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Chapter 6

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Supplement: Short Pulses

Another case in which the frequency spectrum of the source becomes particularly important is that of blue-green generation using short infrared pulses, such as those from mode-locked solid-state lasers or diode lasers emitting picosecond pulses. From Fourier analysis, we know that a short pulse will contain a wide spread of frequency components; thus, we cannot consider the input wave to be monochromatic.

We can analyze the case of short pulses using the same approach we applied previously, which closely follows the treatment given by Glenn[Glenn (1969)] (but see also the treatment by Ahkmanov and colleagues [Ahkmanov, et al. (1969)]). We take the Fourier transform of the input pulse to obtain its frequency-domain spectrum, convolve this spectrum with itself and multiply by the nonlinear susceptibility to obtain the frequency-domain spectrum of the nonlinear polarization. This frequency-domain expression for the nonlinear polarization becomes a driving term in the generation of the second-harmonic wave. Once we find the second-harmonic wave in the frequency domain, we can obtain the time-domain behavior by taking the inverse Fourier transform.

Suppose that our fundamental input wave consists of a pulse of the form $E_1(x,t) = A_1 f(t - x/v_{g1}) \cos(\omega_1 t - k_1 x)$, where $f(t - x/v_{g1})$ is a function giving the envelope of the applied pulse, which travels at the group velocity:

$$v_{g1} = \left(\frac{\partial k}{\partial \omega}\right)\Bigg|_{\omega=\omega_1}^{-1} = \frac{c}{n - \lambda \dfrac{\partial n}{\partial \lambda}}\Bigg|_{\lambda=\lambda_1} \tag{6.1}$$

We can calculate the group velocity from the Sellmeier equations that give the refractive index n as a function of wavelength λ, which are given in Chapter 2 of "CBGL" for a number of important nonlinear materials. We will designate the Fourier transform of

$f(t)$ as $\mathcal{F}(\omega)$. The Fourier transform of $f(t-x/v_{g1}) = f(t-\alpha x)$ is then $e^{-j\alpha\omega x}F(\omega)$, where $\alpha = 1/v_{g1}$. If we take the Fourier transform of $E_1(x,t)$, we obtain

$$\mathcal{E}_1(x,\omega) = \frac{1}{2\pi}A_1 e^{-j\alpha\omega x}\mathcal{F}(\omega)* \left[\pi e^{-jk_1 x}\delta(\omega-\omega_1) + \pi e^{jk_1 x}\delta(\omega+\omega_1)\right] \tag{6.2}$$

$$= \frac{1}{2}A_1\left[e^{-jk_1 x}e^{-j\alpha(\omega-\omega_1)x}\mathcal{F}(\omega-\omega_1) + e^{jk_1 x}e^{-j\alpha(\omega+\omega_1)x}\mathcal{F}(\omega+\omega_1)\right] \tag{6.3}$$

Convolving this spectrum with itself, we obtain (Fig. S-6-1):

$$\mathfrak{F}\{E_1(x,t)\cdot E_1(x,t)\} = \frac{1}{2\pi}\{\mathcal{E}_1(x,\omega) * \mathcal{E}_1(x,\omega)\} \tag{6.4}$$

$$= \frac{1}{2\pi}\left(\frac{A_1}{2}\right)^2\left[e^{-2jk_1 x}e^{-j\alpha(\omega-2\omega_1)x}\mathcal{H}(\omega-2\omega_1)\right. \tag{6.5}$$

$$\left. + e^{2jk_1 x}e^{-j\alpha(\omega+2\omega_1)x}\mathcal{H}(\omega+2\omega_1) + 2e^{-j\alpha\omega x}\mathcal{H}(\omega)\right] \tag{6.6}$$

We have introduced $\mathcal{H}(\omega) = \mathcal{F}(\omega) * \mathcal{F}(\omega)$. Note that the Fourier transform of $f^2(t)$ is thus $\mathcal{H}(\omega)/2\pi$.

