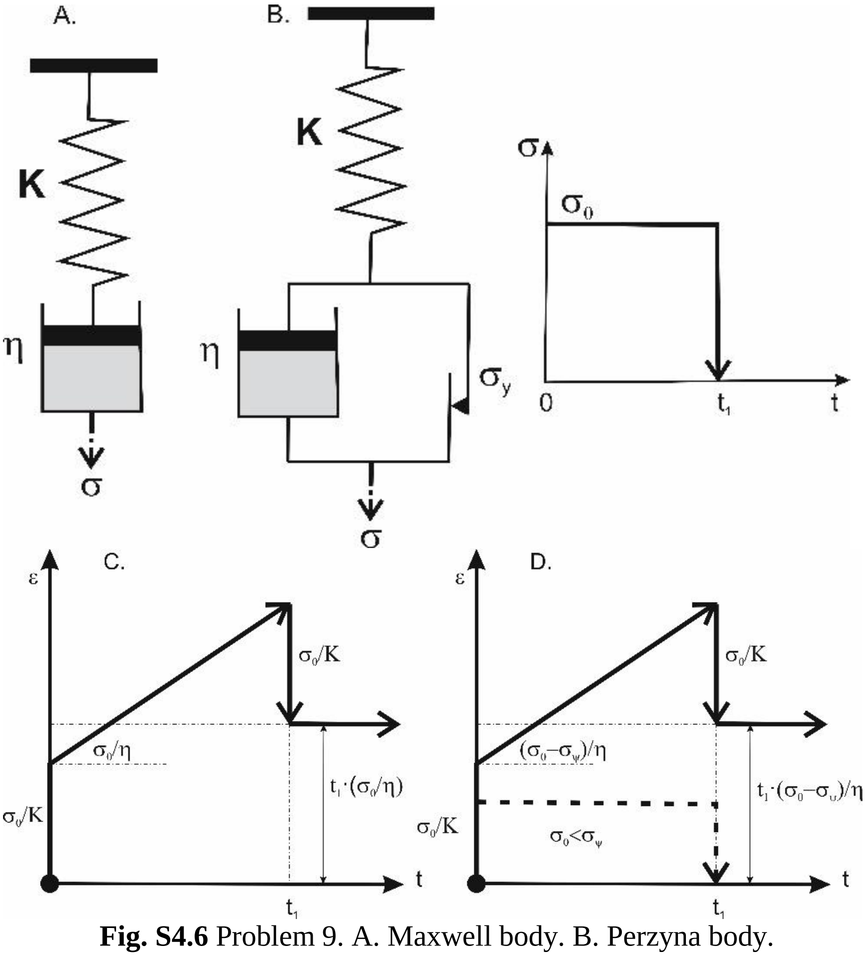


Fig. S4.6 Problem 9. A. Maxwell body. B. Perzyna body.

10. Three components of principle strains are the eigenvalues of the matrix:

$\begin{vmatrix} 0.15-\lambda & -0.05 & 0.05 \\ -\ 0.05 & 0.20-\lambda & 0.03 \\ 0.05 & 0.03 & -0.35-\lambda \end{vmatrix} = 0$, the solution is: $\varepsilon_{1,2,3} = \begin{pmatrix} -0.3571 \\ 0.1262 \\ 0.2309 \end{pmatrix}$. The maximum shear strain is given by $\tau_{max} = \frac{\max(\varepsilon_i)-\min(\varepsilon_i)}{2}$=$\frac{0.2309+0.3571}{2} = 0.294$. The strain in the direction $\vec{n}$=($\frac{1}{2}$,- $\frac{1}{2}$, $-\frac{1}{\sqrt{2}}$) is given by: $\varepsilon_n' = \varepsilon_{11} \cdot \overbrace{cos^2\alpha}^{\frac{1}{2\cdot 2}}$+$\varepsilon_{12} \cdot \overbrace{cos\alpha \cdot cos\beta}^{-\frac{1}{2\sqrt{2}}} + \varepsilon_{13} \cdot \overbrace{cos\alpha \cdot cos\gamma}^{-\frac{1}{2\sqrt{2}}}$ +$\varepsilon_{21} \cdot \overbrace{cos\alpha \cdot cos\beta}^{-\frac{1}{2\sqrt{2}}} + \varepsilon_{22} \cdot \overbrace{cos^2\beta}^{\frac{1}{2\cdot 2}} + \varepsilon_{23} \cdot \overbrace{cos\beta \cdot cos\gamma}^{-\frac{1}{2\sqrt{2}}}$+$\varepsilon_{31} \cdot \overbrace{cos\alpha \cdot cos\gamma}^{-\frac{1}{2\sqrt{2}}} + \varepsilon_{32} \cdot \overbrace{cos\beta \cdot cos\gamma}^{-\frac{1}{2\sqrt{2}}} +$ $\varepsilon_{33} \cdot \overbrace{cos^2\gamma}^{\frac{1}{2}}$= -0.169. The Poisson’s ratio is 0.5, since the volume strain is 0.

Exercises and control questions to Chapter 5:

1. The Darcy's law may only be applied to very small velocities *v*, where the kinetic energy of flow can be neglected and the energy losses are only of a hydraulic nature. Similar to classical fluid dynamics, the Reynolds number *Re* serves as a measure of flow slowness: $v_{max}=\frac{Re_{max}\cdot\eta}{\rho_{water}\cdot d_{capillary}}$. The maximum Reynolds number is *Re=10.* The effective diameter of intergranular capillary for cubic packing may be estimated as follows: per elementary cubic cell having volume d^3, there is a sphere of volume $\pi\cdot d^3/6$. The difference of two volumes is the pore space volume. One considers a cylinder which possesses the volume equal the pore space volume and the length d. Then, the effective cros section area of the cylinder is given by:

$$\frac{\pi\cdot d_{capillary}^2}{4}=\frac{d^3-\frac{\pi\cdot d^3}{6}}{d}=\left(1-\frac{\pi}{6}\right)\cdot d^2\rightarrow d_{capillary}=2d\cdot\sqrt{\frac{1-\frac{\pi}{6}}{\pi}}\approx 0.78\cdot d$$

$$=0.039\ cm$$

For the hexagonal packing the volume of the elementary cell is $3\sqrt{2}\cdot d^3$, the volume of 6 spheres per cell is πd^3 and the length of elementary cell is $2\sqrt{2}/\sqrt{3}\cdot d$ (see Exercise 1 to Chapter 2):

$$\frac{\pi\cdot d_{capillary}^2}{4}=\frac{3\sqrt{2}\cdot \mathrm{d}^3-\pi\cdot d^3}{2\sqrt{2}/\sqrt{3}\cdot d}=\frac{\sqrt{3}}{2}\left(3-\frac{\pi}{\sqrt{2}}\right)\cdot d^2\rightarrow d_{capillary}$$

$$=d\cdot\sqrt{\frac{2\sqrt{3}\cdot\left(3-\frac{\pi}{\sqrt{2}}\right)}{\pi}}\approx 0.926\cdot d=0.046\ cm$$

$$v_{max}=\frac{10\cdot 1.14\cdot 10^{-3}}{10^3\cdot d}=\overbrace{0.029}^{cubic\ packing}\quad\div\quad\overbrace{0.0246}^{hexagonal\ packing}\qquad m/sec$$

2. $k=B\cdot\frac{(\phi-\phi_c)^3}{(1+\phi_c-\phi)^2}\cdot d^2$, where B=15 and ϕ_c=0.035.

$$k=15\times\frac{(0{,}32-0{,}035)^3}{(1+0{,}035-0{,}32)^2}\times(80\cdot 10^{-6}\,m)^2=4\cdot 10^{-9}\,m^2=4000\,D$$

$$k=15\times\frac{(0{,}32-0{,}035)^3}{(1+0{,}035-0{,}32)^2}\times(240\cdot 10^{-6}\,m)^2=36\cdot 10^{-9}\,m^2=36000\,D$$

If $\phi\approx\phi_c$, then $k\approx B\cdot\frac{(\phi-\phi_c)^3}{(1)^2}\cdot d^2$ and n=3.

3. In a cylindrical vessel having diameter D_r = 30 cm , there are free shaken layers of spherical quartz sand particles and water. The quartz sand particles have a volume of V = 108 dm³. The volume V_B of the vessel is 180 dm³. Laminar water flows through the bottom layer. Determine the diameter D_k of the quartz sand particles, if the

velocity of the water level decrease in the vessel is U = 5 cm/s. Viscosity of water $\eta = 1.14\cdot10^{-3}$ Pa sec.

The porosity of particle suspension is given by: $\varphi = \frac{180-108}{180}$=0.4.

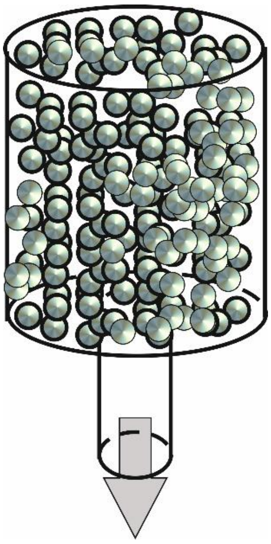


Fig. S5.1 Problem 3.

Physical velocity of water in porous space of suspension is: $v = \frac{U}{\varphi} = 12.5\ cm/s$. The Darcy's law can be written in the form as follows:

$$U = k\frac{\left(\frac{\Delta P}{h}\right)}{\mu} \Rightarrow k = \frac{U\cdot\mu}{\Delta\rho\cdot g} = \frac{5\cdot10^{-2}\cdot1{,}14\cdot10^{-3}}{10^{3}\cdot9{,}872} = 5\cdot10^{-9}\,m^{2}$$

One considers a cubic (a) and hexagonal (b) dense packings of quartz particles:

(a) $k = \frac{\pi}{32}\cdot R^2 = \frac{\pi}{128}\cdot D^2_{cubic}$

(b) $k = \frac{\pi}{16\sqrt{3}}\cdot R^2 = \frac{\pi}{64\sqrt{3}}\cdot D^2_{hexagonal}$

Then, for cubic dense packing:

$$D_{cibic} = \sqrt{\frac{5\cdot10^{-9}\cdot128}{\pi}} \approx 0.45\ mm$$

And for hexagonal dense packing:

$$D_{hexagonal} = \sqrt{\frac{5\cdot10^{-9}\cdot16\sqrt{3}}{\pi}} \approx 0.21\ mm$$

4. One may rewrite the relationship in the form: $\ln(\beta) + 0.5\cdot\ln(\kappa) \propto -n\cdot\ln(\varphi)$. The data from Table S5.1 are fitted to n=3.3 with the correlation coefficient R^2=0.816.

Table S5.1 Permeability κ and non-Darcy parameter β for some porous rocks

Parameter	*κ, 10^{-15} m^2*	*β, 10^8 m^{-1}*	*φ, porosity*	*Reference*
Dakota sandstone	3.5	157.9	0.14	Zeng & Grigg, 2006
Indiana limestone	21.6	36	0.15	Zeng & Grigg, 2006

Berea sandstone	196	2.9	0.18	Zeng & Grigg, 2006
Linyi sandstone	2.4	825.5	0.10	Choi &Song, 2019
Bandera sandstone	31.6	9.6	0.19	Choi &Song, 2019
Buff Berea sandstone	290.8	4.3	0.22	Choi &Song, 2019
Boise sandstone	1091	4.15	0.27	Choi &Song, 2019
Glass beads: 595–707 µm	$2.8\cdot 10^4$	0.033	0.37	Macini et al., 2011
Natural sand: 500–595 µm	$3.35\cdot10^4$	0.034	0.45	Macini et al., 2011

5. One denote by q is the volumetric flow velocity in the slit per channel width. Then, the mean flow velocity is given by: $v_m = \frac{q}{2w}$. The first derivative of the solution is: $\frac{dv(y)}{dy} = \frac{\kappa}{\eta}\cdot\left(-\frac{dP}{dx}\right)\cdot\left(-\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\cdot\frac{1}{w}\cdot\frac{\sinh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\cdot\frac{y}{w}\right)}{\cosh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\right)}\right)$. The second derivative is: $\frac{d^2v(y)}{dy^2} = \frac{\kappa}{\eta}\cdot\left(-\frac{dP}{dx}\right)\cdot\left(-\frac{Da^2}{\frac{\eta'}{\eta}}\cdot\frac{1}{w^2}\cdot\frac{\cosh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\cdot\frac{y}{w}\right)}{\cosh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\right)}\right)$. Then, $\frac{\eta}{\kappa}\cdot v(y) - \eta'\cdot\frac{d^2v(y)}{dy^2} = \left(-\frac{dP}{dx}\right)\cdot\left[1-\frac{\cosh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\cdot\frac{y}{w}\right)}{\cosh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\right)}\right] - \frac{\kappa}{\eta}\left(-\frac{dP}{dx}\right)\cdot\left(-\frac{w^2}{\kappa}\cdot\frac{\eta}{w^2}\cdot\frac{\cosh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\cdot\frac{y}{w}\right)}{\cosh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\right)}\right) = \left(-\frac{dP}{dx}\right)$. The boundary conditions are satisfied due to: $v(\pm w) = \frac{\kappa}{\eta}\cdot\left(-\frac{dP}{dx}\right)\cdot\left[1-\frac{\cosh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\cdot\frac{\pm w}{w}\right)}{\cosh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\right)}\right] \equiv 0$. The dimensionless pressure gradient is: $\frac{\overbrace{\int_{-w}^{w}dy}^{\equiv 1}}{2w}\cdot\frac{\kappa}{\eta}\cdot\left(-\frac{dP}{dx}\right) = \frac{1}{2w}\int_{-w}^{w}v(y)\cdot dy - \frac{\kappa\cdot\eta'}{2w\cdot\eta}\cdot\int_{-w}^{w}\frac{d^2v(y)}{dy^2}\cdot dy = \left(\frac{q}{2w} - \frac{\kappa}{\eta}\left(-\frac{dP}{dx}\right)\frac{\kappa\cdot\eta'}{\eta}\cdot\frac{1}{2w}\cdot\left[\frac{dv(w)}{dy} - \frac{dv(-w)}{dy}\right]\right) = v_m - \frac{\kappa}{\eta^2}\cdot\eta'\cdot\left(-\frac{dP}{dx}\right)\cdot\left(-\frac{1}{\sqrt{\frac{\eta'}{\eta}}}\cdot\frac{1}{w\cdot\sqrt{k}}\cdot\tanh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\right)\right)$. Thus, : $\frac{\kappa}{\eta}\left(-\frac{dP}{dx}\right)\cdot\left(1-\frac{\sqrt{\kappa}}{w}\cdot\sqrt{\frac{\eta'}{\eta}}\cdot\tanh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\right)\right) = v_m$, or

$\frac{w^2}{v_m \cdot \eta} \cdot \left(-\frac{dP}{dx}\right) = \frac{Da^2}{\left(1 - \frac{1}{Da} \cdot \sqrt{\frac{\eta'}{\eta}} \cdot \tanh\left(\frac{Da}{\sqrt{\frac{\eta'}{\eta}}}\right)\right)}$. At $\kappa \rightarrow \infty$, Da$\approx 0$, then $\frac{w^2}{v_m \cdot \eta} \cdot \left(-\frac{dP}{dx}\right) \approx 3 \cdot \frac{\eta'}{\eta}$ the viscous flow limit is reached. At $Da \rightarrow \infty$, $\kappa \approx 0$, then, $\frac{w^2}{v_m \cdot \eta} \cdot \left(-\frac{dP}{dx}\right) \approx Da^2$, and the Darcy flow limit is reached.

6. One uses the Hagen-Poiseille law in the form as follows:

$$q = \frac{10}{\underbrace{60 \cdot 60}_{1\,hour}} m^3/sec = \frac{\pi}{8 \cdot \underbrace{\rho \cdot \nu}_{dynamic\ viscosity}} \cdot \frac{\Delta P}{L} \cdot \left(\frac{D}{2}\right)^4$$

$q = \frac{10}{\underbrace{60 \cdot 60}_{1Stunde}} m^3/\sec = \frac{\pi}{8\,\underbrace{\rho \cdot \nu}_{Viskosität}} \cdot \frac{\Delta P}{L} \cdot \underbrace{\left(\frac{D}{2}\right)^4}_{Radius}$. From this identity it follows:

$$\Delta P = \frac{q \cdot 8\rho\nu L \cdot 16}{\pi D^4} = \frac{10 \cdot 8 \cdot 0{,}9 \cdot 10^3 \cdot 40 \cdot 10^{-6} \cdot 10^3 \cdot 16}{60 \cdot 60 \cdot 3{,}14 \cdot 625 \cdot 10^{-8}} = 2{,}05 MPa = 20{,}5 Atm$$

7. (a) Consider the Darcy's law in the form (see Fig. S5.2): $q = \frac{Q}{A} = K_f \cdot \frac{\Delta h}{L}$. Then, $q = \frac{\overbrace{3}^{K_f}}{\underbrace{60 \cdot 60 \cdot 24}_{1\,day}} \cdot \frac{\overbrace{\Delta h}^{water\ table\ difference}}{\underbrace{L}_{distance\ of\ water\ flow}}$ $=5.8 \cdot 10^{-8}\ m/sec$. The volumetric water flow rate per unit width is: $V = q \cdot 1m \cdot 10m \cdot \underbrace{2}_{in\ two\ opposite\ directions} = 1.16 \cdot 10^{-6}\ m^3/sec.$

(b) Alternatively, the Darcy's law may be written in the form: $q = \frac{k}{\eta} \cdot \frac{\overbrace{\Delta P}^{\rho \cdot g \cdot \Delta h}}{L} = \frac{k}{\eta} \cdot \frac{\rho g \Delta h}{L}$. From this identity the permeability is given by: $k = \frac{\eta}{\rho \cdot g \cdot \Delta h} \cdot L \cdot q = 3{,}5 \cdot 10^{-12} m^2 = 3{,}4D$.

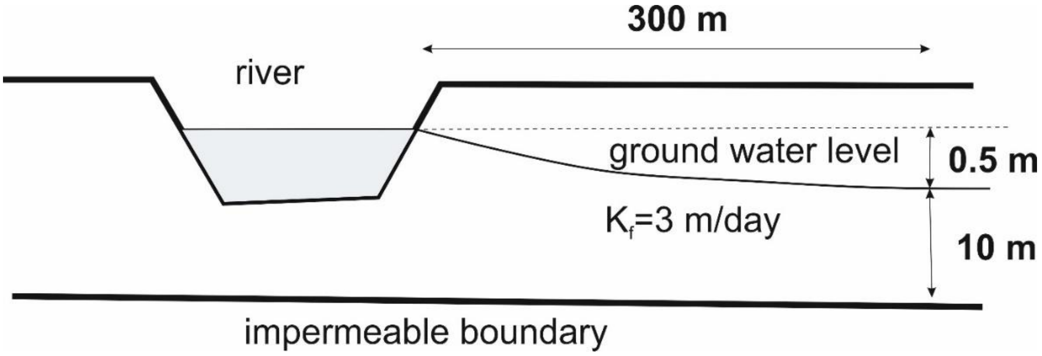


Fig. S5.2 Problem 7.

8. A well with a diameter of 0.25 m has a 30 m long section below the groundwater level. The material of the aquifer has a grain size of d = 2.0 mm (hydraulically effective grain diameter).

Groundwater is pumped out of the well at a rate of 0.3 m^3/s. Is the Darcy law in the near field and at a distance 10 m of the well applicable?

<u>*Note:*</u> For each cylindrical cross-section of a radius *r* according to the continuity law and the Darcy law the following applies: $Q = q \cdot A = 2\pi r \cdot l \cdot k_f \cdot \frac{dh}{dr} \rightarrow \frac{Q}{2\pi l \cdot k_f}\frac{dr}{r} = dh$, and after integration from r_0, the inner radius of a bore-well to the distant point *R* one obtains the *Dupuit-Theme formula:* $Q = 2\pi \cdot l \cdot k_f \cdot \frac{\Delta h}{ln\left(\frac{R}{r_0}\right)}$=$2\pi \cdot l \cdot \frac{\cdot \kappa}{\eta} \cdot \frac{\Delta P}{ln\left(\frac{R}{r_0}\right)}$ for a radial flow rate around a bore-well.

Consider a steady radial flow in unconfined aquifer (see Fig. S5.3). The aquifer is unconfined and underlain by a horizontal confining layer. The well is pumped at a constant rate. The Dupuit-Theme formula is valid. The Darcy's law and the radial flow in the unconfined aquifer can be described as: $Q = 2\pi \cdot R \cdot h(R) \cdot k_f \cdot \frac{dh}{dR}$. Q is the constant pumping rate. After integration, one obtains: $\int_{h-(h_0-L)}^{h} 2h \cdot dh = \frac{Q}{\pi k_f} \cdot \int_{r_0}^{R} \frac{dR}{R} \rightarrow \ (h + h_0 - L) \cdot L = \frac{Q}{\pi k_f} \cdot ln\left(\frac{R}{r_0}\right)$. The gradient a a function of radial distance is: $\left(\frac{dh}{dR}\right) \ = \frac{Q}{\pi L \cdot k_f} \cdot \frac{r_0}{R}$.

