Solutions to A Student's Guide to the Navier-Stokes Equations

Justin W. Garvin
The University of Iowa

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Chapter 1 Solutions

Problems

1.1 Find the mass flow rate of a fluid at a constant density of 1.2 kg/m^3 passing through a surface whose area is 1 m^2 and whose normal is $\vec{n} = \frac{1}{\sqrt{14}}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$. The velocity of the flow is: $\vec{V} = 2\hat{i} + 3\hat{j} + 0\hat{k}$ m/s.

<u>**Solution:**</u> The mass passing through a surface is defined as:

$$\dot{m}_{pass} = \oiint_A \rho\vec{V} \cdot \vec{n} dA$$

We can plug our numbers in:

$$\begin{aligned}
\dot{m}_{pass} &= \oiint_A \rho\vec{V} \cdot \vec{n} dA \\
&= \oiint_A 1.2\frac{\text{kg}}{\text{m}^3}\left(2\hat{i} + 3\hat{j}\right)\text{m/s} \cdot \left(\frac{1}{\sqrt{14}}\left(\hat{i} + 2\hat{j} + 3\hat{k}\right)\right) dA \\
&= \oiint_A 1.2\frac{\text{kg}}{\text{m}^3}\left(2 + 6 + 0\right)\frac{1}{\sqrt{14}}\text{m/s} dA \\
&= 1.2\frac{\text{kg}}{\text{sm}^2}\frac{8}{\sqrt{14}} A\Big|_0^{1\ \text{m}^2} \\
&= \frac{9.6}{\sqrt{14}}\text{kg/s}
\end{aligned}$$

1.2 Approximate how long will it take, in minutes, to fill up a bathtub with dimensions of 1.3 m x 0.6 m x 0.4 m if water is coming out of a 40 mm diameter faucet at a speed of 1.2 m/s. Assume the density of water is 1000 kg/m^3.

Solution: This is a basic mass conservation problem. In words, we have:

$$\begin{array}{c}\text{the change}\\ \text{of mass of water}\\ \text{in the bathtub}\\ \text{in a given}\\ \text{unit of time}\end{array} = \begin{array}{c}\text{time rate of}\\ \text{mass of water}\\ \text{entering the}\\ \text{bathtub}\end{array} - \begin{array}{c}\text{time rate of}\\ \text{mass of water}\\ \text{leaving the}\\ \text{bathtub}\end{array}$$

There is no mass coming out, which leaves us with:

$$\begin{array}{c}\text{the change}\\ \text{of mass of water}\\ \text{in the bathtub}\\ \text{in a given}\\ \text{unit of time}\end{array} = \begin{array}{c}\text{time rate of}\\ \text{mass of water}\\ \text{entering the}\\ \text{bathtub}\end{array}$$

So, the change of the mass in a given unit of time in the tub is just equal to the mass flow rate into the tube:

$$\dot{m}_{in} = \oiint_A \rho \vec{V} \cdot \vec{n} dA \xrightarrow{\text{simplifies to}} \rho V A$$

It simplifies to $\rho V A$ because ρ and $\vec{V}$ are constant and can be pulled out of the area integral. In addition, it is assumed that the velocity and the normal are in the same direction. Plugging in numbers:

$$\dot{m}_{in} = \rho V A = (1000)\,(1.2)\left(\frac{\pi}{4} 0.04^2\right) = 0.302\text{kg/s}$$

Thus the tube fills at a rate of 0.302 kg/s. The total mass of water that the tube ($mass_{total}$) can hold is the volume of the tub multiplied by the density of water:

$$mass_{tube} = (1000)\,(1.3 \times 0.6 \times 0.4) = 312\text{kg}$$

The length of times to fill is:

$$time = \frac{mass_{total}}{\dot{m}_{in}} = \frac{312\text{kg}}{0.302\text{kg/s}} = 1033 \text{ s} = 17 \text{ minutes}$$

1.3 The divergence of the velocity vector in spherical coordinates can be written as:

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2}\frac{\partial\left(r^2 V_r\right)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}\left(V_\theta \sin(\theta)\right) + \frac{1}{r\sin(\theta)}\frac{\partial V_\phi}{\partial\phi}$$

where V_r, V_θ, and V_ϕ are the velocity coordinates in the $r-$, $\theta-$, and $\phi-$ direction, respectively. Determine if a flow with the following flow field velocity is incompressible:

$$V_r = -U\cos(\theta)\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)$$

$$V_\theta = U\sin(\theta)\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)$$

$$V_\phi = 0$$

where R and U are constants (note, R is not the gas constant in this problem).

<u>**Solution:**</u> This one is simple enough. We just have to plug in the V_r, V_{theta}, and V_z expression into the divergence of velocity equation given for spherical coordinates and see if it is equal to zero, since the divergence of velocity is zero for an incompressible flow. The easiest term in the velocity divergence term is the last one since V_ϕ is zero:

$$\frac{1}{r\sin\theta}\frac{\partial V_{phi}}{\partial\phi} = \frac{1}{r\sin\theta}\frac{\partial 0}{\partial\phi} = 0$$

The next term we will look is the first term of the velocity divergence. Before we do anything, let's break out the derivative using the product rule:

$$\begin{aligned}\frac{1}{r^2}\frac{\partial\left(r^2 V_r\right)}{\partial r} &= \underbrace{\frac{1}{r^2}r^2}_{=1}\frac{\partial V_r}{\partial r} + \frac{1}{r^2}V_r\underbrace{\frac{\partial r^2}{\partial r}}_{=2r} \quad \text{(product rule)} \\ &= \frac{\partial V_r}{\partial r} + \frac{2}{r}V_r\end{aligned} \tag{1.1}$$

We can insert V_r into Equation 1.1 to get:

$$\frac{1}{r^2}\frac{\partial\left(r^2V_r\right)}{\partial r} = \frac{\partial V_r}{\partial r} + \frac{2}{r}V_r$$
$$= \frac{\partial}{\partial r}\left(-U\cos(\theta)\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)\right) + \frac{2}{r}\left(-U\cos(\theta)\left(1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right)\right)$$
$$= -U\cos\theta\left(\left(+\frac{3R}{2r^2} - 3\frac{R^3}{2r^4}\right)\right) - U\cos\theta\left(\left(\frac{2}{r} - \frac{3R}{r^2} + \frac{R^3}{r^4}\right)\right)$$
$$= -U\cos\theta\left(\frac{2}{r} - \frac{3}{2r^2} - \frac{1}{2}\frac{R^3}{r^4}\right)$$

Now onto the second term. We can do the same thing for the second term as we did for the first term. That is, we can expand out the derivative using the product rule:

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta\sin(\theta)) = \underbrace{\frac{1}{r\sin\theta}\sin\theta}_{=\frac{1}{r}}\frac{\partial V_\theta}{\partial\theta} + \frac{1}{r\sin\theta}V_\theta\underbrace{\frac{\partial\sin\theta}{\partial\theta}}_{=\cos\theta}$$

So now we can plug in V_θ into the above equation to get:

$$\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(V_\theta\sin(\theta)) = \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\cos\theta}{r\sin\theta}V_\theta$$
$$= \frac{1}{r}\frac{\partial}{\partial\theta}\left(U\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)\right) + \frac{\cos\theta}{r\sin\theta}U\sin\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)$$
$$= \frac{1}{r}U\cos\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right) + \frac{U}{r}\cos\theta\left(1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right)$$
$$= U\cos\theta\left(\frac{2}{r} - \frac{3R}{r^2} - \frac{R^3}{2r^4}\right) \tag{1.2}$$

As you can see, Equation 1.3 and 1.2 are negatives of each other. Therefore, adding them up gives us zero and, hence, constitutes an incompressible flow.

1.4 An incompressible fluid with density $\rho = 1000$ kg/m^3 travels through a channel with a rectangular cross-section of dimensions 25 mm by 30 mm. The average velocity of the flow in this portion of the channel is 1 m/s. If the flow is at a steady state, what will the velocity be if the rectangular cross-section increases to 50 mm by 50 mm? **Start from the integral form of the continuity equation.**

Solution: Starting from the integral form of the continuity equation we have:

$$\iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} + \oiint_A \rho \vec{V} \cdot \vec{n} dA = 0$$

The slow is at steady state, so the time derivative goes away and we are left with:

$$\oiint_A \rho \vec{V} \cdot \vec{n} dA = 0$$

We essentially have two sections (areas) that have flow: an inlet and an exit. So, we can break up the area integral into two pieces:

$$\iint_{inlet} \rho \vec{V} \cdot \vec{n} dA + \iint_{outlet} \rho \vec{V} \cdot \vec{n} dA = 0$$

We are going to assume the density and the velocity do not change with the cross-sectional area. In addition, we will assume the inlet normal is $-\hat{i}$ and the outlet normal is $\hat{i}$, giving us:

$$\rho_{inlet} \vec{V}_{inlet} \cdot -\hat{i} A_{inlet} + \rho_{outlet} \vec{V}_{outlet} \cdot \hat{i} A_{outlet} = 0$$

If the velocity is only in the $x-$direction, then we have:

$$\begin{aligned}
& -\rho_{inlet} u_{inlet} A_{inlet} + \rho_{outlet} u_{outlet} \cdot \hat{i} A_{outlet} = 0 \\
& -u_{inlet} A_{inlet} + u_{outlet} A_{outlet} = 0 \\
& \xrightarrow{\text{solve for } u_{outlet}} u_{outlet} = u_{inlet} \frac{A_{inlet}}{A_{outlet}}
\end{aligned}$$

Plugging in numbers leads to:

$$u_{outlet} = 1\text{m/s} \frac{(30)(25)}{(50)(50)} = 0.3\text{m/s}$$

1.5 Air in a pipe with a diameter of 10 cm starts out at a temperature of 700 K and pressure of 4×10^5 Pa. The initial flow velocity is 10 meters per second. If the pipe diameter contracts to 5 cm with the air speeding up to 115 m/s and the temperature decreases to 500 K, what is the pressure after the contraction? You can assume the molecular weight of air is 28.97 kg/kmol and a steady state. **Start from the integral form of the continuity equation.**

Solution: This one is very similar to the previous problem except now we are no longer assuming a constant density. We are still going to start with the intgral form of the continuity equation:

$$\iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} + \oiint_A \rho \vec{V} \cdot \vec{n} dA = 0$$

Again, like last time, we are going to assume steady state:

$$\oiint_A \rho \vec{V} \cdot \vec{n} dA = 0$$

We have an inlet and an exit section, leaving us with:

$$\iint_{inlet} \rho \vec{V} \cdot \vec{n} dA + \iint_{outlet} \rho \vec{V} \cdot \vec{n} dA = 0$$

We are going to assume the inlet surface has a normal of $-\hat{i}$ and the outlet surface has a normal of $\hat{i}$. In addition, we are going to assume the velocity is only in the $x-$direction, giving us:

$$\begin{aligned} \rho_{inlet} u_{inlet} \hat{i} \cdot \left(-\hat{i}\right) A_{inlet} + \rho_{outlet} u_{outlet} \hat{i} \cdot \left(\hat{i}\right) A_{outlet} &= 0 \\ \xrightarrow{\text{leads to}} -\rho_{inlet} u_{inlet} A_{inlet} + \rho_{outlet} u_{outlet} A_{outlet} &= 0 \end{aligned}$$

Solve for the density at the exit:

$$\begin{aligned} \rho_{outlet} &= \frac{\rho_{inlet} u_{inlet} A_{inlet}}{u_{outlet} A_{outlet}} \\ \xrightarrow{\text{use } \rho = \frac{p}{RT}} &= \frac{p_{inlet}}{RT_{inlet}} \frac{u_{inlet} A_{inlet}}{u_{outlet} A_{outlet}} \end{aligned} \tag{1.3}$$

Next, utilize the fact that (for an ideal gas), $p = \rho RT$ and plug in $\frac{p_{outlet}}{RT_{outlet}}$ for ρ_{outlet} in Equation 1.3 to get:

$$\underbrace{\frac{p_{outlet}}{RT_{outlet}}}_{=\rho_{outlet}} = \frac{p_{inlet}}{RT_{inlet}} \frac{u_{inlet} A_{inlet}}{u_{outlet} A_{outlet}}$$

Solve for p_{outlet} to get:

$$\begin{aligned}
p_{outlet} &= p_{inlet}\frac{T_{outlet}}{T_{inlet}}\frac{u_{inlet}}{u_{outlet}}\frac{A_{inlet}}{A_{outlet}} \\
&= 4\times 10^5\text{Pa}\frac{500\text{K}}{700\text{K}}\frac{10\text{m/s}}{115\text{m/s}}\frac{(10\text{ cm})^2}{(5\text{ cm})^2} \\
&= 99378\text{Pa}
\end{aligned}$$

1.6 The velocity profile of flow in a pipe in the $z-$direction (V_z) is given by:

$$V_z = \frac{\Delta p R^2}{4\mu L}\left(1 - \frac{r^2}{R^2}\right)$$

where r is the radial component (in cylindrical coordinates), R in this case is the radius of the pipe, L is the length of the pipe, Δp is considered to be a pressure difference between the entrance and exit of the pipe, and μ is something called the dynamic viscosity. Obtain an expression for the mass flow rate through the pipe.

Solution: The mass flow is given by:

$$\dot{m} = \iint_A \rho\vec{V}\cdot\vec{n}dA$$

The velocity in the $z-$direction is given and it is assumed that the normals of the surface areas of the flow passing through the pipe is also in the $z-$direction. Therefore, we can replace the dot product of the velocity with the normal with just the velocity in the $z-$direction:

$$\dot{m} = \iint_A \rho V_z dA$$

We are now dealing with cylindrical coordinates. If you recall, the area, dA in cylindrical coordinates will just be given by $rdrd\theta$, therefore we now have:

$$\dot{m} = \iint_A \rho V_z rdrd\theta$$

We can replace V_z with the expression given in the problem:

$$\dot{m} = \iint_A \rho\left(\frac{\Delta p R^2}{4\mu L}\left(1 - \frac{r^2}{R^2}\right)\right)rdrd\theta$$

We can pull out the constant $\rho\frac{\Delta pR^2}{4\mu L}$ from the problem and integrate over r from 0 to R and integrate over θ from 0 to 2π to get:

$$
\begin{aligned}
\dot{m} &= \rho\frac{\Delta pR^2}{4\mu L}\int_0^2 \pi\int_0^R\left(1-\frac{r^2}{R^2}\right)rdrd\theta \\
&= \rho\frac{\Delta pR^2}{4\mu L}2\pi\int_0^R\left(1-\frac{r^2}{R^2}\right)rdr \\
&= \rho\frac{\Delta pR^2}{4\mu L}2\pi\int_0^R\left(r-\frac{r^3}{R^2}\right)dr \\
&= \rho\frac{\Delta pR^2}{4\mu L}2\pi\left(\frac{1}{2}r^2-\frac{1}{4}\frac{r^4}{R^2}\right)\Bigg|_0^R \\
&= \rho\frac{\Delta pR^2}{4\mu L}2\pi\left(\frac{1}{2}R^2-\frac{1}{4}R^2\right)
\end{aligned}
$$

Therefore, after the algebra:

$$\dot{m} = \rho\frac{\Delta pR^4\pi}{8\mu L}$$

1.7 If the velocity profile of flow in a channel is given by: $u = \frac{U_\infty}{H}y$, what is the mass flow rate through the channel per length into the page if the height of the channel is H?

