Solutions to the Tutorial Problems in the book "Magnetohydrodynamics of the Sun" by ER Priest (2014) CHAPTER 11

PROBLEM 11.1. Generalised Kippenhahn-Schlüter Model.
Extend the Kippenhahn-Schlüter model to allow for a small external pressure by imposing boundary conditions $p = p_e$ and $B_z = B_{ze}$ at $x = H$ and linearising about the resulting Kippenhahn-Schlüter model. Show that the horizontal field strength increases with height over a scale L_0 when $l^{1/2} \gg \alpha > 1.7$ (a thin prominence) or $\alpha^2 \gg l < 1$ (a thick weak prominence), where $l = L_0/H$, $\alpha = 2B_{0x}/(2\mu p_e + B_{ze}^2)^{1/2}$.

SOLUTION.
If we impose boundary conditions $p = p_e$ and $B_z = B_{ze}$ at $x = H$, the Kippenhahn-Schlüter model becomes

$$\bar{B}_{0z} = \tanh \bar{x},$$

where $\bar{B}_{0z} = B_{0z}/(2\mu p_e + B_{ze}^2)^{1/2}$, $\bar{x} = x/(\alpha H)$ and $\alpha = 2B_{0x}/(2\mu p_e + B_{ze}^2)^{1/2}$.

Then we follow Ballester and Priest (1987) by linearising the magnetohydrostatic equations

$$\mathbf{0} = -\boldsymbol{\nabla} p - \rho g \hat{\boldsymbol{z}} - \boldsymbol{\nabla}\left(\frac{B^2}{2\mu}\right) + (\mathbf{B}\cdot\boldsymbol{\nabla})\left(\frac{\mathbf{B}}{\mu}\right),$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0,$$

$$p = R\rho T,$$

about this solution to give (with e.g., $B_x = B_{0x} + \epsilon B_{1x}$, where $\epsilon \ll 1$)

$$-B_{1z}\frac{\partial B_{0z}}{\partial x} + B_{0z}\left(\frac{\partial B_{1x}}{\partial z} - \frac{\partial B_{1z}}{\partial x}\right) - \mu\frac{\partial p_1}{\partial x} = 0,$$

$$B_{1x}\frac{\partial B_{0z}}{\partial x} - B_{0x}\left(\frac{\partial B_{1x}}{\partial z} - \frac{\partial B_{1z}}{\partial x}\right) - \mu\rho_1 g - \mu\frac{\partial p_1}{\partial z} = 0,$$

$$\frac{\partial B_{1x}}{\partial x} + \frac{\partial B_{1z}}{\partial z} = 0.$$

Now assume each of the variables is a function of $\bar{x}$ times a decaying function of $\bar{z}$ in the form $\bar{B}_{1x} = \bar{B}_{1x}(\bar{x})\exp(-\bar{z}/l)$, where $\bar{z} = z/(\alpha H)$, $l = L_0/H$ and $\bar{B}_{1x} = B_{1x}/B_{0x}$. Then eliminate $\bar{p}_1$ and $\bar{B}_{1z}$ to give the following basic equation for $\bar{B}_{1x}$

$$l\frac{d^3\bar{B}_{1x}}{d\bar{x}^3} + 2(l-1)\tanh\bar{x}\frac{d^2\bar{B}_{1x}}{d\bar{x}^2} + \left(\frac{\alpha^2}{l} + 2l\,\mathrm{sech}^2\bar{x}\right)\frac{d\bar{B}_{1x}}{d\bar{x}}$$
$$+ 2\tanh\bar{x}\left(\frac{\alpha^2}{l} - \frac{\alpha^2}{l^2} - 2\frac{\alpha}{l}\mathrm{sech}^2\bar{x}\right)\bar{B}_{1x} = 0,$$

