Appendix SA12.1 Earlier Formulation of Energy Input–Output Models

A12.1.1 Introduction

In this appendix we present a more detailed description of the original formulation of the energy input–output model that is summarized in chapter 12. While still widely applied in the literature, this approach suffers from limitations that in some cases should preclude its use. This formulation was initially adopted by Strout (1967) and Bullard and Herendeen (1975a and 1975b), even though the Bullard and Herendeen subsequently developed the more contemporary hybrid units formulation to replace it, perhaps in many cases since the contemporary approach has additional data requirements. While still widely applied in the literature, however, this approach suffers from limitations that in some cases should preclude its use unless the data are unavailable to use the hybrid units approach. In other cases, as we will show, however, the model can be acceptable or even equivalent to the hybrid units formulation presented in chapter 12.

First, recall the $m \times n$ matrix of energy flows, $\mathbf{E}$, which was defined in the text of this chapter and used in the basic accounting relationship

$$\mathbf{Ei} + \mathbf{q} = \mathbf{g} \tag{A12.1.1}$$

The traditional approach to energy input–output analysis is to define a matrix of *direct energy coefficients,* $\mathbf{D} = [d_{kj}]$ where $d_{kj} = e_{kj} / x_j$, that is, the amount of energy type k (in Btus or some other convenient energy units for $k = 1,..., m$) required directly to produce a dollar's worth of each producing sector's output ($j = 1,..., n$). Expressed in matrix terms this is $\mathbf{D} = \mathbf{E}\hat{\mathbf{x}}^{-1}$. This is, of course, directly analogous to the direct input coefficients, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, except that $\mathbf{D}$ will in general not be square since $m < n$.

For purposes that will become clear later, defining a direct energy coefficient is equivalent to first defining a matrix, $\mathbf{P}$, of *implied energy prices,* with elements defined as $p_{kj} = z_{kj} / e_{kj}$ ($k = 1,\ldots, m$; $j = 1,..., n$), defined only for $e_{kj} \neq 0$. The units p_{kj} are then dollars paid per unit of energy of type k delivered to consuming sector j. These prices are "implied" since the prices calculated in this way generally do not necessarily correspond to the price actually paid for energy, but their significance, nonetheless, will become clear shortly. For now, implied prices can be used to derive the direct energy coefficients as $d_{kj} = \frac{a_{kj}}{p_{kj}}$. This is equivalent to our previous definition of $\mathbf{D}$, since $d_{kj} = \frac{a_{kj}}{p_{kj}} - \left(\frac{z_{kj}}{x_j}\right)\left(\frac{e_{kj}}{z_{kj}}\right) = \frac{e_{kj}}{x_j}$ or, in matrix terms, $\mathbf{D} = \mathbf{E}\hat{\mathbf{x}}^{-1}$. It follows directly that $\mathbf{E} = \mathbf{D}\hat{\mathbf{x}}$ and from the original energy transactions balance equation $\mathbf{Ei} + \mathbf{q} = \mathbf{g}$, we obtain $\mathbf{D}\hat{\mathbf{x}}\mathbf{i} + \mathbf{q} = \mathbf{g}$ but, since $\hat{\mathbf{x}}\mathbf{i} = \mathbf{x}$, $\mathbf{Dx} + \mathbf{q} = \mathbf{g}$, which, as noted earlier, is directly analogous to $\mathbf{Ax} + \mathbf{f} = \mathbf{x}$ of the traditional Leontief model. The traditional method continues to develop a matrix of total energy coefficients first by substituting $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ to obtain

$$\mathbf{Dx} = \mathbf{D}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} \tag{A12.1.2}$$