The components of the nonlinear polarization near $\omega = \pm 2\omega_1$ contribute to generation of the second-harmonic pulse. Thus, the relevant nonlinear polarization is:

$$\mathcal{P}(x,\omega) = 2\epsilon_o d_{eff}\left(\frac{1}{2\pi}\right)\left(\frac{A_1}{2}\right)^2\left[e^{-2jk_1 x}e^{-j\alpha(\omega-2\omega_1)x}\mathcal{H}(\omega-2\omega_1)\right. \tag{6.7}$$

$$\left. + e^{2jk_1 x}e^{-j\alpha(\omega+2\omega_1)x}\mathcal{H}(\omega+2\omega_1)\right] \tag{6.8}$$

From Eq. 2.11 of "CBGL", we have

$$\frac{\partial\widetilde{\mathcal{E}}_g(x,\omega)}{\partial x} = \frac{-jk_g(\omega)}{2\epsilon_o\epsilon_{zz}^{(1)}}\mathcal{P}_{NL}(x,\omega)e^{jk_g(\omega)x} \tag{6.9}$$

$$= \frac{-jd_{eff}}{n_3^2}\left(\frac{A_1^2}{8\pi}\right)\left[k_g(\omega)e^{jk_g(\omega)x}e^{-2jk_1 x}e^{-j\alpha(\omega-2\omega_1)x}\mathcal{H}(\omega-2\omega_1)\right. \tag{6.10}$$

$$\left. + k_g(\omega)e^{jk_g(\omega)x}e^{2jk_1 x}e^{-j\alpha(\omega+2\omega_1)x}\mathcal{H}(\omega+2\omega_1)\right] \tag{6.11}$$

In order to simplify the expression further, we expand $k_g(\omega)$ near $\pm 2\omega_1$ (Fig. S-6-2).

Near $+2\omega_1$, we may write $k_g(\omega) = k_g(2\omega_1) + \beta(\omega-2\omega_1) = k_3 + \beta(\omega-2\omega_1)$and near $-2\omega_1$, we may write $k_g(-\omega) = -k_3 + \beta(\omega+2\omega_1)$ where $\beta = \left.\frac{\partial k_g}{\partial\omega}\right|_{\omega=2\omega_1} = \frac{1}{v_{g2}}$, where