with three boundary conditions, namely, $d\bar{B}_{1x}/d\bar{x}(0) = 0$ (which implies $\bar{B}_{1z}(0) = 0$), $\bar{B}_{1z}(1/\alpha) = 1$ and $p_1(1/\alpha) = 1$. Then $\bar{B}_{1z}$ follows from $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, namely,

$$\bar{B}_{1z} = \frac{l}{\alpha}\frac{d\bar{B}_{1x}}{d\bar{x}},$$

while $\bar{p}_1$ follows from the z-component of the force balance, namely,

$$2\bar{B}_{1x}\frac{d\bar{B}_{0z}}{d\bar{x}} + \frac{\alpha^2}{l}\bar{B}_{1x} + \frac{d\bar{B}_{1z}}{d\bar{x}} - \alpha\bar{p}_1 + \frac{\alpha}{l}\bar{p}_1 = 0.$$

Consider first a weak thick prominence for which $\alpha^2 \gg l < 1$, so that the equation for $\bar{B}_{1x}$ reduces to

$$\left(\frac{\alpha^2}{l}\right)\frac{d\bar{B}_{1x}}{d\bar{x}} - 2\left(\frac{\alpha^2}{l^2}\right)\tanh\bar{x}\ \bar{B}_{1x} = 0,$$

with solution

$$\bar{B}_{1x} = C\cosh^{2/l}\bar{x}.$$

The equation for $\bar{p}_1$ reduces to

$$\alpha\bar{B}_{1x} + \bar{p}_1 = 0,$$

and so the boundary condition $\bar{p}_1(1/\alpha) = 1$ implies that $C < 0$. This in turn implies that $\bar{B}_{1x} < 0$ and therefore the horizontal field $B_{0x}[1+\epsilon\bar{B}_{1x}\exp(-\bar{z}/l)]$ decreases in magnitude with height.

Consider next a thin prominence for which $l \gg \alpha^2 > 1$, so that the equation for $\bar{B}_{1x}$ reduces to

$$l\frac{d^3\bar{B}_{1x}}{d\bar{x}^3} + 2l\tanh\bar{x}\frac{d^2\bar{B}_{1x}}{d\bar{x}^2} + 2l\,\mathrm{sech}^2\bar{x}\frac{d\bar{B}_{1x}}{d\bar{x}} = 0,$$

with general solution, after using the boundary condition $d\bar{B}_{1x}/d\bar{x}(0) = 0$,

$$\bar{B}_{1x} = \tfrac{1}{2}C(\bar{x}\tanh\bar{x} - 1) - K.$$

This has two constants of integration determined by the other two boundary conditions, with $\bar{p}_1$ and $\bar{B}_{1x}$ following in this limit ($l \gg \alpha^2$) from

$$2\bar{B}_{1x}\frac{d\bar{B}_{0z}}{d\bar{x}} + \frac{d\bar{B}_{1z}}{d\bar{x}} - \alpha\bar{p}_1 = 0$$

and

$$\bar{B}_{1z} = \tfrac{1}{2}Cl(\bar{x}\ \mathrm{sech}^2\bar{x} + \tanh\bar{x}).$$

Thus, the boundary condition $\bar{B}_{1z}(1/\alpha) = 1$ implies

$$\frac{Cl}{2}\left[\left(\frac{1}{\alpha}\right)\mathrm{sech}^2\left(\frac{1}{\alpha}\right) + \tanh\left(\frac{1}{\alpha}\right)\right] = 1,$$

which determines Cl and implies that $C \ll 1$ since $l \gg 1$.