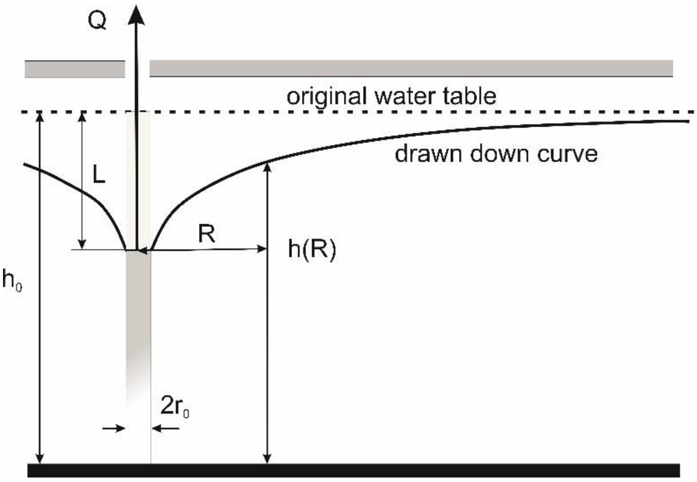


Fig. S5.3 Problem 8.

Thus, the ratio of the Darcy flow velocity at the distance 10 m from the bore-hole and at the wall is the ratio of pressure gradients: $\frac{q_{10}}{q_{0.125}} = \frac{0.125}{10}$. The Darcy velocity at the distance 10 m is given by: $q_{10} = \frac{Q}{2\pi L \cdot R} = 1.59 \cdot 10^{-4}\ m/sec$. The Reynolds number is: $Re = \frac{1.59 \cdot 10^{-4}}{1.8 \cdot 10^{-6}} \cdot 2 \cdot 10^{-3} \approx 0.18 < 1$

9. In the cylindrical sedimentary oil-bearing rock layer of thickness 10 m and radius 1 km located at a depth 2 km. The rock possesses the permeability 0.5 D and porosity 0.2. There is a bore-hole with the diameter 10 cm and a depth of 2 km is located in the centre of the layer. Oil viscosity is 1 Ps s, and its density is 0.87 g/cm^3, absolute oil reservoir pressure 20 MPa. It is necessary to determine whether the well can fountain when it is opened, and what is its the production rate, if at a bottom the bore-hole the pressure is 19 MPa.

$Q = 2\pi \cdot l \cdot k_f \cdot \frac{\Delta h}{ln\left(\frac{R}{r_0}\right)}$=$2\pi \cdot l \cdot \frac{\cdot \kappa}{\eta} \cdot \frac{\Delta P}{ln\left(\frac{R}{r_0}\right)} = 2\pi \cdot 10 \cdot 0.5 \cdot \frac{10^{-12}}{1} \cdot \frac{10^6}{\ln\left(\frac{1000}{0.05}\right)} = 3.17 \cdot 10^{-6} \frac{m^3}{sec}$.

The hydrostatic pressure at the bottom of the oil column is $P_{oil} = 0.87 \cdot 10^3 \cdot 9.81 \cdot 2 \cdot 10^3 = 17.1\ MPa\ P_{lithostatic} = 2.2 \cdot 10^3 \cdot 9.81 \cdot 2 \cdot 10^3 = 43.2\ MPa$ .The pressure losses by the flow of oil in the borehole is: $\Delta P_{loss} = \frac{8 \cdot \eta \cdot l \cdot Q}{\pi \cdot r^4} \approx 0$. The buoyancy is 26.1 MPa and it is larger than the actual pressure at the bottom of the borehole 19 MPa.

10.

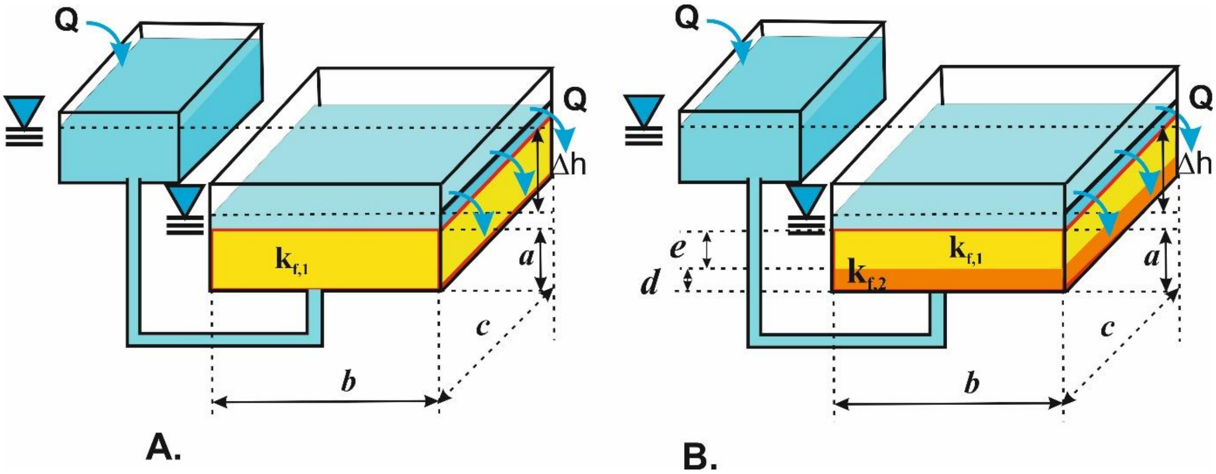


Fig. E5.4 Problem 10.

a) Calculation of the volumetric flow rate Q:

$$Q = v_f \cdot A_{total} = 0.00045\frac{m}{s}\cdot 0.05m \cdot 0.06\, m = 81\, ml/min$$

b) Calculation of the mean physical velocity v_m flow rate:
The effective cross section area of flow is:

$$\sum_i A_i = A_{total}\cdot p = 0.05m \cdot 0.06\, m \cdot 0.1 = 3\cdot 10^{-4}\, m^2$$

The mean physical flow velocity is: $v_m = v_f \cdot \frac{A_{total}}{\sum_i A_i} = \frac{v_f}{p} = 0.0045\, m/sec$

The Darcy flow velocity v_f is based on the entire cross-section of the soil sample and not just on the flow cross-section of the pore channels. The mean physical flow rate in pores is average flow rate by through cross-section area ΣA_i. The Darcy's filter velocity v_f is lower than the physical flow velocity v_m in pore channels.

c) Verification of the Darcy's law applicability:

$$Re = \frac{v_f \cdot d_{50}}{\nu} = \frac{0.00045\cdot 0.001}{10^{-6}} = 0.45 < 1$$

Calculation of the hydraulic conductivity coefficient $\boldsymbol{k_f}$ for the experimental setup A in Fig. S.5.3:

$k_f = \frac{Q}{A_{total}\cdot\frac{\Delta h}{\Delta L}} = v_f \cdot \frac{\Delta L}{\Delta h}$ =0.00045 $\cdot\,\frac{0.04}{0.1}$=1.8$\cdot\, 10^{-4}\, m/s$

d) Calculation of the hydraulic conductivity coefficient $\boldsymbol{k_{f.\,äq}}$ for the experimental setup B in Fig. S.5.4: the setup consists of 2 parallel soil layers. From the continuity of the Darcy's filter velocity it follows:

$$\boldsymbol{v_f = k_{f,1}\cdot\frac{\Delta h_1}{\Delta L_1} = k_{f,2}\cdot\frac{\Delta h_2}{\Delta L_2} = k_{f,2}\cdot\frac{\Delta h - \Delta h_1}{\Delta L_2}}$$

$$\boldsymbol{\Delta h_1 = \frac{k_{f,2}\cdot\frac{\Delta h}{\Delta L_2}}{\frac{k_{f,1}}{\Delta L_1}+\frac{k_{f,2}}{\Delta L_2}} = \frac{3\cdot 10^{-6} m/sek\cdot\frac{0{,}1m}{0{,}01m}}{\frac{1{,}8\cdot 10^{-4}}{0{,}03m}+\frac{3\cdot 10^{-6} m/sek}{0{,}01m}} = 0{,}0048m}$$

$$v_{f,2} = v_{f,1} = k_{f,1} \cdot \frac{\Delta h_1}{\Delta L_1} = 1{,}8 \cdot 10^{-4}\, m/sek \cdot \frac{0{,}0048m}{0{,}03m} = 2{,}88 \cdot 10^{-5}\, m/sek$$

$$k_{f,äq} = v_f \cdot \frac{\Delta L_{gesamte}}{\Delta h} = 2{,}88 \cdot 10^{-5}\, m/sek \cdot \frac{0{,}01m + 0{,}03m}{0{,}1m} = 1{,}152 \cdot 10^{-5}\ m/sek$$

Exercises and control questions to Chapter 6:

1. According to the Gassmann's equation :

$$\frac{K_u}{K_S - K_u} = \frac{K}{K_S - K} + \frac{K_{Fluid}}{\phi \cdot (K_S - K_{Fluid})}$$

A. In the case when the rock is saturated with gas:

$\frac{K_u}{36-K_u} = \frac{12}{36-12} + \frac{0.133}{0.18 \cdot (36-0.133)}$, then, it follows: K_u=12.32 GPa

B. In the case when the rock is saturated with water:

$\frac{K_u}{36-K_u} = \frac{12}{36-12} + \frac{2.2}{0.18 \cdot (36-2.2)}$, then, it follows: K_u=16.66 GPa.

2. Calculate the pore compression modulus as a function of burial depth using the data of porosity of sandstones and shales from Fig. E6.1 (replotted from Krumbein & Sloss, 1953). One denotes by $K_p^P = -v_p \cdot \left(\frac{\partial P}{\partial v_p}\right)_p$ the pressure derivatic eof pore volume at constant pore pressure p, and $K_p^p = v_p \cdot \left(\frac{\partial p}{\partial v_p}\right)_P$ the pore pressure derivative of pore volume at constant confining pressure P. the total relative change of the pore volume is the sum: $-\frac{dv_p}{v_p} = \frac{dP}{K_p^P} - \frac{dp}{K_p^p}$. By analogy one may introduce the bulk volume derivatives: $V_b = V_m + v_p$, where V_m is the volume of the rock matrix. By analogy one ma write the total relative change of bulk volume as follows: $-\frac{dV_b}{V_b} = \frac{dP}{K_b^P} - \frac{dp}{K_b^p}$. $dV_b - dv_p = dV_m$, or $\frac{1}{K_p^P} - \frac{1}{K_p^p} = \frac{1}{K_m} = \frac{1}{K_b^P} - \frac{1}{K_b^p}$. If the matrix material is incompressible, then, $K_p^P \approx K_p^p$, and $K_b^P \approx K_b^p$. Due to the definition of porosity: $\phi = \frac{v_p}{V_b}$, it follows: $\cdot\, \phi \cdot K_b^p = K_p^P$, i.e. $-v_p \cdot \left(\frac{\partial P}{\partial v_p}\right)_p = \phi \cdot V_b \cdot \left(\frac{\partial p}{\partial V_b}\right)_P = -\phi \cdot \left(\frac{\partial p}{\partial \phi}\right)_P$, since for incompressible matrix $\partial v_p = \partial V_b$, and $\frac{\partial v_p}{V_b} = \frac{\partial V_b}{V_b} = d\phi$. Finally, $v_p \cdot \left(\frac{\partial P}{\partial v_p}\right)_p = \phi \cdot \left(\frac{\partial p}{\partial \phi}\right)_P$.

From Fig. S. 6.1: for sandstone in the pressure interval from 500 to 1500 m: ΔP= 26.5 MPa, dϕ≈ 0.05 $\phi = 0.35$, then, $K_p \approx 185.5\ MPa$, for shales dϕ≈ 0.12 $\phi = 0.15$, then, $K_p \approx 33\ MPa$

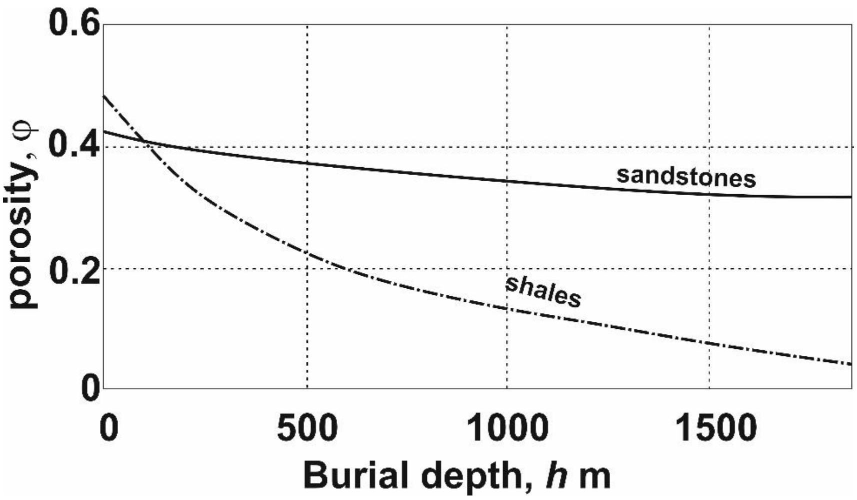


Fig. S6.1 Porosity as a function of burial depth (Problem 2).

3. The vertical axis in Fig. S6.2 is $\frac{\partial v_p}{v_p}$, the horozontasl axis is the overburden pressure P. at P=30 MPa: $K_p = -\frac{v_p \cdot d(P-p)}{dv_p} = \frac{30}{0.02} = 1.5\ GPa$. $K = -\frac{v_p \cdot d(P-p)}{\phi \cdot dv_p} = 7.14\ GPa$

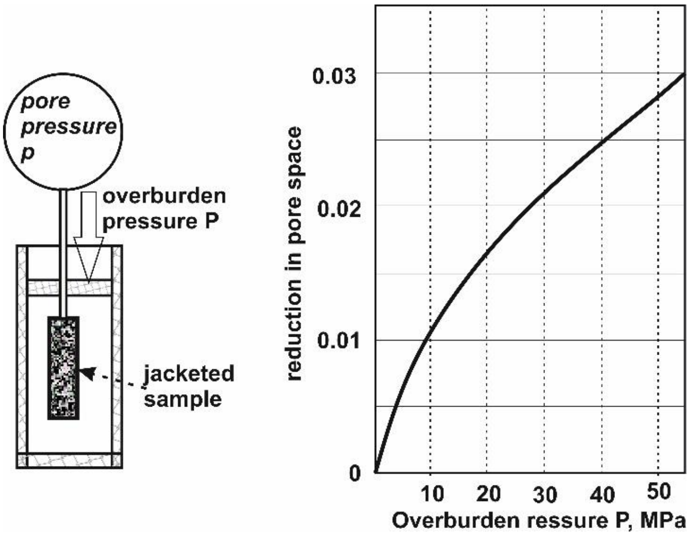


Fig. E6.2 Illustration to Exercise 3.

4. According to the Gassmann equation:
$$\frac{K_{sat}}{K_0 - K_{sat}} = \frac{K_{dry}}{K_0 - K_{dry}} + \frac{K_f}{\varphi \cdot (K_0 - K_f)}.$$

The value of K_{sat}, K_0 and K_{dry} at ϕ=0.15 taken from Fig. S.6.3 provide the relationship:

$\frac{25}{36-25} = \frac{23}{36-23} + \frac{K_{water}}{0.15 \cdot (36 - K_{water})}$. From this identity it follows: K_{wterr}=2.5 GPa

In the case of oil: $\frac{23.5}{36-23.5} = \frac{23}{36-23} + \frac{K_{oil}}{0.15 \cdot (36 - K_{oil})}$, and K_{oil}=0.6 GPa.

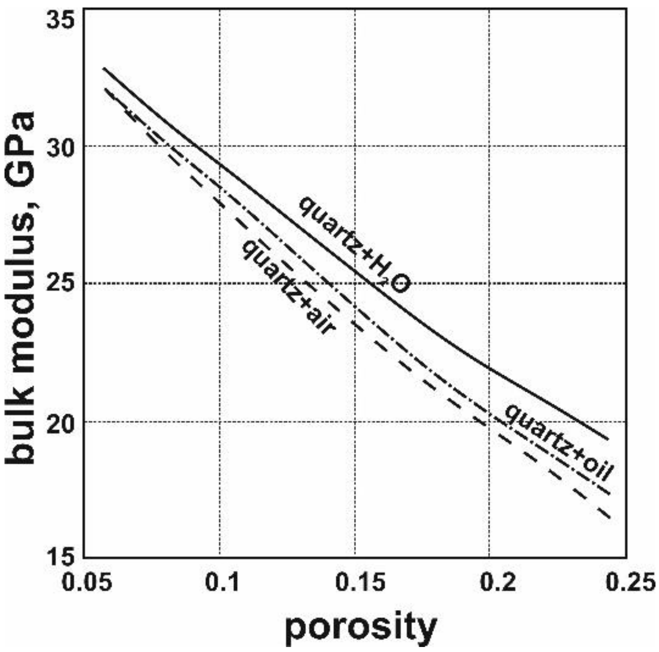


Fig. E6.3 Data curves to Exercise 4 (replotted from Mavko et al., 1998).

5. Determine the sonic velocity after the Reuss averaging in the suspension, which consists of calcite particles, air, and water. Saturation S_r with water is 50%. The properties of calcite, water and air are given below:
   $K_{calcite}$ = 75 GPa and the density $\rho_{calcite}$ = 2.71 g / cm³, K_{water} = 2.2 GPa and the density ρ_{water} = 1 g / cm³, K_{air} = 0 , 000131 GPa and the density ρ_{air} = 0.00119 g/cm³. The porosity of the suspension is ϕ = 0.4.

   The Reuss average is given by the relationship:

$$\widetilde{K}^{Reuss}=\left(\frac{(1-S)\cdot\phi}{K_{air}}+\frac{s\cdot\phi}{K_{water}}+\frac{1-\phi}{K_{calcite}}\right)^{-1}=\left(\frac{0.5\cdot 0.4}{0.131}+\frac{0.5\cdot 0.4}{2.2}+\frac{0.6}{75}\right)^{-1}=0.615\,GPa$$

   The density of calcite suspension is the arithmetic average:

   $\rho=(1-S)\cdot\phi\cdot\rho_{air}+s\cdot\phi\cdot\rho_{water}+(1-\phi)\cdot\rho_{calcite}=0.4\cdot 0.5\cdot 1+0.6\cdot$ 2.71 =1.826 g/cm³.

   The sound velocity is given by:

   Die Schall-Geschwindigkeit der Suspension ist gleich:

$$V=\sqrt{\frac{\widetilde{K}^{Reuss}}{\rho}}=\sqrt{\frac{0.615\cdot 10^{9}}{1.826\cdot 10^{3}}}=580\,m/s$$

6. If a material contains liquid under a hydrostatic pressure p_f, this pressure will counteracts evenly all the principal stresses of the applied load. The normal stress on any surface is reduced by the pressure p_f. In rocks this corresponds to the reduction of the ambient pressure which is the average of the diagonal elements in stress matrix. Point on the Mohr circle, how does a change in the mean normal stress $(\frac{\sigma_1+\sigma_2+\sigma_3}{3})$ offset?
   Draw the original Mohr's circle with radius $\sigma_1-\sigma_2$ having the centrum at $\frac{\sigma_1+\sigma_2}{2}$. By applying pore pressure p_f the center of circle is shifted along the abscissa by the value p_f

without changing its radius. The increase in the fluid pressure reduces the effective stresses and shifts the stress state closer to the Mohr envelope (see Fig. S6.4).

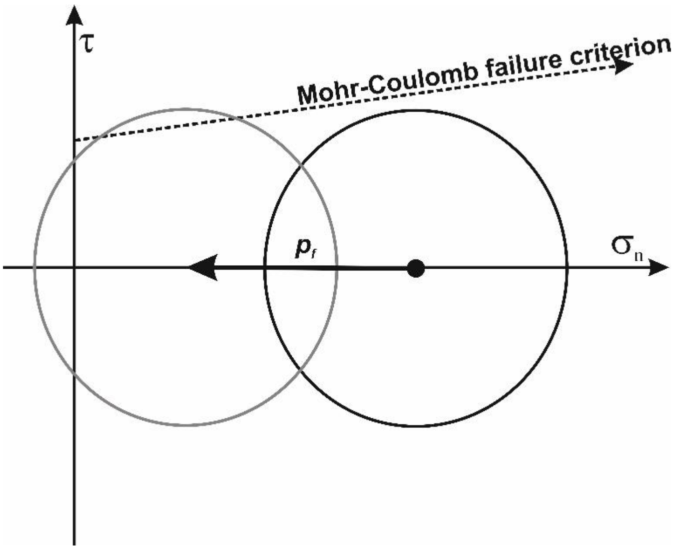


Fig. S6.4 Problem 6.

The values of all normal stresses are reduced by the value of the mean stress which is in the case of pore pressure increased by pf. The values of all shear stresses remain the same, which shows that they are independent of the hydrostatic components. If the liquid pressure is increased enough, the circle moves until it touches the Mohr's envelope as a tangent, and then failure occurs.