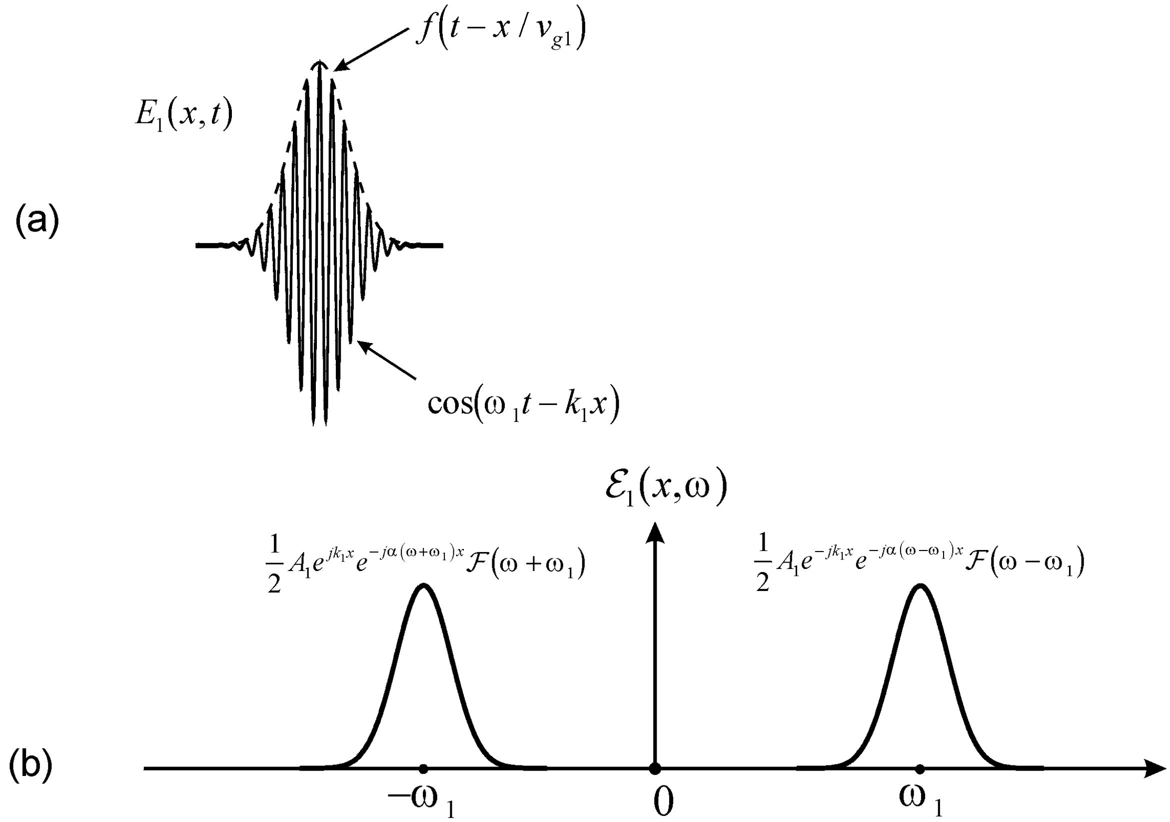

Figure 6.1: Figure S-6-1. (a) Time-domain representation of a short pulse. (b) Corresponding frequency-domain representation

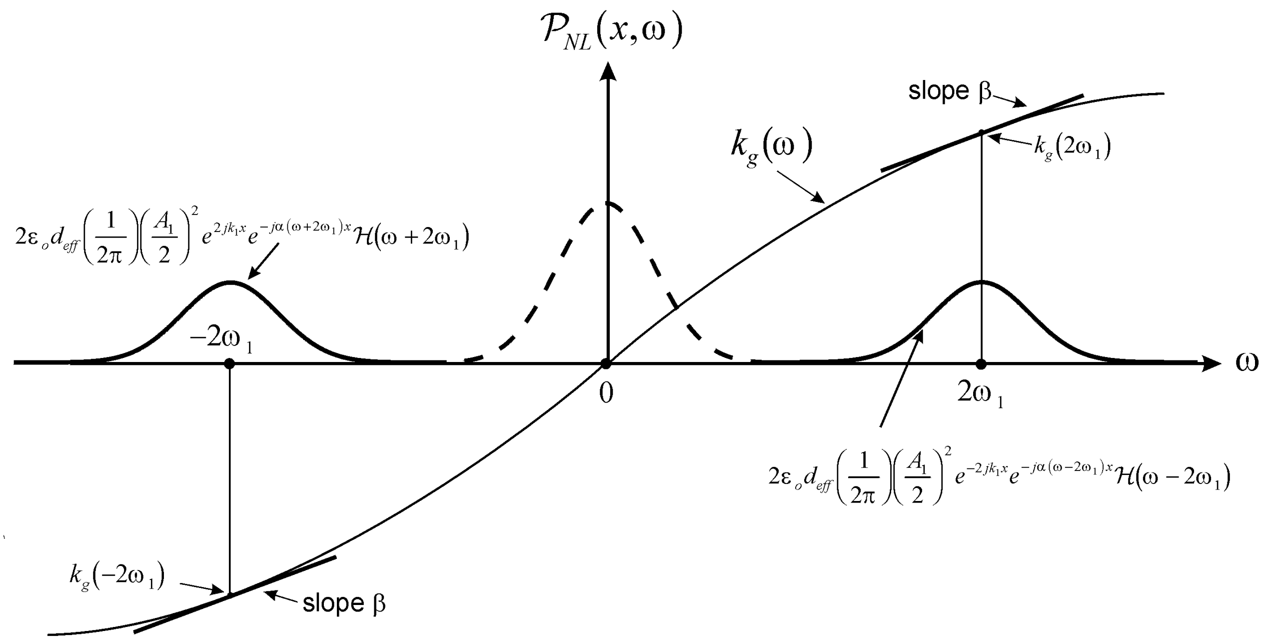

Figure 6.2: Figure S-6-2. Frequency-domain representation of the nonlinear polarization for short pulse SHG. Only the portions shown with a solid line near $\pm 2\omega_1$contribute to second-harmonic generation.