Solution: Assume that the velocity is only in the $x-$direction and that the normal is also in the $x-$direction. Also, assume the density is constant. Therefore:

$$
\begin{aligned}
\iint_A \rho\vec{V}\cdot\vec{n}dA &= \rho\iint_A udA \\
&= \rho\int_0^H udy \\
&= \rho\int_0^H \frac{U_\infty}{H}ydy \\
&= \rho\frac{U_\infty}{H}\frac{1}{2}y^2\Bigg|_0^H = \rho\frac{U_\infty H}{2} \quad \text{(per width into the board)}
\end{aligned}
$$

1.8 Given the following velocity vector: $\vec{V} = Cy\cos(5x)\hat{i} + D\sin(5x)y^2\hat{j}$, what do C and D need to be in order for this vector field to be considered incompressible flow?

Solution: Take the divergence of velocity:

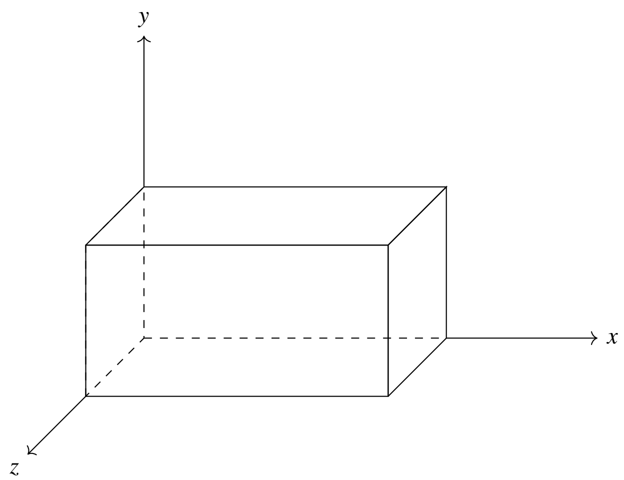
$$\begin{aligned}\vec{\nabla} \cdot \vec{V} &= \frac{\partial}{\partial x}\left(Cy\cos(5x)\right) + \frac{\partial}{\partial y}\left(D\sin(5x)\,y^2\right)\\ &= -5Cy\sin(5x) + 2Dy\sin(5x)\end{aligned}$$

In order for the flow to be incompressible, the above expression must equal zero. Thus:

$$\begin{aligned}&-5Cy\sin(5x) + 2Dy\sin(5x) = 0\\ &\therefore C = \frac{2}{5}D\end{aligned}$$

1.9 The flux of a quantity given by the vector, $\vec{f} = 5xy\hat{\imath} + 10y\hat{\jmath} + 0\hat{k}$, passes through a cube with dimensions (in $x-$, $y-$, and $z-$ directions) of 2 x 1 x 1 (assume the cube "starts" at the origin). Find the value of the area integral, $\oiint_A \vec{f} \cdot \vec{n}dA$.

Solution: The geometry for this problem is simple enough:



The first thing to notice in this problem is that the flux is two-dimensional since the $\vec{f}$ vector does not have a component in the $z-$direction. Therefore, the front and back faces (i.e. the $z-$faces) will not be utilized). With this in mind, we can break out the area integral into the following four separate integrals corresponding to the left, right, bottom, and top faces, i.e.:

$$\oiint_A \vec{f} \cdot \vec{n} dA = \iint_{\text{left}} \vec{f} \cdot \left(-\hat{i}\right) dA + \iint_{\text{right}} \vec{f} \cdot \left(\hat{i}\right) dA + \iint_{\text{bottom}} \vec{f} \cdot \left(-\hat{j}\right) dA + \iint_{\text{top}} \vec{f} \cdot \left(\hat{j}\right) dA$$

Plugging in the expression for the flux vector and taking the dot products, we get:

$$\oiint_A \vec{f} \cdot \vec{n} dA = \iint_{\text{left}} -5xydA + \iint_{\text{right}} 5xydA + \iint_{\text{bottom}} -10ydA + \iint_{\text{top}} 10ydA$$

Now we need to perform these integrals. The limits of the integration depend on which face you are at. For example, the left face lies on the $x = 0$ place with $y-$ and $z-$varying. Therefore, the left face needs to be integrated with respect to y and z. The right face is at the $x = 2$ plane with y and z varying, so the right face needs to be integrated over y and z. The top and bottom faces (which are located at $y = 1$ and $y = 0$, respectively) need to be integrated over the x and z coordinates. Thus:

$$\oiint_A \vec{f} \cdot \vec{n} dA = \left(\int_0^1 \int_0^1 -5xydydz\right)\bigg|_{x=0} + \left(\int_0^1 \int_0^1 5xydydz\right)\bigg|_{x=2} + \left(\int_0^1 \int_0^2 -10ydxdz\right)\bigg|_{y=0} + \left(\int_0^1 \int_0^2 10ydxdz\right)\bigg|_{y=1}$$

Notice that, once the integrals are evaluated, the values of x or y need to be set to match whether the surface is the left ($x = 0$), right ($x = 2$), bottom ($y = 0$), or the top ($y = 1$) surface.

Evaluating the integrals leads to:

$$\oiint_A \vec{f} \cdot \vec{n} dA = \underbrace{\left(-5x\frac{1}{2}y^2\Big|_{y=0}^{y=1} z\Big|_{z=0}^{z=1}\right)\bigg|_{x=0}}_{=0} + \underbrace{\left(5x\frac{1}{2}y^2\Big|_{y=0}^{y=1} z\Big|_{z=0}^{z=1}\right)\bigg|_{x=2}}_{=5} + \underbrace{\left(-10yx\Big|_{x=0}^{x=2} z\Big|_{z=0}^{z=1}\right)\bigg|_{y=0}}_{=0} + \underbrace{\left(10yx\Big|_{x=0}^{x=2} z\Big|_{z=0}^{z=1}\right)\bigg|_{y=1}}_{=20}$$

Adding them up thus equals **25**!

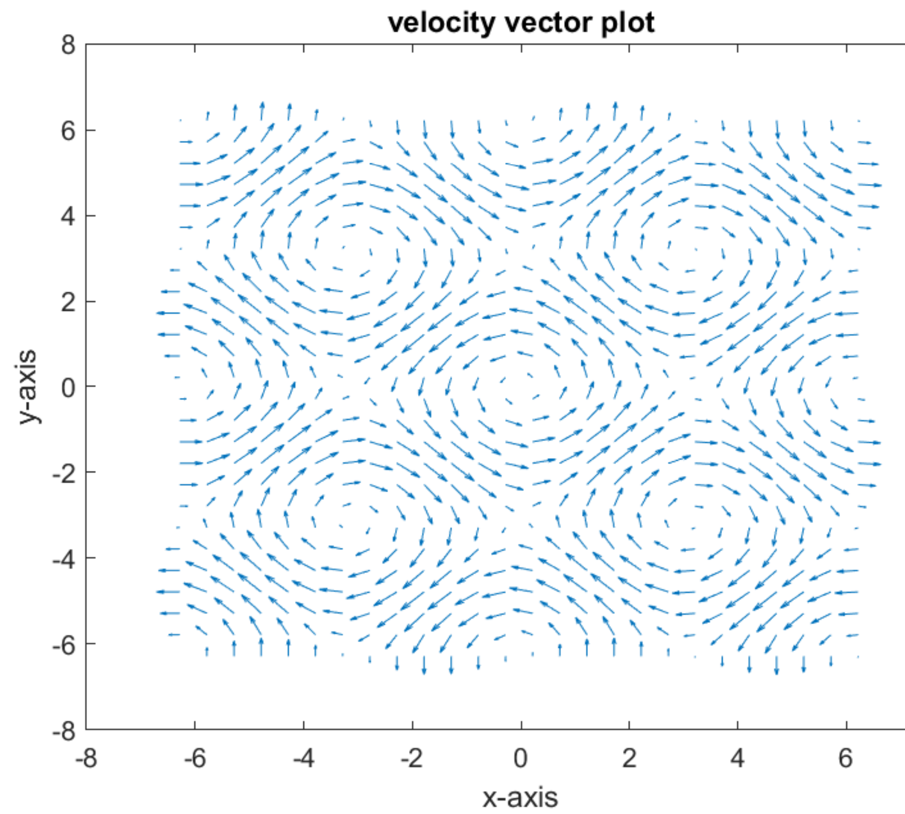
1.10 Sketch the velocity vector given by: $\vec{V} = -\sin(y)\hat{\imath} + \sin(x)\hat{\jmath}$.

Solution: We are asked to plot this velocity vector on an x and y grid. This essentially means calculating the u and v values at different x and y points and plotting the u and v as a vector on an xy grid. The u values are just obtained by calculating $-\sin(y)$ with the $y-$ plugged in and the v values are calculated via $\sin(x)$ with the $x-$ plugged in. This is done for some x and y values below:

x	y	$u = -\sin(y)$	$v = \sin(x)$
0	0	0	0
1	0	0	0.84
1	1	-0.84	0.84
0	1	-0.84	0
-1	-1	0.84	-0.84
-1	0	0	-0.84
0	-1	0.84	0
2	0	0	0.909
2	2	-0.909	0.909
.	.	.	.
.	.	.	.
.	.	.	.

The values of u and v are made into a "vector" and plotted at the different x and y points. This can all be done relatively easy in either a Matlab or Python script. An example Matlab script is given below along with the corresponding plot:

```
clc
clear
x = [-2*pi:0.5:2*pi]; %sets up an x-axis
y = [-2*pi:0.5:2*pi]; %sets up a y-axis
[x,y] = meshgrid(x,y); %takes the axis info and makes a grid
                       %of x and y values
u = -sin(y); %calculates the u velocity
v = sin(x); %calculates the v velocity
quiver(x,y,u,v)  %draws the velocity vectors on the grid
xlabel('x-axis')
ylabel('y-axis')
title('velocity vector plot')
```


velocity vector plot
8
6
4
2
0
-2
-4
-6
-8
y-axis
-8
-6
-4
-2
0
2
4
6
x-axis

2

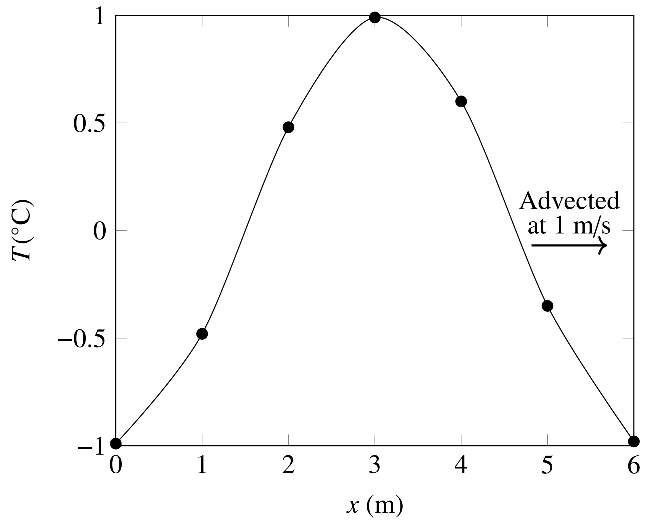
Chapter 2 Solutions

Problems

2.1 Given an initial temperature distribution of $T(x) = \sin(x)$, create a table similar to Table 2.1 for times of 0 s, 0.5 s, 1 s, and 1.5 s when $u = 1$ m/s. Plot the final temperature distribution at $t = 1.5$ s.

Solution: The table is filled out below and the figure at $t = 1.5$ s is given on the next page.

x (m)	T (celsius) (@$t = 0$) (T=T(x,0))	T (celsius) (@$t = 0.5$ s) (T=T(x-u*0.5,0))	T (celsius) (@$t = 1$ s) (T=T(x-u*1,0))	T (celsius) (@$t = 1.5$ s) (T=T(x-u*1.5,0))
0	0	-0.48	-0.84	-0.99
1.0	0.84	0.48	0	-0.48
2.0	0.909	0.99	0.84	0.48
3.0	0.141	0.6	0.909	0.99
4.0	-0.76	-0.35	0.141	0.6
5.0	-0.96	-0.98	-0.76	-0.35
6.0	-0.28	-0.7	-0.96	-0.98



Notice that the sine plot is just shifted to the right by 1.5*1 in the $x-$direction.

2.2 Explain why the characteristic curves for the advection equation are parallel when the velocity is a constant value.

Solution: The reason why the characteristic curves are parallel for constant velocity is because the characteristic curves provide a way for determining how fast the fluid elements are advecting via the inverse of the slope of the $t - x$ curve. Since the velocity of the fluid elements are the same throughout, the various characteristic curves must also have the same slope.

2.3 Given an initial $x-$velocity distribution of:

$$u(x,0) = \begin{cases} 1 & x \leq 0.5 \\ 2x & 0.5 < x < 1 \\ 2 & x \geq 1 \end{cases}$$

Does a shock form when the inviscid Burgers' equation is applied to this initial velocity field? Why or why not? Plot the characteristic curves.

Solution: In order to determine if a shock forms, we can plot the characteristic curves and see if they overlap. The characteristic curves are just the t vs. x curves of various fluid elements as they travel. The velocity of the fluid elements do not change since the inviscid Burgers' equation, in

Lagrangian form, is just $\frac{Du}{Dt} = 0$. Therefore, we can pick a few points on our initial velocity distribution, call them fluid elements, and track their position in time via characteristic curves. If the curves overlap, then a shock is considered to have developed since there is an "overtaking" of one fluid element going past another. The initial points (i.e. initial fluid elements) we will pick will be (you can pick whichever points you would like):

$x = 0,$ which has a velocity of $u = 1$ (Fluid element 1)

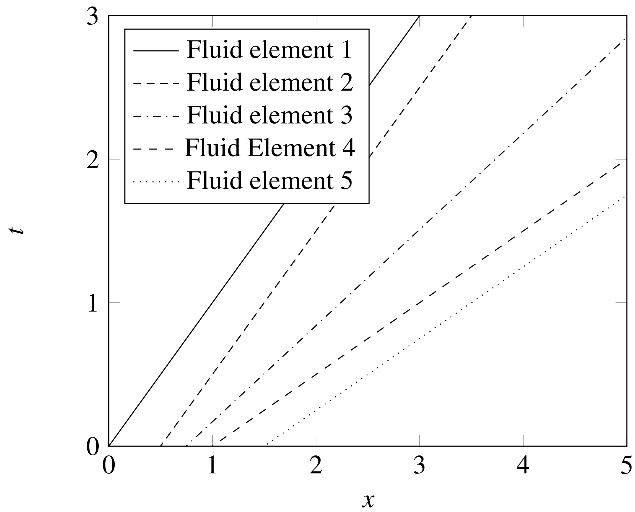
$x = 0.5,$ which has a velocity of $u = 1$ (Fluid element 2)

$x = 0.75,$ which has a velocity of $u = 1.5$ (Fluid element 3)

$x = 1,$ which has a velocity of $u = 2$ (Fluid element 4)

$x = 1.5,$ which has a velocity of $u = 2$ (Fluid element 5)

We can plot the characteristic curves (the t vs x curves) with the slopes being the inverse of velocity, giving us:



Notice that the characteristic curves appear to be diverging from each other...hence no shock will form as time passes. This is called rarefraction.