The matrix $\mathbf{D}(\mathbf{I}-\mathbf{A})^{-1}$ is defined as the matrix of total interindustry energy coefficients. In order to account for the energy consumed directly by final demand, the second term in the energy transactions balance equation (A12.1.1), we return to the notion of implied energy prices, this time for the energy that is delivered to final demand (as done in earlier interindustry transactions when the p were defined – recall that direct energy coefficients were defined only for interindustry energy transactions). Now we have $\mathbf{p_f} = [p_{kf}]$ where

$$p_{kf} = f_k / q_k \qquad \text{(A12.1.3)}$$

Here f_k is the final demand in dollars for the output of energy sector k and p_{kf} is the corresponding implied energy price in units of dollars of final demand per unit of energy type k (for $q_k \neq 0$; for $q_k = 0$ we will define $p_{kf} = 0$). This relationship allows us to express final demand and the corresponding energy requirements in a manner similar to that for interindustry energy requirements associated with interindustry transactions, by rewriting (A12.1.3) as $q_k = (1/p_{kf})f_k$ or in matrix terms as $\mathbf{q} = \tilde{\mathbf{Q}}\mathbf{f}$, where $\tilde{\mathbf{Q}} = [\tilde{q}_k]$ is an $m \times n$ matrix of implied inverse energy prices for final demand whose elements are defined as

$$\tilde{q}_k = \begin{cases} 1/p_{kf}, & \text{when energy sector } k \text{ and industry sector } j \text{ describe the same industrial sector} \\ 0, & \text{otherwise} \end{cases}$$

There will, of course, be at most m nonzero elements in $\tilde{\mathbf{Q}}$ since there are only m elements in $\mathbf{q}$. By constructing $\tilde{\mathbf{Q}}$ of dimension $m \times n$, we can combine it with the interindustry energy coefficients to produce a matrix of total (interindustry plus final-demand) energy coefficients, to obtain $\mathbf{g} = \mathbf{Dx} + \mathbf{q}$ or, substituting $(\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ for $\mathbf{x}$, we have $\mathbf{g} = \mathbf{D}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{f} + \tilde{\mathbf{Q}}\mathbf{f}$ and collecting terms we have

$$\mathbf{g} = [\mathbf{D}(\mathbf{I}-\mathbf{A})^{-1} + \tilde{\mathbf{Q}}]\mathbf{f} \qquad \text{(A12.1.4)}$$

The bracketed quantity, which we denote by $\boldsymbol{\varepsilon}$, is a matrix of *total energy coefficients* analogous to $\boldsymbol{\alpha}$ defined in the course of developing the energy conservation conditions, which expresses the total amount of energy (Btus) required of each energy type, $\mathbf{g}$, both directly and indirectly, as a function of final demand $\mathbf{f}$.

Variations of this approach abound in the literature, sometimes ignoring the energy consumed directly in final demand, sometimes assuming uniform energy prices across all consuming sectors, but almost always defining a set of direct energy coefficients in this manner and thereby ignoring or assuming away the technical energy conservation relationships between primary and secondary energy sectors.

A12.1.2 Illustration of the Implications of the Traditional Approach

The following example illustrates the inconsistencies introduced by using variations of the traditional approach just outlined.

Example 12.5: Energy Input–Output Alternative Formulation Consider a simple three-sector input–output economy where two of the sectors are energy sectors, coal and electricity.

Assume that transactions (in millions of dollars) observed for a given year are as shown in Table A12.1.1. Here **Z**, **f** and **x** are $\mathbf{Z}=\begin{bmatrix}0 & 40 & 0\\ 10 & 10 & 10\\ 0 & 0 & 0\end{bmatrix}$, $\mathbf{f}=\begin{bmatrix}0\\ 30\\ 10\end{bmatrix}$ and $\mathbf{x}=\begin{bmatrix}40\\ 60\\ 100\end{bmatrix}$. Suppose that the corresponding energy flows of this economy, expressed in quadrillions of Btus, are given by Table A12.1.2. Hence, using notation introduced earlier,

$$\mathbf{E}=\begin{bmatrix}0 & 120 & 0\\ 20 & 20 & 20\end{bmatrix},\ \mathbf{q}=\begin{bmatrix}0\\ 60\end{bmatrix} \text{ and } \mathbf{g}=\begin{bmatrix}120\\ 120\end{bmatrix}.$$