7. $S_s = \rho \cdot g \cdot \left(\frac{\varphi}{K_f} + \frac{1}{K}\right) = 1000 \cdot 9.81 \cdot \left(\frac{0.25}{1.96 \cdot 10^9} + \frac{1}{0.1 \cdot 10^9}\right) = 10^{-4}\ 1/m$

   Storage coefficient: S=10^{-3} (dimensionless)

8. Reduction in reservoir volume may considered primarily as a uniaxial compaction (Geertsma, 1957). Using the relations for porous rock: $v = v_p + v_s,\ v_p \approx \varphi \cdot v$ and $dv_p \approx dv$, where φ is the porosity of sample, express the pressure derivative of porosity $\frac{d\varphi}{dP}$ through *K* and *K*$_s$:

   A. For a hydrostatic compaction.

   B. For a uniaxial vertical compaction in a rock having Poisson ratio ν. The uniaxial pore compression modulus may be defined as: $\frac{1}{K_{uni}} = -v \cdot \frac{d(\bar{\sigma}-p)}{dv}$=-$z \cdot \frac{d(\bar{\sigma}-p)}{dz}$, where $\bar{\sigma} = \frac{\sigma_x+\sigma_y+\sigma_z}{3}$, where σ_x, σ_y and σ_z are lateral and vertical components of stress. If the Poisson ratio ν is given, then for the uniaxial deformation case: $\varepsilon_x = \varepsilon_y = 0$, and $\sigma_x = \sigma_y = \frac{\nu}{1-\nu} \cdot \sigma_z$, then $\bar{\sigma} = \frac{1+\nu}{3(1-\nu)} \cdot \sigma_z$.

   Denote the hydrostatic pressure as: P= $P_d + p$, where P$_d$ is the differential pressure and p is the pore pressure. From the given relationships it follows, since $v_p \approx \varphi \cdot v$ and $v_p \approx dv$, then, $\frac{\delta\varphi}{\varphi} = \frac{\delta v}{v} - \frac{\delta v_p}{v_p} = \left(1 - \frac{1}{\varphi}\right) \cdot \frac{\delta v}{v} = \frac{(1-\varphi)}{\varphi} \cdot \left[-\frac{\delta v}{v}\right]$. In another sense, $-\frac{\delta v}{v} = \frac{\delta P_d}{K} + \frac{\delta p}{K_s}$=$\frac{\delta P}{K}$-$\frac{\alpha}{K} \cdot \delta p = \frac{1}{K} \cdot (\delta P - \alpha \cdot \delta p)$, where α=1-$\frac{K}{K_s}$ where $\frac{K}{K_s}$ in the case of a single solid state component of the rock matrix is the ratio of bulk modulus of porous rock to the mineral grain bulk modulus.

In the case A, when the pressure is pure hydrostatic, $i.e.\ \delta p = 0,:\ \left(\frac{\delta\varphi}{\delta P}\right)_0 = \frac{(1-\varphi)}{K}$. If the compressibility of mineral grains $\frac{1}{K_s}$ is not negligible, then, $\delta\varphi = \left[\frac{(1-\varphi)}{K} - \frac{1}{K_s}\right] \cdot \delta(P-p)$.

In the case B, $\delta p = \delta P$, the additional factor $P = \frac{1+\nu}{3(1-\nu)} \cdot \sigma_z$ should be taken into account: $\left(\frac{\delta\varphi}{\delta P}\right)_{\overline{\sigma}} = \frac{1+\nu}{3(1-\nu)} \cdot \frac{(1-\varphi)}{K_{uniaxial}} \cdot (1-\alpha)$.

9. $K_{fluid} = \left(\frac{0.5}{2.2} + \frac{0.5}{1}\right)^{-1} = 1.375\ GPa$. The Reuss average bulk modulus: $K_R = \left(\frac{0.25}{1.375} + \frac{0.75}{37}\right)^{-1} = 4.95\ GPa$. From the Gassmann's equation it follows: $\frac{K_{sat}}{37-K_{sat}} = \frac{15}{37-15} + \frac{4.95}{37-4.95}$ =0.8363. $K_{sat} = 16\ 85\ GPa$ The density is: $2.65 \cdot 0.75 + 0.5 \cdot 0.25 \cdot 1 + 0.5 \cdot 0.25 \cdot 0.91 = 2.23\ g(cm^3$

10. Example calculations for Hynesville sandstone at p_u=17.6 MPa:

$$K_{fluid} = \left(\frac{0.1254}{2.15} + \frac{0.7937}{0.5} + \frac{0.0809}{1}\right)^{-1} = 0.58\ GPa$$

The Gassmann's equation is:

$\frac{12.1}{K_s-12.1} = \frac{7.97}{K_s-7.97} + \frac{0.58}{0.0819\cdot(K_s-0.58)}$. It can be solved by the iteration procedure:

```
Ks0=45; c=12.1;a=7.97;b=0.58;fi=0.0819; Ks2=10;
while abs(Ks0-Ks2)>1e-6
Ks1=c*(1+(a/(Ks0-a)+b/fi/(Ks0-b))^-1);
Ks2=Ks0; Ks0=Ks1;
end
Ks0
```

The solution is K_s=*41.8 GPa*.. So $\alpha = 1 - \frac{K_{drn}}{K_s} = 1 - \frac{7.97}{41.8} \approx 0.81$

Table S6.1 Data to Exercise 10.

Lithology/Sample	Haynesville	Mid Bossier
Effective Porosity (%)	8.19	4.51
Wasser saturation (%)	12.54	26.84
Gas saturation (%)	79.37	69.71
Oil saturation (%)	8.09	3.45

Sample Haynesville		
p_u, MPa	drained K, GPa	undrained, K GPa
17.6	7.97	12.1
51.7	11.49	14.99
113.8	16.22	19.42
155.1	18.39	21.21
186.2	21.55	23.78
224.1	22.98	26.01

Sample Mid Bossier		

183	21.55	25.54
220.6	24.62	27.58
262	26.52	32.07
317.2	27	38.3

Solutions to Chapter 7:

Exercises and control questions to Chapter 7:

1.

The sound velocity is given by:: $V_{Susp} = \sqrt{\dfrac{K_{Susp}}{\rho_{Susp}}}$

$$\rho_{Susp} = (1-\phi)\cdot\rho_{Kazit} + \phi\cdot S\cdot\rho_{Wasser} + \phi\cdot(1-S)\cdot\rho_{Luft} = 0{,}6\cdot 2{,}71 + 0{,}4\cdot 0{,}5\cdot 1 + 0{,}4\cdot 0{,}5\cdot 0{,}00119 =$$
$$= 1{,}73\cdot 10^3\, kg/m^3$$

After the Reuss averaging: $\dfrac{1}{K_{Susp}} = \dfrac{1-\phi}{K_{Kalzit}} + \dfrac{S\cdot\phi}{K_{Wasser}} + \dfrac{(1-S)\cdot\phi}{K_{Luft}} = \dfrac{0{,}6}{75} + \dfrac{0{,}1}{2{,}2} + \dfrac{0{,}1}{0{,}131} = 0{,}82$

K_{Susp}=1.22 GPa, V_{Susp}=840 m/sec

After the Voigt averaging:

$$K_{Susp} = (1-\phi)\cdot K_{Kazit} + \phi\cdot S\cdot K_{Wasser} + \phi\cdot(1-S)\cdot K_{Luft} = 0{,}6\cdot 75 + 0{,}4\cdot 0{,}5\cdot 2{,}2 + 0{,}4\cdot 0{,}5\cdot 0{,}131 =$$
$$= 45{,}23\, GPa$$

V_{Susp}=5.11 km/sec

The iterative arithmetic-harmonic mean of *K* is: $K_{a\text{-}h\text{-}mean}$=7.43 GPa V_{Susp}=2.07 km/sec

2.

$$T = \frac{2\pi}{\omega} = 4\, sec$$

a. *f=1/T*=0.25 Hz, λ=*c*·*T*=14 km

b.

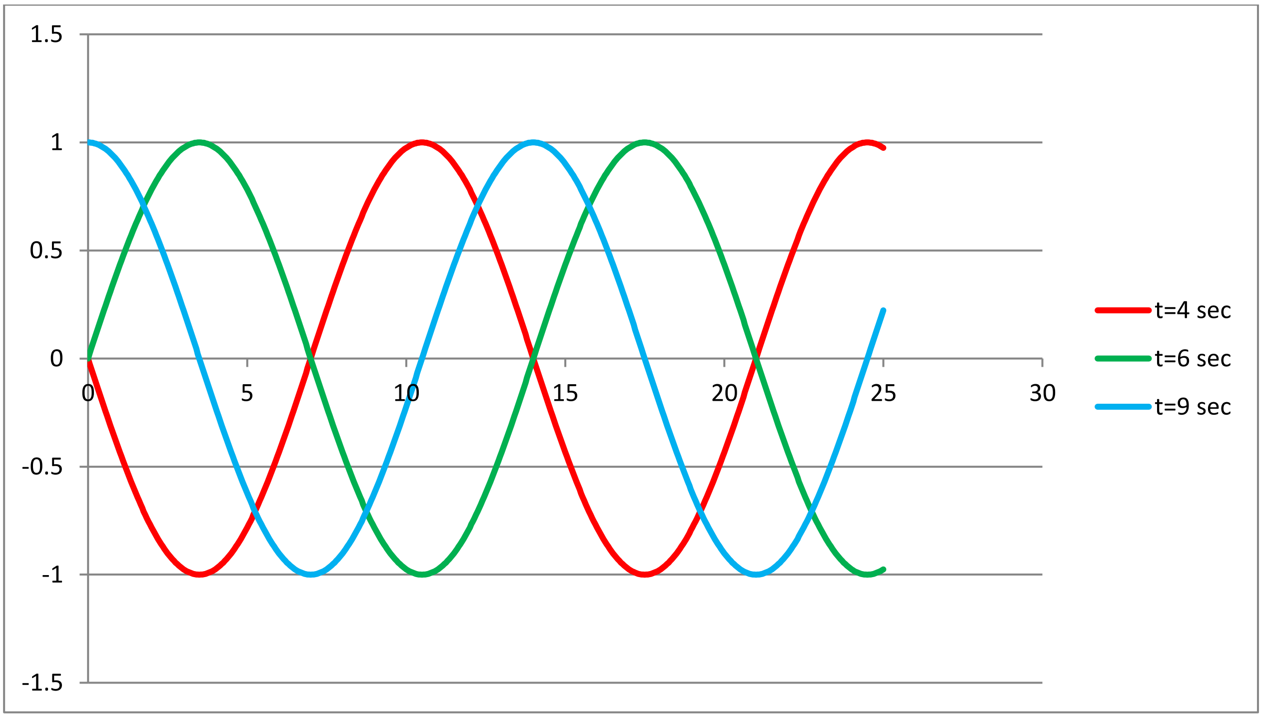


Fig. S.7.1 Problem 2b.

3. *y(t, x) = y*$y(t, x) = y_{max} \cos(\omega\cdot t + k\cdot x + \alpha)$.

$$c=\frac{\omega}{k}=10m/\sec,\lambda=c\cdot\frac{2\pi}{\omega}=2m,y(0{,}25;0{,}8)=2\cdot10^{-3}\cdot\cos\left(10\pi\cdot0{,}25-0{,}8\cdot\pi+\frac{70}{180}\cdot\pi\right)$$

$$y_1=2\cdot10^{-3}\cdot\cos(2{,}5\cdot\pi-0{,}8\cdot\pi+0{,}39\cdot\pi)=-1{,}95\cdot10^{-3}m$$

4. The ratio of shear to bulk moduli ***μ/K*** may be expressed through ***ν***-Poisson's ratio as follows:

$$\mu=\frac{E}{2\cdot(1+\nu)},K=\frac{E}{3\cdot(1-2\nu)}\Rightarrow\frac{\mu}{K}=\frac{3(1-2\nu)}{2(1+\nu)}\approx0{,}52$$

$$\frac{V_S}{V_P}=\sqrt{\frac{\mu}{K+4/3\cdot\mu}}=\sqrt{\frac{0{,}52}{1+\frac{4}{3}\cdot0{,}52}}=0{,}55\Rightarrow V_S=5{,}2\cdot0{,}55=2{,}88km/sec$$

The elastic wave velocity in a porous rock can be estimated from the relationship:

$\frac{1}{V_b}=\frac{\Phi}{V_f}+\frac{1-\Phi}{V_m}\Rightarrow\frac{V_b}{V_m}=\left(\frac{V_m}{V_f}\cdot\Phi+1-\Phi\right)^{-1}$, where Φ is the porosity, V_f is the longitudinal wave velocity in fluid phase, V_m is the elastic velocity in the rock matrix.

For P-waves the propagation velocity is:

$$V_{P,b}=5{,}2\cdot\left(\frac{5{,}2}{1{,}3}\cdot0{,}2+1-0{,}2\right)^{-1}=3{,}25km/sec$$

For S-waves assuming that the shear wave in solid matrix converts in longitudinal waves in fluid phase:

$$V_{S,b}=2{,}88\cdot\left(\frac{2{,}88}{1{,}3}\cdot0{,}2+1-0{,}2\right)^{-1}=2{,}32km/sec$$

5. The wave length should be 5 m, and *f=c/λ,* f=3800/5=760 Hz.

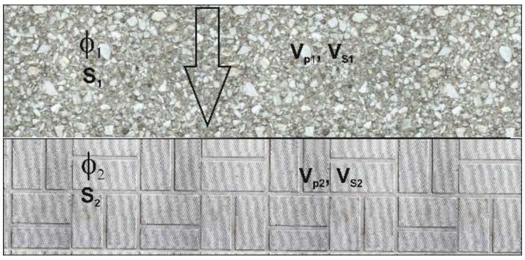


Fig. S7.2 Exercise 6.

6. Calculate the reflection and transmission coefficients of the normally incident seismic waves to the boundary between two layers of sandstones (Fig. E7.1) having porosities

ϕ_1=0.1 and ϕ_2=0.3, and water saturations S_1=0.5 and S_2=1. The density of quartz is 2650 kg/m^3, the density of water is 1000 kg/m^3, the quasi-isotropic polycrystalline elastic constants obtained by Voigt–Reuss–Hill averaging of the bulk modulus of quartz is 38 GPa, of the shear modulus of quartz is 47 GPa, bulk modulus of water is 2.29 GPa. In order to estimate the effective moduli in sandstones use the method of Kunster&Toksöz (1974).

For the upper layer with sandstone 1:
Density $\rho_1 = 2.65 \cdot (1-0.1) + 1 \cdot 0.1 \cdot 0.5 = 2.435\ g/cm^3$
For the lower layer with sandstone 2:
Density $\rho_2 = 2.65 \cdot (1-0.3) + 1 \cdot 0.3 = 2.155\ g/cm^3$
In order to calculate the effective bulk and shear moduli one may use the method of Kunster&Toksöz (1974):
One introduces the elastic parameter for the quartz matrix: $\zeta_q = \frac{\mu_q}{6} \cdot \frac{9K_q+8\mu_q}{K_q+2\mu_q} = 42.6\ GPa.$
In the upper layer, where the inclusions are with water and air two dimensionless coefficients for each phase, water (w) and air (a) are used:
$P_1^{qw}=\frac{K_q+\frac{4}{3}\cdot\mu_q}{K_w+\frac{4}{3}\cdot\mu_q} = 1.534$; $Q_1^{qw}=\frac{\zeta_q+\mu_q}{\zeta_q+\underbrace{0}_{\mu_w}} = 2.1$; $P_1^{qa}=\frac{K_q+\frac{4}{3}\cdot\mu_q}{\underbrace{0}_{K_a}+\frac{4}{3}\cdot\mu_q}$=1.61; $Q_1^{qa}=\frac{\zeta_q+\mu_q}{\zeta_q+\underbrace{0}_{\mu_a}} = 2.1.$
In the lower layer there is only one type of inclusions filled with water:
$P_2^{qw}=\frac{K_q+\frac{4}{3}\cdot\mu_q}{K_w+\frac{4}{3}\cdot\mu_q} = 1.534$; $Q_2^{qw}=\frac{\zeta_q+\mu_q}{\zeta_q+\underbrace{0}_{\mu_w}} = 2.1.$
For the upper layer the effective bulk and shear moduli according to Kuster&Toksöz averaging are:
$\left(K_1^*-K_q\right)\cdot\frac{K_q+\frac{4}{3}\cdot\mu_q}{K_1^*+\frac{4}{3}\cdot\mu_q} = \phi_1\cdot S_1\cdot\left(K_w-K_q\right)\cdot P_1^{qw} + \phi_1\cdot(1-S_1)\cdot\left(\underbrace{0}_{K_a}-K_q\right)\cdot P_1^{qa}$= -5.8 GPa, → $K_1^* = 32.52\ GPa$
$\left(\mu_1^*-\mu_q\right)\cdot\frac{\mu_q+\zeta_q}{\mu_1^*+\zeta_q} = \phi_1\cdot S_1\cdot\left(\underbrace{0}_{\mu_w}-\mu_q\right)\cdot Q_1^{qw} + \phi_1\cdot(1-S_1)\cdot\left(\underbrace{0}_{\mu_a}-\mu_q\right)\cdot Q_1^{qa}$= -9.87$GPa$ → $\mu_1^* = 38.11\ GPa$
In the lower layer
$\left(K_2^*-K_q\right)\cdot\frac{K_q+\frac{4}{3}\cdot\mu_q}{K_2^*+\frac{4}{3}\cdot\mu_q} = \phi_2\cdot\left(K_w-K_q\right)\cdot P_2^{qw}$= -16.4 GPa, → $K_2^* = 23.9\ GPa$
$\left(\mu_2^*-\mu_q\right)\cdot\frac{\mu_q+\zeta_q}{\mu_2^*+\zeta_q} = \phi_2\cdot\left(\underbrace{0}_{\mu_w}-\mu_q\right)\cdot Q_2^{qw}$= -29.6 GPa → $\mu_2^* = 24.76\ GPa$
The P- and S-wave velocities in the upper and lower layers are as follows:
$V_{p,1} = \sqrt{\frac{K_1^*+\frac{4}{3}\cdot\cdot\mu_1^*}{\rho_1}} = 5.85\ km/s$, and $V_{s,1} = \sqrt{\frac{\mu_1^*}{\rho_1}} =$ 3.96 km;
$V_{p,2} = \sqrt{\frac{K_2^*+\frac{4}{3}\cdot\cdot\mu_2^*}{\rho_2}} = 5.14\ km/s$, and $V_{s,1} = \sqrt{\frac{\mu_2^*}{\rho_2}} =$ 3.39 km.
The normal reflection and transmission coefficients for P-wave are:
$R_p = \frac{\rho_2\cdot V_{p,2}-\rho_1\cdot V_{p,1}}{\rho_1\cdot V_{p,1}+\rho_2\cdot V_{p,2}}$= -0.125, $T_p = 2\cdot\frac{\rho_1\cdot V_{p,1}}{\rho_1\cdot V_{p,1}+\rho_2\cdot V_{p,2}}$=1.125,

and for S-waves are: $R_S = \frac{\rho_2 \cdot V_{S,2} - \rho_1 \cdot V_{S,1}}{\rho_1 \cdot V_{S,1} + \rho_2 \cdot V_{S,2}}$ = -0.138 , $T_p = 2 \cdot \frac{\rho_1 \cdot V_{S,1}}{\rho_1 \cdot V_{S,1} + \rho_2 \cdot V_{S,2}}$ =1.138. (Note: the sign minus in the reflection coefficient is due to the fact that the particle displacement amplitude is measured relative to the direction of the wave propagation).

7. a) The relationship between shear and bulk moduli is as follows: $\mu = K \cdot \frac{3\cdot(1-2\nu)}{2\cdot(1+\nu)} \approx 0.52,$

then, $V_p = \sqrt{\frac{K + \frac{4}{3}\cdot 0.52 \cdot K}{\rho}} \approx 1.3 \cdot \sqrt{\frac{K}{\rho}}$. Front her side $$\boldsymbol{V_p = \frac{2300}{300} = 7{,}67 km/s}$$

Resolving relative to K one obtains: $$\boldsymbol{K = \rho \cdot \left(\frac{V_p}{1{,}3}\right)^2 = 104{,}3\, GPa}$$

(b) The sound velocity in water is: $$\boldsymbol{V_{p,Wasser} = \sqrt{\frac{2{,}3\cdot 10^9}{10^3}} = 1.517 m/s = 1{,}52 km/s}$$

The traveling time of the sound wave in water:is: $$\boldsymbol{t = \frac{2300}{1{,}52} = 25{,}27 min}$$

8. From the expression of P- and S-waves it follows:

$$\boldsymbol{\frac{K}{\mu} = \left(\frac{V_p}{V_S}\right)^2 - \frac{4}{3}} \qquad \boldsymbol{\nu = \frac{1}{2} \cdot \frac{\left(\frac{V_p}{V_S}\right)^2 - 2}{\left(\frac{V_P}{V_S}\right)^2 - 1}}$$

For sediments: V_P/V_S=1.84, fort he Earth’s crust: V_P/V_S=1.71. Then, for sediments: K/µ=2.06, für die Erdkruste K/µ =1.61. The Poisson’s ratio for sediments: ν=0.29, and for the Earth’s crust ν=0.24.