v_{g2} is the group velocity of the second-harmonic pulse. We use this expansion for $k_g(\omega)$ in the exponential. Where $k_g(\omega)$ appears simply as a multiplicative factor, we use $k_g(\omega) \approx k_g(2\omega_1) = k_3$:

$$\frac{\partial \widetilde{\mathcal{E}}_g(x,\omega)}{\partial x} = \frac{-jd_{eff}k_3}{n_3^2}\left(\frac{A_1^2}{8\pi}\right)\left[e^{jk_3x}e^{j\beta(\omega-2\omega_1)x}e^{-2jk_1x}e^{-j\alpha(\omega-2\omega_1)x}\mathcal{H}(\omega-2\omega_1)\right. \tag{6.12}$$

$$\left. -e^{-jk_3x}e^{j\beta(\omega+2\omega_1)x}e^{2jk_1x}e^{-j\alpha(\omega+2\omega_1)x}\mathcal{H}(\omega+2\omega_1)\right] \tag{6.13}$$

Assuming that phasematching is achieved at the center frequency of the pulse spectrum, $k_3 = 2k_1$, we have:

$$\frac{\partial \widetilde{\mathcal{E}}_g(x,\omega)}{\partial x} = \frac{-jd_{eff}k_3}{n_3^2}\left(\frac{A_1^2}{8\pi}\right)\left[e^{j(\beta-\alpha)(\omega-2\omega_1)x}\mathcal{H}(\omega-2\omega_1) - e^{j(\beta-\alpha)(\omega+2\omega_1)x}\mathcal{H}(\omega+2\omega_1)\right] \tag{6.14}$$

Performing the integration over length, we obtain:

$$\widetilde{\mathcal{E}}_g(l,\omega) = \frac{-jd_{eff}k_3}{n_3^2}\left(\frac{A_1^2}{8\pi}\right) l \left\{ e^{j(\beta-\alpha)(\omega-2\omega_1)\frac{l}{2}}\,\mathrm{sinc}\left[(\beta-\alpha)(\omega-2\omega_1)\frac{l}{2}\right]\mathcal{H}(\omega-2\omega_1) \right.$$

$$\left. -e^{j(\beta-\alpha)(\omega+2\omega_1)\frac{l}{2}}\,\mathrm{sinc}\left[(\beta-\alpha)(\omega+2\omega_1)\frac{l}{2}\right]\mathcal{H}(\omega+2\omega_1)\right\} \tag{6.15}$$

We can transform this expression back to the time domain to determine the shape of the second-harmonic pulse emitted by the crystal. For clarity, let us first consider the case where $\beta = \alpha$; that is, where the group velocities for the input pulse at the fundamental frequency and the output pulse at the second harmonic are the same. Then we obtain:

$$\widetilde{\mathcal{E}}_g(l,\omega) = \frac{-jd_{eff}k_3}{n_3^2}\left(\frac{A_1^2}{8\pi}\right) l \left\{\mathcal{H}(\omega-2\omega_1) - \mathcal{H}(\omega+2\omega_1)\right\} \tag{6.16}$$

Recall that the frequency spectrum $\mathcal{E}_g(l,\omega) = \widetilde{\mathcal{E}}_g(l,\omega)e^{-jk_g(\omega)x}$, so that the expression which we desire to inverse-Fourier transform is:

$$\mathcal{E}_g(l,\omega) = \frac{-jd_{eff}k_3}{n_3^2}\left(\frac{A_1^2}{8\pi}\right) l \left\{e^{-jk_3l}e^{-j\beta(\omega-2\omega_1)l}\mathcal{H}(\omega-2\omega_1) - e^{+jk_3l}e^{-j\beta(\omega+2\omega_1)l}\mathcal{H}(\omega+2\omega_1)\right\} \tag{6.17}$$

We could actually perform the integration involved in the inverse Fourier transform, but a little bit of reflection will enable us to deduce the answer from things we already know. We can re-write this expression as:

$$\mathcal{E}_g(l,\omega) = \frac{d_{eff}k_3}{n_3^2}\left(\frac{A_1^2}{8\pi}\right) l\left\{\left[e^{-j\beta\omega l}\mathcal{H}(\omega)\right] * -j\left[e^{-jk_3 l}\delta(\omega-2\omega_1) - e^{jk_3 l}\delta(\omega+2\omega_1)\right]\right\} \tag{6.18}$$