2.4 Find an expression for the material derivative of temperature if the temperature is given by the following equation $T = e^{-t}(\sin(2x) + \cos(y))$ and if the velocity vector is: $\vec{V} = \hat{\imath} + 2\hat{\jmath}$.

Solution: The material derivative of the temperature in Cartesian coordinates is given by:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

There does not appear to be a z−coordinate, so the z−will go away. The local time derivative of the temperature is given by:

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left(e^{-t}(\sin(2x) + \cos(y))\right) = -e^{-t}(\sin(2x) + \cos(y)) \tag{2.1}$$

The x−derivative is:

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left(e^{-t}(\sin(2x) + \cos(y))\right) = 2e^{-t}\cos(2x) \tag{2.2}$$

The y−derivative is:

$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y}\left(e^{-t}(\sin(2x) + \cos(y))\right) = -e^{-t}\sin(y) \tag{2.3}$$

The material derivative can now easily be obtained:

$$\begin{aligned}\frac{DT}{Dt} &= \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \underbrace{w\frac{\partial T}{\partial z}}_{=0}\\ &= \underbrace{-e^{-t}(\sin(2x) + \cos(y))}_{\text{from Equation 2.1}} + u\underbrace{e^{-t}\cos(2x)}_{\text{from Equation 2.2}} + v\left(\underbrace{-e^{-t}\sin(y)}_{\text{from Equation 2.3}}\right)\end{aligned}$$

Plug in 1 for u and 2 for v to get:

$$\frac{DT}{Dt} = -e^{-t}(\sin(2x) + \cos(y)) + 2e^{-t}\cos(2x) - 2e^{-t}\sin(y)$$

2.5 For the temperature equation used in Problem 2.4, what is the advective transport term of temperature if the velocity field is given by $\vec{V} = 2xy\hat{i} - y^2\hat{j}$ when $x = 2$, $y = 1$, and $t = 1$?

Solution: The advection transport term is considered the spatial terms in the material derivative (i.e. everything but the local time derivative). In the previous example, the x−derivative of temperature was:

$$\frac{\partial T}{\partial x} = 2e^{-t}\cos(2x)$$

and the $y-$derivative was:

$$\frac{\partial T}{\partial y} = -e^{-t}\sin(y)$$

The $u-$velocity is given by $2xy$ and the $v-$velocity is given by $-y^2$. For $x = 2$, $y = 1$, and $t = 1$, the advection term becomes:

$$\begin{aligned} u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= 2xy\left(2e^{-t}\cos(2x)\right) - y^2\left(-e^{-t}\sin(y)\right) \\ &= 2\,(2)\,(1)\left(2e^{-1}\cos(2\cdot 2)\right) + 1^2\left(e^{-1}\sin(1)\right) \\ &= -1.614 \end{aligned}$$

2.6 Given the velocity field vector: $\vec{V} = y\,(A\cos(2t) + B\sin(3t))\,\hat{i} + 6xyt\,\hat{j}$ m/s, what is the acceleration of the fluid at $x = 0.5$ m and $y = 0.5$ m at time, $t = 1$ s?

Solution: Another material derivative question. In order to find the acceleration, the material derivative of velocity needs to be calculated. Again, it appears there is only an $x-$velocity and a $y-$velocity in our velocity vector:

$$u = y\,(A\cos(2t) + B\sin(3t))\ \text{ m/s}$$

and

$$v = 6xyt \text{ m/s}$$

The material derivative of the velocity vector is given by (Equation 2.13 without the $z-$component):

$$\begin{pmatrix} \frac{Du}{Dt} \\ \\ \frac{Dv}{Dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{pmatrix}$$

The local time derivative is calculated to be:

$$\begin{pmatrix} \frac{\partial u}{\partial t} \\ \\ \frac{\partial v}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial t}\left(y\left(A\cos(2t) + B\sin(3t)\right)\right) \\ \\ \frac{\partial}{\partial t}\left(6xyt\right) \end{pmatrix} = \begin{pmatrix} y\left(-2A\sin(2t) + 3B\cos(3t)\right) \\ \\ 6xy \end{pmatrix} \tag{2.4}$$

The $x-$derivative term of advection:

$$\begin{aligned} \begin{pmatrix} u\frac{\partial u}{\partial x} \\ \\ u\frac{\partial v}{\partial x} \end{pmatrix} &= \underbrace{\left(y\left(A\cos(2t) + B\sin(3t)\right)\right)}_{u} \begin{pmatrix} \frac{\partial}{\partial x}\left(y\left(A\cos(2t) + B\sin(3t)\right)\right) \\ \\ \frac{\partial}{\partial x}\left(6xyt\right) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \\ 6yt\left(y\left(A\cos(2t) + B\sin(3t)\right)\right) \end{pmatrix} \end{aligned} \tag{2.5}$$

The $y-$derivative term of advection is:

$$\begin{aligned} \begin{pmatrix} v\frac{\partial u}{\partial y} \\ \\ v\frac{\partial v}{\partial y} \end{pmatrix} &= \underbrace{6xyt}_{v} \begin{pmatrix} \frac{\partial}{\partial y}\left(y\left(A\cos(2t) + B\sin(3t)\right)\right) \\ \\ \frac{\partial}{\partial y}\left(6xyt\right) \end{pmatrix} \\ &= \begin{pmatrix} 6xyt\left(A\cos(2t) + B\sin(3t)\right) \\ \\ 36x^2yt^2 \end{pmatrix} \end{aligned} \tag{2.6}$$

Adding up Equations 2.5, 2.6, and 2.4, we get the following expression for the material derivative of velocity:

$$\begin{pmatrix} \frac{Du}{Dt} \\ \\ \frac{Dv}{Dt} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} y\,(-2A\sin(2t) + 3B\cos(3t)) + 0 + 6xyt\,(A\cos(2t) + B\sin(3t)\,) \\ \\ 6xy + 6yt\,(y\,(A\cos(2t) + B\sin(3t))) + 36x^2yt^2 \end{pmatrix}$$

$$\xrightarrow{\text{plug in values}} = \begin{pmatrix} 0.5\,(-2A\sin(2) + 3B\cos(3)) + 0 + 6(0.5)(0.5)(1)\,(A\cos(2) + B\sin(3)) \\ \\ 6(0.5)(0.5) + 6(0.5)(1)\,((0.5)\,(A\cos(2) + B\sin(3))) + 36(0.5)^2(0.5)(1)^2 \end{pmatrix}$$

$$= \begin{pmatrix} -A(0.909) + 1.5B(-0.98999) + 1.5A(-0.416) + 1.5B(0.1411) \\ \\ 1.5 + 1.5A(-0.416) + 1.5B(0.1411) + 4.5 \end{pmatrix}$$

$$= \begin{pmatrix} -1.533A - 1.2733B \\ \\ 6 - 0.624A + 0.21165B \end{pmatrix} \xrightarrow[\text{if } A = 1 \text{ and } B = 1]{} \begin{pmatrix} -2.81 \\ \\ 5.59 \end{pmatrix}$$

2.7 The Lagrangian and non-conservation form of the continuity equation can be obtained by applying mass conservation to a moving fluid element, which states that the mass of a moving fluid element does not change in time. Mathematically, this can be written as:

$$\frac{d}{dt}\iiint_{\mathcal{V}(t)} \rho d\mathcal{V} = 0$$

Using the ideas from Section 2.4, where we applied a time derivative to the momentum of a moving fluid element, derive the Lagrangian form of the mass continuity equation as well as the non-conservation form of the continuity equation.

Solution: We can sneak in the time derivative into the volume integral of ρ and utilizing the product rule:

$$\begin{aligned}\frac{d}{dt}\iiint_{\mathcal{V}(t)}\rho d\mathcal{V} &= \iiint_{\mathcal{V}(\sqcup)}\frac{\rho d\mathcal{V}}{Dt}\\ &= \iiint_{\mathcal{V}(t)}\left(\frac{D\rho}{Dt}d\mathcal{V}+\rho\frac{D\left(d\mathcal{V}\right)}{Dt}\right)\end{aligned} \tag{2.7}$$

Recalling the the material derivative of volume is given by:

$$\frac{D\left(d\mathcal{V}\right)}{Dt} = \vec{\nabla}\cdot\vec{V}d\mathcal{V}$$

We can plug in the equation above for the material derivative of the infinitesimal volume into Equation 2.7 to get:

$$\begin{aligned}\frac{d}{dt}\iiint_{\mathcal{V}(t)}\rho d\mathcal{V} &= \iiint_{\mathcal{V}(t)}\left(\frac{D\rho}{Dt}d\mathcal{V}+\rho\vec{\nabla}\cdot\vec{V}d\mathcal{V}\right)\\ &= \underbrace{\iiint_{\mathcal{V}(t)}\left(\frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{V}\right)d\mathcal{V}}_{=0}\end{aligned}$$

Since the integral needs to equal zero for continuity and since the volume is completely arbitrary (meaning that the limits of the volume integral could be anything), then the integrand must also be zero, therefore:

$$\frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{V} = 0$$

Which is the Lagrangian form of the continuity equation. We can also expand the material derivative out into the non-conservation form of the continuity equation:

$$\frac{\partial\rho}{\partial t}+\vec{V}\cdot\vec{\nabla}\rho+\rho\vec{\nabla}\cdot\vec{V} = 0$$

2.8 Suppose the density of a fluid element is given by the expression: $\rho = e^{-0.005t}+1$. It travels with a velocity of 2 m/s in the $x-$direction. What is the value of the material derivative after it has gone 60 meters?

Solution: The density of a fluid element as it moves changes in time given the expression in the problem. Assuming the starting time is $t = 0$ s, the time when the fluid element has gone 60 meters is (60 meters)/(2 meters per second) = 30 seconds. Thus, the material derivative at 30 seconds is nothing but:

$$\frac{D\rho}{Dt} = -0.005e^{-0.005*30} = -0.004\text{kg/s}$$

2.9 What is the summation force vector on a cube fluid element that is 1 meter by 1 meter by 1 meter in size if the density is 1 kg/m^3 and $\vec{V} = 2x\hat{\imath} + 4y\hat{\jmath}$?

Solution: The force balance on a fluid element is given by:

$$\iiint_{\mathcal{V}(t)} \rho \frac{D\vec{V}}{Dt} d\mathcal{V} = \Sigma\vec{F}$$

Since the velocity given contains spatial coordinates, the material derivative of velocity should be calculated in an Eulerian description. Thus:

$$\frac{D\vec{V}}{Dt} = \frac{\partial\vec{V}}{\partial t} + \vec{V}\cdot\vec{\nabla}\vec{V}$$

The velocity vector given does not appear to be a function of time, therefore the system must be in a steady state. Thus, only the advective term needs to be calculated:

$$\vec{V}\cdot\vec{\nabla}\vec{V} = \begin{pmatrix} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x\cdot 2 + 4y\cdot 0 \\ 2x\cdot 0 + 4y\cdot 4 \end{pmatrix} = \begin{pmatrix} 4x \\ 16y \end{pmatrix}$$

Integrating over volume, we get:

$$\iiint_{\mathcal{V}(t)} \rho \frac{D\vec{V}}{Dt} d\mathcal{V} = \int_0^1\int_0^1\int_0^1 \rho \begin{pmatrix} 4x \\ 16y \end{pmatrix} dxdydz$$
$$= \rho \begin{pmatrix} 2x^2\Big|_0^1 y\Big|_0^1 z\Big|_0^1 \\ 8y^2\Big|_0^1 x\Big|_0^1 z\Big|_0^1 \end{pmatrix} = \begin{pmatrix} 2\text{ kg m/s} \\ 8\text{ kg m/s} \end{pmatrix}$$

The summation of forces equals the above expression.

2.10 The gradient of a scalar function, f, is defined as:

$$\vec{\nabla}f \equiv \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Show that $\left(\vec{V}\cdot\vec{\nabla}\right)f = \vec{V}\cdot\left(\vec{\nabla}f\right)$

Solution: Given how $\vec{\nabla}f$ is defined above, $\vec{V}\cdot\left(na\vec{b}laf\right)$ is just:

$$\begin{aligned}\vec{V}\cdot\left(\vec{\nabla}f\right) &= \left(u\hat{i}+v\hat{j}+w\hat{k}\right)\cdot\left(\hat{i}\frac{\partial f}{\partial x}+\hat{j}\frac{\partial f}{\partial y}+\hat{k}\frac{\partial f}{\partial z}\right)\\ &= u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}\end{aligned}$$

This is the same as $\left(\vec{V}\cdot\vec{\nabla}\right)f$ operating on f, observe:

$$\begin{aligned}\left(\vec{V}\cdot\vec{\nabla}\right)f &= \left(\left(u\hat{i}+v\hat{j}+w\hat{k}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\right)f\\ &= \left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)f\\ &= u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}+w\frac{\partial f}{\partial z}\end{aligned}$$

3
Chapter 3 Solutions

Problems

3.1 Find the stress vector acting on a surface if the stress tensor is:

$$\vec{\vec{T}} = \begin{pmatrix} 5 & -3 & 10 \\ -3 & 2 & 4 \\ 10 & 4 & 7 \end{pmatrix}$$

and the normal of the surface is given by: $\vec{n} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + 0\hat{k}$.

Solution: The stress vector can be obtained by simply doing a matrix multiplication of $\vec{\vec{T}}$ with $\vec{n}$. Before we do that, we should check to ensure $\vec{n}$ is a unit normal, i.e. : $||\vec{n}|| = \sqrt{\vec{n}\cdot\vec{n}} = \sqrt{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}} = 1$. Check. Now let's perform the matrix multiplication to find the stress vector (i.e. traction vector), $\vec{\tau}$:

$$\vec{\tau} = \begin{pmatrix} 5 & -3 & 10 \\ -3 & 2 & 4 \\ 10 & 4 & 7 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{\sqrt{2}} - \frac{3}{\sqrt{2}} \\ -\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ \frac{10}{\sqrt{2}} + \frac{4}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{14}{\sqrt{2}} \end{pmatrix}$$

3.2 Do you spot any potential issue with the stress tensor below?

$$\vec{\vec{T}} = \begin{pmatrix} 5x & 4x & 10z \\ -3y & 2xy & 4x \\ 2z & 4y & 7xyz \end{pmatrix}$$

Solution: Yes, the stress tensor is not symmetric. In general, the stress tensor should be symmetric due to the conservation of angular momentum. We illustrated in the book that not having a symmetric tensor can lead to an imbalance of forces (and hence torque) on an infinitesimal fluid element. Since an infinitesimal fluid element cannot resist motion at all (since there is no moment of inertia), the imbalance of torque would cause the fluid element to infinitely accelerate.