9. If one denotes the displacement of a mass m in the horizontal or vertical directions by u, then the displacements of masses locating at the diagonals of square lattice are $\sqrt{2} \cdot u$. The absolute velocity of each mass in the M*N lattice is then $|\dot{u}|$. The total kinetic energy of the lattice is given by: $E_K = \frac{m}{2} \cdot \dot{u}^2 \cdot M \cdot N$. The potential energy of the lattice consists of the $M\cdot N$ springs with stiffness K_1 having deformations 2·u in the horizontal direction plus $2\cdot M\cdot N$ diagonal springs with stiffness K_2 having deformations: $\sqrt{2} \cdot u$. The potential energy is: $E_p = 2 \cdot (K_1 + K_2) \cdot u^2 \cdot M \cdot N$, and the total energy is $E_K + E_p = (\frac{m}{2} \cdot \dot{u}^2 + 2 \cdot (K_1 + K_2) \cdot u^2) \cdot M \cdot N$. A simple mass µ hanging on a spring K possesses the total energy: E= $(\frac{\mu}{2} \cdot \dot{u}^2 + \frac{1}{2} \cdot K \cdot u^2)$ and the frequency of free oscillations: $\omega = \sqrt{\frac{K}{\mu}}$. By analogy with this simple case one may

write for the longitudinal mode of square lattice oscillations: $\omega_L = 2\sqrt{\frac{K_1+K_2}{m}}$. For the transverse mode of oscillations the number of deformed diagonal springs is 2M*N and each of them shortened or elongated by: $\sqrt{2}\cdot u$. The total kinetic energy of moving masses plus potential energy of elastic springs is: $E_K + E_p = (\frac{m}{2}\cdot\dot{u}^2 + \frac{1}{2}K_2\cdot(\sqrt{2}\cdot u)^2\cdot 2M\cdot N$. Therefore, for the frequency of the transverse mode of lattice oscillations one obtains $\omega_L = 2\sqrt{\frac{K_2}{m}}$ (Michály and Martin, 2008).

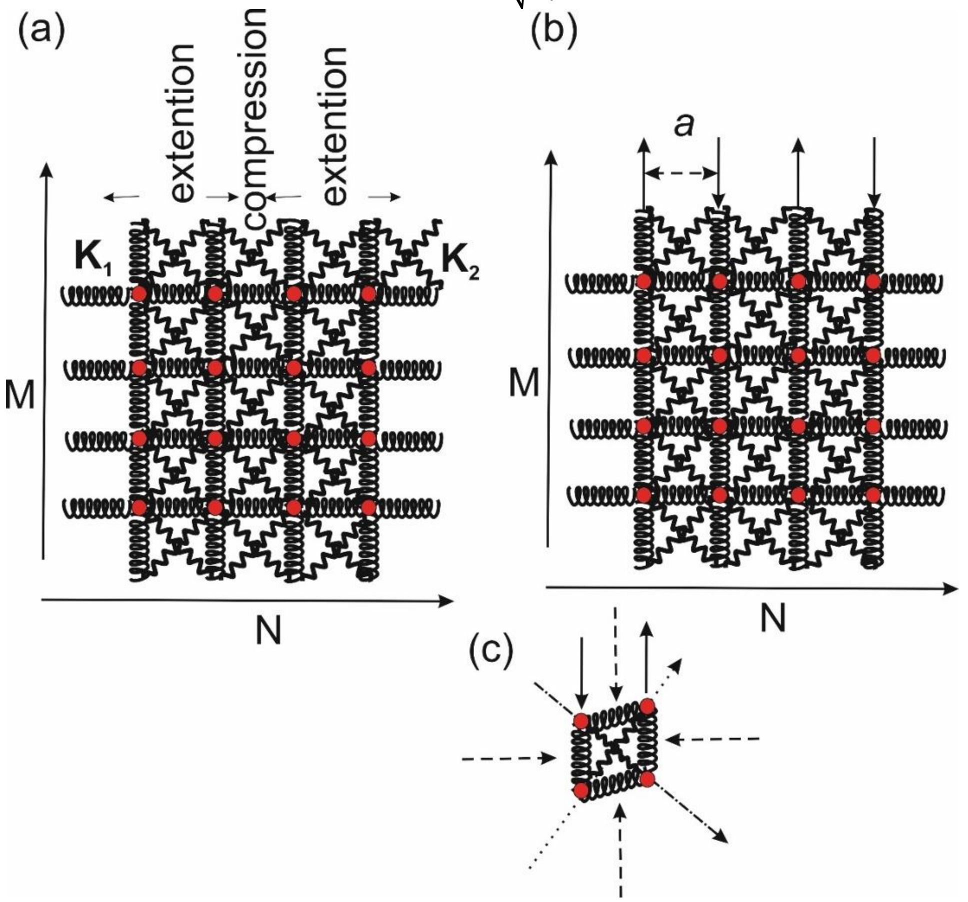


Fig. S7.2 Exercise 9. (a) The compression-dilation mode of lattice vibrations. (b) The transversal mode of lattice vibrations. (c) In the case of transversal mode vibrations there is no deformation of springs in directions indicated by dashed arrows, but there is compression of a spring in the direction of dashed-dotted spring, and extention of a spring in the direction of dotted arrow.

10. Solution is presented in Table below:

Q=1000 X, km	Frequency in Hz 0.1	1	10
1	0.99993717	0.99937188	0.99373651
10	0.99937188	0.99373651	0.93910137
100	0.99373651	0.93910137	0.53348813
1000	0.93910137	0.53348813	0.00186744

Q=100 X, km	Frequency in Hz 0.1	1	10
1	0.999371887	0.993736591	0.939102105
10	0.993736591	0.939102105	0.533492281
100	0.939102105	0.533492281	0.001867589
1000	0.533492281	0.001867589	0

Q=10 X, km	Frequency in Hz 0.1	1	10
1	0.99374432	0.93917517	0.53390752
10	0.93917517	0.53390752	0.00188218
100	0.53390752	0.00188218	0
1000	0.00188218	0	0

Q=1 X, km	Frequency in Hz 0.1	1	10
1	0.94703997	0.58034077	0.00433342
10	0.58034077	0.00433342	0
100	0.00433342	0	0
1000	0	0	0

Exercises and control questions to Chapter 8:

1. According to the Archie's law the effective electric conductivity may be expressed through porosity and degree of saturation as follows: $\sigma_{eff} = \frac{\sigma_{fluid}}{\varphi^{m}\cdot S_{w}^{n}}$. Assuming that the fluid electric conductivity is on pressure independent, after differentiation one obtains: $\frac{\partial\sigma_{eff}}{\partial P} = \sigma_{fluid}\cdot\varphi^{m}\cdot S_{w}^{n}\cdot\left[m\cdot\frac{\partial ln\varphi}{\partial P} + n\cdot\frac{\partial lnS_{w}}{\partial P}\right]$, or $\frac{\partial\ln(\sigma_{eff})}{\partial P} = \frac{\partial\ln(\varphi^{m}\cdot S_{w}^{n})}{\partial P}$. (a) When the total volume of fluid is constant: $\varphi\cdot S_{w} = const$, then, $\frac{\partial\ln(\sigma_{eff})}{\partial P} = \frac{\partial\ln(\varphi^{m-n})}{\partial P} = \frac{\partial\ln({S_{w}}^{n-m})}{\partial P}$. (b) When S_w ~1, then, $\frac{\partial\ln(\sigma_{eff})}{\partial P} = \frac{\partial\ln(\varphi^{m})}{\partial P}$.
2. One has to plot the dependence of R vs. T:

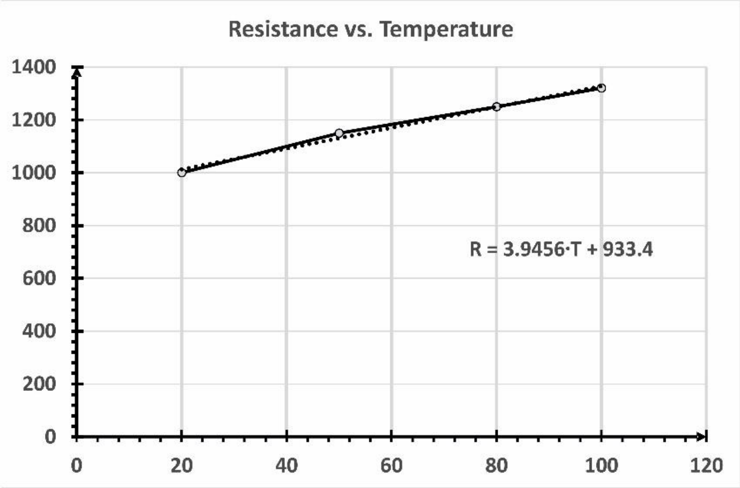


Fig. S8.1 Problem 2.

The thermal coefficient of electric resistance is α≈3.9456/933.4=423·10^{-3} K^{-1}

The specific resistance is: $$\rho = \frac{1000\cdot\pi\cdot(0{,}01)^{2}}{0{,}1} = 3{,}14 Ohm\cdot m$$

3. *(a)* Without rock slab inside the capacitor*:* for the parallel plate capacitor: $$C_{1} = \varepsilon_{0}\cdot 1\cdot\frac{A}{d}$$

The induced electric charge, when the voltage U_0 is applied, is: $$Q_{1} = \frac{\varepsilon_{0}\cdot A\cdot U_{0}}{d}$$

(b) When the rock slab is inside the capacitor: $$C_{2} = \left(\left(\varepsilon_{0}\cdot 1\cdot\frac{A}{d-b}\right)^{-1} + \left(\varepsilon_{0}\cdot\varepsilon_{r}\cdot\frac{A}{b}\right)^{-1}\right)^{-1} = \frac{\varepsilon_{0}\cdot A}{\left(d - b\cdot\frac{\varepsilon_{r}-1}{\varepsilon_{r}}\right)}$$

The induced electric charge when the voltage U_0 is applied, is: $$Q_2 = \frac{\varepsilon_0 \cdot A \cdot U_0}{\left(d - b \cdot \frac{\varepsilon_r - 1}{\varepsilon_r}\right)}$$

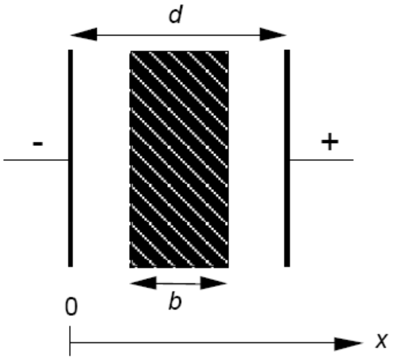


Fig. S8.2 Problem 3.

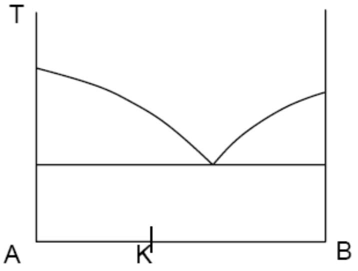


Fig. S8.3 Problem 4.

4. Reuss –average is: $\frac{1}{\varepsilon_{Reuss}} = \frac{0,6}{7} + \frac{0,4}{10} \Rightarrow \varepsilon_{Reuss} = 7,95$

   Voigt- average is: $\varepsilon_{Voigt} = 0,6 \cdot 7 + 0,4 \cdot 10 = 8,2$

5. (a) According to the Coulomb law the electric field E of a point charge Q is given by the relationship:

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r^2} = \frac{1}{\varepsilon_0} \cdot \frac{Q}{4\pi r^2} \Rightarrow \frac{Q}{4\pi r^2} = \varepsilon_0 \cdot E = 8,85 \cdot 10^{-12} \frac{C^2}{N \cdot m^2} \cdot 100 \frac{N}{C} = 8,85 \cdot 10^{-10} \frac{C}{m^2}$$

This surface density of electric charges corresponds to the number n per square meter: $n = \frac{8.85 \cdot 10^{-10}}{1.6022 \cdot 10^{-19}} = 5,52 \cdot 10^9$, where $1.6022 \cdot 10^{-19} C$ is the electric charge of electron.

(b) The electric field close to the Earth's surface (height d << R_E radius of the Earth). The electric potential in a parallel plate capacitor having thickness 1.6 m is given by the relationship: $\varphi = E \cdot d = 100 \cdot 1.6 = 160V$ (!)

(c) If the human body is ideally conductive, then, the whole body is under the same electric potential f from bottom to the top.

(d) If between two points at the distance d there is a potential difference φ, the energy per square meter is the work W per surface A to move the electrical charge Q in the field E by distance d is given by:

$$\overbrace{\frac{W}{A}}^{specific\ surface\ energy} = \overbrace{\frac{Q}{A}}^{surface\ charge\ density} \cdot \varphi = \varepsilon_0 \cdot E \cdot E \cdot d = 8.85 \cdot 10^{-12} \cdot 10^4 \cdot 100 = 8.85 \cdot 10^{-6}\, Joul.$$

This is a negligible amount of energy

6. The parameters of the Arrhenius equation used for the lelectrical conductivity calculations as a function of temperature are presented in Table below:

Table S8.1 Problem 6

Mineral	ϕ, Vol.%	Pre-exponential factor σ_0, S/m	Activation enthalpy ΔH, eV
Olivine	60	490	1,62
Orthopyroxene	25	5248	1,80
Klinopyroxene	15	1778	1,87

The electrical conductivity of individual minerals at 1200°C according to the given in Table S8.1 parameters are as follows:

$$\sigma_{Ol}(1200\,°C) = 490 \cdot e^{-\frac{1,62}{8,617 \cdot 10^{-5} \cdot 1473}} = 1,4 \cdot 10^{-3}\, S/m = 14 \cdot 10^{-4}\, S/m,$$

$$\sigma_{Opx}(1200\,°C) = 5248 \cdot e^{-\frac{1,8}{8,617 \cdot 10^{-5} \cdot 1473}} = 3,6 \cdot 10^{-3}\, S/m = 36 \cdot 10^{-4}\, S/m, \ \textit{(Maximale Wert)}$$

$$\sigma_{Cpx}(1200\,°C) = 1778 \cdot e^{-\frac{1,87}{8,617 \cdot 10^{-5} \cdot 1473}} = 7,1 \cdot 10^{-4}\, S/m, \textit{ (Minimale Wert).}$$

Solution 1:

$$A_1 = \frac{\phi_{Ol}}{\left(\sigma_{Ol} - \sigma_{Cpx}\right)^{-1} + \left(3 \cdot \sigma_{Cpx}\right)^{-1}} + \frac{\phi_{Opx}}{\left(\sigma_{Opx} - \sigma_{Cpx}\right)^{-1} + \left(3 \cdot \sigma_{Cpx}\right)^{-1}} =$$

$$= \frac{0{,}6}{\left(14 \cdot 10^{-4} - 7{,}1 \cdot 10^{-4}\right)^{-1} + \left(3 \cdot 7{,}1 \cdot 10^{-4}\right)^{-1}} + \frac{0{,}25}{\left(36 \cdot 10^{-4} - 7{,}1 \cdot 10^{-4}\right)^{-1} + \left(3 \cdot 7{,}1 \cdot 10^{-4}\right)^{-1}} \approx 6{,}2 \cdot 10^{-4}\, S/m$$

$$A_2 = \frac{\phi_{Ol}}{\left(\sigma_{Ol} - \sigma_{Opx}\right)^{-1} + \left(3 \cdot \sigma_{Opx}\right)^{-1}} + \frac{\phi_{Cpx}}{\left(\sigma_{Cpx} - \sigma_{Opx}\right)^{-1} + \left(3 \cdot \sigma_{Opx}\right)^{-1}} =$$

$$= \frac{0{,}6}{\left(14 \cdot 10^{-4} - 36 \cdot 10^{-4}\right)^{-1} + \left(3 \cdot 36 \cdot 10^{-4}\right)^{-1}} + \frac{0{,}15}{\left(7{,}1 \cdot 10^{-4} - 36 \cdot 10^{-4}\right)^{-1} + \left(3 \cdot 36 \cdot 10^{-4}\right)^{-1}} \approx -22{,}5 \cdot 10^{-4}\, S/m$$

$$\sigma^{HS-} = \sigma_{min} + \frac{3 \cdot \sigma_{min} \cdot A_1}{3 \cdot \sigma_{min} - A_1} = 7{,}1 \cdot 10^{-4} + \frac{3 \cdot 7{,}1 \cdot 10^{-4} \cdot 6{,}2 \cdot 10^{-4}}{\underbrace{3 \cdot 7{,}1 \cdot 10^{-4} - 6{,}2 \cdot 10^{-4}}_{8{,}7 \cdot 10^{-4}}} \approx 15{,}8 \cdot 10^{-4}\, S/m$$

$$\sigma^{HS+} = \sigma_{max} + \frac{3 \cdot \sigma_{max} \cdot A_2}{3 \cdot \sigma_{max} - A_2} = 36 \cdot 10^{-4} + \frac{3 \cdot 36 \cdot 10^{-4} \cdot (-22{,}5 \cdot 10^{-4})}{\underbrace{3 \cdot 36 \cdot 10^{-4} - (-22{,}5 \cdot 10^{-4})}_{-18{,}6 \cdot 10^{-4}}} \approx 17{,}4 \cdot 10^{-4}\, S/m$$

Solution 2:

$$\sigma^{HS-} = \frac{1}{\left(\frac{X_{Cpx}}{3 \cdot \sigma_{Cpx}}\right) + \left(\frac{X_{Ol}}{\sigma_{Ol} + 2 \cdot \sigma_{Cpx}}\right) + \left(\frac{X_{Opx}}{\sigma_{Opx} + 2 \cdot \sigma_{Cpx}}\right)} - 2 \cdot \sigma_{Cpx} =$$

$$= \left[\underbrace{\left(\frac{0{,}15}{3 \cdot 7{,}1} + \frac{0{,}6}{14 + 2 \cdot 7{,}1} + \frac{0{,}25}{36 + 2 \cdot 7{,}1}\right)^{-1}}_{30{,}0} - 2 \cdot 7{,}1\right] \cdot 10^{-4}\, S/m \approx 15{,}8 \cdot 10^{-4}\, S/m$$

$$\sigma^{HS+} = \frac{1}{\left(\frac{X_{Cpx}}{\sigma_{Cpx} + 2 \cdot \sigma_{Opx}}\right) + \left(\frac{X_{Ol}}{\sigma_{Ol} + 2 \cdot \sigma_{Opx}}\right) + \left(\frac{X_{Opx}}{3 \cdot \sigma_{Opx}}\right)} - 2 \cdot \sigma_{Opx} =$$

$$= \left[\underbrace{\left(\frac{0{,}15}{7{,}1 + 2 \cdot 36} + \frac{0{,}6}{14 + 2 \cdot 36} + \frac{0{,}25}{3 \cdot 36}\right)^{-1}}_{89.4} - 2 \cdot 36\right] \cdot 10^{-4}\, S/m \approx 17{,}4 \cdot 10^{-4}\, S/m$$

7. The Voigt average is: $\varepsilon_V = \underbrace{0.65 \cdot 4}_{soil\ matrix} + \underbrace{0.35 \cdot 0.45 \cdot 81}_{soil\ water} + \underbrace{0.35 \cdot 0.55 \cdot 1}_{soil\ air} = 15.55$

   The Reuss average is: $\varepsilon_R = \left(\underbrace{0.65/4}_{soil\ matrix} + \underbrace{0.35 \cdot 0.45/81}_{soil\ water} + \underbrace{0.35 \cdot 0.55/1}_{soil\ air}\right)^{-1} = 2.8$

   The Hill mean value is: $\varepsilon_H = \frac{\varepsilon_V + \varepsilon_R}{2} = 9.18$.

8. The electric charge **Q** on the surface of a hole metal sphere generates the electric field which will be corrugated by the field of electric charges in the surrounding space, for example, by the surface charges on the Earth. That is why the voltage potential **U** is traditionally measured relative to the potential of the Earth's surface. If the electrical charges close to the metal sphere are rearranged, the electric charge distribution of the sphere is also changed as well as the electric potential **U** of the sphere. If the metal sphere is located far away from all conductive surfaces, then the spherical capacity is given by the relationship: $C = \frac{Q}{U} = \frac{Q}{\frac{Q}{4\pi\varepsilon_0 \cdot r}} = 4\pi\varepsilon_0 \cdot r$. If the metal hole sphere is located concentric inside another metal sphere, which is grounded, then from the second sphere flows away exactly the same amount of electric charge that has been stored in the first sphere. This phenomena is called the *Influence*, which results in the absence of electric field outside of two spheres and the presence of electric field only between two metal surfaces. The second metal sphere is called shielding, and with this one can achieve two things: the inside electric field does not interact or corrugate the electric field in the outer space, and the space of inner electric field is limited by two metal surface that results in the increase of capacity. The capacity of two concentric metal spheres or simply spherical capacitor is given by:
$$\frac{1}{C} = \frac{1}{C_{Aussen}} + \frac{1}{C_{Innen}} = \left(-\frac{U_{Aussen}}{Q} + \frac{U_{Innen}}{Q}\right)^{-1} = \left(\frac{1}{4\pi\varepsilon_0 r_2} - \frac{1}{4\pi\varepsilon_0 r_1}\right)^{-1} \quad \text{or}$$
$C = 4\pi\varepsilon_0 \cdot \frac{r_1 \cdot r_2}{r_1 - r_2}$. If one fills the inner space between two concentric metal spheres with dielectric material, then the capacity will be increase by factor ε_r
$C = 4\pi\varepsilon_0\varepsilon \cdot \frac{r_1 \cdot r_2}{r_1 - r_2}$. Plugging in this expression the temperature dependence of the dielectric constant and the dependence of temperature with the depth z, one obtains for the capacity of the Earth's mantle: $C = 4\pi\varepsilon_0 \cdot \left(8.12 + 6.4 \cdot 10^{-3} \cdot \left[3500 \cdot \frac{z}{\sqrt{1+z^2}} - 300\right]\right) \cdot 7638$, where $z = \frac{d(km)+125}{1500}$. The mean temperature of the Earth's mantle is $T \approx 1500°C \approx 1800K$. Then, the whole mantle capacity is:
$$C = 4\pi \cdot 8{,}854 \cdot 10^{-12} \cdot \underbrace{(8{,}12 + 6{,}4 \cdot 10^{-3} \cdot 1500)}_{17{,}72} \cdot \frac{6372 \cdot 3474}{2898} \cdot 10^{3} = 0{,}015F$$

9. At the Earth's surface the electric field strength is on average E(R) = 130 V/m. Consider a spherical symmetrical charge distribution with the total charge Q at the origin. The charge density $\rho = \rho_0 \exp(-r/R) \cdot R^2/r^2$ should decrease exponentially up to the Earth's surface at r = R. Calculate the total charge Q of the Earth and the charge density ρ_0. Estimate the electric field E(r) and the electrostatic potential Φ (r) of this charge distribution in the far field (i.e. for distances r >> R). Radius of the Earth $R = 6.372\ 10^6$ m, $\varepsilon_0 \approx 8.8654 \cdot 10^{-12}$ A·s/V·m, and inside the Earth $\varepsilon_r = 18$, outside the earth $\varepsilon_r = 1$.

