Chapter 7: Sequences and Series

Part B: Series

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Table of Contents

Sum of a Series

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$$a_1 + a_2 + a_3 + \cdots \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n$$

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- **Convergent** if $\lim_{n\to\infty} S_n$ exists,
- **Divergent** if $\lim_{n\to\infty} S_n$ does not exist.

Geometric Series

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If $|r| > 1$, then $|r|^n \to \infty$, so the series diverges.

If $|r| < 1$, then $r^n \to 0$, so the series converges and its sum is $\dfrac{1}{1-r}$.

Examples of Divergence

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- $\displaystyle\sum_{n=1}^{\infty} 1$: $S_n = 1 + \cdots + 1 = n \to \infty$.
- $\displaystyle\sum_{n=1}^{\infty} (-1)^n$: $S_n = \begin{cases} 0 & \text{if } n \text{ even}, \\ -1 & \text{if } n \text{ odd}, \end{cases}$ diverges.
- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$: $S_n = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \frac{n}{\sqrt{n}} = \sqrt{n} \to \infty$.

The last of these is the most important. It shows that terms that are decreasing to zero can still accumulate in an unbounded way.

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Hence,

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} \to 1.$$

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Therefore $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

This is called a **telescoping series** due to the cancellations in the partial sums.

Harmonic Series

$\sum_{n=1}^{\infty} 1/n$ is called the **harmonic series**. Here are some of its partial sums:

$$S_1 = 1, \quad S_2 = 1.5, \quad S_4 = 2.1, \quad S_8 = 2.7, \quad S_{16} = 3.4.$$

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We see that while the growth of the partial sums is slowing down, there seems to be an increase of at least $1/2$ every time the number of terms doubles. First, we verify this:

$$S_{2n} - S_n = \frac{1}{n+1} + \cdots + \frac{1}{2n} > \frac{n}{2n} = \frac{1}{2}.$$

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Task 1

Show that $\sum_{k=1}^{n} 1/k > \int_1^{n+1} dx/x = \log(n+1)$. *Use this observation for another proof that the harmonic series diverges.*

Algebra of Series

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Theorem 1

Let $\sum_{n=1}^{\infty} a_n = L$, $\sum_{n=1}^{\infty} b_n = M$ *and* $c \in \mathbb{R}$. *Then:*

1. $\sum_{n=1}^{\infty} (c\, a_n) = cL.$
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Task 2

Do the following series converge?

1. $\sum_{n=1}^{\infty}\frac{2^n}{3^{n+1}}.$
2. $\sum_{n=1}^{\infty}\frac{2^n+3^n}{5^n}.$
3. $\sum_{n=1}^{\infty}\Big(\frac{1}{n}-\frac{1}{2^n}\Big).$

Tail of a Series

The notation $\sum_{n=k}^{\infty} a_n$ refers to the expression $a_k + a_{k+1} + \cdots$. This is also a series, with first term $b_1 = a_k$, second term $b_2 = a_{k+1}$, and so on. Given an initial series $\sum_{n=1}^{\infty} a_n$, a series of the form $\sum_{n=k}^{\infty} a_n$ is called its **tail**.

Task 3

Show that $\sum_{n=1}^{\infty} a_n$ *converges if and only if the* **tail** $\sum_{n=k}^{\infty} a_n$ *converges. Further,*

$$\sum_{n=1}^{\infty} a_n = S_{k-1} + \sum_{n=k}^{\infty} a_n.$$

When we are only discussing convergence and not the actual sum we can drop the range of the index and just write $\sum_n a_n$, or even $\sum a_n$.

Divergence Test

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Theorem 2

If $\sum a_n$ *is a convergent series then* $a_n \to 0$.

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However, if $a_n \to 0$ then we do not learn anything. The series may converge (e.g. $\sum 1/2^n$ and $\sum 1/n(n+1)$) or diverge (e.g. $\sum 1/n$ and $\sum 1/\sqrt{n}$).