3.3 A stress tensor (in pascals) of a flow is given by:

$$\vec{\vec{T}} = \begin{pmatrix} 5x & -3y & 10z \\ -3y & 5y & 4x \\ 10z & 4x & 5z \end{pmatrix}$$

What is the acceleration of a fluid particle with density of $1.2\frac{kg}{m^3}$ (assuming no body force)?

Solution: Here we must use Cauchy's first law of motion:

$$\rho\frac{D\vec{V}}{Dt} = \vec{\nabla} \cdot \vec{\vec{T}} + \rho\vec{g}$$

If there is no body force, then:

$$\rho\frac{D\vec{V}}{Dt} = \vec{\nabla} \cdot \vec{\vec{T}}$$

The acceleration of the fluid element is just the material derivative of velocity of the fluid element, thus:

$$\text{acceleration} = \frac{D\vec{V}}{Dt} = \frac{1}{\rho}\vec{\nabla} \cdot \vec{\vec{T}}$$

Therefore, the acceleration of the fluid element in this case is:

$$\begin{aligned} \text{acceleration} &= \frac{1}{1.2}\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}\begin{pmatrix} 5x & -3y & 10z \\ -3y & 5y & 4x \\ 10z & 4x & 5z \end{pmatrix} \\ &= \frac{1}{1.2}\begin{pmatrix} 5-3+10 & 0+5+0 & 0+0+5 \end{pmatrix} \end{aligned}$$

The acceleration is then (written in column vector form):

$$\text{acceleration} = \begin{pmatrix} 10 \\ 4.17 \\ 4.17 \end{pmatrix} \text{m/s}^2$$

3.4 If the velocity field of a given flow is given by:

$$\vec{V} = A\cos(x)\hat{i} + B\sin(y)\hat{j} + C\tan(z)\hat{k}$$

Determine an expression for the total force (per volume) acting on the fluid at a given (x, y, z) point.

Solution: This is another Cauchy first law problem. Again, Cauchy's first law is:

$$\rho\frac{D\vec{V}}{Dt} = \vec{\nabla}\cdot\vec{\vec{T}} + \rho\vec{g}$$

The right-hand side is the total force (per volume). Thus, if we found an expression for the left-hand side by finding the material derivative of the velocity and multiplied by density, we would have an expression for the total force.

The material derivative of velocity can be written in Cartesian coordinates (in an Eulerian description) as:

$$\frac{D\vec{V}}{Dt} = \begin{pmatrix} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{pmatrix}$$

In our example, $u = A\cos(x)$, $v = B\sin(y)$, $w = C\tan(z)$. In addition, the velocity field appears to be at a steady state, so the time derivatives go away. We can start crossing off terms in the material derivative:

$$\frac{D\vec{V}}{Dt} = \begin{pmatrix} A\cos(x)\frac{\partial(A\cos(x))}{\partial x} + B\sin(y)\cancelto{0}{\frac{\partial(A\cos(x))}{\partial y}} + C\tan(z)\cancelto{0}{\frac{\partial(A\cos(x))}{\partial z}} \\ \\ A\cos(x)\cancelto{0}{\frac{\partial(B\sin(y))}{\partial x}} + B\sin(y)\frac{\partial(B\sin(y))}{\partial y} + C\tan(z)\cancelto{0}{\frac{\partial(B\sin(y))}{\partial z}} \\ \\ A\cos(x)\cancelto{0}{\frac{\partial(C\tan(z))}{\partial x}} + B\sin(y)\cancelto{0}{\frac{\partial(C\tan(z))}{\partial y}} + C\tan(z)\frac{\partial(C\tan(z))}{\partial z} \end{pmatrix}$$

This leads to the following expression for the force per volume after the derivatives have been taken:

$$\text{force per volume} = \rho\frac{D\vec{V}}{Dt} = \rho\begin{pmatrix} -A^2\cos(x)\sin(x) \\ \\ B^2\sin(y)\cos(y) \\ \\ C^2\tan(z)\sec^2(z) \end{pmatrix}$$

3.5 For the velocity field given in Problem 3.4, if there are no body forces and if the dynamic viscosity is considered a constant, find an expression for the pressure gradient (given $A = 1$, $B = 1$, and $C = 0$). Be sure to check if this flow is incompressible or not. Is this flow at a steady state?

Solution: For $A = 1,\ B = 1,\ C = 0$, the force per volume (given from the previous problem) is:

$$\text{force per volume} = \rho\frac{D\vec{V}}{Dt} = \rho\begin{pmatrix} -\cos(x)\sin(x) \\ \\ \sin(y)\cos(y) \\ \\ 0 \end{pmatrix} \tag{3.1}$$

Since there is no time variation in the velocity field, given by: $\vec{V} = \cos(x)\hat{i} + \sin(y)\hat{j}$, the problem is **at a steady state.** We should also determine if incompressible:

$$\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix}\begin{pmatrix} \cos(x) \\ \sin(y) \end{pmatrix} = -\sin(x) + \cos(y) \neq 0 \tag{3.2}$$

Thus, the flow is **not an incompressible flow.** Now we need to find the pressure gradient. If we assume the compressible Navier-Stokes equations in Lagrangian form with no body force and a constant viscosity, we have:

$$\rho\frac{D\vec{V}}{Dt} = -\vec{\nabla}p - \frac{2}{3}\mu\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) + \mu\vec{\nabla}\cdot\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)$$

Solving for the pressure gradient gives us:

$$\vec{\nabla}p = -\left(\rho\frac{D\vec{V}}{Dt} + \frac{2}{3}\mu\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) - \mu\vec{\nabla}\cdot\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)\right)$$

We already obtained $\rho\frac{D\vec{V}}{Dt}$ from Equation 3.1. We can now find an expression for $\frac{2}{3}\mu\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right)$. From Equation 3.39, we have:

$$\frac{2}{3}\mu\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) = \frac{2}{3}\mu\begin{pmatrix}\frac{\partial\left(\vec{\nabla}\cdot\vec{V}\right)}{\partial x}\\ \frac{\partial\left(\vec{\nabla}\cdot\vec{V}\right)}{\partial y}\\ \frac{\partial\left(\vec{\nabla}\cdot\vec{V}\right)}{\partial z}\end{pmatrix}$$

We can plug our velocity divergence from Equation 3.2 into the above equation to get:

$$\frac{2}{3}\mu\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) = \frac{2}{3}\mu\begin{pmatrix}-\cos(x)\\ -\sin(y)\\ 0\end{pmatrix} \tag{3.3}$$

Next up we need to calculate the $\mu\vec{\nabla}\cdot\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)$ term. To calculate this term, we need to first find the velocity gradient. The velocity gradient is just determined by the following expression:

$$\vec{\nabla}\vec{V} = \begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}\end{pmatrix}$$

Plugging in our velocity field of $\vec{V} = \cos(x)\hat{\imath} + \sin(y)\hat{\jmath}$ leads to:

$$\vec{\nabla}\vec{V} = \begin{pmatrix} -\sin(x) & 0 & 0 \\ 0 & \cos(y) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, $\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}$ becomes:

$$\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger} = \begin{pmatrix} -\sin(x) & 0 & 0 \\ 0 & \cos(y) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -\sin(x) & 0 & 0 \\ 0 & \cos(y) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2\sin(x) & 0 & 0 \\ 0 & 2\cos(y) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The diffusive transport term then becomes:

$$\mu\vec{\nabla} \cdot \left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) = \mu \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} -2\sin(x) & 0 & 0 \\ 0 & 2\cos(y) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \mu \begin{pmatrix} -2\cos(x) & -2\sin(y) & 0 \end{pmatrix} \tag{3.4}$$

Putting together Equations 3.1, 3.3, and 3.4 (written as a column vector) leads to the following for the pressure gradient:

$$\vec{\nabla}p = -\left(\rho\frac{D\vec{V}}{Dt} + \frac{2}{3}\mu\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right) - \mu\vec{\nabla}\cdot\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)\right)$$

$$= -\left(\rho\begin{pmatrix} -\cos(x)\sin(x) \\ \\ \sin(y)\cos(y) \\ \\ 0 \end{pmatrix} + \frac{2}{3}\mu\begin{pmatrix} -\cos(x) \\ \\ -\sin(y) \\ \\ 0 \end{pmatrix} - \mu\begin{pmatrix} -2\cos(x) \\ \\ -2\sin(y) \\ \\ 0 \end{pmatrix}\right)$$

3.6 Determine the Newtonian stress tensor if the pressure is given by $10x$ pascals and a velocity field of $\vec{V} = 2x\sin(2y)\hat{\imath} + x\cos(2y)\hat{\jmath}$ m/s at location $x = 1$ m, $y = 2$ m. Assume a dynamic viscosity of 10^{-4} Pa·s.

Solution: The Newtonian stress tensor is given by:

$$\vec{\vec{T}} = -p\vec{\vec{I}} - \frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)\vec{\vec{I}} + \mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) =$$

$$\begin{pmatrix} -p - \frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right) + 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & -p - \frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right) + 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \\ \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & -p - \frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right) + 2\mu\frac{\partial w}{\partial z} \end{pmatrix}$$

The velocity field has no z−component, so we can just write the stress tensor in two-dimensions:

$$\vec{\vec{T}} = -p\vec{\vec{I}} - \frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)\vec{\vec{I}} + \mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) =$$

$$\begin{pmatrix} -p - \frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right) + 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & -p - \frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right) + 2\mu\frac{\partial v}{\partial y} \end{pmatrix}$$

The pressure term is simple:

$$-p\vec{\vec{I}} = \begin{pmatrix} -10x & 0 \\ 0 & -10x \end{pmatrix}\Bigg|_{x=1} = \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix} \tag{3.5}$$

Next we need to find the divergence of velocity:

$$\vec{\nabla} \cdot \vec{V} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} 2x\sin(2y) \\ x\cos(2y) \end{pmatrix} = 2\sin(2y) - 2x\sin(2y)$$

Now that we have the divergence of velocity, we can find $-\frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{V}\right)\vec{I}$ at $x = 1$ m, $y = 2$ m and a dynamic viscosity of 10^{-4} Pa via:

$$\begin{aligned}
& -\frac{2}{3}\mu\left(\vec{\nabla} \cdot \vec{V}\right)\vec{I} \\
&= \left.\begin{pmatrix} -\frac{2}{3}\mu\left(2\sin(2y) - 2x\sin(2y)\right) & 0 \\ 0 & -\frac{2}{3}\mu\left(2\sin(2y) - 2x\sin(2y)\right) \end{pmatrix}\right|_{x=1,y=2,\mu=10^{-4}} \\
&= \begin{pmatrix} -\frac{2}{3}10^{-4}\left(2\sin(4) - 2\sin(4)\right) & 0 \\ 0 & -\frac{2}{3}10^{-4}\left(2\sin(4) - 2\sin(4)\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned} \tag{3.6}$$

Next up is the $\mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)$ term, The velocity gradient in 2D is:

$$\begin{aligned}
\vec{\nabla}\vec{V} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} &= \begin{pmatrix} \frac{\partial(2x\sin(2y))}{\partial x} & \frac{\partial(x\cos(2y))}{\partial x} \\ \frac{\partial(2x\sin(2y))}{\partial y} & \frac{\partial(x\cos(2y))}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} 2\sin(2y) & \cos(2y) \\ 4x\cos(2y) & -2x\sin(2y) \end{pmatrix}
\end{aligned}$$

Using the velocity gradient above, the $\mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)$ turns out to be:

$$\mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)$$

$$= \mu\begin{pmatrix} 2\sin(2y) & \cos(2y) \\ 4x\cos(2y) & -2x\sin(2y) \end{pmatrix} + \begin{pmatrix} 2\sin(2y) & 4x\cos(2y) \\ \cos(2y) & -2x\sin(2y) \end{pmatrix}$$

$$= \mu\begin{pmatrix} 4\sin(2y) & 4x\cos(2y) + \cos(2y) \\ \cos(2y) + 4x\cos(2y) & -4x\sin(2y) \end{pmatrix}\Bigg|_{x=1,y=2,\mu=10^{-4}}$$

$$= \begin{pmatrix} 4\sin(4) & 5\cos(4) \\ 5\cos(4) & -4\sin(4) \end{pmatrix} = \begin{pmatrix} -3.027 & -3.27 \\ -3.27 & 3.027 \end{pmatrix} \tag{3.7}$$

Adding up Equations 3.5, 3.6, and 3.7, the stress tensor becomes:

$$\vec{\vec{T}} = \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -3.027 & -3.27 \\ -3.27 & 3.027 \end{pmatrix} = \begin{pmatrix} -13.027 & -3.27 \\ -3.27 & -6.973 \end{pmatrix}$$

3.7 What is the pressure, in pascals, a person experiences when they have dived 5 meters below the water surface? Assume atmospheric pressure is 10^5 pascals.

Solution: This problem deals with hydrostatic pressure. The equation of which is just:

$$p = \rho|g_y|depth + p_{atm}$$

Plugging in the numbers from the problem gives us (assume water density is 1000 kg/m^3 and the gravitational acceleration is 9.8 m/s^2:

$$p = \rho|g_y|depth + p_{atm} = 1000 * 9.8 * 5 + 10^5 = 149000\text{Pa}$$

3.8 Estimate the shear stress (i.e. viscous force) on a surface defined by $\vec{n} = \hat{\imath}$ where the velocity distribution is given by:

$$u = \frac{U}{H} y$$

$$v = w = 0$$

where U is a velocity parameter and H is a length scale parameter.

Solution: The shear stress comes from the $\mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)$ term in the stress tensor, which is given by:

$$\mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) = \begin{pmatrix} 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & 2\mu\frac{\partial w}{\partial z} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \mu\frac{U}{H} & 0 \\ \mu\frac{U}{H} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The shear stress vector ($\vec{\tau}_{shear}$) can be written as:

$$\vec{\tau}_{shear} = \mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) \cdot \vec{n} = \begin{pmatrix} 0 & \mu\frac{U}{H} & 0 \\ \mu\frac{U}{H} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mu\frac{U}{H}\hat{j}$$

The shear stress on the $\vec{n} = \hat{i}$ surface is $\mu\frac{U}{H}$ in the $y-$direction.

3.9 Show that $\left(\vec{V} \cdot \vec{\nabla}\right)\vec{V} = \vec{V} \cdot \left(\vec{\nabla}\vec{V}\right)$ in two dimensional Cartesian coordinates.