Note some of the special characteristics of this energy economy. First, the coal sector delivers all of its product to the electricity sector, another energy sector. Hence, as discussed earlier, the coal sector is known as a *primary* energy sector and electricity is a *secondary* energy sector. Note also that the total amount of coal used is the same as the amount of electricity consumed in the economy, which seems reasonable since the electricity sector received all of its primary energy from coal (excluding conversion efficiencies, for the moment). Another important peculiarity of this example is the matrix of *implied energy prices,*

$\mathbf{P}=[z_{kj}/e_{kj}]=\begin{bmatrix}0 & 40/120 & 0\\ 10/20 & 10/20 & 10/20\end{bmatrix}=\begin{bmatrix}0 & 0.333 & 0\\ 0.5 & 0.5 & 0.5\end{bmatrix}$ and $\mathbf{p_f}=[f_k/q_k]\begin{bmatrix}0\\ 0.5\end{bmatrix}$. Note that the price of electricity

Table A12.1.1 Dollar Transactions for Example 12.5 (millions of dollars)

	Coal	Electric Power	Autos	Final Demand	Total Output
Coal	0	40	0	0	40
Electric Power	10	10	10	30	60
Automobiles	0	0	0	100	100

Table A12.1.2 Energy Flows for Example 12.5 (10^{15} Btus)

	Coal	Electric Power	Autos	Final Energy Demand	Total Energy Output
Coal	0	120	0	0	120
Electric Power	20	20	20	60	120

is the same across all consuming sectors, including final demand (0.5). Hence, by the earlier development, $\tilde{\mathbf{Q}}$, the matrix of inverse energy prices for final demand, and **D**, the direct energy coefficients matrix are $\tilde{\mathbf{Q}}=\begin{bmatrix}q_1 & 0 & 0\\ 0 & q_2 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 0\end{bmatrix}$ and

$$\mathbf{D}=\begin{bmatrix}0 & 120 & 0\\ 20 & 20 & 20\end{bmatrix}\begin{bmatrix}1/40 & 0 & 0\\ 0 & 1/60 & 0\\ 0 & 0 & 1/100\end{bmatrix}=\begin{bmatrix}0 & 2 & 0\\ 0.5 & 0.333 & 0.2\end{bmatrix}$$. Finally, we can compute

$$\mathbf{A}=\mathbf{Z}\hat{\mathbf{x}}^{-1}=\begin{bmatrix}0 & 0.667 & 0\\ 0.25 & 0.167 & 0.1\\ 0 & 0 & 0\end{bmatrix} \text{ and } (\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}1.25 & 1.00 & 0.10\\ 0.38 & 1.50 & 0.15\\ 0 & 0 & 1.00\end{bmatrix}$$. Knowing $\mathbf{D}, (\mathbf{I}-\mathbf{A})^{-1}$ and $\tilde{\mathbf{Q}}$ and using (A12.1.4) we can find the total energy requirements,

$$\boldsymbol{\varepsilon}=\mathbf{D}(\mathbf{I}-\mathbf{A})^{-1}+\tilde{\mathbf{Q}}=\begin{bmatrix}0.75 & 3 & 0.3\\ 0.75 & 3 & 0.3\end{bmatrix}.$$

It should not be surprising, at least for this example, that the rows of ε are identical, because of the peculiarities of this energy economy noted earlier. Suppose, however, we change the example only slightly to remove the uniformity of energy prices across consuming sectors.

Example 12.6: Energy Input–Output Example (Revised) In modifying Example 12.5, only slightly, we redefine only E and Q as shown in Table A12.1.3, i.e., increasing the amount of electricity consumed by the autos sector from 20 to 30 quads and reducing the amount of electricity consumed by final demand from 60 to 50 quads (denoted in bold face in the table).