Next, the boundary condition $\bar{p}_1(1/\alpha) = 1$ implies

$$2\bar{B}_{1x}(1/\alpha)\frac{d\bar{B}_{0z}(1/\alpha)}{d\bar{x}} + \frac{d\bar{B}_{1z}(1/\alpha)}{d\bar{x}} - \alpha = 0,$$

where

$$\bar{B}_{1x}(1/\alpha) = \tfrac{1}{2}C[(1/\alpha)\tanh(1/\alpha) - 1] - K \sim -K,$$

since $C \ll 1$,

$$\frac{d\bar{B}_{0z}(1/\alpha)}{d\bar{x}} = \mathrm{sech}^2(1/\alpha),$$

and

$$\frac{d\bar{B}_{1z}(1/\alpha)}{d\bar{x}} = Cl\ \mathrm{sech}^2(1/\alpha) - Cl(1/\alpha)\mathrm{sech}^2(1/\alpha)\tanh(1/\alpha).$$

Thus, after substituting for Cl, the equation for K becomes

$$-K\ \mathrm{sech}^2(1/\alpha) + \frac{\mathrm{sech}^2(1/\alpha) - (1/\alpha)\ \mathrm{sech}^2(1/\alpha) + \tanh(1/\alpha)}{(1/\alpha)\ \mathrm{sech}^2(1/\alpha) + \tanh(1/\alpha)} - \frac{\alpha}{2} = 0,$$

or

$$-K = \cosh^2(1/\alpha)\left(\frac{\tanh(1/\alpha) - \alpha}{1 + \alpha\sinh(1/\alpha)\cosh(1/\alpha)} + \frac{\alpha}{2}\right).$$

Thus, we note that (for positive α), as $\alpha \to 0$ this behaves like

$$\frac{\cosh(1/\alpha)}{\alpha \sinh(1/\alpha)},$$

which is positive, whereas, as $\alpha \to \infty$, it behaves like $-\alpha/2$, which is negative. Indeed, for $\alpha > 1.7$, we find $K > 0$.

However, the value of $\bar{B}_{1x}$ at $\bar{x} = 0$ is $-C/2 - K \sim -K$ and so is negative when $\alpha > 1.7$. Thus, we have established as required that the horizontal field $B_{0x}[1 + \epsilon \bar{B}_{1x} \exp(-\bar{z}/l)]$ decreases in magnitude with height when $l \gg \alpha^2 > 1.7$.

PROBLEM 11.2. Nonisothermal Kippenhahn-Schlüter Model. For an isothermal Kippenhahn-Schlüter model, rewrite the standard solution in terms of $\beta_1 = 2\mu p_1/B_x^2$, $H_1 = k_B T_1/(mg)$ and $p_0 = B_{z\infty}^2/(2\mu)$. Next obtain the corresponding solution when the temperature is a given function $T(x)$ and the boundary conditions are imposed to be $p = p_1$ and $T = T_1$ at $x = \pm x_1$. Deduce that there exists a maximum allowable plasma beta (β_1) for the equilibrium to exist.

SOLUTION. The isothermal solution is

$$B_z = B_{z\infty} \tanh \frac{B_{z\infty} x}{2 B_x H}, \qquad p = \frac{B_{z\infty}^2}{2\mu} \text{sech}^2 \frac{B_{z\infty} x}{2 B_x H},$$

which may be written in terms of $\beta_1 = 2\mu p_1/B_x^2$ and $p_0 = B_{z\infty}^2/(2\mu)$ as

$$B_z = (2\mu p_0)^{1/2} \tanh \left[\left(\frac{\beta_1 p_0}{p_1} \right)^{1/2} \frac{l(x)}{2} \right], \qquad p = p_0 \text{sech}^2 \left[\left(\frac{\beta_1 p_0}{p_1} \right)^{1/2} \frac{l(x)}{2} \right],$$

where $l(x) = (T_1/T)(x/H_1)$ and $H_1 = k_B T_1/(mg)$.

When the temperature is instead imposed to be a given function $T(x)$, the above solution remains the same except that now $l(x)$ is defined to be

$$l(x) = \frac{T_1}{H_1} \int_0^x \frac{dx}{T(x)}.$$

Then substitution of the boundary condition $p = p_1$ at $x = \pm x_1$ gives

$$p_0^{1/2} = p_1^{1/2} \cosh \left[\left(\frac{\beta_1 p_0}{p_1} \right)^{1/2} \frac{l_1}{2} \right].$$

If p_1, β_1 and l_1 are imposed, then, by sketching the left- and right-hand sides as functions of $p_0^{1/2}$, this equation determines two values for p_0 provided β_1 is less than a certain maximum value (approximately 1.7 l_1^{-2}). Otherwise there is no solution.