Solution: The $\vec{V} \cdot \vec{\nabla}$ operator in 2D Cartesian coordindates is given by:

$$\vec{V} \cdot \vec{\nabla} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

Applying this operator to $\vec{V} = u\hat{i} + v\hat{j}$ yields:

$$\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\left(u\hat{i} + v\hat{j}\right)$$
$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)\hat{i} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)\hat{j}$$

The $\vec{V}\cdot\left(\vec{\nabla}\vec{V}\right)$ is calculating the dot product of velocity with the velocity gradient. This is given by:

$$\vec{V}\cdot\left(\vec{\nabla}\vec{V}\right) = \begin{pmatrix} u & v \end{pmatrix}\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ & \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} & u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{pmatrix}$$

The above result, while written as a row vector, yields the same result as when we calculated $\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}$. Thus, the calculation order turns out to not matter in this instance.

3.10 Please explain, in your own words, why an equation of state is not necessarily needed for an incompressible flow.

Solution: The reason for not needing an equation of state for an incompressible flow is simply because the incompressible Navier-Stokes equations are closed by including the divergence-free velocity condition. Thus, no thermodynamics is needed to be able to solve for the flow field of an incompressible flow under usual circumstances. Even more to the point, often incompressible flows assume a constant density throughout the flow.

4

Chapter 4 Solutions

4.1 Chapter 4 Solutions

Problems

4.1 Find $\vec{A} \otimes \vec{B}$ if $\vec{A} = 3\hat{i} + 5\hat{j} - 10\hat{k}$ and $\vec{B} = 5\hat{i} - 2\hat{j} - \hat{k}$.

Solution: Matrix multiply the column vector form of $\vec{A}$ with the row vector form of $\vec{B}$:

$$\vec{A} \otimes \vec{B} = \begin{pmatrix} 3 \\ 5 \\ -10 \end{pmatrix} \begin{pmatrix} 5 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 15 & -6 & -3 \\ 25 & -10 & -5 \\ -50 & 20 & 10 \end{pmatrix}$$

4.2 Determine $\oiint_A \rho \vec{V} \otimes \vec{V} \cdot \vec{n} dA$ for a 1 cm x 1 cm x 1 cm Cartesian element if $\vec{V} = 5\sqrt{x}\,\hat{i}$ m/s and the density is 1000 kg/m^3.

Solution: First thing first, let's determine $\vec{V} \otimes \vec{V}$:

$$\vec{V} \otimes \vec{V} = \begin{pmatrix} 5\sqrt{x} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 5\sqrt{x} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 25x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus:

$$\rho \vec{V} \otimes \vec{V} = \begin{pmatrix} 25000x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Next, instead of using the form given in the problem, we can use the divergence theorem to get:

$$\oiint_A \rho\vec{V}\otimes\vec{V}\cdot\vec{n}dA = \iiint_{\mathcal{V}} \vec{\nabla}\cdot\left(\rho\vec{V}\otimes\vec{V}\right)d\mathcal{V}$$

Performing the divergence gives:

$$\vec{\nabla}\cdot\left(\rho\vec{V}\otimes\vec{V}\right) = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{pmatrix}\begin{pmatrix}25000x & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}25000 & 0 & 0\end{pmatrix}$$

Thus, the integral, $\iiint_{\mathcal{V}} \vec{\nabla}\cdot\left(\rho\vec{V}\otimes\vec{V}\right)d\mathcal{V}$ becomes (written in column vector form):

$$\begin{aligned}\iiint_{\mathcal{V}} \vec{\nabla}\cdot\left(\rho\vec{V}\otimes\vec{V}\right)d\mathcal{V} &= \iiint_{\mathcal{V}}\begin{pmatrix}25000\\0\\0\end{pmatrix}d\mathcal{V}\\ &= \int_{\mathrm{x=0}}^{0.01\ \mathrm{m}}\int_{\mathrm{y=0}}^{0.01\ \mathrm{m}}\int_{\mathrm{z=0}}^{0.01\ \mathrm{m}}\begin{pmatrix}25000\\0\\0\end{pmatrix}dxdydz\\ &= \begin{pmatrix}25000\,(0.01)\,(0.01)\,(0.01)\\0\\0\end{pmatrix}\\ &= \begin{pmatrix}0.025\\0\\0\end{pmatrix}\mathrm{kg\ m\ /s^2}\end{aligned}$$

4.3 Consider incompressible flow between two parallel plates a distance, H, apart. If the bottom plate moves with a velocity of U_B and the top is fixed, find an expression of the velocity profile assuming there is no pressure gradient.

Solution: We will make the following assumptions: steady state, one-dimensional flow, no gravity, and no pressure difference. The continuity equation is:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Setting $v = 0$ leads us to:

$$\frac{\partial u}{\partial x} = 0$$

Next we look the Navier-Stokes equations. We can start with the incompressible Navier-Stokes equations in the $x-$direction:

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}}\right) = \cancel{-\frac{\partial p}{\partial x}} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

After our assumptions we are left with:

$$0 = \frac{d^2 u}{dy^2}$$

This is the governing equation for Couette flow. The general solution to this equation is:

$$u = C_1 y + C_2$$

where C_1 and C_2 are constants of integration. In order to find values for C_1 and C_2, boundary conditions need to be applied. The no-slip boundary condition for u is utilized at the top and bottom plates. Recall that the no-slip boundary condition implies that the velocity of the fluid is the same as the velocity of the adjacent solid surface. Thus, in mathematical form, the boundary conditions are:

$$\text{at } y = 0, \quad u = U_B$$

$$\text{at } y = H, \quad u = 0$$

Applying the first boundary condition (i.e. at $y = 0, \ u = U_B$) to our general solution, we get:

$$U_B = C_1 0 + C_2$$
$$\therefore C_2 = U_B$$

Next apply the second boundary condition (i.e. at $y = H, u = 0$) to our general solution, i.e.:

$$0 = C_1 H + U_B$$
$$\therefore C_1 = \frac{-U_B}{H}$$

The solution for the x-velocity profile is now obtained by plugging in the expressions for C_1 and C_2 into the general solution:

$$u = \frac{-U_B}{H} y + U_B$$

4.4 Find an expression for the shear stress on the bottom and top plates from Problem 4.3.

Solution: We can obtain the shear stress on the top and bottom plates by recognizing that the shear term of the stress tensor is given by $\mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)$, i.e.: term in the stress tensor, which is given by:

$$\mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) = \begin{pmatrix} 2\mu\frac{\partial u}{\partial x} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 2\mu\frac{\partial v}{\partial y} & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & 2\mu\frac{\partial w}{\partial z} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\mu\frac{U_B}{H} & 0 \\ -\mu\frac{U_B}{H} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Therefore, the shear stress (i.e. the stress vector, $\vec{\tau}_{shear}$) on the bottom (given by normal, $\vec{n}_b = -\hat{j}$can be written as:

$$\vec{\tau}_{shear,bottom} = \mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)\cdot\vec{n}_b = \begin{pmatrix} 0 & -\mu\frac{U_B}{H} & 0 \\ -\mu\frac{U_B}{H} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = \mu\frac{U_B}{H}\hat{i}$$

Likewise, the shear stress at the top is given by:

$$\vec{\tau}_{shear,top} = \mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)\cdot\vec{n}_t = \begin{pmatrix} 0 & -\mu\frac{U_B}{H} & 0 \\ -\mu\frac{U_B}{H} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\mu\frac{U_B}{H}\hat{i}$$

4.5 Consider incompressible flow between two parallel plates a distance, $H = 0.5$ cm, apart. The top plate is moving with a velocity to the right of 1 m/s. There is also a constant pressure gradient that is resisting the flow in the $x-$direction with a magnitude of 2 pascals per meter. The dynamics viscosity of the fluid is 2x10^{-5} Pa · s. Determine the velocity at the midpoint of the parallel plates.

Solution: Like before, we will make the following assumptions: incompressible flow, steady state, one-dimensional flow, and no gravity. The continuity equation is (after setting $v = 0$):

$$\frac{\partial u}{\partial x} = 0$$

The incompressible Navier-Stokes equations in the $x-$direction is:

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + u\cancel{\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

After our assumptions we are left with:

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{d^2 u}{dy^2}$$

This is the governing equation for Couette flow. The general solution to this equation is:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + C_1 y + C_2$$

where C_1 and C_2 are constants of integration. In order to find values for C_1 and C_2, boundary conditions need to be applied. The boundary conditions are:

$$\text{at } y = 0, \quad u = 0$$

$$\text{at } y = H, \quad u = U_T$$

Applying the first boundary condition (i.e. at $y = 0, \ u = 0$) to our general solution, we get:

$$0 = \frac{1}{2\mu}\frac{\partial p}{\partial x}0^2 + C_1 0 + C_2$$
$$\therefore C_2 = 0$$

Next apply the second boundary condition (i.e. at $y = H, u = U_T$) to our general solution, i.e.:

$$U_T = \frac{1}{2\mu}\frac{\partial p}{\partial x}H^2 + C_1 H$$
$$\therefore C_1 = \frac{U_T}{H} - \frac{1}{2\mu}\frac{\partial p}{\partial x}H$$

Thus the solution for the x-velocity profile is:

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + \left(\frac{U_T}{H} - \frac{1}{2\mu}\frac{\partial p}{\partial x}H\right)y$$

To find the value of the velocity at the midpoint, we plug in $y = \frac{H}{2}$ to get:

$$u_{midpoint} = \frac{1}{2\mu}\frac{\partial p}{\partial x}\frac{H^2}{4} + \left(\frac{U_T}{2} - \frac{1}{4\mu}\frac{\partial p}{\partial x}H^2\right) = \left(\frac{U_T}{2} - \frac{1}{8\mu}\frac{\partial p}{\partial x}H^2\right)$$

Plugging in $\frac{\partial p}{\partial x} = 2$ Pa/m, $\mu = 2\text{x}10^{-5}\text{Pa} \cdot \text{s}$, and $U_B = 1$ m/s, we get:

$$u_{midpoint} = \left(\frac{U_T}{2} - \frac{1}{8\mu}\frac{\partial p}{\partial x}H^2\right) = \frac{1}{2} - \frac{H^2}{8\text{x}10^{-5}} \text{ m/s}$$

If $H = 0.005$ meters (or 0.5 centimeters), then:

$$u_{midpoint} = 0.1875 \text{ m/s}$$

4.6 In your own words, describe the various forms of the Navier-Stokes equations discussed in this book and how they were obtained.

Solution: A quick discussion of this is on the video podcasts.

4.7 Show that the conservation form and the non-conservation form of the Navier-Stokes equations are equivalent. You may use Cartesian coordinates.

Solution: The non-conservation form of the Navier-Stokes equations is:

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\vec{\nabla}\vec{V}\right) = -\vec{\nabla}p - \vec{\nabla}\left(\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)\right) + \vec{\nabla}\cdot\left(\mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right)\right) + \rho\vec{g}$$

The conservation form of the Navier-Stokes equations is:

$$\frac{\partial\left(\rho\vec{V}\right)}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\otimes\vec{V}\right) = -\nabla p - \vec{\nabla}\left(\frac{2}{3}\mu\vec{\nabla}\cdot\vec{V}\right) + \vec{\nabla}\cdot\left(\mu\vec{\nabla}\vec{V} + \mu\left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) + \rho\vec{g}$$

As you can tell, the main difference between the two versions is the left-hand side. Thus, we need to show that:

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\vec{\nabla}\vec{V}\right) = \frac{\partial\left(\rho\vec{V}\right)}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{V}\otimes\vec{V}\right)$$

The easiest way to show this equivalency is by using Cartesian coordinates. For example, recall that $\rho\vec{V}\otimes\vec{V}$ equals:

$$\rho\vec{V}\otimes\vec{V} = \begin{pmatrix} \rho uu & \rho uv & \rho uw \\ \rho vu & \rho vv & \rho vw \\ \rho wu & \rho wv & \rho ww \end{pmatrix}$$

Taking the divergence we have:

$$\vec{\nabla}\cdot\left(\rho\vec{V}\otimes\vec{V}\right) = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \rho uu & \rho uv & \rho uw \\ \rho vu & \rho vv & \rho vw \\ \rho wu & \rho wv & \rho ww \end{pmatrix}$$

This leads to (in column vector form):

$$\vec{\nabla} \cdot \left(\rho \vec{V} \otimes \vec{V}\right) = \begin{pmatrix} \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z} \\ \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho wv)}{\partial z} \\ \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho ww)}{\partial z} \end{pmatrix}$$

Adding the time derivative:

$$\frac{\partial\left(\rho\vec{V}\right)}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V} \otimes \vec{V}\right) = \begin{pmatrix} \frac{\partial \rho u}{\partial t} \\ \frac{\partial \rho v}{\partial t} \\ \frac{\partial w}{\partial t} \end{pmatrix} + \begin{pmatrix} \frac{\partial(\rho uu)}{\partial x} + \frac{\partial(\rho vu)}{\partial y} + \frac{\partial(\rho wu)}{\partial z} \\ \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho vv)}{\partial y} + \frac{\partial(\rho wv)}{\partial z} \\ \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho ww)}{\partial z} \end{pmatrix}$$

We can take the result from above (for simplicity, let's just take the $x-$direction) and expand it out via product rule:

$$\frac{\partial\,(\rho u)}{\partial t} + \frac{\partial\,(\rho uu)}{\partial x} + \frac{\partial\,(\rho vu)}{\partial y} + \frac{\partial\,(\rho wu)}{\partial z}$$

$$= \underbrace{\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t}}_{=\frac{\partial(\rho u)}{\partial t}} + \underbrace{\rho u\frac{\partial u}{\partial x} + u\frac{\partial\,(\rho u)}{\partial x}}_{=\frac{\partial(\rho uu)}{\partial x}} + \underbrace{\rho v\frac{\partial u}{\partial y} + u\frac{\partial\,(\rho v)}{\partial y}}_{=\frac{\partial(\rho vu)}{\partial y}} + \underbrace{\rho w\frac{\partial u}{\partial z} + u\frac{\partial\,(\rho w)}{\partial z}}_{=\frac{\partial(\rho wu)}{\partial z}}$$

$$\xrightarrow{\text{group terms}} = \underbrace{\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)}_{\text{non-conservative form}} + \underbrace{u\left(\frac{\partial \rho}{\partial t} + \frac{\partial\,(\rho u)}{\partial t} + \frac{\partial\,(\rho v)}{\partial y} + \frac{(\rho w)}{\partial z}\right)}_{=0 \text{ from continuity}}$$

$$= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

Thus, the conservation and the non-conservation forms of the equations are equivalent. The same procedure can be done for the other directions.

5
Chapter 5 Solutions

Problems

5.1 Evaluate the viscous dissipation term at an (x, y) location of $(1, 5)$ meters with a dynamic viscosity of 10^{-3} Pa · s if velocity field is given by: $\vec{V} = 2y\cos(2x)\hat{\imath} + y^2\sin(2x)\hat{\jmath}$ m/s.