Inside the Earth the electric potential *Φ(r)* as a function of the Earth's radius is given by:

$$\Phi(r)=\frac{1}{4\pi\varepsilon_0}\cdot\frac{\int\limits_0^r \frac{\rho_0\cdot R^2\cdot e^{-\frac{r}{R}}}{r^2}\cdot 4\pi r^2\cdot dr}{r}=\frac{\rho_0\cdot R^3}{\varepsilon_0\cdot\varepsilon}\cdot\frac{1-e^{-\frac{r}{R}}}{r}$$

. The electric field strength *E* inside the Earth is given by the relationship:

$$E(r)=-grad(\Phi(r))=-\frac{d\Phi(r)}{dr}=\frac{\rho_0\cdot R^3}{\varepsilon_0\cdot\varepsilon\cdot r^2}\left(1-\left[1-\frac{r}{R}\right]\cdot e^{-\frac{r}{R}}\right)$$

. At the Earths surface R=r the electric field strength satisfies the identity

$$130=\frac{\rho_0\cdot R}{\varepsilon_0}\Rightarrow\rho_0=\frac{\varepsilon_0\cdot 130}{R}=1{,}81\cdot 10^{-16}C/m^3$$

. The total electric charge of the Earth is the integral:

$$Q=\int\limits_0^R \frac{\rho_0\cdot R^2\cdot e^{-\frac{r}{R}}}{r^2}\cdot 4\pi r^2\cdot dr=4\pi\cdot\rho_0\cdot R^3\cdot\left(1-e^{-1}\right)=3{,}7\cdot 10^5 C.$$

Far away from the Earth's surface ***r***>>***R*** the electric potential varies as a function of distance r: $\Phi(r)=\frac{\rho_0\cdot R^3}{\varepsilon_0}\cdot\frac{1-e^{-1}}{r}$ and the electric field strength as: $E(r)=\frac{\rho_0\cdot R^3\cdot(1-e^{-1})}{\varepsilon_0\cdot r^2}$.

10. (a) If the electric charge of the Earth is Q, then the electric field strength on the Earth's surface is:

$$E_1=\frac{Q}{4\pi\varepsilon_0\cdot R^2}=\frac{\sigma}{\varepsilon_0}\Rightarrow\sigma=\varepsilon_0\cdot E_1=8{,}8654\cdot 10^{-12}\cdot 130=1{,}15\cdot 10^{-9}C/m^2$$

At the height h=10 km above the Earth's surface in vacuum (in the absence of space atmospheric charges) the electric field strength is:

$$E_2^*=\frac{Q}{4\pi\varepsilon_0\cdot(R+h)^2}=\frac{\sigma}{\varepsilon_0}\cdot(1-2\underbrace{\frac{h}{R}}_{\approx 0{,}003})\approx\frac{\sigma}{\varepsilon_0}$$

(b) Due to the influence of space atmospheric charges of density ρ the electric charge *Q* of the Earth must be corrected by the electric charge of the spherical shell of thickness *h* (shielding effect):

$$E_2=E_1-\frac{4\pi R^2\cdot h\cdot\rho}{4\pi\varepsilon_0(R+h/2)^2}\approx E_1-\frac{\rho\cdot h}{\varepsilon_0}\rightarrow\rho=\frac{(130-4)\cdot 8{,}8654\cdot 10^{-12}}{10\cdot 10^3}=1{,}12\cdot 10^{-13}C/m^3$$

11. The rescaling of the triangular lattice are shown in Fig. S8.4. Instead of three positions in the original lattice a single site position in super lattice is generated. The site occupancy in the original lattice is marked as solid black circle. The site occupancy in super lattice is shown as a solid grey circle. Empty sites are marked as X. By p one denotes the probability of an occupied site in the original lattice and by p' is the probability of the occupied site in the super lattice. The probability that in the original lattice all three sites are occupied is p^3. The probability that two

sites are occupied and one is empty is $p^2 \cdot (1-p)$, but there are 3 possible arrangements of this configuration. So the total probability that in the super lattice the site is occupies is given by the equation: $p' = p^3 + 3 \cdot p^2 \cdot (1-p)$. The fixed points, i.e. the roots of the equation when $p' = p$, are (0, 0.5 and 1). The non-trivial solution is: $p_c = 0.5$, this is the lattice percolation threshold for the site occupancy. In the original lattice the distance between two neighbor rows of sites is 1 (see insert in Fig. S8.4). The distance between points is: $\frac{2}{\sqrt{3}}$. In the super lattice the distance between neighbor points is 2. This means that the scaling parameter b=$\sqrt{3}$. One denotes by $R(p) = p^3 + 3 \cdot p^2 \cdot (1-p)$, then $\frac{dR(p)}{dp} = 6 \cdot p - 6 \cdot p^2$

According to the definition of the critical exponent: $\tau = \frac{ln b}{ln\left(\frac{dR(p)}{dp}\right)_{p=0.5}} = \frac{ln(\sqrt{3})}{\ln(1.5)} = 1.355$.

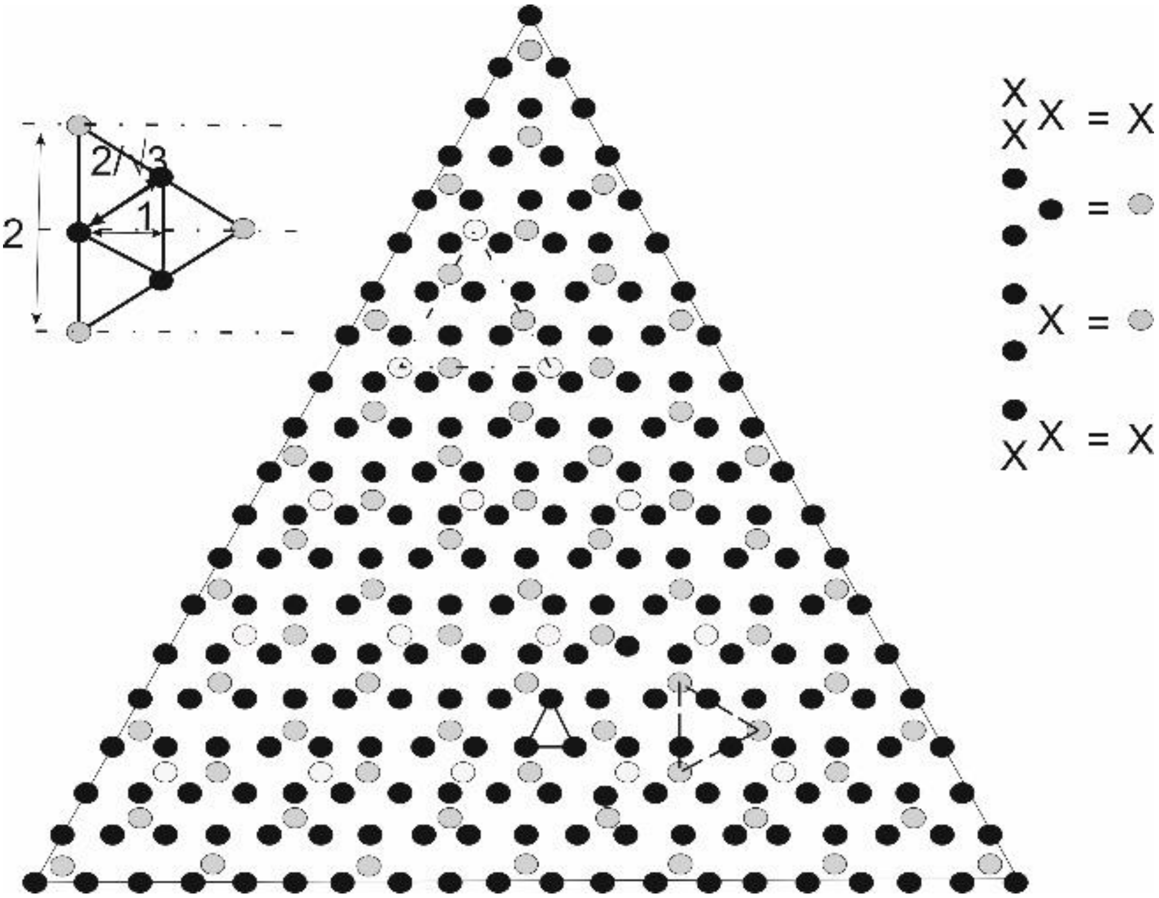


Fig. S8.4 The triangular resistor network: filled circles are the original lattice (the cell is denoted by the solid line triangle), grey circles are the new lattice (dashed line triangle), open circles represent the next step of scaling (dashed-dotted line triangle). The scaling factor *b* is calculated as a ratio of distances between the nearest-neighbors in two lattices (ratio of sides of open and filled circle triangles) (adapted from Stauffer, 1979).

Exercises and control questions to Chapter 9:

1. One denotes the width of parallel capacitor plates as *a*.

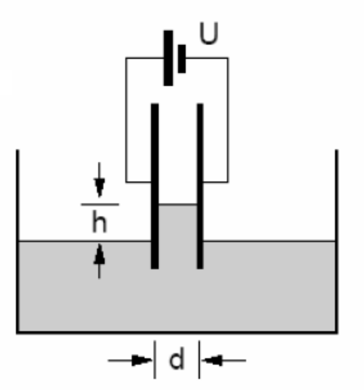


Fig. S9.1 Problem 1.

The electrostatic energy of the empty parallel plate capacitor is: $W_{el}^{1} = \frac{1}{2}\varepsilon_0 \cdot \frac{L*a}{d} \cdot U^2$,

where *a* ist he width and *L* is the height of capacitor plates. When the liquid rises in the space between the metallic plates the capacity changes and consists of two parallel connected capacitors: $C=C_1+C_2$, wher C_1 is the empty part and C_2 is with liquid filled part of capacitor C. So the electrostatic energy of capacitor is given by::

$W_{el}^{2} = \frac{1}{2} \cdot \varepsilon_0 \cdot \left[\frac{(L-h)\cdot a}{d} + \varepsilon_r \cdot \frac{h \cdot a}{d}\right] \cdot U^2$. The increment of capacitor energy between the empty and partially filled with liquid states is:

$\Delta W_{el} = W_{el}^{2} - W_{el}^{1} = \frac{1}{2} \cdot \varepsilon_0 \cdot (\varepsilon_r - 1) \cdot \frac{h \cdot a}{d} \cdot U^2$. This energy increment may be considered as a change of the potential energy of system and must equal to the increase of liquid potential and kinetic energies. The increase of liquid potential energy due to rise by height *h* in the space between capacitor plates is: $\Delta W_{pot} = \frac{1}{2} h \cdot mg = \frac{1}{2} h \cdot \rho \cdot g \cdot \underbrace{h \cdot a \cdot d}_{Volumen}$. From the condition that the new equilibrium state of the system must correspond to the minimum of the potential energy increment means that : $\Delta W_{el} - \Delta W_{pot} \Rightarrow min$, from which the relationship follows:

$\frac{d}{dh}\left(\frac{1}{2}\varepsilon_0 \cdot (\varepsilon_r - 1)\frac{h \cdot a}{d} \cdot U^2 - \frac{1}{2}\rho \cdot g \cdot h^2 \cdot a \cdot d\right) = 0 \rightarrow h = \frac{\varepsilon_0 \cdot (\varepsilon_r - 1) \cdot U^2}{2 \cdot \rho \cdot g \cdot d^2}$. The difference between the liquid potential energy and the electrostatic energy of capacitor when liquid raised to the height *h* is:

$$\Delta(W_{el}-W_{pot})=\frac{1}{2}\varepsilon_0\cdot(\varepsilon_r-1)\frac{\overbrace{h}^{\frac{\varepsilon_0\cdot(\varepsilon_r-1)\cdot U^2}{2\cdot\rho g\cdot d^2}}\cdot a}{d}\cdot U^2-\frac{1}{2}\rho g\underbrace{h^2}_{\left(\frac{\varepsilon_0\cdot(\varepsilon_r-1)\cdot U^2}{2\cdot\rho g\cdot d^2}\right)^2}\cdot a\cdot d=$$

$$=\frac{1}{8}\cdot\frac{\varepsilon_0^2\cdot(\varepsilon_r-1)^2\cdot U^4\cdot a}{\rho\cdot g\cdot d^3}$$

This is the half of electrostatic capacitor energy increment, if one plugged h in ΔW_{el} the other half equals the lost kinetic energy of fluid flow. Liquid raised to the height h with certain flow velocity and due to viscosity the kinetic energy converted in thermal losses. Thus, the temperature of liquid increased by ΔT:

$$\underbrace{m}_{\rho\cdot a\cdot h\cdot d}\cdot C_p\cdot\Delta T=\frac{1}{8}\cdot\frac{\varepsilon_0^2\cdot(\varepsilon_r-1)^2\cdot U^4\cdot a}{\rho\cdot g\cdot d^3}=\frac{1}{4}\cdot\frac{\varepsilon_0\cdot(\varepsilon_r-1)\cdot U^2\cdot a\cdot h}{d}\Rightarrow\Delta T=\frac{1}{4}\cdot\frac{\varepsilon_0\cdot(\varepsilon_r-1)\cdot U^2}{\rho\cdot C_p\cdot d^2}$$

In takes d=10μm, $\varepsilon_r=81$ (water), U=100 V, $C_p=4.19$ kJ/kg/°K, ρ=1000 kg/m^3, then ΔT≈0.004° (!).

2. (a) The equivalent circuit of the complex capacitor is shown in Fig: S9.2:.

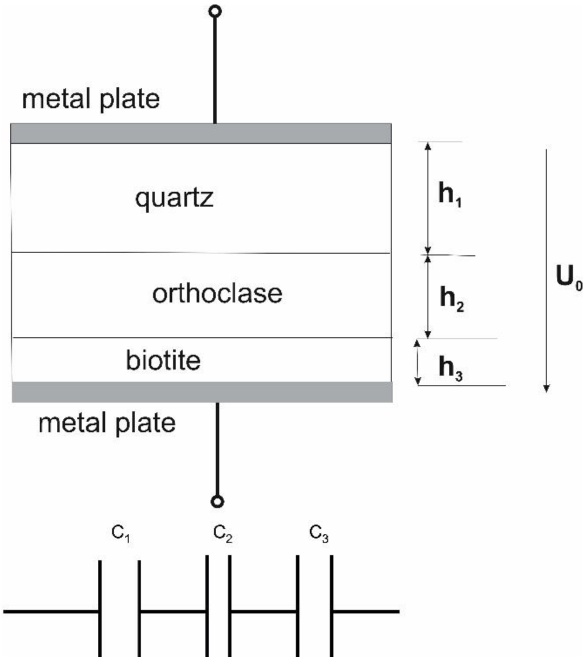


Fig. S9.2 Problem 2.

The total capacity of three condensators is the harmonic sum:

$$C=\left(\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}\right)^{-1}=\left(\frac{h_1}{\varepsilon_0\varepsilon_1 bt}+\frac{h_2}{\varepsilon_0\varepsilon_2 bt}+\frac{h_3}{\varepsilon_0\varepsilon_3 bt}\right)^{-1}=$$

$$=8{,}85\cdot 10^{-12}\cdot 90\cdot 10^{-3}x60\cdot 10^{-3}\cdot\left(\frac{4\cdot 10^{-3}}{4{,}5}+\frac{3\cdot 10^{-3}}{5{,}6}+\frac{1{,}5\cdot 10^{-3}}{10{,}3}\right)^{-1}=3{,}04\cdot 10^{-11}F=30{,}4\,pF$$

(b) If the applied voltage is U_o=10 kV then on metal plated of capacitors will be induced the same electric charge Q: $Q=C\cdot U=C_1\cdot U_1=C_2\cdot U_2=C_3\cdot U_3$. The strength of electric field in each capacitor is given by the relationships

$$E_1=\frac{C\cdot U_0}{h_1\cdot C_1};E_2=\frac{C\cdot U_0}{h_2\cdot C_2};E_3=\frac{C\cdot U_0}{h_3\cdot C_3};\rightarrow E_1=56\cdot 10^3 V/m;E_2=34\cdot 10^3 V/m;E_3=0{,}93\cdot 10^3 V/m.$$

3. The equivalent electric circuit, which consists of 5 capacitors, is shown in Fig. S9.3.

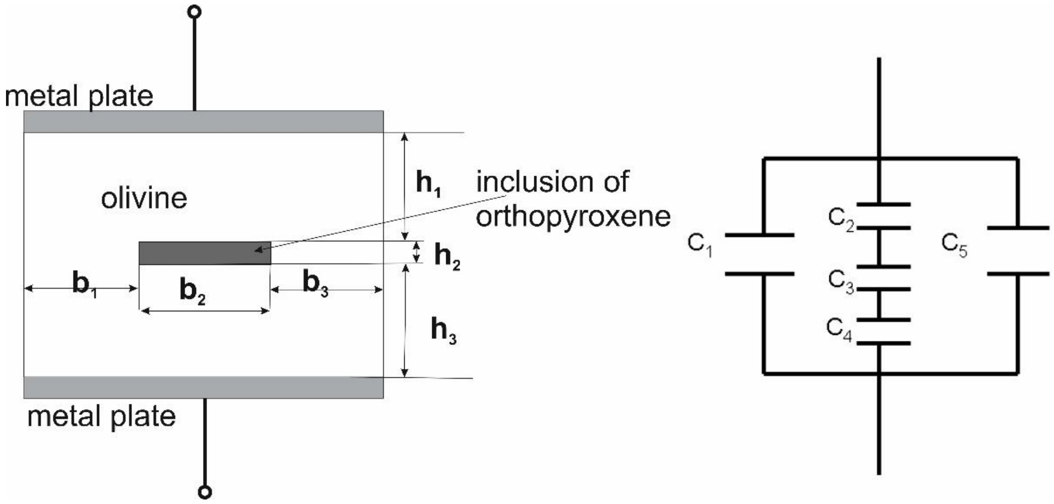


Fig. S9.3 Problem 3.