PROBLEM 11.3. Oscillation of a Kuperus-Raadu Model.
Show that vertical oscillations of a Kuperus-Raadu prominence model give a period of $2\pi(h/g)^{1/2}$.

SOLUTION. Kuperus and Raadu (1974) model a prominence as a line current I and mass $m = \pi R^2$ at height h in equilibrium

$$0 = \frac{\mu\ I^2}{4\ \pi\ h} - m\ g,$$

between the force of gravity and the repulsion between the line current and its image $-I$ a distance h below the photosphere.

Suppose the prominence is perturbed by moving it up by a distance z with the force of gravity remaining the same and the repulsion decreasing to

$$\frac{\mu\ I^2}{4\ \pi\ (h+z)}.$$

Then its vertical equation of motion becomes

$$m\frac{d^2z}{dt^2} = \frac{\mu\ I^2}{4\ \pi\ (h+z)} - m\ g,$$

or substituting for mg,

$$m\frac{d^2z}{dt^2} = \frac{\mu\ I^2}{4\ \pi\ (h+z)} - \frac{\mu\ I^2}{4\ \pi\ h}.$$

After Taylor expanding the first term for $z \ll h$ and keeping only the linear term, this becomes

$$m\frac{d^2z}{dt^2} = -\frac{\mu\ I^2}{4\ \pi\ h}\frac{z}{h},$$

or, substituting for m from the initial force balance,

$$\frac{d^2z}{dt^2} = -\frac{g}{h}.$$

This implies that the initial equilibrium is stable and that the prominence performs small vertical oscillations of frequence $(g/h)^{1/2}$ and period $2\pi(h/g)^{1/2}$, which is about 20 min for a prominence height of $h = 10$ Mm.

PROBLEM 11.4. Flux-Rope Model.
Seek solutions for current-sheet support in a force-free flux rope with $B_z = cA$.

SOLUTION. Following Ridgway, Priest and Amari (1991), we model a cylindrical flux rope of radius a and consider a force-free field independent of z that is expressed in terms of a flux function $A(r, \theta)$ as

$$\mathbf{B} = \left(\frac{1}{r}\frac{\partial A}{\partial \theta}, -\frac{\partial A}{\partial r}, B_z(A) \right),$$

where the force-free equation reduces to

$$\nabla^2 A + F(A) = 0,$$

with $F(A) = d/dA(\frac{1}{2}B_z^2)$. When $B_z = cA$, this becomes

$$\nabla^2 A + c^2 A = 0,$$

Separable solutions may be found of the form

$$A(r, \theta) = R(r)\cos(K\theta),$$

where

$$r\frac{d}{dr}\left(r\frac{dR}{dr} \right) + (c^2 r^2 - K^2)R = 0.$$

This is Bessel's equation of order K and so the solution that is nonsingular at $r = 0$ is

$$A(r, \theta) = b_K J_K(cr)\cos(K\theta),$$

where $J_K(cr)$ is the Bessel function of the first kind of order K.

Since the differential equation is linear, the general solution is the sum over possible values K_n of K, namely,

$$A(r, \theta) = \frac{B_0}{c}\left[J_0(cr) + \sum_{n=0}^{\infty} b(K_n) J_{K_n}(cr)\cos(K_n\theta) \right],$$

where B_0 is the axial field strength at $r = 0$ and b represents the departure from cylindrical symmetry ($b = 0$). Singularities of B_r, B_θ can be avoided by taking $K \geqslant 1$ and a sign change of B_θ for small r by taking $K \geqslant 2$.