Table A12.1.3 Energy Flows for Example 1 Revised (10^{15} Btus)

	Coal	Electric Power	Autos	Final Energy Demand	Total Energy Output
Coal	0	120	0	0	120
Electric Power	20	20	**30**	**50**	120

Note that we do not change the total energy consumption of 120 quads in Table A12.1.2 and we do not change the economic transactions measured in dollars, **Z**. However, since some of the energy transactions measured in quads change, the corresponding relative interindustry and final-demand energy prices change as well. The new implied energy prices for interindustry and final-demand sales along with the matrix of implied inverse energy prices to final demand are then, respectively, given by $\mathbf{P}=\begin{bmatrix}0 & 0.333 & 0\\ 0.5 & 0.5 & 0.333\end{bmatrix}$, $\mathbf{p_f}=\begin{bmatrix}0\\ 0.599\end{bmatrix}$ and $\tilde{\mathbf{Q}}=\begin{bmatrix}0 & 0 & 0\\ 0 & 1.67 & 0\end{bmatrix}$. Note that the prices are no longer uniform. Finally $\boldsymbol{\varepsilon}$, the matrix of total energy coefficients, becomes $\boldsymbol{\varepsilon}=\begin{bmatrix}0.75 & 3 & 0.3\\ 0.75 & 2.667 & 0.4\end{bmatrix}$. Looking at the elements in the third column, this new total energy requirements matrix specifies that one dollar's worth of automobiles requires 0.4×10^{15} Btus of electricity to produce that output, but only 0.3×10^{15} Btus of coal. This violates the energy conservation condition for this example, since the electricity-producing sector received all its primary energy from coal (electricity is a pass-through sector for coal). In other words, by design for this example the two total energy requirements matrix rows should be the same.

It should be readily apparent that application of this energy input–output formulation simply yields the output of the traditional Leontief model multiplied by a set of conversion factors – the implied energy prices. Such formulations are frequently applied in the literature, but in the following we show more generally that this formulation provides internally consistent results only when these energy prices are the same across *all* consuming sectors (including final demand) for each energy type or when a new final demand presented to the economy is very close to that from which the input–output model was originally derived. Only under such circumstances will the model always faithfully reproduce the original data. Griffin (1976) shows that the condition of uniform prices across all energy-consuming sectors does not hold at all historically for the US economy. Similar results are illustrated in Weisz and Duchin (2006). Possible cases where it could be more acceptable are discussed later. For reference, Table A12.1.4 summarizes the alternative energy formulation just described compared with the analogous "hybrid units" formulation developed in this chapter and so-called physical input–output models.

Table A12.1.4 Summary of Energy Input–Output Relationships

	Economic Model	Hybrid Units Energy Model Method II	Alternative Energy Model Method I
Transactions	$\mathbf{Z}$ $\mathbf{Zi}+\mathbf{f}=\hat{\mathbf{x}}$	$\mathbf{Z}^*, \mathbf{E}$ $\mathbf{Z}^*\mathbf{i}+\mathbf{f}^*=\mathbf{x}^*$ $\mathbf{Ei}+\mathbf{q}=\mathbf{g}$	$\mathbf{E}$ $\mathbf{g}=\mathbf{Ei}+\mathbf{q}$
Direct Requirements	$\mathbf{A}=\mathbf{Z}\hat{\mathbf{x}}^{-1}$ $\mathbf{Ax}+\mathbf{f}=\mathbf{x}$	$\mathbf{A}^*=\mathbf{Z}^*\hat{\mathbf{x}}^{-1}$; $\boldsymbol{\delta}=\mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{A}^*$ $\mathbf{A}^*\mathbf{x}+\mathbf{f}^*=\mathbf{x}^*$; $\boldsymbol{\delta}\mathbf{x}^*+\mathbf{q}=\mathbf{g}$	$\mathbf{D}=\mathbf{E}\hat{\mathbf{x}}^{-1}$ $\mathbf{Dx}+\mathbf{q}=\mathbf{g}$
Total Requirements	$(\mathbf{I}-\mathbf{A})^{-1}$ $\mathbf{x}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}$	$(\mathbf{I}-\mathbf{A}^*)^{-1}$; $\boldsymbol{\alpha}=\mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I}-\mathbf{A}^*)^{-1}$ $\mathbf{x}^*=(\mathbf{I}-\mathbf{A}^*)^{-1}\mathbf{f}^*$; $\mathbf{g}=\boldsymbol{\alpha}\mathbf{f}^*$	$\boldsymbol{\varepsilon}=\mathbf{D}(\mathbf{I}-\mathbf{A})^{-1}+\tilde{\mathbf{Q}}$ $\mathbf{g}=\boldsymbol{\varepsilon}\mathbf{f}$

