February 2, 2015

Problems for Chapters 3 and 9 of *Advanced Mathematics for Applications*

The Fourier Series

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1 General

1. Find the Fourier series representing over $0 < x < \pi$ the function

$$u(x) \;=\; \begin{cases} \sin x + \cos x & 0 < x \le \frac{1}{2}\pi \\ \sin x - \cos x & \frac{1}{2}\pi \le < x < \pi \end{cases} \;.$$

2. Over the interval $0 < x < 2\pi$ sketch the function

$$u(x) \;=\; \begin{cases} x & 0 \le x \le \frac{1}{2}\pi \\ \frac{1}{2}\pi & \frac{1}{2}\pi \le x \le \pi \\ \pi - \frac{1}{2}x & \pi \le x \le 2\pi \end{cases}$$

and find its Fourier series.

3. Over the interval $0 < x < 2\pi$ find the Fourier series of the function

$$u(x) \;=\; \begin{cases} \sin\frac{x}{2} & 0 \le x \le \pi \\ -\sin\frac{x}{2} & \pi \le x \le 2\pi \end{cases} \;.$$

4. Over the interval $0 < x < \pi$ expand (a) in a sine and (b) a cosine series the function

$$u(x) \;=\; \begin{cases} \frac{1}{3}\pi & 0 < x < \frac{1}{3}\pi \\ 0 & \frac{1}{3}\pi < x < \frac{2}{3}\pi \\ -\frac{1}{3}\pi & \frac{2}{3}\pi < x < \pi \end{cases} \;.$$

5. Prove the Fourier series expansion

$$e^{ax} \;=\; \frac{\sinh \pi a}{\pi}\left[\frac{1}{a} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\left(a\,\cos nx - n\,\sin nx\right)\right].$$

Hence, deduce the sums of the two series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+a^2}\,, \qquad \sum_{n=1}^{\infty}\frac{1}{n^2+a^2}\,.$$

6. Prove, for $-\pi < x < \pi$, the Fourier series expansions

$$\sin \alpha x \;=\; \frac{2}{\pi}\,\sin \alpha\pi \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n \sin nx}{n^2 - \alpha^2}\,, \qquad \cos \alpha x \;=\; \frac{2}{\pi}\,\sin \alpha\pi \left(\frac{1}{2\alpha} + \sum_{n=0}^{\infty} \frac{(-1)^n n \cos nx}{n^2 + \alpha^2} \right).$$

7. Over the interval $-\pi < x < \pi$ find the Fourier series expansion of the function

$$u(x) \;=\; \begin{cases} \pi^2 & -\pi < x \le 0 \\ (x-\pi)^2 & 0 \le x < \pi \end{cases}\,.$$

Hence, deduce the sums of the two series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\,, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\,.$$

8. In $-\pi < x < \pi$ consider the ordinary differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + 4u \;=\; \frac{\pi^2}{12} - \frac{1}{4}x^2 - p(x)\,.$$

Find conditions on the function $p(x)$ which will ensure the existence of a periodic solution $u(x)$.

9. Consider the ordinary differential equation

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + a\frac{\mathrm{d}u}{\mathrm{d}x} + bu \;=\; f(x)$$

where a and b are given constants and $f(x)$ is periodic of period 2π. Expand u in a Fourier series and discuss conditions on a, b and f which ensure that u is also periodic with the same period.

2 Partial differential equations

1. Solve the equation

$$\frac{\partial u}{\partial t} \;=\; D\frac{\partial^2 u}{\partial x^2}\,, \qquad 0 < x < \frac{1}{4}\pi\,,$$

with $D > 0$ a given constant. The initial condition is $u(x,0) = x(\pi/4 - x)$ and the boundary conditions $u(0,t) = u(\pi/4,t) = 0$.

2. Solve the diffusion equation

$$\frac{\partial u}{\partial t} \;=\; \frac{\partial^2 u}{\partial x^2}\,, \qquad 0 < x < \pi$$

subject to the boundary conditions $\partial u/\partial x|_{x=0} = g(t)$, $\partial u/\partial x|_{x=\pi} = 0$, and to the initial condition $u(x,0) = 0$.

3. On $0 < x < \pi$ solve the diffusion equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \;=\; 0$$

subject to the boundary conditions $\partial u/\partial x|_{x=0} = g(t)$, $u(\pi,t) = 0$ and to the initial condition $u(x,0) = 0$.

4. A taut string (section 3.3) with mass μ per unit length and tension T is fixed ad its end points $x = 0$ and $x = L$ is initially at rest. At time $t = 0$ a uniform load q is applied to it and remains constant for $t > 0$. Determine the motion of the string.

5. A point mass m is attached at the center point of a string of length $2L$, mass μ per unit length and tension T fixed at its two ends. At time $t = 0$ the mass is displaced by a small amount h. Find the subsequent motion of the string if its initial velocity vanishes.

6. Use the method demonstrated in section 3.6 to solve the two-dimensional Laplace equation

$$\nabla^2 u = 0$$

inside the circle of radius a subject to $\mathbf{n} \cdot \boldsymbol{\nabla} u = f$, given, on the circle and to the condition of regularity at the center of the circle.

7. Use the method demonstrated in section 3.6 to solve the two-dimensional Poisson equation

$$\nabla^2 u = -2\pi f(\mathbf{x})$$

inside the circle $0 < r < a$ subject to $u = 0$ on the circle, and u regular at the center.