As a particular example, consider

$$A(r, \theta) = B_0\{J_0(cr) + b_K J_K(cr)\cos(K\theta)\}/c,$$

with $K \geqslant 2$, for which the field components are

$$B_r = -B_0 K b_K J_K(cr)\sin(K\theta)/(cr),$$

$$B_\theta = B_0\{J_1(cr) - b_K[KJ_K(cr)/(cr) - J_{K+1}(cr)]\cos(K\theta)\},$$
$$B_z = B_0\{J_0(cr) + b_K J_K(cr)\cos(K\theta)\}.$$

To ensure that a field line dip is present at $\theta = \pm\pi$ we need $B_r(r, \pi) > 0$, which implies $b_K < 0$ for $2i < K < 2i+1$ and $b_K > 0$ for $2i+1 < K < 2i+2$, where $i = 1, 2, 3....$ Examples of the resulting fields can be found in Ridgway, Priest and Amari (1991), as well as other examples when $B_z = cA^{1/2}$.

PROBLEM 11.5. Linear Force-Free Flux-Rope Model.
Set up a model for a cylindrical linear force-free flux rope with no axial field reversal.

SOLUTION. Following Rust and Kumar (1994), consider a cylindrically symmetric flux tube with field components $[B_\theta(r)B_z(r)]$ in cylindrical polars, satisfying the linear force-free equation

$$(\nabla^2 + \alpha^2)\mathbf{B} = \mathbf{0},$$

where $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. The appropriate solution is

$$B_\theta = B_0 J_1(r/r_0), \qquad B_z = B_0 J_0(r/r_0),$$

where J_0 and J_1 are Bessel functions, $r_0 = 1/\alpha$ and B_0 is the axial field at $r = 0$.

In order to avoid a field reversal we need to suppose this solution holds for $r \leqslant R$, say, and surround the force-free field by a potential field, namely,

$$B_\theta = cB_0R/r, \qquad B_z = 0.$$

Then, equating the field components are $r = R$ implies that

$$c = J_1(R/r_0) \qquad \text{and} \qquad J_0(R/r_0) = 0,$$

and so, in order to avoid reversals in the axial field, we take R/r_0 to be the first zero of $J_0(x)$, namely, 2.2.

PROBLEM 11.6. A Dip in a Potential Field.
Show that, for a 2.5D potential arcade with the fundamental solution plus the nth harmonic, inverse polarity is not possible and that a dip needs parasite polarity.

SOLUTION. Consider a 2.5D potential coronal arcade field of the form

$$B_x = -\cos kx \; e^{-kz} + b_n \cos nkx \; e^{-nkz},$$

$$B_y = 0,$$

$$B_z = \sin kx \; e^{-kz} - b_n \sin nkx \; e^{-l_n z},$$

Thus, on the z-axis the horizontal field is

$$B_x = -e^{-kz} + b_n e^{-nkz},$$

while on the x-axis the vertical field is

$$B_z = \sin kx - b_n \sin nkx.$$

The presence of a dip on the z-axis needs $dB_x^2/dz > 0$ there, where

$$\frac{dB_x^2}{dz} = 2k(-e^{-kz} + b_n e^{-nkz})(e^{-kz} - n b_n e^{-nkz}).$$

This in turn implies that

$$\frac{1}{n} e^{(n-1)kz} < b_n < e^{(n-1)kz},$$

and so

$$b_n > \frac{1}{n},$$

which is just the condition that there be parasite flux since then $B_z < 0$ on the positive part of the x-axis near $x = 0$ and then changes sign as x increases.

PROBLEM 11.7. Current Sheet Models using Complex-Variable Theory.

Use complex variable theory to build current sheet models of infinite or finite height by modifying the field given by $B_y + iB_x = B_0(p^2 + z^2)^{1/2}/z$, where $z = x + iy$.