We now explore further the conditions of energy conservation and the conditions under which the alternative model can be applied, first through an example and then more generally. First, however, for reference, in Table A12.1.4 we summarize the relationships developed so far for the traditional Leontief model, the hybrid units interindustry model defined in the text of this chapter, which we will refer to as Method II and, finally, the alternative energy model just defined, which we will refer to as Method I.

Extensions of Example 12.1 Recall the two-sector economy given in Example 12.1 of the text of this chapter where we constructed the following hybrid units energy input–output relationships using Method II: $\mathbf{Z}^*=\begin{bmatrix}10 & 20\\ 60 & 80\end{bmatrix}$ and $\mathbf{x}=\begin{bmatrix}100\\ 240\end{bmatrix}$. The direct and total energy requirements matrices (Method II) for this example are $\mathbf{A}^*=\mathbf{Z}^*\left(\hat{\mathbf{x}}^*\right)^{-1}=\begin{bmatrix}0.100 & 0.083\\ 0.600 & 0.333\end{bmatrix}$ and

$\mathbf{L}^* = \begin{bmatrix} 1.212 & 1.515 \\ 1.091 & 1.636 \end{bmatrix}$ so that $\boldsymbol{\delta} = \mathbf{G}\left(\hat{\mathbf{x}}^*\right)^{-1}\mathbf{A}^* = \begin{bmatrix} 0.600 & 0.333 \end{bmatrix}$ and

$\boldsymbol{\alpha} = \mathbf{G}\left(\hat{\mathbf{x}}^*\right)^{-1}\mathbf{L}^* = \begin{bmatrix} 1.091 & 1.636 \end{bmatrix}$. The analogous information for the alternative energy input–output formulation, using Method I, is given by the energy transactions $\mathbf{E} = \begin{bmatrix} 60 & 80 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} 100 \end{bmatrix}$ and the interindustry dollar transactions and total outputs by $\mathbf{Z} = \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$. Hence, the direct and total energy requirements matrices are

$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0.100 & 0.167 \\ 0.300 & 0.333 \end{bmatrix}$ and $\left(\mathbf{I} - \mathbf{A}\right)^{-1} = \begin{bmatrix} 1.212 & 0.303 \\ 0.546 & 1.636 \end{bmatrix}$. Using **q,** we have

$\tilde{\mathbf{Q}} = \begin{bmatrix} 0 & 100/50 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix}$ and $\mathbf{D} = \mathbf{E}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 60 & 80 \end{bmatrix}\begin{bmatrix} 1/100 & 0 \\ 0 & 1/120 \end{bmatrix} = \begin{bmatrix} 0.600 & 0.667 \end{bmatrix}$.

It follows directly from (A12.1.4) that $\boldsymbol{\varepsilon} = \mathbf{E}\hat{\mathbf{x}}^{-1}(\mathbf{I} - \mathbf{A})^{-1} + \tilde{\mathbf{Q}} = \mathbf{D}(\mathbf{I} - \mathbf{A})^{-1} + \tilde{\mathbf{Q}}$, which for the example is $\boldsymbol{\varepsilon} = \begin{bmatrix} 0.600 & 0.667 \end{bmatrix}\begin{bmatrix} 1.212 & 0.303 \\ 0.546 & 1.636 \end{bmatrix} + \begin{bmatrix} 0 & 2 \end{bmatrix} = \begin{bmatrix} 1.091 & 3.273 \end{bmatrix}$. Note that **ε** is identical to **α**, except that the elements involving energy consumption in **α** are simply multiplied by the relevant energy price. This is reasonable because **ε** is used in conjunction with **f** and not with **f***. That is, $\mathbf{f}^* = \begin{bmatrix} 70 \\ 100 \end{bmatrix}$ with an energy price of 2 (10^{15} Btus/\$$10^6$) is equivalent to $\mathbf{f} = \begin{bmatrix} 70 \\ 50 \end{bmatrix}$, so, we have $\boldsymbol{\varepsilon}\mathbf{f} = \begin{bmatrix} 1.091 & 3.272 \end{bmatrix}\begin{bmatrix} 70 \\ 50 \end{bmatrix} = 240$ and $\boldsymbol{\alpha}\mathbf{f}^* = \begin{bmatrix} 1.091 & 1.636 \end{bmatrix}\begin{bmatrix} 70 \\ 100 \end{bmatrix} = 240$. The first expression, **εf**, generates the total energy requirement (240×10^{15} Btus) needed to support final demand **f**. The second expression, **αf***, yields the same result but in terms of supporting the equivalent final demand, **f***, measured in hybrid units.