Since $\mathcal{H}(\omega)/2\pi$ is the Fourier transform of $f^2(t)$, then $e^{-j\beta\omega l}\mathcal{H}(\omega)$ must be the Fourier transform of $2\pi f^2(t-\beta l)$. We can also deduce that $-j\left[e^{-jk_3 l}\delta(\omega-2\omega_1) - e^{jk_3 l}\delta(\omega+2\omega_1)\right]$ is the transform of $\sin(2\omega_1 t - 2k_1 l)/\pi$. Putting this all together, we have:

$$E_g(l,t) = \frac{4\pi d_{eff}k_3}{n_3^2}\left(\frac{A_1^2}{8\pi}\right) l\left\{f^2(t-\beta x)\sin(2\omega_1 t - 2k_1 x)\right\} \tag{6.19}$$

$$\frac{d_{eff}k_3 A_1^2 l}{2n_3^2}\left\{f^2(t-\beta l)\sin(2\omega_1 t - 2k_1 x)\right\} \tag{6.20}$$

When $\alpha \neq \beta$, the situation is a little more complicated. Returning to Equation 6.15, we can write the corresponding $\mathcal{E}_g(l,\omega)$ in the following form:

$$\mathcal{E}_g(l,\omega) = \frac{-jd_{eff}k_3}{n_3^2}\left(\frac{A_1^2}{8\pi}\right) l \tag{6.21}$$

$$\left\{e^{-jk_3 l}e^{-j\beta(\omega-2\omega_1)l}e^{j(\beta-\alpha)(\omega-2\omega_1)\frac{l}{2}}\operatorname{sinc}\left[(\beta-\alpha)(\omega-2\omega_1)\frac{l}{2}\right]\mathcal{H}(\omega-2\omega_1)\right. \tag{6.22}$$

$$\left.-e^{jk_3 l}e^{-j\beta(\omega+2\omega_1)l}e^{j(\beta-\alpha)(\omega+2\omega_1)\frac{l}{2}}\operatorname{sinc}\left[(\beta-\alpha)(\omega+2\omega_1)\frac{l}{2}\right]\mathcal{H}(\omega+2\omega_1)\right\} \tag{6.23}$$

$$= \frac{d_{eff}k_3}{n_3^2}\left(\frac{A_1^2}{8\pi}\right) l\left\{\left[e^{-j\beta\omega l}\mathcal{H}(\omega)\right]\left[e^{j(\beta-\alpha)\omega\frac{l}{2}}\operatorname{sinc}\left[(\beta-\alpha)\omega\frac{l}{2}\right]\right]\right\} \tag{6.24}$$

$$* - |\left\{e^{-jk_3 l}\delta(\omega-2\omega_1) - e^{jk_3 l}\delta(\omega+2\omega_1)\right\} \tag{6.25}$$

Again, by reasoning a little we can determine what function has this Fourier transform. We have already observed that $e^{-j\beta\omega l}\mathcal{H}(\omega)$ is the Fourier transform of $2\pi f^2(t-\beta l)$ and that $-j\left[e^{-jk_3 l}\delta(\omega-2\omega_1) - e^{jk_3 l}\delta(\omega+2\omega_1)\right]$ is the Fourier transform of $\dfrac{\sin(2\omega_1 t - k_3 l)}{\pi}$. We have saw in the supplemental notes on power and energy that the sinc function in the frequency-domain corresponds to finite duration in the time-domain. In this case, $e^{j(\beta-\alpha)\omega\frac{l}{2}}\operatorname{sinc}\left[(\beta-\alpha)\omega\dfrac{l}{2}\right]$ corresponds to a function that has the value