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Task 4

Show that the following series diverge.

1. $\sum_{n=1}^{\infty} \sin(n\pi/2)$.
2. $\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n}$.

Comparison Test

Theorem 3

Let $\sum a_n$ *and* $\sum b_n$ *satisfy* $0 \leq a_n \leq b_n$ *for every* n. *Then*

1. *If* $\sum b_n$ *converges, so does* $\sum a_n$.
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Proof. Consider the partial sums $S_n = \sum_{i=1}^{n} a_i$ and $T_n = \sum_{i=1}^{n} b_i$. The non-negativity of the terms implies that (S_n) and (T_n) are increasing sequences.

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First, suppose that $\sum b_n$ is convergent. Then $T_n \to T = \sup\{T_n\}$. Now $S_n \le T_n \le T$ for each n, so (S_n) is increasing and bounded above. Hence it is convergent.

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The second claim is just a rewording of the first. □

Example: $\sum 1/n^2$

Consider $\sum_{n=1}^{\infty} 1/n^2$. We shall compare it with the telescoping series $\sum_{n=1}^{\infty} 1/n(n+1)$, which converges. As n increases, the contribution of 1 becomes tiny compared to that of n and so we expect $\sum 1/n^2$ and $\sum 1/n(n+1)$ to have the same behaviour.

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Now let us expand these two series,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots ,$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots .$$

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Suppose we drop the first term of the $\sum 1/n^2$ series, and also shift the index of the telescoping series:

$$\sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \cdots$$

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The Comparison Test applies and implies that $\sum_{n=2}^{\infty} 1/n^2$ converges. Hence $\sum_{n=1}^{\infty} 1/n^2$ converges.

Examples

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Example 4

Consider the following comparisons for $\sum_{n=1}^{\infty} 1/n^{3/4}$:

$$0 < \underbrace{\frac{1}{n^2}}_{\sum \text{ converges}} \leq \underbrace{\frac{1}{n}}_{\sum \text{ diverges}} \leq \frac{1}{n^{3/4}} \leq \underbrace{\frac{1}{\sqrt{n}}}_{\sum \text{ diverges}} .$$

$\sum 1/n^2$ does not help because it is a smaller converging series. $\sum 1/\sqrt{n}$ does not help because it is a larger diverging series. However, $\sum 1/n$ is a useful combination: a smaller series that diverges. So $\sum 1/n^{3/4}$ diverges.

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Example 5

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$. We know $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges. And $\displaystyle 0 < \frac{1}{2^n + 3^n} \le \frac{1}{2^n}$. So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n}$ converges.

Limit Comparison Test

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Theorem 6

Let $\sum a_n$ and $\sum b_n$ be series whose terms are all positive, and suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0.$$

Then $\sum b_n$ converges if and only if $\sum a_n$ converges.

Limit Comparison Test

Theorem 6

Let $\sum a_n$ *and* $\sum b_n$ *be series whose terms are all positive, and suppose*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \neq 0.$$

Then $\sum b_n$ *converges if and only if* $\sum a_n$ *converges.*

Proof. Take two positive numbers m, M such that $m < c < M$. There is $N \in \mathbb{N}$ such that $m < \dfrac{a_n}{b_n} < M$ for every $n \geq N$. So $m\, b_n < a_n$ and $M\, b_n > a_n$ for every $n \geq N$. Now apply the Comparison Test.

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If $\sum b_n$ converges, then $\sum(M\, b_n)$ converges, so $\sum a_n$ converges.

If $\sum b_n$ diverges, then $\sum(m\, b_n)$ diverges, so $\sum a_n$ diverges. □

Example

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Example 7

Consider $\sum_{n=1}^{\infty} \frac{\pi}{n^2+4n+3}$. Let us compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$:

$$\lim_{n\to\infty} \frac{\pi/(n^2+4n+3)}{1/n^2} = \lim_{n\to\infty} \frac{\pi n^2}{n^2+4n+3} = \pi.$$

Therefore $\sum_{n=1}^{\infty} \frac{\pi}{n^2+4n+3}$ converges.