The result should not be surprising at all, since under conditions of uniform interindustry energy prices, the computation of **ε** is simply a price adjustment of **α**. To reflect this in our notation, we define a two-element vector $\mathbf{r} = \begin{bmatrix} 1 & 2 \end{bmatrix}$ where the first element is the value that converts the nonenergy units of the original model to the nonenergy units of the hybrid units model. Clearly these units are the same, so the value of this element is always unity. The second element is the interindustry inverse energy price.

Given this vector **r**, we can easily write $\mathbf{f}^* = \hat{\mathbf{r}}\mathbf{f}$. For the example this is $\mathbf{f}^* = \hat{\mathbf{r}}\mathbf{f} = \begin{bmatrix} 70 \\ 100 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 70 \\ 50 \end{bmatrix}$. Also, $\mathbf{x}^* = \hat{\mathbf{r}}\mathbf{x}$ or $\mathbf{x} = \hat{\mathbf{r}}^{-1}\mathbf{x}^*$. For the example this is $\mathbf{x}^* = \hat{\mathbf{r}}\mathbf{x} = \begin{bmatrix} 100 \\ 240 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 100 \\ 120 \end{bmatrix}$. The implications of this are as follows. For the case of uniform interindustry energy prices, there is no need to account for energy in Btus at all, since this is equivalent to deriving outputs in dollars and converting to Btus by simply multiplying by the energy price. However, as we found before, if prices are not uniform for all consumers (both interindustry and final-demand consumers), such procedures are inappropriate.

It is important to note that the above result, i.e., that a vector **r** exists such that $\mathbf{x}^* = \hat{\mathbf{r}}\mathbf{x}$ and $\mathbf{f}^* = \hat{\mathbf{r}}\mathbf{f}$, will in general be true *only* under conditions of uniform energy prices, which we will illustrate in the following. Recall that in the case of using the alternative formulation in Example 12.1, when this condition was not met, the model gave inappropriate results. We can test to see if the hybrid units model fares better when we relax the condition of uniform energy prices by considering, once again, the two-sector model of Example 12.1 with new energy flows and corresponding energy prices.

Note that the dollar quantities, **Z**, **f**, **x**, **A** and $(\mathbf{I} - \mathbf{A})^{-1}$ do not change at all from the earlier case. However, the hybrid units quantities change since the energy transactions have changed by reducing the amount of energy delivered to final demand by 20 quadrillion Btus and increasing the amount of energy consumed by the energy sector itself by the same amount, thus keeping total energy output the same. With a change in energy flows but no change in the corresponding dollar transactions, the energy prices change and are no longer uniform for all consumers, as shown in Table A12.1.6.