Solution: The viscous dissipation term is given by:

$$-\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)^2 + \mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) : \vec{\nabla}\vec{V}$$

For an incompressible flow, it is just $\mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) : \vec{\nabla}\vec{V}$

Let's first see if the flow is incompressible by calculating the divergence of velocity (with $u = 2y\cos(2x)$, $v = y^2\sin(2x)$, and $w = 0$:

$$\begin{aligned}\vec{\nabla}\cdot\vec{V} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\\ &= \frac{\partial\left(2y\cos(2x)\right)}{\partial x} + \frac{\partial\left(y^2\sin(2x)\right)}{\partial y}\\ &= -4y\sin(2x) + 2y\sin(2x) = -2y\sin(2x)\end{aligned}$$

Therefore, the $-\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)^2$ portion is:

$$-\frac{2}{3}\mu\left(\vec{\nabla}\cdot\vec{V}\right)^2 = -\frac{2}{3}\mu\left(-2y\sin(2x)\right)^2 = -\frac{8}{3}\mu y^2\sin^2(2x)$$

Plugging in the numbers for the problem yields:

$$-\frac{8}{3}\mu y^2 \sin^2(2x) = -\frac{8}{3}\left(10^{-3}\,(5)^2\,(0.909)^2\right) = -0.055\,\frac{\mathrm{J}}{\mathrm{m^3 s}} \tag{5.1}$$

Next up we need to calculate $\mu\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) : \vec{\nabla}\vec{V}$. This is essentially a 2D scenario, so the velocity gradient is:

$$\vec{\nabla}\vec{V} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} -4y\sin(2x) & y^2\cos(2x) \\ 2\cos(2x) & 2y\sin(2x) \end{pmatrix}$$

The $\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}$ portion turns out to be:

$$\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger} = \begin{pmatrix} -4y\sin(2x) & y^2\cos(2x) \\ 2\cos(2x) & 2y\sin(2x) \end{pmatrix} + \begin{pmatrix} -4y\sin(2x) & 2\cos(2x) \\ y^2\cos(2x) & 2y\sin(2x) \end{pmatrix}$$

$$= \begin{pmatrix} -8y\sin(2x) & y^2\cos(2x) + 2\cos(2x) \\ 2\cos(2x) + y^2\cos(2x) & 4y\sin(2x) \end{pmatrix}$$

Next we calculate the $\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) : \vec{\nabla}\vec{V}$:

$$\left(\vec{\nabla}\vec{V} + \left(\vec{\nabla}\vec{V}\right)^{\dagger}\right) : \vec{\nabla}\vec{V}$$

$$= \begin{pmatrix} -8y\sin(2x) & y^2\cos(2x) + 2\cos(2x) \\ 2\cos(2x) + y^2\cos(2x) & 4y\sin(2x) \end{pmatrix} : \begin{pmatrix} -4y\sin(2x) & y^2\cos(2x) \\ 2\cos(2x) & 2y\sin(2x) \end{pmatrix}$$

$$\begin{aligned} =&64y^2\sin^2(2x) + y^4\cos^2(2x) + 2y^2\cos^2(2x) \\ &+ 4\cos^2(2x) + 2y^2\cos^2(2x) + 16y^2\sin^2(2x) \\ &= 1322.9 + 108.234 + 8.658 + 0.693 + 8.659 + 330.729 = 1779.87 \end{aligned}$$

Multiplying this result by *mu* gives 1.77987 and adding to Equation 5.1 leads to the value for the viscous dissipation: **1.72** $\mathbf{\frac{J}{m^3 s}}$.

5.2 Calculate the power (per volume) if the stress tensor is:

$$\vec{\vec{T}} = \begin{pmatrix} 10x & 5y & 6x \\ 5y & 2y & 25y \\ 6x & 25y & 3z \end{pmatrix}$$

with a velocity vector is $\vec{V} = \hat{i} + 2\hat{j} - 2\hat{k}$ and no body force.

Solution: The power (per volume) without body forces can be written as:

$$\dot{W} = -\vec{\nabla} \cdot \left(\vec{V} \cdot \vec{\vec{T}}\right)$$

The velocity dotted with the stress tensor leads to:

$$\vec{V} \cdot \vec{\vec{T}} = \begin{pmatrix} 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 10x & 5y & 6x \\ 5y & 2y & 25y \\ 6x & 25y & 3z \end{pmatrix}$$

$$= \begin{pmatrix} 10x + 10y - 12x & 5y + 4y - 50y & 6x + 50y - 6z \end{pmatrix}$$

$$= \begin{pmatrix} -2x + 10y & -41y & 6x + 50y - 6z \end{pmatrix}$$

The power per volume is therefore:

$$\dot{W} = -\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left((-2x + 10y)\,\hat{i} + -41y\,\hat{j} + (6x + 50y - 6z)\,\hat{k}\right)$$

$$= -(-2 - 41 - 6) = 49\frac{\text{watts}}{\text{m}^3}$$

5.3 Does the follow expression satisfy the Laplace equation?

$$T = T_0 + T_1\frac{\sin(\pi x)\sinh(\pi y)}{\sinh(\pi L)}$$

You may assume T_0, T_1, and L are constants.

Solution: The Laplace equation is:

$$\nabla^2 T = \vec{\nabla} \cdot \vec{\nabla} T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

Taking the derivative of T with respect to x gives:

$$\frac{\partial T}{\partial x} = T_1\frac{\cos(\pi x)\sinh(\pi y)}{\sinh(\pi L)}\pi$$

Taking the derivative with respect to x again gives:

$$\frac{\partial^2 T}{\partial x^2} = -T_1 \frac{\sin(\pi x)\sinh(\pi y)}{\sinh(\pi L)}\pi^2$$

Taking the derivative of T with respect to y gives (the derivative of hyperbolic sine is hyperbolic cosine):

$$\frac{\partial T}{\partial y} = T_1 \frac{\sin(\pi x)\cosh(\pi y)}{\sinh(\pi L)}\pi$$

The second derivative is (the derivative of hyperbolic cosine is hyperbolic sine):

$$\frac{\partial^2 T}{\partial y^2} = T_1 \frac{\sin(\pi x)\sinh(\pi y)}{\sinh(\pi L)}\pi^2$$

which is the negative of the second derivative with respect to x. Thus, addding them up leads to zero and this solution for temperature is a solution to Laplace's equation.

5.4 Consider a 2D flow whose velocity vector is given by $\vec{V} = -10x\hat{\imath} + 10y\hat{\jmath}$ m/s and whose temperature is $T = 50e^{-0.03t}\sin(2x)\cosh(5y)$ kelvin. The fluid has a thermal conductivity of 0.6 $\frac{\text{W}}{\text{mK}}$, a density of 1000 kg/m^3, and a specific heat of 4180 $\frac{\text{J}}{\text{kgK}}$. What is the value of the heat generation at a time of 10 seconds and (x, y) coordinates of $(1, 1)$ meters?

Solution: Let's first check to see if the flow in incompressible or not:

$$\vec{\nabla} \cdot \vec{V} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(-10x\hat{\imath} + 10y\hat{\jmath}\right) = -10 + 10 = 0$$

Yes, the flow is incompressible. So, we need to use the 2D incompressible energy equation:

$$\rho c_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi + \dot{q}_{gen}$$
$$\text{where } \Phi = \mu\left(2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2\right)$$

With $u = -10x$ and $v = 10y$, the viscous dissipation becomes:

$$\begin{aligned}\Phi &= \mu\left(2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2\right)\\ &= \mu\left(2\,(-10)^2 + (0+0)^2 + 2\,(10)^2\right)\\ &= 400\mu\end{aligned} \tag{5.2}$$

The $x-$ and $y-$derivatives of T are:

$$\begin{aligned}\frac{\partial T}{\partial x} &= 100e^{-0.03t}\cos(2x)\cosh(5y)\\ &= 100e^{-0.3}\cos(2)\cosh(5)\\ &= 5467 \text{ kelvin/m}\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}\frac{\partial T}{\partial y} &= 250e^{-0.03t}\sin(2x)\sinh(5y)\\ &= 250e^{-0.3}\sin(2)\sinh(5)\\ &= 12496 \text{ kelvin/m}\end{aligned} \tag{5.4}$$

The second-order derivatives are:

$$\begin{aligned}\frac{\partial^2 T}{\partial x^2} &= -200e^{-0.03t}\sin(2x)\cosh(5y)\\ &= -200e^{-0.3}\sin(2)\cosh(5)\\ &= -9997.9 \text{ kelvin/m}^2\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\frac{\partial T}{\partial y} &= 1250e^{-0.03t}\sin(2x)\cosh(5y)\\ &= 1250e^{-0.3}\sin(2)\cosh(5)\\ &= 62487 \text{ kelvin/m}^2\end{aligned} \tag{5.6}$$

The time derivative is:

$$\begin{aligned}\frac{\partial T}{\partial t} &= -0.03\,(50)\,e^{-0.03t}\sin(2x)\cosh(5y)\\ &= -1.5e^{-0.3}\sin(2)\cosh(5)\\ &= -75 \text{ kelvin/s}\end{aligned} \tag{5.7}$$

Putting it all together yields:

$$\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \Phi + \dot{q}_{gen}$$

$$\xrightarrow{\text{leads to}} (1000)(4180)\left(-75 + (-10)(5467) + (10)(12496)\right) = 0.6\left(-9997.9 + 62487\right) + 400\mu + \dot{q}_{gen}$$

$$\dot{q}_{gen} = 2.9e11 - 400\mu \text{ watts per meter cubed}$$

5.5 Find the steady state temperature in the middle of a 10 cm bar whose sides in the $y-$ and $z-$directions can be ignored and the temperature of the right (at $x = 10$ cm) is fixed at 0 degrees Celsius and the flux on the left is 1 W/m^2. The thermal conductivity is 0.6 $\frac{\text{W}}{\text{mK}}$.

Solution: The temperature distribution for this problem can be obtained by simply solving the following equation:

$$\frac{\partial^2 T}{\partial x^2} = 0$$

This is just the equation for diffusion in one dimension at steady state. Simple enough. The general solution for this problem is just a straight line:

$$T = C_1 x + C_2$$

We now need to apply boundary conditions:

$$\text{at } x = 0, \quad \vec{q}'' \cdot \vec{n} = 1 \text{ watt per meter squared}$$
$$\text{at } x = 10 \text{ cm}, \quad T = 0°\text{C}$$

Applying the first boundary condition (at $x = 10$ cm, $T = 0$°C) leads to:

$$0 = C_1 0.1 + C_2 \quad \text{(notice we switched to using meters)}$$
$$\therefore C_2 = -0.1 C_1$$

Applying the next boundary condition (i.e. at $x = 0$, $\vec{q}'' \cdot \vec{n} = 1$ watt per meter squared) yields (with $\vec{n} = -\hat{\imath}$ since it is considered the left face):

$$\vec{q}'' \cdot \vec{n} = -k\left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j} + \frac{\partial T}{\partial z}\hat{k}\right) \cdot -\hat{i} = -k\frac{\partial T}{\partial x}$$

$$\xrightarrow{\text{leads to}} -kC_1 = 1\ \frac{\mathrm{W}}{\mathrm{m}^2}$$

$$\therefore C_1 = -\frac{1}{k} = -\frac{1}{0.6} = -1.67\frac{{}^\circ\mathrm{C}}{\mathrm{m}}$$

The temperature in the middle of the bar (i.e. at $x = 5$ cm or $x = 0.05$ m) leads to:

$$T = -\frac{1}{0.6}(0.05) + \left(-0.1\frac{-1}{0.6}\right) = 0.0833^\circ\mathrm{C}$$

5.6 Does the following function for temperature satisfy the heat equation when there is no heat generation (α is thermal diffusivity):

$$T = T_0 + T_m \exp\left(\frac{-2\pi^2\alpha}{L^2}t\right)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)$$

You may assume L, T_0, and T_m are also constant.

Solution: The heat equation is given by (in 2D):

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

The time derivative of the given temperature function is:

$$\frac{\partial T}{\partial t} = T_m\left(\frac{-2\pi^2\alpha}{L^2}\right)\exp\left(\frac{-2\pi^2\alpha}{L^2}t\right)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)$$

The second derivative of T with respect to x is:

$$\frac{\partial^2 T}{\partial x^2} = T_m\left(\frac{-\pi^2}{L^2}\right)\exp\left(\frac{-2\pi^2\alpha}{L^2}t\right)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)$$

The second derivative of T with respect to y is:

$$\frac{\partial^2 T}{\partial y^2} = T_m\left(\frac{-\pi^2}{L^2}\right)\exp\left(\frac{-2\pi^2\alpha}{L^2}t\right)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)$$

Adding the second derivative with respect to x with the second derivative with respect to y and multiplying with α gives:

$$\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\alpha\left(2T_m\left(\frac{-\pi^2}{L^2}\right)\exp\left(\frac{-2\pi^2\alpha}{L^2}t\right)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{L}\right)\right)$$

Upon inspection, the above equation is equal to the time derivative. Hence, this equation satisfies the heat equation.

5.7 Consider a system of water with a pinch of salt centered in the middle. If we were to model the "stirring" of the system of water using the advection equation, what would the result be? What would we need to do in order to more accurately model the physical reality of the situation?

Solution: If the pinch of salt was very small so that the velocity at the different parts of the salt would be essentially the same, then the salt would stay relatively the same shape and size and just move with the stirring of the water. If the pinch of salt was somewhat big to begin with, then the various areas of the salt pinch might experience differing velocities, and hence cause the salt to get stretched. Either way, the salt will just move with the velocity vectors of the flow but will not spread out and diffuse into the water like you would expect. In order to more accurately model the situation, a diffusion term would need to be added. In other words, the convection-diffusion equation should be used.

5.8 Consider incompressible flow between two parallel plates. The flow is driven by the bottom plate moving to the right with a velocity of 100 m/s. The viscosity of the fluid is given by 0.001 Pa · s and the thermal conductivity is 0.6 $\frac{\mathrm{W}}{\mathrm{mK}}$. If both the bottom and the top temperatures are fixed at 0°C, what is the maximum temperature of the flow? You may assume the flow is in a steady-state and is only in the $x-$direction. In addition, you may ignore any pressure gradient as well as any heat generation. How is the maximum temperature of the flow different than the plate temperatures? What accounts for the difference?