8. Use the method demonstrated in section 3.6 to solve the two-dimensional Laplace equation

$$\nabla^2 u = 0$$

outside the circle of radius a subject to $u = f$ given on the circle, $u \to 0$ at infinity.

9. Find the equilibrium configuration of a semicircular membrane of radius a, subjected to a tension T and clamped on its boundary, under the action of a time-independent load $f(\mathbf{x})$.

10. A substitution of the form $v(x,t) = e^{-\alpha t} u(x,t)$, with a suitable value of α, transforms the equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{k}{c^2}\frac{\partial u}{\partial t} + \frac{k^2}{4c^2} u = \frac{\partial^2 u}{\partial x^2}$$

into the one-dimensional telegrapher equation (see p. 9). Find the solution of this equation when

$$u(x,0) = \cos mx\,, \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0\,, \qquad u(0,t) = e^{kt/2}\,,$$

where m is an integer.

11. Find the spherically symmetric solution of the diffusion equation

$$\frac{\partial u}{\partial t} - \nabla^2 u = 0$$

inside a sphere of radius a subject to the initial condition $u(r,0) = u_0(r)$ and to the boundary conditions of regularity at the center of the sphere and $u(a,t) = 0$. This represents the cooling of a sphere having an initial temperature u_0.

12. Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0$$

where c is a given constant, inside a circle of radius a. u is required to vanish on the circle and to be regular at $r = 0$. The initial conditions are $u(\mathbf{x}, t=0) = 0$, $\partial u/\partial t|_{t=0} = \delta(\mathbf{x} - \mathbf{x}_0)$. This may be interpreted as describing the oscillations of a circular membrane (see p. 18) subjected to a unit impulsive load at the initial time.

13. Solve the system of equations

$$\frac{\partial u}{\partial t} = \nabla^2 u + b\nabla^2 v \,, \qquad \frac{\partial v}{\partial t} = b\nabla^2 u + \nabla^2 v \,,$$

where $b > 0$ is a given constant, inside the unit square. The boundary conditions are $u = v = 0$ on the boundary while, at $t = 0$, $u(x, 0) = f(x)$, $v(x, 0) = 0$. For what values of b is the solution bounded for all times?

14. A bar of length L, density ρ and cross-sectional area S is fixed at $x = 0$ and stretched by a force F applied at the other end $L = 0$. Determine the longitudinal oscillations of the bar if the force is suddenly removed at $t = 0$.

15. A force $F(t)$ is applied at $t = 0$ to the left end of bar of length L, density ρ and cross-sectional area S. Determine the subsequent longitudinal motion of the bar if it is initially at rest and undeformed. The bar is fixed at the other end $x = L$.

16. A bar simply supported at $x = 0$ and $x = L$ is in equilibrium under the action of a concentrated force F applied at the point $x = a$. Calculate the transverse motion of the bar after the force is suddenly removed at $t = 0$.

17. Solve, by means of a Fourier series in the angular variable, the analog of the wave guide problem of section 3.5 p. 72 for a domain consisting of a semi-infinite cylinder of radius a. At the base of the cylinder $u(r, \phi, z = 0) = f(r, \phi)$ given. Find the solution containing only outgoing waves (solution of the radial equation requires the use of Bessel functions).

18. Find the solution of the two-dimensional non-homogeneous Helmholtz equation

$$\nabla^2 u + k^2 u = -2\pi\delta^{(2)}(\mathbf{x} - \mathbf{y})$$

inside a circle of radius a with $u = 0$ on the circle and u regular at the center; the point $\mathbf{y}$ is fixed and given.

19. Set up a suitable coordinate system and solve the equation

$$\nabla^2 u - 2\beta\frac{\partial u}{\partial x} = -D\delta(\mathbf{x} - \mathbf{x}_s)$$

where $\beta > 0$ and D are given constants, in an infinite two-dimensional strip of width L parallel to the x axis. The solution is required to vanish as $x \to \pm\infty$ while, on the edges of the strip, the normal component of its gradient vanishes. The location of the point source $\mathbf{x}_s$ is midway between the two edges of the strip. After finding the solution of the problem, take the limit $\beta \to 0$ and disregard the infinite constant that arises with this operation. Explain (without necessarily doing it) how you would sum the remaining series.

3 Fourier series combined with Bessel functions

1. Solve the diffusion equation

$$\frac{\partial u}{\partial t} = \nabla^2 u$$

inside the circle of radius a centered at the origin; u is regular at the center of the circle and vanishes on the circle while, at $t = 0$, $u(\mathbf{x}, 0) = f(\mathbf{x})$.

2. Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0$$

where c is a given constant, inside a square of side L u vanishes on the boundary. The initial conditions are $u(\mathbf{x}, t = 0) = 0$, $\partial u / \partial t|_{t=0} = \delta(\mathbf{x} - \mathbf{x}_0)$. This may be interpreted as describing the oscillations of a square membrane (see p. 18) subjected to a unit impulsive load at the initial time.

3. Find eigenvalues and eigenfunctions of the Laplacian operator inside the portion of the unit circle bounded by the rays $\theta = 0$ and $\theta = \alpha$, with $0 < \alpha \leq 2\pi$. The eigenfunctions are required to vanish on the boundary and to be regular at the origin. Give an approximate explicit expression for the eigenvalues of high order.