Table A12.1.5 Energy and Dollar Flows for Example 12.1 (Revised)

	Widgets	Energy	Final Demand	Total Output
	Value Transactions in Millions of Dollars			
Widgets	10	20	70	100
Energy	30	40	50	120
	Energy Transactions in Quadrillions of Btus			
Energy	60	**100**	**80**	240

Table A12.1.6 Implied Energy Prices for Example 12.1 (Revised)

10^{15} Btus/\$$10^6$	Widgets	Energy	Final Demand	Total Output
Energy	2	2.5	1.6	2

Table A12.1.7 Results for Example 12.1 (Revised)

Method I: Alternative Formulation	Method II: Hybrid units Model
$\mathbf{Z} = \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 100 \\ 120 \end{bmatrix}$	$\mathbf{Z}^* = \begin{bmatrix} 10 & 20 \\ 60 & 100 \end{bmatrix} \mathbf{x}^* = \begin{bmatrix} 100 \\ 240 \end{bmatrix}$
$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0.100 & 0.167 \\ 0.300 & 0.333 \end{bmatrix}$	$\mathbf{A}^* = \mathbf{Z}^*\left(\hat{\mathbf{x}}^*\right)^{-1} = \begin{bmatrix} 0.100 & 0.083 \\ 0.600 & 0.417 \end{bmatrix}$

$$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.212 & 0.303 \\ 0.546 & 1.636 \end{bmatrix} \qquad \mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 1.228 & 0.175 \\ 1.263 & 1.895 \end{bmatrix}$$

As before, from the conventions of the alternative formulation (Method I) and of the hybrid units formulation (Method II) we can derive the results given in Table A12.1.7. We can now calculate the total energy coefficients by the two methods.

Method 1

$$\mathbf{D} = \mathbf{E}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 60 & 100 \end{bmatrix} \begin{bmatrix} 1/100 & 0 \\ 0 & 1/120 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.833 \end{bmatrix}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}(\mathbf{I} - \mathbf{A})^{-1} + \tilde{\mathbf{Q}} = \begin{bmatrix} 0.6 & 0.833 \end{bmatrix} \begin{bmatrix} 1.212 & 0.303 \\ 0.546 & 1.636 \end{bmatrix} + \begin{bmatrix} 0 & 8/5 \end{bmatrix} = \begin{bmatrix} 1.182 & 3.145 \end{bmatrix}$$

From this we can verify that, since $\mathbf{f} = \begin{bmatrix} 70 \\ 50 \end{bmatrix}, \boldsymbol{\varepsilon}\mathbf{f} = \begin{bmatrix} 1.182 & 3.145 \end{bmatrix} \begin{bmatrix} 70 \\ 50 \end{bmatrix} = 240$.

Method 2

$$\mathbf{G}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0 & 240 \end{bmatrix} \begin{bmatrix} 1/100 & 0 \\ 0 & 1/240 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1.228 & 0.175 \\ 1.263 & 1.895 \end{bmatrix} = \begin{bmatrix} 1.263 & 1.895 \end{bmatrix}$$

From this we can verify that, since $\mathbf{f}^* = \begin{bmatrix} 70 \\ 80 \end{bmatrix}, \boldsymbol{\alpha}\mathbf{f}^* = \begin{bmatrix} 1.263 & 1.895 \end{bmatrix} \begin{bmatrix} 70 \\ 80 \end{bmatrix} = 240$. Both methods thus yield the same total energy requirements for the basic data from which the models were originally formulated. However, this is not generally true. Consider two cases of *new* final-demand vectors for which we wish to compute the total energy requirement by both Methods I and II.

Case 1. Consider two final demand vectors, **f** and **f** *, which describe the same final demand since the energy price to final demand is 8/5, so that $\mathbf{f} = \begin{bmatrix} 100 \\ 333.1 \end{bmatrix}$ and $\mathbf{f}^* = \begin{bmatrix} 100 \\ 533 \end{bmatrix}$. That is, the relationship between f_2^* and f_2 is $f_2^* = f_2(8/5) = (333.1)(8/5) = 533$. Computing the total energy requirement by the two methods:

Method I	Method II
$\boldsymbol{\varepsilon}\mathbf{f} = \begin{bmatrix} 1.182 & 3.145 \end{bmatrix} \begin{bmatrix} 100 \\ 333.1 \end{bmatrix} = 1{,}166$	$\boldsymbol{\alpha}\mathbf{f}^* = \begin{bmatrix} 1.263 & 1.895 \end{bmatrix} \begin{bmatrix} 100 \\ 533 \end{bmatrix} = 1{,}136$