Solution: This problem starts out like the other cases between two plates. In particular, the continuity equation simplifies to:

$$\frac{du}{dx}=0$$

And the Navier-Stokes equations (in the $x-$direction) simplifies to:

$$\frac{d^2u}{dy^2} = 0$$

The general solution of the velocity profile is:

$$u = C_1 y + C_2$$

We now need to apply the following velocity boundary conditions:

$$\text{at } y = 0, \quad u = U_B$$
$$\text{at } y = H, \quad u = 0$$

Applying the first boundary condition leads to:

$$U_B = C_1 0 + C_2$$
$$\therefore C_2 = U_B$$

The next velocity boundary condition is:

$$0 = C_1 H + U_B$$
$$\therefore C_1 = \frac{-U_B}{H}$$

Thus, the velocity profile is:

$$u = \frac{-U_B}{H} y + U_B$$

Next, we need to find the temperature profile. The temperature profile is obtained from the energy equation. Making the assumption that the flow is only in the $x-$direction, steady state, the derivative of temperature with respect to x is zero, there is no heat generation, and an incompressible flow, we have:

$$0 = k\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2$$

Solving for the general solution for temperature, with $\frac{du}{dy} = \frac{-U_B}{H}$, we get:

$$T = -\frac{\mu}{k}\left(\frac{U_B}{H}\right)^2 y^2 + C_3 y + C_4$$

We now need to apply the following velocity boundary conditions:

$$\text{at } y = 0, \quad T = 0$$
$$\text{at } y = H, \quad T = 0$$

Applying the first boundary condition leads to:

$$0 = -\frac{\mu}{k}\left(\frac{U_B}{H}\right)^2 0^2 + C_3 0 + C_4$$
$$\therefore C_4 = 0$$

The next temperature boundary condition is:

$$0 = -\frac{\mu}{k}\left(\frac{U_B}{H}\right)^2 H^2 + C_3 H$$
$$\therefore C_3 = \frac{\mu}{k}\left(\frac{U_B}{H}\right)^2 H$$

The temperature profile is thus:

$$T = -\frac{\mu}{k}\left(\frac{U_B}{H}\right)^2 y^2 + \frac{\mu}{k}\left(\frac{U_B}{H}\right)^2 Hy$$

The maximum temperature of the flow is located at the point where the derivative of the temperature with y is zero, thus:

$$\frac{dT}{dy} = -2\frac{\mu}{k}\left(\frac{U_B}{H}\right)^2 y + \frac{\mu}{k}\left(\frac{U_B}{H}\right)^2 H = 0$$
$$\xrightarrow{\text{leads to}} y = \frac{H}{2}$$

Thus, the maximum occurs (as expected) at $y = H/2$. Plugging in $y = H/2$ into the temperature equation leads to:

$$T_{max} = -\frac{\mu}{k}\left(\frac{U_B}{2}\right)^2 + \frac{\mu}{2k}U_B^2 = \frac{\mu}{4k}U_B^2$$

Plugging in values gives:

$$T_{max} = \frac{\mu}{4k}U_B^2 = \frac{0.001}{4(0.6)}100^2 = 4.16°\text{C}$$

The maximum temperature is greater than the sides of the wall because viscous dissipation (i.e. frictional heating due to shearing) has increased the flow temperature.

5.9 In your own words, what is the definition of a boundary layer?

Solution: There is a short discussion on boundary layers in the video podcasts.

6

Chapter 6 Solutions

Problems

6.1 A metal block, whose dimensions in the $x-$, $y-$, and $z-$directions are respectively L x H x W, is in a steady state with each of its sides all held fixed at various temperatures. If L is much smaller than both H and W, show, by scaling, that the governing equation for this problem is simply the conduction term in the $x-$direction is equal to zero.

Solution: The **first step** in this problem is to simplify and write down the governing equation. This problem deals with steady state conduction, thus the governing equation is:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

The **second step** is to introduce scaled variables into the equation. The scaling we will use is the following:

$$\theta^* = \frac{T - T_{shift}}{T_s}, \quad x^* = \frac{x}{L}, \quad y *^= \frac{y}{H}, \quad z* = \frac{z}{W}$$

where the * superscript indicates the non-dimensional version of the variable in question, T_{shift} and T_s are general shift and scale factors, respectively, for the temperature.

We can introduce this scaling into the governing equation to get:

$$\frac{T_s}{L^2}\frac{\partial^2 \theta^*}{\partial x^{*2}} + \frac{T_s}{H^2}\frac{\partial^2 \theta^*}{\partial y^{*2}} + \frac{T_s}{W^2}\frac{\partial^2 \theta^*}{\partial z^{*2}} = 0$$

Step 3 would just be to divide out by a dimensional coefficient in front of one of the terms. Typically, the largest term is the one that is advised to be divided out. In this case, since L is less than both H and W, we can divide out by $\frac{T_s}{L^2}$ to get:

$$\frac{\partial^2\theta^*}{\partial x^{*2}} + \frac{L^2}{H^2}\frac{\partial^2\theta^*}{\partial y^{*2}} + \frac{L^2}{W^2}\frac{\partial^2\theta^*}{\partial z^{*2}} = 0$$

We now have a non-dimensional equation.

In the **fourth step** we can simplify this equation by recognizing that $\frac{L^2}{H^2}$ and $\frac{L^2}{W^2}$ is really small and can thus be ignored, i.e.:

$$\frac{\partial^2\theta^*}{\partial x^{*2}} = 0$$

This is the simplified scaled equation, which is just the conduction term in x equals zero.

Step 1: Simplify the problem and equations (and boundary/initial conditions) as much as possible.

Step 2: Introduce scaled (i.e. non-dimensional) variables and plug them into the simplified equations and conditions.

Step 3: Divide the whole equation by a dimensional "coefficient." This dimensional coefficient is usually determined to be a coefficient in front of one of the terms in the equation. If there are multiple terms in the equation and each one has a coefficient in front, it is usually advised to divide by the biggest coefficient.

Step 4: Make any additional simplifications with the new scaled variables if necessary.

Step 5: Set any characteristic scale factors not previously defined such that the coefficient terms are of order one (this is the hardest part and will be explained as we go). Note, this may not show up in all problems.

Step 6: Solve the resulting equation if applicable.

6.2 Scale the one dimensional heat equation with the heat generation term included. Write the non-dimensional heat equation in terms of Fourier number (Fo), a non-dimensional time parameter defined as:

$$Fo = \frac{\alpha t}{L^2}$$

where α is the thermal diffusivity and L is the length scale.

<u>**Solution:**</u> The one-dimensional heat equation is (**first step**):

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}_{gen}}{\rho c_p}$$

Introducing scaled variables (**second step**):

$$\theta^* = \frac{T - T_{shift}}{T_s}, \quad x^* = \frac{x}{L}, \quad t*^= \frac{t}{t_s}, \quad z* = \frac{z}{W}$$

where T_{scale} and t_{scale} and the temperature and time characteristic scales, respectively. Their values (or expression) has not yet been defined and may change depending on the specifics of the problem. In addition, we are not going to scale the density, specific heat, and heat generation term as we will just assume those to be a constant. With the scaled variables, the heat equation becomes:

$$\frac{T_s}{t_s}\frac{\partial \theta^*}{\partial t^*} = \alpha \frac{T_s}{L^2}\frac{\partial^2 \theta^*}{\partial x^{*2}} + \frac{\dot{q}_{gen}}{\rho c_p}$$

We now need to divide out by a dimensional coefficient (**third step**). The term we need to divide out by is not as obvious because T_s and t_s have still not been defined. At this point, we will just pick a term to divide out by and so we will just divide out by the $\frac{T_s}{t_s}$:

$$\frac{\partial \theta^*}{\partial t^*} = \alpha \frac{t_s}{L^2}\frac{\partial^2 \theta^*}{\partial x^{*2}} + \frac{t_s}{T_s}\frac{\dot{q}_{gen}}{\rho c_p}$$

We have a scaled equation. Now the question is can we make any simplifications (**step 4**) and/or define any characteristics scale not previously defined (**step 5**)? The answer is yes. For one thing, we have not defined a characteristic time scale. To do so, we will set the coefficient in front of the diffusion term to be one, thus:

$$\alpha \frac{t_s}{L^2} = 1 \xrightarrow{\text{gives us}} t_s = \frac{L^2}{\alpha}$$

We now have a scale factor for time.

Introducing this scale factor for time into our scaled equation we get:

$$\frac{\partial \theta^*}{\partial t^*} = \frac{\partial^2 \theta^*}{\partial x^{*2}} + \frac{L^2}{\alpha T_s}\frac{\dot{q}_{gen}}{\rho c_p}$$

We can simplify further by setting the last term to one such that:

$$\frac{L^2}{\alpha T_s}\frac{\dot{q}_{gen}}{\rho c_p} = 1 \xrightarrow{\text{gives us}} T_s = \frac{L^2\dot{q}_{gen}}{\alpha\rho c_p}$$

Introducing the temperature scale into the scaled equation gives us:

$$\frac{\partial\theta^*}{\partial t^*} = \frac{\partial^2\theta^*}{\partial x^{*2}} + 1$$

We have one more thing to do. We still need to put this equation in terms of an often used non-dimensional parameter, the Fourier number. If we plug the time scale, t_s, expresssion that we obtained earlier into the scaling the time, notice that we get:

$$t^* = \frac{t}{t_s} = \frac{t}{\frac{L^2}{\alpha}} = \frac{\alpha t}{L^2}$$

Comparing non-dimensional time, t_s, to the Fourier number, we notice that they are the same thing. In fact, the Fourier number is nothing but a non-dimensional time parameter for conduction problems. So, we can replace the non-dimensional time with Fourier number (Fo) in our final expression to get:

$$\frac{\partial\theta^*}{\partial Fo} = \frac{\partial^2\theta^*}{\partial x^{*2}} + 1$$

Note, depending on the problem, it may have been more fruitful to use different scale factors for either temperature or time. However, with such little information, the scaling done here was the easiest.

6.3 Consider a conduction problem similar to the one given in Figure 5.5, except the boundary condition of the right side (at $x = L$) is no longer held at a fixed temperature and is instead in contact with a fluid at temperature T_∞. The boundary condition on the right side is given by a convection boundary condition, i.e.:

$$\vec{q}'' \cdot \vec{n} = h\,(T - T_\infty)$$

where h is a parameter known as the heat transfer coefficient. Obtain an expression for the non-dimensional steady state temperature in terms of a non-dimensional parameter known as the Biot number (Bi), given by: $Bi = \frac{hL}{k}$.

Solution: The **first thing** to do is to write down the simplified governing equation and boundary conditions:

$$\frac{d^2T}{dx^2} = 0 \tag{6.1}$$

Notice the boundary condition on the right side (i.e. at $x = L$) contains a heat flux. The heat flux vector dotted with the normal on the right side (i.e. $\vec{n} = \hat{\imath}$ becomes:

$$\vec{q}'' \cdot \vec{n} = -k\left(\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z}\right) \cdot \hat{\imath} = -k\frac{\partial T}{\partial x}$$

Thus, the boundary conditions become (the partial is replaced by an ordinary derivative):

$$\begin{aligned} &\text{at } x = 0, \quad T = T_{left} \\ &\text{at } x = L, \quad -k\frac{dT}{dx} = h\left(T - T_\infty\right) \end{aligned} \tag{6.2}$$

We now want to introduced scaled variables for temperature and x (**step 2**):

$$x^* = \frac{x}{L}, \quad \theta^* = \frac{T - T_\infty}{T_s}$$

Notice that we are shifting by T_∞ instead of T_{left}. We could have shifted by T_{left}, however, as you will see, shifting by T_∞ will make the right boundary condition a little easier to handle. Shifting by T_{left} would make the left boundary condition a little easier. So, you will need to make a decision. In addition, the characteristic scale factor for temperature, T_s, will be determined later.

Introducing the scaling into the governing equation yields:

$$\frac{T_s}{L^2}\frac{d^2\theta^*}{dx^{*2}} = 0$$

Introducing the scaling into boundary conditions yields:

$$\text{at } \underbrace{Lx^*}_{x} = 0, \quad \underbrace{T_s\theta^* + T_\infty}_{T} = T_{left}$$

$$\text{at } \underbrace{Lx^*}_{x} = L, \quad \underbrace{-k\frac{T_s}{L}\frac{d\theta^*}{dx^*}}_{-k\frac{dT}{dx}} = h\left(\underbrace{T_s\theta^* + T_\infty}_{=T} - T_\infty\right)$$

The governing equation and boundary conditions still are not non-dimensional. To do that we need to divide out by a dimensional coefficient (**step 3**). For the governing equation, this is easy, we just divide out by $\frac{T_s}{L^2}$ to get:

$$\frac{d^2\theta^*}{dx^{*2}} = 0$$

Likewise, the boundary conditions are also not yet non-dimensional. We can first consider the boundary condition at $x = 0$ (or $Lx^* = 0$):

$$\text{at } \underbrace{Lx^* = 0}_{\substack{\text{divide}\\ \text{by } L}}, \qquad \underbrace{T_s\theta^* + T_\infty = T_{left}}_{\text{subtract by } T_\infty \text{ and divide by } T_s}$$

$$\rightarrow x^* = 0, \quad \theta^* = \underbrace{\frac{T_{left} - T_\infty}{T_s}}_{=1}$$

If we set the boundary condition on the right to 1, we get an expression now for the characteristic scale scale, namely:

$$T_s = T_{left} - T_\infty$$

Scaling the next boundary condition looks like this :

$$\text{at } \underbrace{Lx^* = L}_{\substack{\text{divide}\\ \text{by } L}}, \qquad \underbrace{-k\frac{T_s}{L}\frac{d\theta^*}{dx^*} = h\left(T_s\theta^* + T_\infty - T_\infty\right)}_{\text{divide out by } -k\frac{T_s}{L} \text{ and subtract out the } T_\infty \text{ on the right}}$$

$$\rightarrow x^* = 1, \quad \frac{d\theta^*}{dx^*} = -\underbrace{\frac{hL}{k}}_{=Bi}\theta^*$$

Thus the scaled boundary conditions are:

$$\text{at } x^* = 0, \quad \theta^* = 1$$
$$\text{at } x^* = 1, \quad \frac{d\theta^*}{dx^*} = -Bi\theta^*$$

Notice that we introduced a new parameter, the Biot number (Bi).

Solving our scaled governing equation (**step 6**) leads to the general solution of:

$$\theta^* = C_1 x^* + C_2$$

Applying our first scaled boundary condition (i.e. at $x = 0, \quad \theta^* = 1$) to our scaled general solution leads to:

$$1 = C_1 0 + C_2$$
$$\therefore C_2 = 1$$

Applying the next boundary condition gives us:

$$\left.\frac{d\theta^*}{dx^*}\right|_{x^*=1} = -Bi\theta^*\Big|_{x^*=1}$$
$$\rightarrow C_1 = -Bi\left(C_1 1 + \underbrace{1}_{=C_2}\right)$$
$$\therefore C_1 = \frac{-Bi}{1 + Bi}$$

So our final expression for the non-dimensional temperature is:

$$\theta^* = \frac{-Bi}{1 + Bi}x^* + 1$$

6.4 The book provided the boundary layer equations in scaled form. Rescale the boundary layer equations in dimensional form.