Example

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Example 8

Consider $\sum_{n=1}^{\infty} \frac{2^n}{3^n - 2^n}$. Compare with $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$:

$$\lim_{n\to\infty} \frac{2^n/(3^n - 2^n)}{2^n/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n - 2^n} = \lim_{n\to\infty} \frac{1}{1-(2/3)^n} = 1.$$

So $\sum_{n=1}^{\infty} \frac{2^n}{3^n - 2^n}$ converges.

Integral Test

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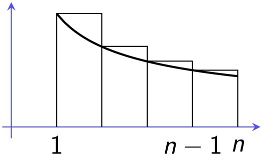
Theorem 9

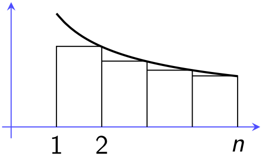
Consider a series $\sum_{n=k}^{\infty} a_n$ *whose terms can be expressed as* $a_n = f(n)$, *where* $f : [k, \infty) \to \mathbb{R}$ *is positive and decreasing. Then* $\sum_{n=k}^{\infty} a_n$ *converges if and only if the improper integral* $\int_k^{\infty} f(x)\, dx$ *converges.*

Integral Test: Proof

Proof. We may assume $k = 1$. Since f is decreasing it is integrable on every closed interval $[1, b]$ with $b > 1$. The partial sums of $\sum_{n=k}^{\infty} a_n$ give upper and lower sums for the integral of f over $[1, n]$:



$S_{n-1} \geq \int_1^n f(x)\,dx$



$S_n - a_1 \leq \int_1^n f(x)\,dx$

First, suppose $\displaystyle\int_1^{\infty} f(x)\,dx$ diverges. Then

$$S_{n-1} = \sum_{i=1}^{n-1} a_i \cdot 1 \geq \int_1^n f(x)\,dx \to \infty \implies S_n \to \infty.$$

(continued ...)

Integral Test: Proof

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(...continued)

Next, suppose $\int_1^\infty f(x)\,dx$ converges.

1. $a_n = f(n) > 0$ implies the partial sums (S_n) are increasing.
2. $S_n = a_1 + \sum_{i=2}^{n} a_i \cdot 1 \leq a_1 + \int_1^n f(x)\,dx \leq a_1 + \int_1^\infty f(x)\,dx.$

Since (S_n) is increasing and bounded above, it is convergent. □

Example

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Example 10

Consider $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$. The corresponding function $f(x) = \frac{\log x}{x^2}$ is positive on $[2, \infty)$. Is it decreasing?

$$f'(x) = \frac{1 - 2\log x}{x^3} \implies f'(x) \leq 0 \text{ for } x > \sqrt{e} = 1.6\ldots.$$

So $f(x)$ is decreasing on $[2, \infty)$. Apply the Integral Test:

$$\int_2^{\infty} \frac{\log x}{x^2}\, dx = \frac{1}{2}(1 + \log 2) \implies \sum_{n=2}^{\infty} \frac{\log n}{n^2} \text{ converges.}$$

p-Series Test

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Theorem 11 (The p-Series Test)

*The p-**series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.*

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Theorem 11 (The p-Series Test)

*The p-**series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.*

Proof. If $p \leq 0$ the terms are greater than or equal to 1, so the series diverges.

For $p > 0$ the function $\frac{1}{x^p}$ is positive and decreasing. And,

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1, \\ \infty & \text{if } 0 < p \leq 1. \end{cases}$$

□

Table of Contents

Alternating Series Test

Theorem 12

Consider an **alternating series** $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ *with each* $b_n \geq 0$*. Suppose the sequence* (b_n) *is decreasing and has limit 0. Then the following hold.*

1. $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ *converges.*
2. *Let* $S = \sum_{n=1}^{\infty}(-1)^{n+1}b_n$ *and* $S_k = \sum_{n=1}^{k}(-1)^{n+1}b_n$*. Then* $|S - S_k| \leq b_{k+1}$.