Case 2. Consider another equivalent pair of final demands, defined as $\mathbf{f} = \begin{bmatrix} 1{,}000 \\ 10 \end{bmatrix}$ and $\mathbf{f}^* = \begin{bmatrix} 1{,}000 \\ 16 \end{bmatrix}$, for which the total energy requirement by the two and methods are:

Method I	Method II
$\boldsymbol{\varepsilon}\mathbf{f} = \begin{bmatrix} 1.182 & 3.145 \end{bmatrix} \begin{bmatrix} 1{,}000 \\ 10 \end{bmatrix} = 1{,}031.90$	$\boldsymbol{\alpha}\mathbf{f}^* = \begin{bmatrix} 1.263 & 1.895 \end{bmatrix} \begin{bmatrix} 1{,}000 \\ 16 \end{bmatrix} = 1{,}293.32$

Note that in Case 1, using Method I results in a higher total energy requirement than using Method II and a lower amount in Case 2. In the following we will show that Method II always computes the total energy requirement correctly. We can then conclude that in Cases 1 and 2, Method I overestimates and underestimates, respectively, the total energy requirement.

A12.1.3 General Limitations of the Alternative Formulation

We return briefly to the alternative formulation of total energy coefficients derived earlier and defined in (A12.1.4): $\mathbf{g} = [\mathbf{D}(\mathbf{I} - \mathbf{A})^{-1} + \tilde{\mathbf{Q}}]\mathbf{f}$. With an arbitrary final demand, denoted as $\mathbf{f}^{new}$, and the corresponding total energy requirement as $\mathbf{g}^{new}$, then we define the total output vector used in defining the total energy coefficients as $\mathbf{x}^{old}$. $\mathbf{D}$ is computed as $\mathbf{D} = \mathbf{E}(\hat{\mathbf{x}}^{old})^{-1}$. Combining the expressions for $\mathbf{g}^{new}$ and $\mathbf{D}$ we obtain

$$\mathbf{g}^{new} = \mathbf{E}\left(\hat{\mathbf{x}}^{old}\right)^{-1}\left(\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{f}^{new} + \tilde{\mathbf{Q}}\mathbf{f}^{new} = \mathbf{E}\left(\hat{\mathbf{x}}^{old}\right)^{-1}\mathbf{x}^{new} + \tilde{\mathbf{Q}}\mathbf{f}^{new} \qquad \text{(A12.1.5)}$$

If $\mathbf{x}^{old} = \mathbf{x}^{new}$, then the product $\left(\hat{\mathbf{x}}^{old}\right)^{-1}\mathbf{x}^{new}$ will be a column vector of ones. In addition, by definition, $\mathbf{q} = \tilde{\mathbf{Q}}\mathbf{f}^{new}$, and, hence, (A12.1.5) becomes $\mathbf{g}^{new} = \mathbf{E}\mathbf{i} + \mathbf{q}$, which is (A12.1.1) from which the total energy coefficients were originally derived. If $\mathbf{x}^{old} \neq \mathbf{x}^{new}$ however which is the case for most applications, the model does not reduce to (A12.1.1) and does not accurately reflect the energy flows generated by a new final demand.

We can conclude that while Method II (the hybrid units formulation) correctly computes in all cases the total energy requirement for any arbitrary vector of final demands consistent with our energy conservation condition, Method I yields correct results only for the base case of final demands from which the model was originally derived, or, as it turns out, if the new final-demand vector is a linear combination of the reference case of final demands (the same scalar multiplied by every element of the final-demand vector, which might be interpreted as uniform economic growth). Hence, in general, if the necessary data are available, the only defense for using Method I in practice is when impact analysis involves new final demands that are not substantially different from the basic data from which the model was derived or when there are uniform interindustry energy prices throughout the economy.[1]

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[1] Herendeen (1974) suggested an ad hoc modification procedure for enforcing consistency.