Each capacitor may be calculated from ist geometry:

$$C_1=\varepsilon_0\varepsilon_1\frac{b_1\cdot t}{(h_1+h_2+h_3)}=C_5=\varepsilon_0\varepsilon_1\frac{b_3\cdot t}{(h_1+h_2+h_3)}=3{,}73\,pF$$

$$C_2=\varepsilon_0\varepsilon_1\frac{b_2\cdot t}{h_1}=C_4=\varepsilon_0\varepsilon_1\frac{b_2\cdot t}{h_3}=5{,}08\,pF$$

$$C_3=\varepsilon_0\varepsilon_2\frac{b_2\cdot t}{h_2}=240{,}72\,pF$$

The total capacity oft he equivalent circuit is given by:

$$C = C_1 + \left(\frac{1}{C_2} + \frac{1}{C_3} + \frac{1}{C_4} \right)^{-1} + C_5 = 9{,}97\,pF\,.$$

4. (a) The total radiation power L coming from the Sun at a distance *r* equals the radiant power density (s) times the surface area of the sphere $4\pi \cdot r^2$:

$L = s \cdot 4\pi r^2 = 3{,}93 \cdot 10^{26} W\,.$

(b) The radiation power of the Sun per unit of its surface area is the intensity of electro-magnetic radiation at solar surface I_{el} : $I_{el} = \dfrac{L}{4\pi R_S^2} = 6{,}38 \cdot 10^7 W/m^2$. The electro-magnetic waves propagate with the speed of light *c*. As it can be seen in Fig. S9.4a) per surface area *A* over the time Δt the Sun emitted the energy *W*: $W = Volume \times energy\ density = V \cdot w_{el} = A \cdot c \cdot \Delta t \cdot w_{el}$, where w_{el} is the emitted energy density. The mean energy density of electro-magnetic waves is given by: $w_{el} = \dfrac{1}{2}\varepsilon_0 E^2 = \dfrac{1}{2}\mu_0 H^2$. By its definition the intensity of electro-magnetic waves on the Sun surface is the emitted energy per area A and per time Dt: $W = A \cdot \Delta t \cdot$Ie , or $I_{el} = \dfrac{Ac\Delta t \cdot w_{el}}{A\Delta t} = \dfrac{1}{2}c\varepsilon_0 E^2 = \dfrac{1}{2}c\mu_0 H^2$ where ε_0=8.85x10^{-12} *F/m* and μ_0=1.257x10^{-6} *V·s/A·m*. The mean radiation intensity I_{el} on the Sun surface defines the strength of electric and magnetic fields. Resolving these two relationships relative to E and H, one obtains: $E = \sqrt{\dfrac{2 \cdot 6{,}38 \cdot 10^7}{8{,}85 \cdot 10^{-12} \cdot 3 \cdot 10^8}} = 2{,}19 \cdot 10^5\ V/m;$

$H = \sqrt{\dfrac{2 \cdot 6{,}38 \cdot 10^7}{1{,}257 \cdot 10^{-6} \cdot 3 \cdot 10^8}} = 582\ \text{A/m}\,.$

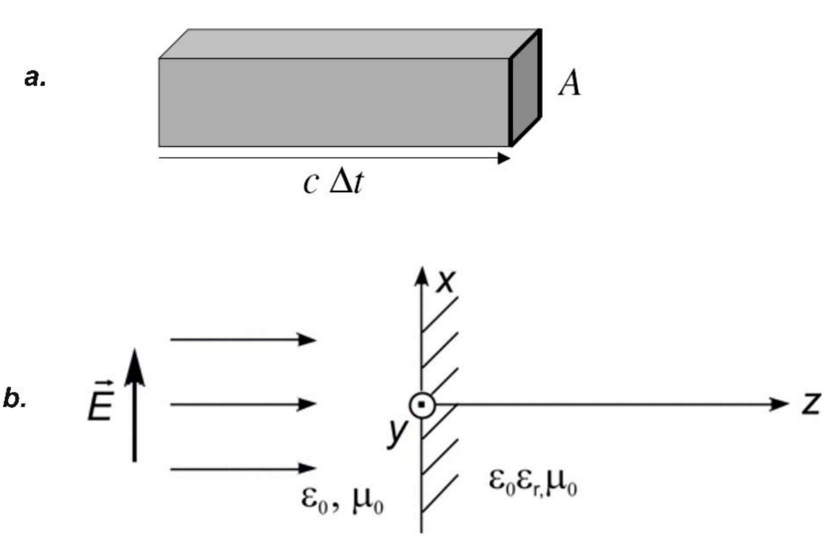


Fig. S9.4 (a) Problem 4. (b) Problem 9.

5. (a) $C = \varepsilon_0 \dfrac{A}{h} = 8{,}85 \cdot 10^{-12} \cdot \dfrac{17 \cdot 10^6}{900} = 0{,}167\ microFarad$

(b) $U_{\max} = h \cdot E_{\max} = 900 \cdot 10^6 = 9 \cdot 10^8 V$.

The maximum charge is: $Q_{\max} = C \cdot U_{\max} = 0{,}167 \cdot 10^{-6} \cdot 9 \cdot 10^8 = 1{,}5 Coulomb$

(c) Maximum electric current: $I_{\max} = \dfrac{Q_{\max}}{\Delta t} = \dfrac{1{,}5}{10^{-3}} = 1500\,A$.

(d) Maximum electric power: $W = I_{\max} \cdot U_{\max} = 1500 \cdot 9 \cdot 10^8 = 1{,}35 \cdot 10^{12} W$

6. If one assumes that the earth is a conductive sphere having the radius R_E and the surface density of electric charges ρ_E, then, for $r=R_E$:

$$E_1 = \frac{Q_E}{4\pi\varepsilon_0 \cdot R_E^2} = \frac{\rho_E}{\varepsilon_0} \to \rho_E = \varepsilon_0 E_1 = 8{,}85 \cdot 10^{-12} \cdot 300 = 2{,}6 \cdot 10^{-9} \frac{Coulomb}{m^2}.$$

The electric charges in the atmosphere Q_A compensate partially the surface charges Q_E. The volume density of electric charges in the atmosphere at the height H is: $\rho_A = \dfrac{Q_A}{4\pi R_E^2 \cdot H}$.

Then, it follows

$$E_2 = \frac{Q_E - Q_A}{4\pi\varepsilon_0 \cdot R_E^2} = E_1 - \frac{\rho_A \cdot 4\pi R_E^2 \cdot H}{4\pi\varepsilon_0 \cdot R_E^2} = E_1 - \frac{\rho_A \cdot H}{\varepsilon_0} \to \rho_A = \varepsilon_0 \frac{\left(E_1 - E_2\right)}{H} = 1{,}62 \frac{Coulomb}{m^3}$$

7. (a) The equation of electric field E indicates that the wave is propagating in the direction of x-axis.

(b) The mathematical expression of the plane wave is: $E = E_0 \cos(\omega \cdot t - k \cdot x)$. The comparison of the given expression with the plane wave equation results in:

Wave amplitude: E_0=30 V/m;

Frequency: $f = \frac{\omega}{2\pi} = 10^8\, Hz$;

Wavelength: $\lambda = \frac{2\pi}{k} = 3\, m$.

8. Phasors are the complex amplitudes of waves, i.e. phason is the complex number factor $\hat{A}$ which stands by a harmonic function $e^{i\cdot\omega\cdot t}$.

(a) Since cos (ω t + a·x)=$\frac{e^{i\cdot(\omega\cdot t + a\cdot x)} + e^{-i\cdot(\omega\cdot t + a\cdot x)}}{2}$, the phasons are $\hat{A}$ =E_o· $e^{-b\cdot y + i\cdot a\cdot x}$ and $\hat{A}$ =E_o· $e^{-b\cdot y - i\cdot a\cdot x}$.

(b) Since cos (ω·t)· cos (a ·x)=$\frac{1}{2} \cdot$ [cos (ω· t − $a \cdot$ x) − cos (ω· t + $a \cdot$ x)], and sin (ω·t) ·sin (a ·x)= $\frac{1}{2} \cdot$ [cos (ω· t − $a \cdot$ x) + cos (ω· t + $a \cdot$ x)], then, E (x, t) = E_o· cos (ω· t - a·x). The phasons are $\hat{A}$ =E_o· $e^{\pm i\cdot a\cdot x}$.

9. A homogeneous plane wave with the electric field hits perpendicular to a dielectric half-space z> 0 with the dielectric constant ε_r (see Fig. E9.4).

a) Which components of the electric and magnetic field must be continuous at the interface?

b) Calculate the phasors of the reflected wave E_r and the transmitted wave E_t!

c) Determine the EM-power transmitted into the half space z> 0 per unit area!

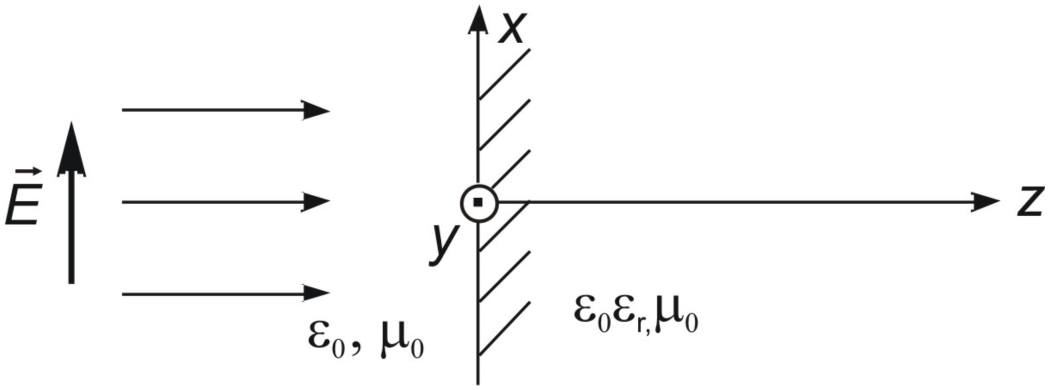


Fig. S9.4 Problem 9.

One writes for a plane wave the equation: $\vec{E}(x,y,z,t) = \hat{E} \cdot e^{i\cdot(\omega\cdot t - \vec{k}\cdot\vec{r})}$, and in the case depicted in Fig. S9.4 $\vec{k} = (0\,,0,\ k_z)$ and $\hat{E} = (\hat{E}_x, 0, 0)$. One assumes that the plane wave is propagating in a medium without losses, for which the wave vector $\vec{k}$ may be expressed a follows: $k = \omega \cdot \sqrt{\varepsilon_0 \cdot \varepsilon_r \cdot \mu_0} = \frac{2\pi}{\lambda} \cdot \sqrt{\varepsilon_r}$. For the y-component of magnetic field H, according the Maxwell equations one may write: $H_y = \frac{i}{\omega \cdot \mu_0} \cdot \frac{\partial \hat{E}_x}{\partial z}$. After differentiation one obtains the identity: $H_y = \frac{k_z}{\omega \cdot \mu_0} \cdot \hat{E}_x$.

(a) The normal components of the displacement vector $\vec{D}$ and magnetic induction vector $\vec{B}$ at the interface should be continuous as well as the tangential components of vectors $\vec{E}$ und $\vec{H}$.

(b) This means that at interface z=0 the boundary conditions are as follows:

$$\boldsymbol{E_0 - E_R = E_T}$$

$$\boldsymbol{\sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot (E_0 + E_R) = \sqrt{\frac{\varepsilon_0 \cdot \varepsilon_r}{\mu_0}} \cdot E_T}$$

, where E_0 is the electric field amplitude of the incident wave E_R and E_T are the electric field components of reflected and transmitted waves, respectively. From these two relationships the reflection and transmission coefficients are as follows:

$$\boldsymbol{\frac{E_R}{E_0} = \frac{\sqrt{\varepsilon} - 1}{\sqrt{\varepsilon} + 1}}$$

$$\boldsymbol{\frac{E_T}{E_0} = \frac{2}{\sqrt{\varepsilon} + 1}}.$$

(c) According to the definition of the Poynting vector $\vec{S}$:

$\vec{S} = \vec{E} \times \vec{H}$, which is parallel to $\vec{k}$. The averaged value of $\vec{S}$ over one period of field vector oscillation results in: $\langle \vec{S} \rangle = \frac{1}{2} \cdot c \cdot \varepsilon_0 \cdot \varepsilon_r \cdot \hat{E}_x^2$. The transmitted energy is the normal component of vector $\vec{S}$ can be expressed through E_T as follows:

$$\boldsymbol{\vec{S} = \frac{1}{2} \cdot \frac{c^2}{4\pi \cdot \omega} \cdot \frac{4}{\left(\sqrt{\varepsilon} + 1\right)^2} \cdot k_0 \cdot E_0^2 \cdot \vec{n}_z = \frac{c}{2\pi} \frac{E_0{}^2}{\left(\sqrt{\varepsilon} + 1\right)^2}}$$

10. Plugging $E = E_0 \cdot e^{-\vartheta \cdot x}$, in the Maxwell equation for the electric field: $\nabla^2 \frac{\partial E}{\partial t} = (i \cdot \omega \cdot \sigma \cdot \mu - \omega^2 \cdot \varepsilon \cdot \mu) \cdot \frac{\partial E}{\partial t}$, one obtains the identity $\vartheta^2 = i \cdot \omega \cdot \sigma \cdot \mu - \omega^2 \cdot \varepsilon \cdot \mu$. One denotes $Re[\vartheta] = a, Im[\vartheta\} = b$, then, the identity may be rewritten: $\vartheta^2 = (a + i \cdot b)^2 = i \cdot \omega \cdot \sigma \cdot \mu - \omega^2 \cdot \varepsilon \cdot \mu$, which results in two relationships for real and imaginary components of ϑ. $a^2 - b^2 = -\omega^2 \cdot \varepsilon \cdot \mu$, and $2 \cdot a \cdot b = \omega \cdot \sigma \cdot \mu$. After solving these two equations relative to a, one gets the biquadratic equation: $a^4 + \omega^2 \cdot \varepsilon \cdot \mu \cdot a^2 - \frac{\omega^2 \cdot \sigma^2 \cdot \mu^2}{4}$. The positive root of this equation is given by: $a^2 = \frac{\omega^2 \cdot \varepsilon \cdot \mu}{2} \cdot$

$\left(\sqrt{1+\frac{\sigma^2}{\omega^2\cdot\varepsilon^2}}-1\right)$. This general solution indicates the linear relationship between ϑ and frequency ω. For very conductive rocks: $\frac{\sigma}{\omega\cdot\varepsilon} \gg 1$, $\vartheta = \sqrt{\frac{\omega\cdot\mu\cdot\sigma}{2}}$. For non-conductive rocks at high frequencies: $\frac{\sigma}{\omega\cdot\varepsilon} \ll 1$ the square root may be expanded in the Taylor series, and, then, $\vartheta = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$.

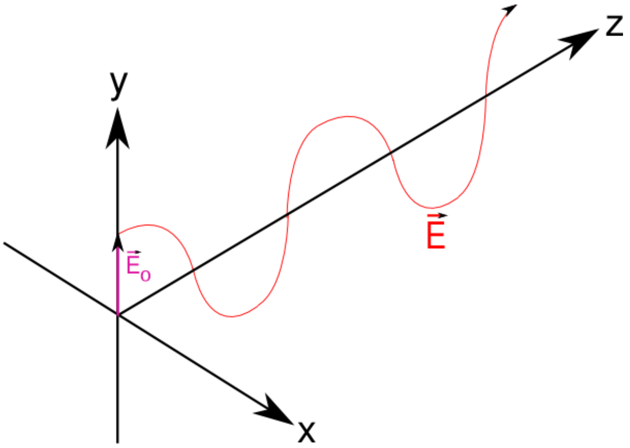


Fig. S9.5 Problem 11.

a. The maximum of the magnetic field induction is: ***$B_0=E_0/c=6.67\ \mu T$.***

b. At the indicated point the magnetic field induction is one fourth of the maximum value: ***$B(z_1)=E(z_1)/c=1.67\ \mu T$***. The direction of the EM-wave propagation vector $\vec{k}$ builds with the vectors $\vec{B}$ and $\vec{E}$ the right-handed coordinate system. After the right hand rule $B(z_1)$ is directed to the positive x-axis.

c. If the coordinate system oriented in such a way that at z=0 the induction vector is at maximum, then, the wave equation for $\vec{B}$ vector may be presented as a cosine function: ***$\vec{B}(z) = B_0\cdot\cos(k\cdot z)$.*** The wave number ***k*** is calculated from the given frequency: ν=10 MHz and EM-wave velocity c, ***$k=2\pi\nu/c$***. The point z_1 corresponds to the distance from the maximum of $\vec{B}$: **$z_1 = c/(2\pi\cdot\nu)\cdot\arccos(B(z_1)/B_0)=6{,}3$ m.**

Exercises and control questions to Chapter 10:

1.

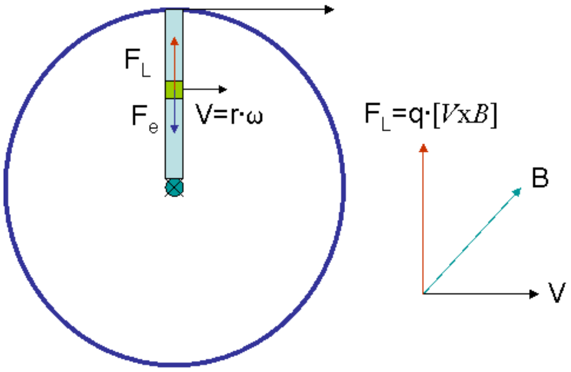


Fig. S10.1 Problem 1.

Consider the length element *dr* of the bar. There is a Lorenz force acting on the electrical charge *q*: $F_L = q\cdot[\vec{V}\times\vec{B}]$. The direction of the Lorenz force (red arrow in Fig. S10.1) is orthogonal tot he directions of charge movement V and the magnetic field induction vector B. This force acts on negative and positive charges in two opposite directions and the electric field E builds up along the length element *dr:*

$F_e = q\cdot E = q\cdot\frac{dU}{dr}$, where *dU* is the voltage difference between two ends of the length element *dr*. In steady-state these two forces are in balance, and one obtains the differential equation in respect to the voltage *U*: $q\cdot 2\pi f\cdot r\cdot B = q\cdot\frac{dU}{dr}$. After integration this identity by *r* from *0* to *L* it follows:

$$dU = 2\pi f\cdot B\cdot r\cdot dr \Rightarrow U = \pi f\cdot B\cdot L^2 = 3{,}14\cdot 60\cdot 0{,}25\cdot 4\cdot 10^{-5} \approx 1{,}9mV$$

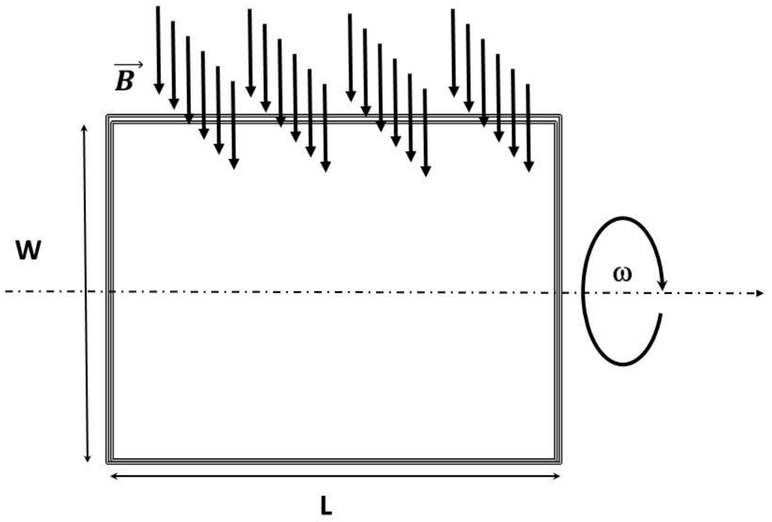


Fig. S10.2 Problem 2.