Alternating Series Test: Proof

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Proof. Consider the consecutive odd partial sums:

- $S_{2k+1} - S_{2k-1} = (-1)^{2k+1} b_{2k} + (-1)^{2k+2} b_{2k+1} = b_{2k+1} - b_{2k} \leq 0.$
- $S_{2k+1} = (b_1 - b_2) + (b_3 - b_4) + \cdots + (b_{2k-1} - b_{2k}) + b_{2k+1} \geq 0.$

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Thus, the odd partial sums are decreasing and bounded below by 0, hence convergent to some L.

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Similarly the even partial sums are increasing and bounded above (by b_1), hence convergent to some M.

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Similarly the even partial sums are increasing and bounded above (by b_1), hence convergent to some M.

Finally,

$$S_{2k+1} - S_{2k} = b_{2k+1} \implies L - M = 0 \implies L = M = S.$$

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Thus, the odd partial sums are decreasing and bounded below by 0, hence convergent to some L.

Similarly the even partial sums are increasing and bounded above (by b_1), hence convergent to some M.

Finally,

$$S_{2k+1} - S_{2k} = b_{2k+1} \implies L - M = 0 \implies L = M = S.$$

For the error estimate we note that the limit S lies between S_n and S_{n+1}. Hence $|S - S_n| \leq |S_{n+1} - S_n| = |b_{n+1}|$. □

Alternating Harmonic Series

The **alternating harmonic series** is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots .$$

It has the form $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with $b_n = 1/n$.

The sequence b_n satisfies $b_n \geq 0$, is decreasing, and converges to 0.

By the alternating series test, this series converges.

Example

Example 13

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$.

It has the form $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with $b_n = 1/n! > 0$.

The sequence (b_n) decreases and converges to 0, hence by the Alternating Series Test the given series converges.

Suppose we want to estimate the sum to 3 decimal places, i.e., with an error less than 0.0005. Observe that $1/7! = 0.0002$. So

$S_6 = \displaystyle\sum_{n=1}^{6} \frac{(-1)^{n+1}}{n!} = 0.368$ is accurate to 3 decimal places.

Absolute Convergence

A series $\sum a_n$ is called **absolutely convergent** if $\sum |a_n|$ converges.

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Example 14

Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Hence $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent.

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Theorem 15

If a series is absolutely convergent then it is also convergent.

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Theorem 15

If a series is absolutely convergent then it is also convergent.

Proof. Suppose $\sum |a_n|$ converges. Then $0 \leq a_n + |a_n| \leq 2|a_n|$. Now,

$$\sum |a_n| \text{ converges } \implies \sum 2|a_n| \text{ converges } \implies \sum (a_n + |a_n|) \text{ converges}$$
$$\implies \sum a_n = \sum ((a_n + |a_n|) - |a_n|) \text{ converges.}$$

□

Example

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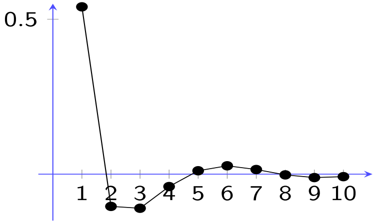
Example 16

Consider $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$. Here is a plot of the *terms* of the series:



The terms are neither always positive, nor always negative, nor alternating. No convergence test applies directly.

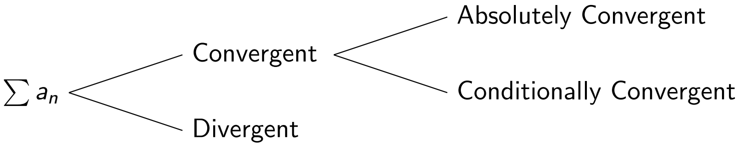
But we can use $0 \leq \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$ to show absolute convergence!

Conditional Convergence

A series is called **conditionally convergent** if it is convergent but not absolutely convergent.