SOLUTION. More details of the solution may be found in the paper by Malherbe and Priest (Astron. Astrophys. 123,80-88,1983). Potential fields containing current sheets may be set up in terms of complex variable theory, since, for any function of the form $B_y + iB_x = f(z)$, where $f(z)$ is an analytical function, the functions $B_x(x, y)$ and $B_y(x, y)$ satisfy Laplace's equation. If $f(z)$ contains cuts in the complex plane, those cuts represent current sheets. Shear may be added to this 2D field by adding a uniform field out of the plane, or by modifying the theory to include force-free fields: for example, Ridgway et al (1991) show how to use constant-current force-free fields.

The solution

$$B_y + iB_x = B_0 \frac{(p^2 + z^2)^{1/2}}{z},$$

represents the field of a magnetic arcade surmounted by a current sheet stretching upwards from a base at $z = ip$, but the field either side of the sheet is oppositely directed and purely vertical (Fig.1a in Malherbe and Priest, 1983).

First of all, for Inverse-Polarity sheet models of infinite height the above function may be modified to add a horizontal field at the current sheet (Fig.3a) by putting

$$B_y + iB_x = B_0 \frac{(p^2 + z^2)^{1/2}}{z} + iB_1 \frac{z - ih}{z},$$

where $h \leqslant p$. It possesses an X-type neutral point below the prominence at

$$y = \frac{h + (B_0^4 p^2/B_1^4 + (p^2 - h^2)B_0^2/B_1^2)^{1/2}}{1 + B_0^2/B_1^2}.$$

When $h = p$, the X-point coincides with the base of the sheet. The singularity at $z = 0$ may be avoided since the photosphere may be regarded as being at a finite height above $z = 0$.

Next, for Inverse-Polarity models having a finite height, consider instead the solution given by

$$B_y + iB_x = B_0 \frac{[(p^2+z^2)(q^2+z^2)]^{1/2}}{z},$$

which is similar to the previous one with a vertical field both sides of a current sheet, except that now the sheet is finite in height, stretching from $z = ip$ to $z = iq$ (Fig.1b in Malherbe and Priest, 1983). This may be modified to give three different types of Inverse Polarity model, namely,

$$B_y + iB_x = B_0 \frac{[(p^2+z^2)(q^2+z^2)]^{1/2}}{z} + B_1(z - ip),$$

which has open field lines towards the corona (Fig.3c),

$$B_y + iB_x = -B_0 \frac{[(p^2+z^2)(q^2+z^2)]^{1/2}}{z(z+ih)^2} - B_1 \frac{z-ip}{z(z+ih)},$$

which has closed field lines down to to the photosphere when $q \leqslant 2p$ (Fig.3d), and

$$B_y + iB_x = -B_0 \frac{[(p^2+z^2)(q^2+z^2)]^{1/2}}{z(z+ih)^2} + B_1 \frac{z-ip}{z(z-iq)},$$

which has helical field lines when $q/p < B_1/B_0$ (Fig.3e).

Normal-Polarity arcade models may be obtained from

$$B_y + iB_x = B_0 \frac{(p^2+z^2)^{1/2}}{z(z+ih)} - \frac{B_1}{z},$$

with $p/h < B_1/B_0$, which has an infinite current sheet (Fig.4a), or

$$B_y + iB_x = -B_0 \frac{[(p^2+z^2)(q^2+z^2)]^{1/2}}{z(z+ih)^2} - \frac{B_1}{z},$$

with $pq/h^2 < B_1/B_0$, which has a current sheet of finite length (Fig.4b). When $p/h \geqslant B_1/B_0$ for the first solution or $pq/h^2 \geqslant B_1/B_0$ for the second solution, they change their polarity from normal to inverse.

Normal-Polarity models with helical structure (Fig.4c) may be modelled by

$$B_y + iB_x = -B_0 \frac{[(p^2+z^2)(q^2+z^2)]^{1/2}}{z(z+ih)^2} + \frac{B_1}{z}\frac{z+ik}{z-iq},$$

when $pq^2/(h^2k) < B_1/B_0 < 1$ and $kq > h^2$.