Solution: The scaled boundary layer equations are:

Continuity for an incompressible flow past a flat plate:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$x-$**momentum for a steady incompressible flow past a flat plate (with a constant free stream velocity):**

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{\partial^2 u^*}{\partial y^{*2}}$$

$y-$**momentum for a steady incompressible flow past a flat plate:**

$$\frac{\partial p^*}{\partial y^*} = 0$$

To rescale, all that we have to do is reintroduce the expressions for the scaled variables into the equations:

$$u^* = \frac{u}{U_\infty}, \quad v^* = \frac{v}{V_s}, \quad p^* = \frac{p}{p_s}, \quad x^* = \frac{x}{\mathbb{X}}, \quad y^* = \frac{y}{\delta}$$

where:

$$V_s = \frac{\delta U_\infty}{\mathbb{X}}, \quad p_s = \rho U_\infty^2, \quad \delta = \sqrt{\frac{\mu \mathbb{X}}{\rho U_\infty}}$$

Thus, the continuity equation becomes:

$$\frac{\partial \left(\frac{u}{U_\infty} \right)}{\partial \left(\frac{x}{\mathbb{X}} \right)} + \frac{\partial \left(\frac{v}{\frac{\delta U_\infty}{\mathbb{X}}} \right)}{\partial \left(\frac{y}{\delta} \right)} = 0$$
$$\rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The Navier-Stokes in $x-$becomes:

$$\frac{u}{U_\infty}\frac{\partial\left(\dfrac{u}{U_\infty}\right)}{\partial\left(\dfrac{x}{\mathbb{X}}\right)}+\frac{v}{\dfrac{\delta U_\infty}{\mathbb{X}}}\frac{\partial\left(\dfrac{u}{U_\infty}\right)}{\partial\left(\dfrac{y}{\delta}\right)}=\frac{\partial^2\left(\dfrac{u}{U_\infty}\right)}{\partial\left(\dfrac{y}{\delta}\right)^2}$$

$$\xrightarrow{\text{divide out by } \frac{\mathbb{X}}{U_\infty^2}} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{U_\infty\delta^2}{\mathbb{X}}\frac{\partial^2 u}{\partial y^2}$$

$$\xrightarrow{\text{introduce } \delta} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{U_\infty\left(\frac{\mu\mathbb{X}}{\rho U_\infty}\right)}{\mathbb{X}}\frac{\partial^2 u}{\partial y^2}$$

$$\xrightarrow{\text{with } \nu=\mu/\rho} u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

The momentum equation in $y-$ is simply:

$$\frac{\partial\left(\dfrac{p}{p_s}\right)}{\partial\left(\dfrac{y}{\delta}\right)}=0$$

$$\rightarrow \frac{\partial p}{\partial y}=0$$

6.5 Scale Cauchy's momentum equation.

Solution: The Cauchy momentum equation (i.e. Cauchy's first law) is:

$$\rho\left(\frac{\partial\vec{V}}{\partial t}+\vec{V}\cdot\vec{\nabla}\vec{V}\right)=\vec{\nabla}\cdot\vec{\vec{T}}+\rho\vec{g}$$

Introduce the following scaled variables (we will assume density is a constant):

$$\vec{V}^*=\frac{\vec{V}}{U_\infty},\quad t^*=\frac{t}{t_s},\quad \vec{\nabla}^*=\frac{\vec{\nabla}}{1/L},\quad \vec{\vec{T}}^*=\frac{\vec{\vec{T}}}{\sigma_s},\quad \vec{g}^*=\frac{\vec{g}}{g_0}$$

where U_∞ is a velocity scale (we will just assume is known), g_0 is a gravity scale, t_s is a time scale to be determined, and σ_s is a stress scale to be determined. Plugging this scaling into Cauchy's equation leads to:

$$\rho\left(\frac{U_\infty}{t_s}\frac{\partial \vec{V}^*}{\partial t^*} + \frac{U_\infty^2}{L}\vec{V}^* \cdot \vec{\nabla}^* \vec{V}^*\right) = \frac{\sigma_s}{L}\vec{\nabla} \cdot \vec{\vec{T}}^* + \rho g_0 \vec{g}^*$$

We can pull out the $\frac{U_\infty^2}{L}$ term from the parentheses on the left-hand side to get:

$$\rho\frac{U_\infty^2}{L}\left(\frac{U_\infty}{t_s L}\frac{\partial \vec{V}^*}{\partial t^*} + \vec{V}^* \cdot \vec{\nabla}^* \vec{V}^*\right) = \frac{\sigma_s}{L}\vec{\nabla}^* \cdot \vec{\vec{T}}^* + \rho g_0 \vec{g}^*$$

We can divide out by $\rho\frac{U_\infty^2}{L}$ to get:

$$\frac{U_\infty}{t_s L}\frac{\partial \vec{V}^*}{\partial t^*} + \vec{V}^* \cdot \vec{\nabla}^* \vec{V}^* = \frac{\sigma_s}{\rho U \infty^2}\vec{\nabla}^* \cdot \vec{\vec{T}}^* + \frac{g_0 L}{U_\infty^2}\vec{g}^*$$

We can now set the first term on the left to be one (making $t_s = \frac{L}{U_\infty}$) and the first term on the right to be one (making $\sigma_s = \rho U_\infty^2$). In addition, the gravity term coefficient ($\frac{g_0 L}{U_\infty^2}$) is just the inverse of the Froude number squared. Thus, the scaled Cauchy equation is:

$$\frac{\partial \vec{V}^*}{\partial t^*} + \vec{V}^* \cdot \vec{\nabla}^* \vec{V} = \vec{\nabla}^* \cdot \vec{\vec{T}}^* + \frac{1}{Fr^2}\vec{g}^*$$

Note that there are other ways of generally scaling Cauchy's equation and the exact scaling will depend on the problem at hand. For instance, some may scale the stress tensor with a general known pressure scale, p_0, and also include a scaling for density of $\rho^* = \frac{\rho}{\rho_0}$, where ρ_0 is a known density scale, leading to a Cauchy first law equation of:

$$\frac{\partial \vec{V}^*}{\partial t^*} + \vec{V}^* \cdot \vec{\nabla}^* \vec{V} = \frac{p_0}{\rho_0 U_\infty^2}\frac{1}{\rho^*}\vec{\nabla}^* \cdot \vec{\vec{T}}^* + \frac{1}{Fr^2}\frac{\vec{g}^*}{\rho^*}$$

The $\frac{p_0}{\rho_0 U_\infty^2}$ is often given the name, Euler number, Eu, thus:

$$\frac{\partial \vec{V}^*}{\partial t^*} + \vec{V}^* \cdot \vec{\nabla}^* \vec{V} = \frac{Eu}{\rho^*}\vec{\nabla}^* \cdot \vec{\vec{T}}^* + \frac{1}{Fr^2}\frac{\vec{g}^*}{\rho^*}$$

6.6 Consider Couette flow between two plates a distance of 2 millimeters apart whose top plate moves with a velocity of 1 m/s and the bottom plate is held fixed. The temperature of both plates are held fixed at zero degrees Celsius. The thermal conductivity is 0.6 $\frac{\mathrm{W}}{\mathrm{mK}}$ and the dynamic viscosity is 10^{-3} Pa·s. Find a value for the non-dimensional temperature in the middle in the flow. Unscale the result and obtain a value for the temperature in the middle of the flow in Celsius.

Solution: The **first step** is to write down the $x-$momentum and energy equations for this problem, which was obtained in Chapter 5:

$$\frac{d^2u}{dy^2} = 0 \qquad (x\text{−momentum})$$
$$\frac{d^2T}{dy^2} = -\frac{\mu}{k}\left(\frac{du}{dy}\right)^2 \qquad (\text{energy})$$

There is no $y-$momentum equation for this problem because we are going to assume only a one-dimensional flow.

The boundary conditions are:

$$\text{at } y = 0, \quad u = 0 \quad \text{and} \quad T = T_B \ (0°\text{C})$$
$$\text{at } y = H \ (2 \text{ mm}), \quad u = U_T \ (1 \text{ m/s}) \quad \text{and} \quad T = T_T \ (0°C)$$

For the **next step**, we can introduce non-dimensional variables:

$$y^* = \frac{y}{H}, \quad u* = \frac{u}{U_T}, \quad \theta^* = \frac{T - T_B}{Ts}$$

Notice that we are not defined the temperature scale as $T_T - T_B$. The reason for not doing so is that, for this particular problem, it would result in dividing by zero. Plugging these scaled variables into our simplified governing equations give:

$$\frac{U_T}{H^2}\frac{\partial^2 u^*}{\partial y^{*2}} = 0 \qquad (x\text{−momentum})$$
$$\frac{T_s}{H^2}\frac{\partial^2 \theta^*}{\partial y^{*2}} = -\frac{\mu U_T^2}{kH^2}\left(\frac{\partial u^*}{\partial y^*}\right)^2 \qquad (\text{energy})$$

When we plug the scaled variables into the boundary conditions we get:

$$\text{at } Hy^* = 0, \;\; U_T u^* = 0 \quad \text{and} \quad T_s\theta^* + T_B = T_B$$
$$\text{at } Hy^* = H, \;\; U_T u^* = U_T \quad \text{and} \quad T_s\theta^* + T_B = T_T$$

Our **third step** is to properly scale the equations by dividing out by a coefficient in front of one of the terms. In the case of the $x-$momentum equation, the only coefficient is the $\frac{U_T}{H^2}$. In the case of the temperature equation, there are two terms. The term on the left (the diffusion term) has a coefficient of $\frac{T_s}{H^2}$ in front of it and the term on the right (the viscous dissipation term) has the coefficient $\frac{\mu U_T^2}{kH^2}$. We will divide out by $\frac{T_s}{H^2}$. Thus, dividing out the $x-$momentum equation by $\frac{U_T}{H^2}$ and the energy by $\frac{T_s}{H^2}$ yields the following result for the scaled governing equations:

$$\frac{d^2u^*}{dy^{*2}} = 0 \qquad \text{(scaled } x\text{−momentum)}$$
$$\frac{d^2\theta^*}{dy^{*2}} = -\frac{\mu U_T^2}{kT_s}\left(\frac{du^*}{dy^*}\right)^2 \qquad \text{(scaled energy)}$$

The boundary conditions from Equation also need to be scaled by dividing out by a coefficient. Since the temperatures at both the top and the bottom are zero, the scaled temperature at the top and bottom will also be zero. The scaled velocity at the top will be 1 and the scaled velocity at the bottom will be zero. Thus:

$$\text{at } y^* = 0, \;\; u^* = 0 \quad \text{and} \quad \theta^* = 0$$
$$\text{at } y^* = 1, \;\; u^* = 1 \quad \text{and} \quad \theta^* = 0$$

Note now that we can find an expression for T_s (**step 5**) by setting the coefficient in front of the viscous dissipation term on the right hand side of the energy equation to be zero, i.e.:

$$\frac{\mu U_T^2}{kT_s} = 1 \xrightarrow{\text{implies}} T_s = \frac{\mu U_T^2}{k}$$

Thus, our new energy equation is:

$$\frac{d^2\theta^*}{dy^{*2}} = -\left(\frac{du^*}{dy^*}\right)^2$$

So, the final scaled equations and boundary conditions we are to solve are:

$$\frac{\partial^2 u^*}{\partial y^{*2}} = 0 \qquad \text{(scaled } x\text{−momentum)}$$
$$\frac{\partial^2 \theta^*}{\partial y^{*2}} = -\left(\frac{\partial u^*}{\partial y^*}\right)^2 \qquad \text{(scaled energy)}$$

With boundary conditions:

$$\begin{aligned} &\text{at } y^* = 0: \; u^* = 0 \quad \text{and} \quad \theta^* = 0 \\ &\text{at } y^* = 1: \; u^* = 1 \quad \text{and} \quad \theta^* = 0 \end{aligned}$$

We can now solve this problem (**step 6**). The x−momentum has a general solution that is linear. That is:

$$u^* = C_1 y^* + C_2$$

where C_1 and C_2 are the unknowns. Applying the boundary conditions for u^* leads to:

$$\begin{aligned} &\text{at } y^* = 0, \quad u^* = 0 \quad \rightarrow \quad 0 = C_1 0 + C_2 \\ &\therefore C_2 = 0 \\ &\text{at } y^* = 1, \quad u^* = 1 \quad \rightarrow \quad 1 = C_1 1 + 0 \\ &\therefore C_1 = 1 \end{aligned}$$

So the final x−velocity profile is just a line given by:

$$u^* = y^*$$

Next up is the energy equation. The energy equation requires the derivative of the x−velocity with respect to y, which is nothing but 1. Thus, the general solution to the energy equation is:

$$\frac{d^2\theta^*}{dy^{*2}} = -\underbrace{\left(\frac{du^*}{dy^*}\right)^2}_{=1} \xrightarrow{\text{solve}} \theta^* = -\frac{1}{2}y^{*2} + C_3 y^* + C_4$$

Applying the boundary conditions for the energy equation leads to:

$$\text{at } y^* = 0, \quad \theta^* = 0 \quad \rightarrow \quad 0 = -\frac{1}{2}0^2 + C_3 0 + C_4$$
$$\therefore C_4 = 0$$
$$\text{at } y^* = 1, \quad \theta^* = 0 \quad \rightarrow \quad 0 = -\frac{1}{2}1^2 + C_3 1$$
$$\therefore C_3 = \frac{1}{2}$$

So, our final energy equation is (with constants C_3 and C_4 included):

$$\theta^* = -\frac{1}{2}y^{*2} + \frac{1}{2}y^*$$

The non-dimensional temperature in the middle of the flow is obtained by setting $y^* = 1/2$ to get:

$$\theta^*_{midpoint} = -\frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\frac{1}{2} = \frac{1}{8}$$

The unscaled temperature in the middle of the flow is obtained simply by multiplying by the temperature scale, i.e.:

$$T_{midpoint} = \theta^*_{midpoint} T_s = \frac{1}{8}\frac{\mu U_T^2}{k} = \frac{1}{8}\frac{(10^{-3})(1)^2}{0.6} = 0.0002°C$$

6.7 Consider flow past a flat plate. At what point downstream from the leading edge does the boundary layer double in size compared to the size of the boundary layer at $x = 1$ meter? You may assume the Reynolds number never reaches 10^5, which is the transition to turbulence.

Solution: The boundary layer thickness is given by:

$$\delta_{BL} = 5\frac{\mathbb{X}}{\sqrt{Re_x}}$$

Which just equals:

$$\delta_{BL} = 5\frac{\mathbb{X}}{\sqrt{\frac{U_\infty \mathbb{X}}{\nu}}} = 5\sqrt{\frac{\nu \mathbb{X}}{U_\infty}}$$

Thus the expression for the boundary layer thickness when $x = \mathbb{X} = 1$ m is given by:

$$\delta_{BL} = 5\sqrt{\frac{\nu}{U_\infty}}$$

The distance downstream when the boundary layer thickness is twice as thick can be obtained by:

$$5\sqrt{\frac{\nu\mathbb{X}}{U_\infty}} = 2\left(5\sqrt{\frac{\nu}{U_\infty}}\right)$$

Solving for $\mathbb{X}$ yields:

$$\mathbb{X} = 4 \text{ meters}$$