For example, the alternating harmonic series is conditionally convergent.

Any series has three possibilities, as depicted below.



Rearrangements

Consider the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} \cdots$.

By adding the first three terms we can see that its sum is between 0.5 and 0.83. (We'll see later that it is exactly $\log 2 \approx 0.69$)

Let us rearrange it as follows, by moving the positive terms forward:

$$1 + \underbrace{\frac{1}{3} - \frac{1}{2} + \frac{1}{5}}_{>0} + \underbrace{\frac{1}{7} - \frac{1}{4} + \frac{1}{9}}_{>0} + \underbrace{\frac{1}{11} - \frac{1}{6} + \frac{1}{13}}_{>0} \cdots .$$

So the rearrangement is either divergent or has a sum > 1.

Changing the order of the terms can affect the convergence of a series!

Riemann Rearrangement Theorem

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Theorem 17

1. *A conditionally convergent series can be rearranged so that its sum equals any given real number or even diverges.*
2. *Any rearrangement of an absolutely convergent series is absolutely convergent and converges to the same sum.*

We'll not prove these statements.

Ratio Test

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Theorem 18

Consider a series $\sum a_n$ *with non-zero terms. Let* $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

1. $L < 1$ *implies absolute convergence.*
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Next, let $L > 1$. There is $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_{n+1}| > |a_n|$. So $n > N \implies |a_n| > |a_N| > 0$. Therefore $|a_n| \not\to 0$. Hence $a_n \not\to 0$ and $\sum a_n$ diverges.

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For the inconclusiveness of $L = 1$ consider $\sum 1/n$ and $\sum 1/n^2$. □

Example

Example 19

Consider the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, for any given $x \in \mathbb{R}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \left|\frac{x^{n+1}n!}{x^n(n+1)!}\right| = \left|\frac{x}{n+1}\right| \to 0.$$

$L = 0 < 1$ implies the series converges absolutely.

Root Test

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Theorem 20 (Root Test)

Consider a series $\sum a_n$. *Let* $\lim_{n\to\infty} |a_n|^{1/n} = L$.

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Now assume $L > 1$. There is $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n|^{1/n} > 1$, hence $|a_n| > 1$. Therefore $a_n \not\to 0$ and $\sum a_n$ diverges.

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Example

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Example 21

Consider the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| \frac{n^{2/n}}{2} \right| = \frac{1}{2}.$$

$L = 1/2 < 1$ implies the series converges absolutely.

Example

Example 22

Consider the series $\sum_{n=1}^{\infty} a_n$ with $a_n = \begin{cases} n/2^n & \text{if } n \text{ odd,} \\ 1/2^n & \text{if } n \text{ even.} \end{cases}$

$$\text{Ratio Test : } \left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} 1/(2n) & \text{if } n \text{ odd,} \\ (n+1)/2 & \text{if } n \text{ even,} \end{cases} \quad \text{diverges.}$$

So the Ratio Test gives no result.

$$\text{Root Test : } |a_n|^{1/n} = \begin{cases} n^{1/n}/2 & \text{if } n \text{ odd} \\ 1/2 & \text{if } n \text{ even} \end{cases} \to \frac{1}{2}.$$

$L = 1/2 < 1$ implies the series converges absolutely.

This example illustrates a general fact: If the ratio test works for a particular series, so will the root test. But it is possible that the root test works and the ratio test is inconclusive.

Use of Stirling's Approximation

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In a Root Test limit calculation, $(n!)^{1/n}$ can be replaced by n/e due to Stirling's approximation.

Consider the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$.

Apply the Root Test and Stirling's Approximation,

$$\lim_{n\to\infty} \left[\frac{(n!)^2}{2^{n^2}}\right]^{1/n} = \lim_{n\to\infty} \frac{(n!)^{2/n}}{2^n} = \lim_{n\to\infty} \frac{(n/e)^2}{2^n} = 0.$$

So the series converges.