2. The rectangular coil rotates along the axis parallel to L. The surface area of the rectangular loop is $A=L\cdot x=L\cdot W\cdot\cos(\omega\cdot t)$, where x is the projection of W on the axis normal to L, and w is the rotational frequency. The magnetic flux Φ passing through one coil loop is given by: $= \iint_A \vec{B}\cdot d\vec{A}$. The magnetic field in the Frankfurt area is B≈ 49.8 μT The induced electric voltage is: $U_{ind} = \oint_C \vec{E}\cdot d\vec{r} = -\frac{d}{dt}\iint_A \vec{B}\cdot d\vec{A}$, or per one loop of wire: $U_{loop} = -\frac{d}{dt}\left(A\cdot\vec{B}\right) = -\frac{d}{dt}\left(L\cdot W\cdot cos(\omega\cdot t)\cdot\vec{B}\right)=L\cdot W\cdot\omega\cdot sin(\omega\cdot t)\cdot\vec{B}$. The peak voltage in the coil of n loops is $U_{peak} = n\cdot L\cdot W\cdot\omega\cdot\vec{B}$, and resolving in respect to ω one obtains: $\omega = \frac{U_{peak}}{n\cdot L\cdot W\cdot B} = \frac{38\cdot 10^{-3}}{280\cdot 0.112\cdot 0.078\cdot 49.8\cdot 10^{-6}} = 312\ rad/sec.$

3. The energy density of magnetic field having strength H is: $\boldsymbol{\rho_{Magn}} = \frac{1}{2}\mu_0\cdot\mu\cdot \boldsymbol{H}^2$

   The kinetic energy density is: $\boldsymbol{\rho_{Kin}} = \frac{1}{2}\cdot\frac{m_H\cdot N}{10^{-6}}\cdot \boldsymbol{V}^2$, where $m_H=1.66\cdot 10^{-27}$ kg is the mass of proton, V is the mean velocity and N is the concentration of protons. In thermal equilibrium: $\rho_{kin}=\rho_{Mag}$. Resolving the identity in respect to H one obtains the relationship:.

$$\boldsymbol{H} = \sqrt{\frac{m_H\cdot N}{\mu_0\cdot 10^{-6}}}\cdot \boldsymbol{V} = \sqrt{\frac{1{,}66\cdot 10^{-27}\cdot 1}{1{,}2566\cdot 10^{-6}\cdot 10^{-6}}}\cdot 10^3 \approx 3{,}6\cdot 10^{-5}\, A/m$$

4. If an observer moves from the latitude of Mainz ϕ_0 to the latitude of Frankfurt $\phi_0+\Delta\phi$, then, the two components of the Earth's magnetic field, i.e. $\propto sin(\phi_0+\Delta\phi)$ and $\propto cos(\phi_0+\Delta\phi)$:may be expanded in the Taylor series:

$$H_h=\frac{M\cdot cos(\phi_0+\Delta\phi)}{R^3}=\frac{M}{R^3}\left[cos\,\phi_0-sin\,\phi_0\cdot\Delta\phi-\frac{1}{2}\cdot cos\,\phi_0\cdot\Delta\phi^2+.....\right].$$

$$H_v=\frac{2M\cdot sin(\phi_0+\Delta\phi)}{R^3}=\frac{2M}{R^3}\left[sin\,\phi_0+cos\,\phi_0\cdot\Delta\phi-\frac{1}{2}\cdot sin\,\phi_0\cdot\Delta\phi^2+.....\right]$$

The small increments in magnetic field components are:

$$\Delta H_h=-\frac{M}{R^3}\left[sin\,\phi_0\cdot\Delta\phi-\frac{1}{2}\cdot cos\,\phi_0\cdot\Delta\phi^2+.....\right]$$

$$\Delta H_v=\frac{2M}{R^3}\left[cos\,\phi_0\cdot\Delta\phi-\frac{1}{2}\cdot sin\,\phi_0\cdot\Delta\phi^2+.....\right]$$. The relative increments are:

$$\frac{\Delta H_h}{H_h}=-\left[\frac{sin\,\phi_0}{cos\,\phi_0}\cdot\Delta\phi-\frac{1}{2}\cdot\Delta\phi^2+.....\right]$$ und $$\frac{\Delta H_v}{H_v}=\left[\frac{cos\,\phi_0}{sin\,\phi_0}\cdot\Delta\phi-\frac{1}{2}\cdot\Delta\phi^2+.....\right]$$. The difference in latitude between Main and Frankfurt is:

$$\Delta\phi=10'54{,}14''=2\pi\left[\frac{10}{360\cdot60}+\frac{54{,}14}{360\cdot60\cdot60}\right]\approx3{,}17\cdot10^{-3}\,rad$$. Taking into account that

$\tan\phi_0\approx1.19$, then, $\frac{\Delta H_h}{H_h}=-\left[\frac{sin\,\phi_0}{cos\,\phi_0}\cdot\Delta\phi.\right]\approx-0{,}38\%;\ \frac{\Delta H_v}{H_v}=\left[\frac{cos\,\phi_0}{sin\,\phi_0}\cdot\Delta\phi.\right]\approx+0{,}27\%$

5. The specific saturation magnetization of Ni is given by: $M_S=N\cdot\mu_{Ni}$, where N is the number of Ni atoms per unit volume, and μ_{Ni} is the magnetic dipole moment of a single Ni-atom. $N=N_A\cdot\frac{\rho}{m_{Ni}}$, where N_a is the Avogadro number:

$$\mu_{Ni}=\frac{M_S\cdot m_{Ni}}{N_A\cdot\rho_{Ni}}=\frac{470\cdot10^3\cdot58{,}71}{6{,}022\cdot10^{23}\cdot8{,}9\cdot10^6}=5{,}15\cdot10^{-24}A\cdot m^2\approx0{,}56\mu_B$$

6. The molar magnetic susceptibility χ_m according to the definition is as follows:

$$\chi_M=\chi\cdot V_m=\chi\cdot\frac{m}{\rho}=-7{,}2\cdot10^{-7}\frac{62}{2{,}55\cdot10^6}=-1{,}75\cdot10^{-11}m^3/mol$$

7. Assuming that in rock the atoms possess only $+\mu_B$ and $-\mu_B$ magnetic dipole moments, then one introduces the variable<. $x=\frac{\mu_B\cdot B}{kT}$. The magnetization of rock material is:

$$M = \underbrace{\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)}_{x\to\infty} \cdot \overbrace{N\cdot\mu_B}^{M_S} \approx \overbrace{N\cdot\mu_B}^{M_S} = \frac{N_A\cdot\rho}{m}\cdot\mu_B = \frac{6{,}022\cdot10^{23}\cdot2{,}6\cdot10^{6}}{60}\cdot9{,}27\cdot10^{-24} = 242kA/m$$

$$\chi = \frac{N\cdot{\mu_B}^2\mu_0}{kT} = \frac{M_S\cdot\mu_B\cdot\mu_0}{kT} = \frac{242\cdot10^3\cdot9{,}27\cdot10^{-24}\cdot1{,}257\cdot10^{-6}}{1{,}38\cdot10^{-23}\cdot293} = 7\cdot10^{-4}(SI).$$

If one assumes from magnetic induction B independent magnetic susceptibility χ, then

$$B = \mu_0\cdot H \approx \frac{\mu_0\cdot M_S}{\chi} = \frac{1{,}257\cdot10^{-6}}{7\cdot10^{-4}}\cdot242\cdot10^3 = 4{,}35Tesla$$

In ordert to get 96,4% of saturation magnatization at T=293K, one has to plug in the identity for *M*: $x \approx 2 = \frac{\mu_B\cdot B}{k\cdot T} \Rightarrow B = \frac{2k\cdot T}{\mu_B} = \frac{2\cdot1{,}38\cdot10^{-23}\cdot293}{9{,}27\cdot10^{-24}} = 87Tesla$

8. Consider a magnetic sample in a form of cylinder of length *l* with the cross section area *A* hanging in magnetic field with the vertical gradient: *dB/dx*. In the gradient magnetic field there is a vertical force F_x acting at the element volume *dV* in *x*-axis direction: $F_x = \frac{dU}{dx}; dF_x = M\cdot dV\cdot\frac{dB}{dx}$

Plugging *dV=A·dx* into this identity, one obtains:

$dF_x = A\cdot dx\cdot M\cdot\frac{dB}{dx} = \underbrace{A\cdot dx}_{dV}\cdot\frac{\chi}{\mu_0}\cdot B\cdot\frac{dB}{dx}$. The total force F_x can be calculated by integration over the length of specimen from x_1 to x2 (see Fig. S10.3):

$F = \frac{A\cdot\chi}{\mu_0}\int_{X_1}^{X_2} B\cdot dB = \frac{A\cdot\chi}{\mu_0}\left(B_1^2 - B_2^2\right)$, where B_1 and B_2 is the magnetic field at points x1 and x2, respectively. For a sufficiently long sample x2 will be far outside the centre of magnetic field, so $B_2 << B1$, and finally, one obtains the identity: $F = \frac{1}{2\mu_0}A\cdot\chi\cdot B^2$. The balance is measuring the weight difference between zero magnetic field and B, $\Delta m\cdot g$. Paramagnetic samples (χ>0) the additional vertical force acts downward of *x*-axis, i.e. the sample is forced into the magnetic field and $\Delta m\cdot g > 0$. For diamagnetic samples the additional force acts upward and the sample is pushed out of the magnetic field. For a constant gradient magnetic field one may write::

$F \approx V\frac{\chi}{\mu_0}B\cdot\frac{dB}{dx} = 10^{-6}\frac{10^{-5}}{1{,}257\cdot10^{-6}}\cdot1\cdot100 = 0{,}8\cdot10^{-3}N$, which corresponds to c. 0.08g of the weight increment.

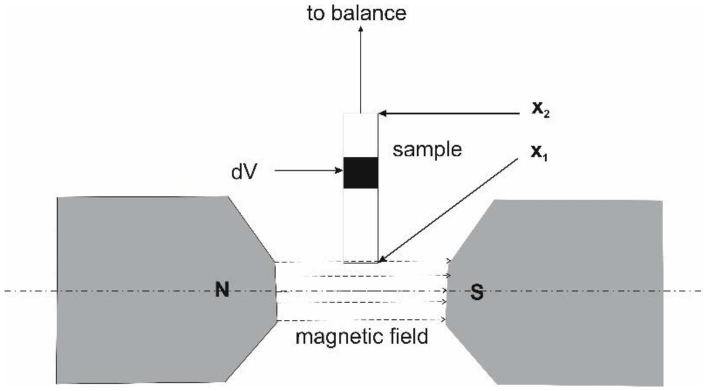


Fig. S10.3 Problem 8.

9. The temperature at depth z is given by: $\boldsymbol{T(z) = 20°C + z(km) * 30}$. So, a magnetite crystal reaches the Curie temperature at depth Z: $\boldsymbol{Z = \frac{580-20}{30} \approx 18{,}7km}$, for a pyrrhotite crystal: $\boldsymbol{Z = \frac{325-20}{30} \approx 10{,}2km}$.

10. The magnetic field of the earth is to a large degree the dipole field. So with the depth the magnetic field varies as $\vec{H}_{Earth}(d) \propto \frac{1}{(R-d)^3}$, where R is the radius of the Earth. For the basalt cube located at the depth *d* the magnetic field is: $\vec{H}_{Earth}(d) = \vec{H}_{Earth,0} \cdot \frac{R^3}{(R-d)^3} \approx \vec{H}_{Earth,0} \cdot \left(1 + 3 \cdot \frac{d}{R}\right)$. When *d*=5 m the effect is of depth is negligible. The magnetic dipole moment is: $\vec{m} = \vec{M} \cdot V$, and the basalt cube magnetization is given by: $\vec{M} = \chi_{mag} \cdot \vec{H}_{Earth,0}$, The cube volume is $V=l^3$, plugging in $\vec{m}$: $\vec{m} = \chi_m \cdot V \cdot \vec{H}_{Earth} = 0.03 \cdot 125 \cdot \vec{H}_{Earth}$=3.75$\cdot \vec{H}_{Earth,0}$. At the North magnetic pole $\vec{H}_{Earth,0} \approx 65\mu T$, then $\vec{m} \approx 2.4 \cdot 10^{-4}$ A·m²/kg. The magnetic dipole moment of basalt cube contributes to the additional magnetic field on the surface: $\vec{H}_B = \vec{m} \cdot \frac{1}{2\pi \cdot d^3} \approx 0.005 \cdot \vec{H}_{Earth,0}$. The total field on the surface is $\vec{H}_B + \vec{H}_{Earth,0} = 1.005 \cdot 65\mu T$.

11. For the saturation magnetization one may write the identity: $M_S = \frac{N}{V} \cdot \mu_B$. The factor $\frac{N}{V}$ is given by: $\frac{N}{V} = \frac{N_A \cdot \rho}{m_{mol}} = 8,5 \cdot 10^{28} \frac{1}{m^3}$. Plugging this into M_S one obtains: $M_S = 7,9 \cdot 10^5 \frac{A}{m}$. The saturation magnetization of iron in the magnetic field B is: $M_S = \frac{B}{\mu_0} = \frac{2{,}1T}{\mu_0} = 1,7 \cdot 10^6 \frac{A}{m}$. For the magnetic moment of iron μ in the magnetic field B one may write: $\mu = \frac{M_S \cdot V}{N} = 1,96 \cdot 10^{-23} \frac{T \cdot A}{m}$. The factor $\frac{N}{V} = 8.5 \cdot 10^{28}\ m^{-3}$ here is the same. The ratio between the magnetic moment of iron atom μ and the Bohr's magneton μ_B is: $\frac{\mu}{\mu_B} \approx 2,12$.

Solutions to Chapter 11:

1.

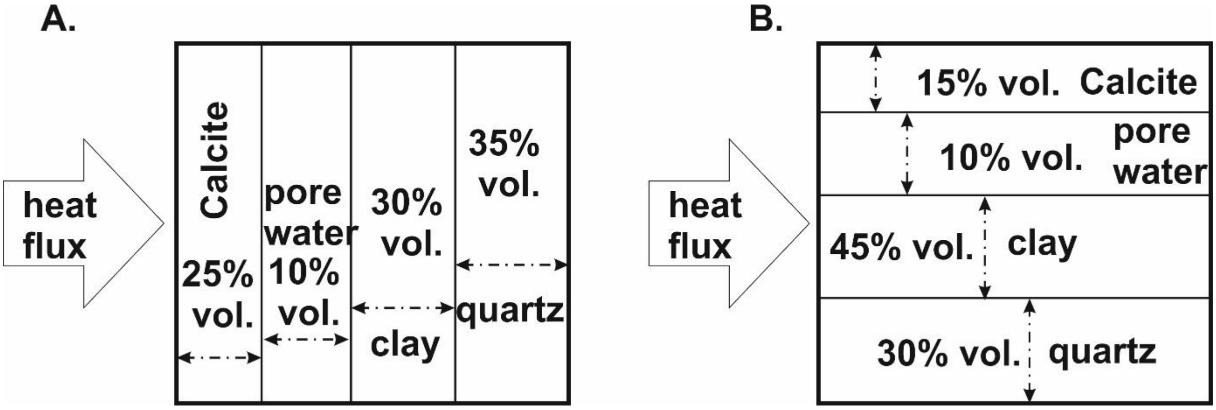


Fig. S11 .1 Problem 1.

Solution in the case A: If the heat flux is in the direction normal to mineral interfaces, the, the heat flux passing through each phase is the same:

$\frac{\Delta T}{L}=\frac{\Delta T_1+\Delta T_2+...}{L}=\frac{\Delta T_1}{\Delta x_1}\cdot\overbrace{\frac{\Delta x_1}{L}}^{\varphi_1}+\frac{\Delta T_2}{\Delta x_2}\cdot\overbrace{\frac{\Delta x_2}{L}}^{\varphi_2}+...$ Since the heat flux is the same passing through is constant and equals Q, one may write: $Q=-\lambda\frac{\Delta T}{L}=-\lambda_1\cdot\frac{\Delta T_1}{\Delta x_1}=-\lambda_2\cdot\frac{\Delta T_2}{\Delta x_2}=...$It follows that: $\frac{Q}{\lambda}=\frac{Q}{\lambda_1}\cdot\varphi_1+\frac{Q}{\lambda_2}\cdot\varphi_2+...\Rightarrow\frac{1}{\lambda}=\frac{1}{\lambda_1}\cdot\varphi_1+\frac{1}{\lambda_2}\cdot\varphi_2+.....$

$\frac{1}{\lambda}=\frac{0,25}{3,3}+\frac{0,30}{0,63}+\frac{0,25}{7,1}+\frac{0,20}{2,3}=0,674\Rightarrow\lambda=\frac{1}{0,674}$ λ=1,48 W/m/K

Solution in the case B: If the heat flux is in the direction parallel tot he mineral interfaces, then, the temperature gradient ist he same in each layer of mineral phases, because the temperature difference and the length of each phase are the same:

$$Q=\lambda\cdot\frac{\Delta T}{L}=Q_1+Q_2+....=\varphi_1\cdot\lambda_1\cdot\frac{\Delta T}{L}+\varphi_2\cdot\lambda_2\cdot\frac{\Delta T}{L}+...\Rightarrow\lambda=\varphi_1\cdot\lambda_1+\varphi_2\cdot\lambda_2+.$$

λ=0,2·2,3+0,3·0,63+0,3·7,1+0,2·3,3=3,44 W/m/K.

2. One denotes the total surface area of the black body by *A*. The absorbed radiation power is given by: $\dot{Q}_a=S\cdot A$, where *S =1.3608 ± 0.0005 kW/m²* is the solar constant . The emitted radiation power is: $\dot{Q}_e=2A\cdot\sigma T^4$, where σ=5,67·10^{-8} W/m²/K^4 is the Stefan-Boltzmann constant. The factor 2 is due to face and rear surfaces *A*. In thermal equilibrium two radiation powers are in balance: $S\cdot A=2A\cdot\sigma T^4\Rightarrow T=\sqrt[4]{\frac{S}{2\sigma}}=\sqrt[4]{\frac{1380}{2\cdot 5,67\cdot 10^{-8}}}=332\,K$

3. The Lorenz constant *L* is calculated and listed in Table below:

Minerals	K [W/K·cm]	ρ, [Ω·cm]	L, W·Ω/K²
Diamond	23.2	10^{16}	$7.7\cdot10^{14}$
Quartz	7.0	10^{15}	$2.3\cdot10^{13}$
Forsterite	5.15	10^{7}	$1.7\cdot10^{5}$
Dolomite	3.5	10^{10}	$1.2\cdot10^{8}$
Diopside	5.76	10^{12}	$1.9\cdot10^{10}$
Hornblende	2.54	10^{14}	$8.5\cdot10^{11}$
Graphite	4.5	10^{-3}	$1.5\cdot10^{-5}$

4. The heat flux trough the rock specimen is given by:

$$\frac{\dot{Q}}{S} = -\lambda\frac{(0^\circ - T_1)}{L} \Rightarrow T_1 = \frac{L\cdot\dot{Q}}{S\cdot\lambda} = \frac{3\cdot 0{,}1}{50\cdot10^{-4}\cdot 2{,}4} = 25^\circ C$$

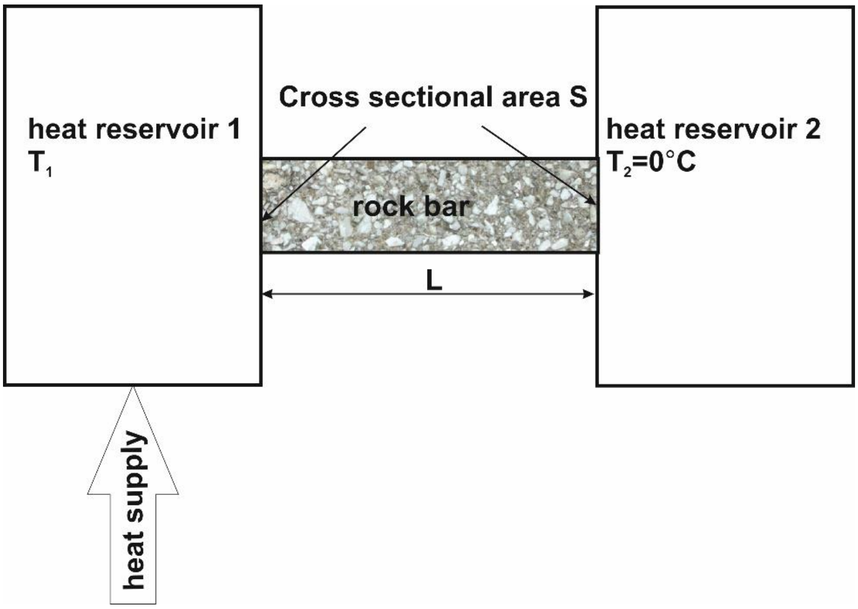


Fig. S11.2 Problem 4.

5. (a) The heat exchange between basaltic rock specimen and water is given by the equation: :

$$m_{Basalt}\cdot C_p \cdot(95^\circ - 22{,}5^\circ C) = m_{wasser}4{,}19\cdot10^3\cdot(22{,}5-20) \Rightarrow C_p = \frac{4{,}19\cdot10^3\cdot2{,}5}{72{,}5} = 144\,J/kg/K$$

The molar heat capacity may be expressed via mean atomic weight of basalt:

$$C_n = C_p\cdot A_r = 144\cdot62\cdot10^{-3} = 8{,}92\,J/(mol*K)$$

The Dulong-Petit law states, that the molar heat capacity of a monoatomic solid is the universal constant equals 3 times universal gas constant R: $C_n = 3 * R$ = 3 * 8,3 J/(mol*K) = 24,9 J/(mol*K).

(b) When the thermal elongation of the basaltic specimen is given, then the needed temperature increase may be calculated as follows:

$$0{,}01 = 3\alpha \cdot \Delta T \Rightarrow \Delta T = \frac{0{,}01}{3 \cdot 5{,}5 \cdot 10^{-5}} = 60{,}7° .$$

6.

Magnesite	*Calcite*
$\kappa\,[001]$ = 6,55 mm^2/s	$\kappa\,[001]$ = 2,02 mm^2/s
$\kappa\,[100]$ = 2,36 mm^2/s	$\kappa\,[100]$ = 1,82 mm^2/s
$\kappa\,[010]$ = 2,39 mm^2/s	$\kappa\,[010]$ = 1,85 mm^2/s

The average value of κ in three crystallographic axis directions [001], [010], and [100] is determined from the equations as follows:

for magnesite:

$$\tilde{\kappa}_{Voigt} = \frac{1}{3} \cdot 6{,}55 + \frac{1}{3} \cdot 2{,}36 + \frac{1}{3} \cdot 2{,}39 = 3{,}77 \cdot 10^{-6}\, m^2/s$$

$$\tilde{\kappa}_{\mathrm{Re}uss} = \left(\frac{1}{3} \cdot \frac{1}{6{,}55} + \frac{1}{3} \cdot \frac{1}{2{,}36} + \frac{1}{3} \cdot \frac{1}{2{,}39} \right)^{-1} = 3{,}02 \cdot 10^{-6}\, m^2/s$$

$$\tilde{\kappa}_{Hill} = \frac{1}{2} \cdot \left(\tilde{\kappa}_{Voigt} + \tilde{\kappa}_{\mathrm{Re}uss} \right) = 3{,}4 \cdot 10^{-6}\, m^2/s$$

and for calcite:

$$\tilde{\kappa}_{Voigt} = \frac{1}{3} \cdot 2{,}02 + \frac{1}{3} \cdot 1{,}82 + \frac{1}{3} \cdot 1{,}85 = 1{,}90 \cdot 10^{-6}\, m^2/s$$

$$\tilde{\kappa}_{\mathrm{Re}uss} = \left(\frac{1}{3} \cdot \frac{1}{2{,}02} + \frac{1}{3} \cdot \frac{1}{1{,}82} + \frac{1}{3} \cdot \frac{1}{1{,}85} \right)^{-1} = 1{,}89 \cdot 10^{-6}\, m^2/s$$

$$\tilde{\kappa}_{Hill} = \frac{1}{2} \cdot \left(\tilde{\kappa}_{Voigt} + \tilde{\kappa}_{\mathrm{Re}uss} \right) = 1{,}9 \cdot 10^{-6}\, m^2/s$$

7. The heat flux equation provides:

$$Q = -\lambda \cdot \frac{dT}{dx} \Rightarrow \lambda = \frac{Q \cdot L}{\Delta T} = \frac{40 \cdot 10^{-3} \cdot 5 \cdot 10^{3}}{200} = 1 W/m/°K$$

8. The heat flux of magma chamber having volume V by cooling from the temperature T_0=1100°C to T_m are $Q_1 = \rho \cdot V \cdot C_p \cdot (1100 - T_m)$. The heat flux over 1 year through the surface area *A* in the case (a) is $Q_2 = t \cdot \Lambda \cdot A \cdot \left(\frac{1100+T_m}{2} - 300\right)$. In the case (b) $Q_2 = t \cdot \Lambda \cdot A \cdot \left(\frac{1100+T_m}{2} - \frac{500+100}{2}\right)$. The heat losses and the total heat over 1 year time should be equal: $Q_1 = Q_2$. From this identity the temperature T_m can be calculated: T_m=821°C

9. On the surface of a granite spherical body there is a constant heat flux q = 50 mW/m². Calculate the radius of sphere considering the heat production in granite rocks *A* = 6· 10^{-6} W/m³ and the granite density 2.65 g/cm³.