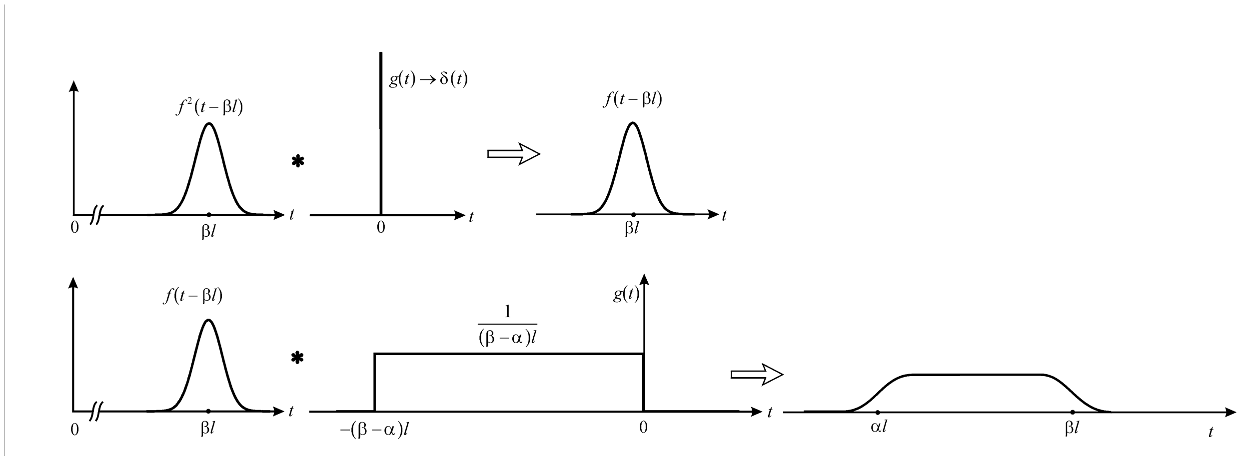

Figure 6.3: Figure S-6-3. Convolution of the pulse envelope $f(t-\beta l)$ with the "smearing function" $g(t)$

$1/(\beta-\alpha)\,l$ between $-(\beta-\alpha)\,l$ and 0 and is zero elsewhere (Fig. S-6-3). The other pieces of information we need are that $\mathcal{F}\left\{A(t)\,B(t)\right\} = \frac{1}{2\pi}\left\{\mathcal{A}(\omega) * \mathcal{B}(\omega)\right\}$ and that $\mathcal{F}\left\{A(t) * B(t)\right\} = \mathcal{A}(\omega)\,\mathcal{B}(\omega)$. Thus, the function that has the expression above as its Fourier transform must be:

$$\begin{aligned} E_g(l,t) &= \frac{dk_3}{n_3^2}\left(\frac{A_1^2}{8\pi}\right) l\left\{4\pi\right\}\left\{f^2\left(t-\beta l\right) * g\left(t\right)\right\}\sin\left(2\omega_1 t - k_3 l\right) && (6.26) \\ &= \frac{dk_3 A_1^2 l}{2n_3^2}\left\{f^2\left(t-\beta l\right) * g\left(t\right)\right\}\sin\left(2\omega_1 t - k_3 l\right) && (6.27) \end{aligned}$$

where $g(t)$is the function depicted in Fig. S-6-3.

The second-harmonic output pulse has an envelope given by the convolution of f^2 with g. As $\beta-\alpha \rightarrow 0$, $g(t) \rightarrow \delta(t)$ and $f^2\left(t-\beta l\right) * g\left(t\right) \rightarrow f^2\left(t-\beta l\right)$, producing the same result as in Eq.6.19. For $\beta \neq \alpha$, the envelope is made longer than f^2 by the convolution. Consider the case where $\beta > \alpha$ and $(\beta-\alpha)\,l \gg \tau_p$, where τ_p is the duration of the fundamental pulse. Here, the fundamental pulse travels faster than the second harmonic pulse, and the difference in their arrival times at the output face of the crystal is greater than the length of the input pulse. Some blue light is generated near the input face as the fundamental pulse first enters the crystal. This light propagates through the crystal and arrives at the output at $t \approx \beta l$. In the meantime, the fundamental pulse has propagated through the crystal and arrived at the output face at the earlier time $t \approx \alpha l$. Therefore, the first bit of second harmonic light to reach the output face is that which is generated near the output face of the crystal, and the last bit of second harmonic light to emerge from the crystal is that which was generated near the input face. The envelope of the output pulse is essentially rectangular with duration $(\beta - \alpha)\, l$.

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