The specific heat production in granite per unit volume due to radioactive nuclides is: A[W/m³]≈$1{,}03 \cdot 10^{-9}$ [W/kg]·$2{,}65 \cdot 10^3$[kg/m³]= $2{,}73 \cdot 10^{-6}$ W/m³. The total heat production *A** per second in the granite sphere of radius *R* is given by: $A^* = A \cdot 4/3 \pi R^3$.

Through the outer surface of the sphere per one second there are heat losses: $Q = q \cdot 4\pi R^2$. In balance: $Q = A^*$, or $R = 3 \cdot q/A = 150 \cdot 10^{-3}/2{,}73 \cdot 10^{-6}$=40.214,5 m≈40,2 km

10. In order to solve this problem, one considers a general case, when fluid with the temperature *T* flows in a long channel, the walls of which are set at constant temperature T_0 (see Fig. S11.3). The solution of the problem is to determine how the temperature of the fluid varies with the flow distance *x* in the channel, *T (*x), when *by x = 0* the inlet temperature is T_1 and by x = L the outlet fluid temperature is T_2. An expected variation of temperature *T(x)* along the channel is shown in the upper part of Fig. S11.3. When the fluid is cooled due to heat exchange with channel walls ($T_0 < T$), then, the heat flux from the channel wall of length dx is given by: $\dot{q}\cdot \pi D\cdot dx = h\cdot \pi D\cdot \left(T - T_0\right)\cdot dx$, where D is the cylindrical channel diameter. The heat, which has been brought in the channel element of length dx is:

$$\dot{m}\cdot C_p \cdot dT = \rho \cdot \frac{\pi D^2}{4}\cdot \overbrace{\frac{dx}{dt}}^{V}\cdot C_p \cdot \overbrace{dT}^{<0}$$

, where *m* is the mass of fluid in the channel element of length *x*, ρ is the fluid density, *V* is the mean velocity of fluid flow, C_p is the heat capacity. At small Nusselt-numbers the heat losses of fluid and heat flux through the wall should be equal. In the integral form it corresponds to the identity:

$$\int_{T_1}^{T}\frac{dT}{T_0 - T} = \int_0^x \frac{4h}{\rho\cdot V\cdot C_p\cdot D}dx$$

, after integration, and taking into account that at x = 0, T(0) = T_1, it follows:

$$-\frac{4hx}{\rho V C_p D} = \int_{T_1}^{T}\frac{dT}{T - T_0} = \ln\left(T - T_0\right)\Big|_{T_1}^{T} = \ln\left(\frac{T - T_0}{T_1 - T_0}\right)$$

. This is the temperature variation of fluid along the channel. The outlet temperature T_2 at *x = L* is given by:

$$T_2 = T_0 + (T_1 - T_0)\cdot e^{-\frac{4hL}{\rho V C_p D}} = 600 + 400\cdot e^{-\frac{40\cdot 1000}{2650\cdot 1\cdot 800\cdot 0{,}02}} = 756°C$$

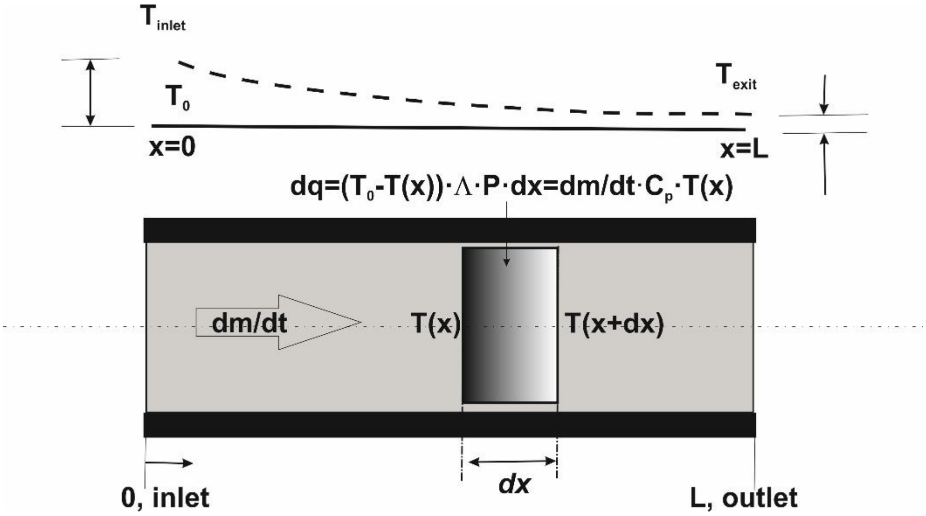


Fig. S11.3 Problem 10.

Solutions of problems to Chapter 12:

1. The age of the rock may be calculated using the equation 12.7

$$t = \frac{1}{\lambda} \cdot ln \left\{ \frac{\left({}^{87}Sr / {}^{86}Sr \right)_t - \left({}^{87}Sr / {}^{86}Sr \right)_{t=0}}{\left({}^{87}Rb / {}^{86}Sr \right)_t} + 1 \right\}$$

From the graphic on Fig. 12.15 one can determine a slope of the straight line

$$\frac{\left({}^{87}Sr / {}^{86}Sr \right)_t - \left({}^{87}Sr / {}^{86}Sr \right)_{t=0}}{\left({}^{87}Rb / {}^{86}Sr \right)_t} = \frac{0.717-0.707}{0.41} = 24.39 \cdot 10^{-3}$$

$$\lambda = \frac{\ln(2)}{\tau_{1/2}} = 1.41 \cdot 10^{-11} a$$

. Then the age is as follows

$$t_{age} = \frac{\ln(1+24.39 \cdot 10^{-3})}{1.41 \cdot 10^{-11}} = 1.71\ Ga$$

2. The solution of this problems is based on the tabular constants of heat production of the three main radioactive elements on rocks U (235 and 238), Th (232) and K (40):

$$\begin{aligned} A^* &= \left(A^*_{235_U} \cdot C_{235_U} + A^*_{238_U} \cdot C_{238_U} + A^*_{232_{Th}} \cdot C_{232_{Th}} + A^*_{40_K} \cdot C_{40_K}\right) = \\ &= (56 \cdot 10^{-5} \cdot 0.7 \cdot 10^{-2} \cdot 4 \cdot 10^{-6})_{from\ 235_U} + (9.6 \cdot 10^{-5} \cdot 99.3 \cdot 10^{-2} \cdot 4 \cdot 10^{-6})_{from\ 238_U} + \\ &+ (2.6 \cdot 10^{-5} \cdot 20 \cdot 10^{-6})_{from\ 232_{Th}} + (2.8 \cdot 10^{-5} \cdot 0.01 \cdot 10^{-2} \cdot 4 \cdot 10^{-2})_{from\ 40_K} = \\ &= 1.03 \cdot 10^{-9} Watt/kg \end{aligned}$$

3. By plotting the disintegration chain reaction of ^{238}U αββαααααββαββα on a graph p-n (number of protons vs number of neutrons we get the chain as follows

$${}^{238}_{92}U \overset{\alpha}{\Rightarrow} {}^{234}_{90}Th \overset{\beta}{\Rightarrow} {}^{234}_{91}Pr \overset{\beta}{\Rightarrow} {}^{234}_{92}U \overset{\alpha}{\Rightarrow} {}^{230}_{90}Th \overset{\alpha}{\Rightarrow} {}^{226}_{88}Ra \overset{\alpha}{\Rightarrow} {}^{222}_{86}Rn \overset{\alpha}{\Rightarrow} {}^{218}_{84}Po \overset{\alpha}{\Rightarrow} {}^{214}_{82}Pb \overset{\beta}{\Rightarrow} {}^{214}_{83}Bi \overset{\beta}{\Rightarrow} {}^{214}_{84}Po \overset{\alpha}{\Rightarrow} {}^{210}_{82}Pb \overset{\beta}{\Rightarrow} {}^{210}_{83}Bi \overset{\beta}{\Rightarrow} {}^{210}_{84}Po \overset{\alpha}{\Rightarrow} {}^{206}_{82}Pb$$

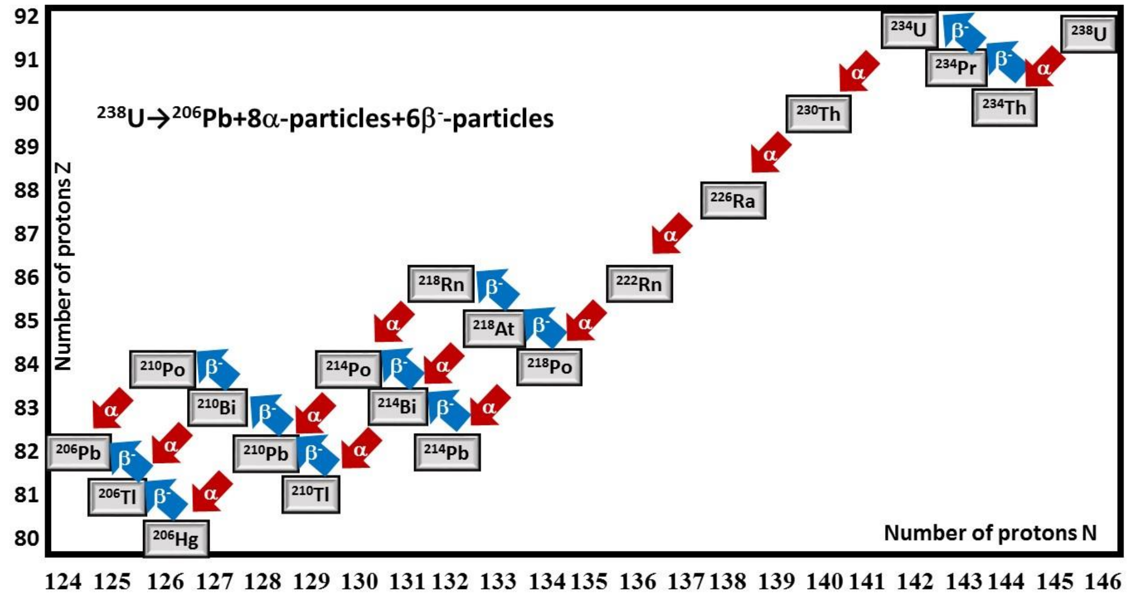


Fig. S12.1 To the graphical solution of problem 3.

4. The half-life time $t_{1/2}$ refers to the lifetime of a decay product τ for ${}^{238}_{92}U$ through the relationship: $\boldsymbol{t}_{\frac{1}{2}} = \mathbf{ln}(2) \cdot \tau \to \tau = \frac{\boldsymbol{t}_{\frac{1}{2}}}{\mathbf{ln}(2)} = \frac{4{,}5 \cdot 10^{9}}{0{,}693} = 6{,}49 \cdot 10^{9}\,\boldsymbol{years}$

5. Two disintegration equations for uranium isotopes are given by: $\begin{array}{l} N_{238}(t) = N_{238}(0) \cdot e^{-\lambda_{238} \cdot t} \\ N_{235}(t) = N_{235}(0) \cdot e^{-\lambda_{235} \cdot t} \end{array}$

We are looking for time t, at which the proportion between two isotopes is given by: $\frac{N_{238}(t)}{N_{235}(t)} = \frac{99{,}28}{0{,}72} = 137{,}89$. Assuming that the abundancies of two isotopes at the beginning were about the same $N_{238}(0) = N_{235}(0)$, one gets:

$$137{,}89 = \frac{\boldsymbol{e}^{-\lambda_{238} \cdot \boldsymbol{t}}}{\boldsymbol{e}^{-\lambda_{235} \cdot \boldsymbol{t}}} = \boldsymbol{e}^{(\lambda_{235} - \lambda_{238}) \cdot \boldsymbol{t}} \Rightarrow \boldsymbol{t} = \frac{\mathbf{ln}(137{,}89)}{\lambda_{235} - \lambda_{238}} = \frac{4{,}93}{\frac{1}{\tau_{235}} - \frac{\mathbf{ln}(2)}{\boldsymbol{t}_{238,\frac{1}{2}}}} = 5{,}93 \cdot 10^{9}\,\boldsymbol{years}$$

6. After the time ***t*** from starting *N(0)* nuclides of ${}^{238}_{92}U$ there are still remaining N (t) nuclei: $N_{238}(t) = N_{238}(0) \cdot e^{-\lambda \cdot t}$. The proportion P of non-decaying nuclei is

$P = 1 - \frac{N_{238}(t)}{N_{238}(0)} = 1 - e^{-\lambda \cdot t} = 1 - e^{-\frac{\ln(2)}{4{,}5 \cdot 10^{9}} \cdot 4{,}2 \cdot 10^{9}} = 0{,}476$. P can also be regarded as a probability of a non-disintegrated nucleus.

7. Per 200 g of living bones per 1 minute there are $\boldsymbol{A} = (15 \cdot 200) = 3000\,\boldsymbol{Min}^{-1}$ decays. After the first half-life, only half of the nuclei remained: $\frac{N(t_{\frac{1}{2}})}{N(0)} = \frac{1}{2} = e^{-\lambda \cdot t_{\frac{1}{2}}}$. The decay rate

$A(t) = \frac{dN(t)}{dt} = -\lambda \cdot N(0) \cdot e^{-\lambda \cdot t}$ changes after n half-lives as follows

$\frac{A(0)}{A(n \cdot t_{\frac{1}{2}})} = \frac{1}{e^{-n \cdot \lambda \cdot t_{\frac{1}{2}}}} = \left(\frac{1}{e^{-\lambda \cdot t_{\frac{1}{2}}}} \right)^{n} = 2^{n}$. After *n* half-lives, the decay rate has dropped to a value that is a factor *1/2^n^* lower. Which means that $(2)^{n} = \frac{3000}{400} = 7{,}5 \Rightarrow n = \frac{\ln(7{,}5)}{\ln(2)} = 2{,}91$.

The total age of the animal bones equals to $\boldsymbol{t} = \boldsymbol{n} \cdot \boldsymbol{t}_{\frac{1}{2}} = 2{,}91 \cdot 5730 = 16\,656{,}5\,\boldsymbol{years}$

8. The half-life period of ${}^{226}_{88}Ra$ is given as follows

$$\boldsymbol{t}_{\frac{1}{2}} = \frac{\mathbf{ln}(2)}{\lambda} \to \lambda = \frac{\mathbf{ln}(2)}{\boldsymbol{t}_{\frac{1}{2}}} = \frac{0{,}693}{1620 \cdot 365 \cdot 24 \cdot 60 \cdot 60} = 1{,}36 \cdot 10^{-11}\,\boldsymbol{Sec}^{-1}$$

9. The disintegration constant of ${}^{24}_{11}Na$ is $\lambda = \frac{\ln(2)}{t_{\frac{1}{2}}} = \frac{0{,}693}{14{,}8 \cdot 60 \cdot 60} = 1{,}3 \cdot 10^{-5}\,\sec^{-1}$. A probability or percentage of disintegrated nuclei is calculated below

$P\% = \left(1 - \cdot e^{-\lambda \cdot t}\right) \cdot 100\% \Rightarrow \left(1 - \frac{P\%}{100\%}\right) = e^{-\lambda \cdot t} \Rightarrow t = -\frac{\ln\left(1 - \frac{P\%}{100\%}\right)}{\lambda}$. For the given percentage of disintegration the decay time is calculated as follows: P=90%: t=49,2 hours P=99%: t=98,4 hours P=99,9%: t=147,6 hours.

10. The percentage P of non-decayed after 100 years ^{14}C nuclei in % is given by:

$$P\% = \left(1 - \cdot e^{-\lambda \cdot t}\right) \cdot 100\% = \left(1 - e^{-\frac{\ln(2)}{5730} \cdot 100}\right) \cdot 100 = 1{,}73\%$$

11. For the disintegration of ^{40}K nuclides, the decay equation is applied: $N^{40_K}(t) = N^{40_K}(0) \cdot e^{-\lambda \cdot t}$, where λ is the decay constant of ^{40}K, $\lambda = \frac{\ln(2)}{1{,}29 \cdot 10^9}$ (year^{-1}). For the decay rate of ^{40}Ar nuclides the differential equilibrium equation is applied:

$\frac{dN^{40_{Ar}}}{dt} = \lambda_{Ar} \cdot N^{40_K}(t) = \underbrace{0{,}11 \cdot \lambda}_{\lambda_{Ar}} \cdot N^{40_K}(t) = 0{,}11 \cdot \lambda \cdot \underbrace{N^{40_K}(0) \cdot e^{-\lambda \cdot t}}_{N^{40K}(t)}$. After integrating this equation, you get: $N^{40_{Ar}}(t) = const - 0{,}11 \cdot N^{40_K}(0) \cdot e^{-\lambda \cdot t}$, with an unknown constant of integration. At the moment t=0 there were no ^{40}Ar nuclei $N^{40_{Ar}}(0) = 0$. From this initial condition, the constant of integration may be determined:

$N^{40_{Ar}}(t) = 0{,}11 \cdot N^{40_K}(0) - 0{,}11 \cdot N^{40_K}(0) \cdot e^{-\lambda \cdot t} = 0{,}11 \cdot N^{40_K}(0) \cdot \left(1 - e^{-\lambda \cdot t}\right)$. From the first equation for the decay of ^{40}K nuclides we get: $N^{40_K}(0) = N^{40_K}(t) \cdot e^{\lambda \cdot t}$. After the replacement in equation *N(t)* for ^{40}Ar, we get: $N^{40_{Ar}}(t) = 0{,}11 \cdot N^{40_K}(t) \cdot \left(e^{\lambda \cdot t} - 1\right)$, or the desired numerical ratio $\frac{\boldsymbol{N}^{40\,Ar}}{\boldsymbol{N}^{40\,K}}$ results from the relationship as follows:

$$N^{40_{Ar}}(t) \Big/ N^{40_K}(t) = 0{,}11 \cdot \left(e^{\lambda \cdot t} - 1\right) = 0{,}11 \cdot \left(e^{\frac{\ln(2) \cdot 1{,}76 \cdot 10^6}{1{,}29 \cdot 10^9}} - 1\right) = 0{,}11 \cdot 0{,}000946 = 1{,}04 \cdot 10^{-4} \approx 0{,}01\%$$

12. A. The decay constant λ is given through the half-life time as follows:

$$\lambda = \frac{\ln(2)}{\tau_{1/2}} = 5.42 \cdot 10^{-10}\ \boldsymbol{years}^{-1}$$ or 1,65·10^{-17} sec^{-1}.

B. In one kilogram of rock there is 0.2·10^{-3} gram of ^{40}K. In 40 gram of ^{40}K there is 6.022 10^{23} atoms of the isotope. The number of ^{40}K nuclei in one kilo of rock is:

$$\boldsymbol{N}^{40\,K} = \frac{0.2 \cdot 10^{-3}}{40} \cdot 6.022 \cdot 10^{23} \approx 3 \cdot 10^{18}$$

C. The activity of the rock sample is equal to the decay rate of ^{40}K at the moment. So, the activity equals to: $\frac{\boldsymbol{dN}}{\boldsymbol{dt}} = \lambda \cdot \boldsymbol{N}^{40\,K} \approx 50\ \boldsymbol{Bq}$

13. If the ratio of a daughter isotope to the parent isotope is given, then we may write the relationship of two isotopes at any time presuming that at t=0 there was no daughter isotope presented: $\frac{\boldsymbol{N}^{Pb}}{\boldsymbol{N}^{U}} = \frac{\boldsymbol{N}^{U}(0) \cdot (1 - \boldsymbol{e}^{-\lambda \cdot t})}{\boldsymbol{N}^{U} \cdot \boldsymbol{e}^{-\lambda \cdot t}} = \boldsymbol{e}^{\lambda \cdot t} - 1$. From this relationship it follows that the age of the rock may be estimated as follows:

$$\boldsymbol{t} = \frac{1}{\lambda}\mathbf{ln}(15+1) = \frac{\tau_{1/2}}{\mathbf{ln}\,2}\mathbf{ln}\,16 = 4\cdot 7.038\cdot 10^{8}\ \boldsymbol{years} \approx \boldsymbol{2.815\ Gy}$$