SAMPLE EXAM QUESTIONS (OR ADDITIONAL EXERCISES), AND DERIVED ANSWERS

1. Consider a seismic line laid out along the x axis. The x axis points in a direction 25^o east of north. The z axis points downwards into the Earth. The seismometers are $3C$ (i.e., they each produce 3 traces, corresponding to the u_x, u_y, and u_z components of displacement $\mathbf{u}$). It is desired to rotate the data about the z axis into an $x'y'z'$ coordinate system in which the x' axis points directly north.

(a). Calculate the rotation matrix needed to do this.

(b). If, at a given time, the displacement $\mathbf{u} = (-2, 1, -1)$, as measured on the original 3 traces, what is $\mathbf{u}'$ (i.e., what is $\mathbf{u}$ in the new coordinate system)?

2. The components of a $3D$ tensor t_{ij} are all zero except for $t_{11} = 5$, $t_{33} = -2$, and $t_{13} = 4$. What is the value of the quantity $L = \sum_{i=1}^{3}\sum_{j=1}^{3} t_{ij}\delta_{ij}$? δ_{ij} is the Kronecker delta.

3. Let a be a constant, and let $g(t) = 1$ for $0 \le t \le a$, -1 for $-a \le t < 0$, and 0 for all other t. Sketch a graph of $g(t)$. Calculate the amplitude and phase spectra of $g(t)$, sketch graphs of them, and briefly discuss them. Hint: after calculating the Fourier transform, also calculate $\sin^2\theta$ from the formula $\sin\theta = \left(e^{i\theta} - e^{-i\theta}\right)/2i$.

4. *(a)*. Suppose a coordinate system is rotated about the x_3 axis by an angle ϕ_1 and that this is followed by a second rotation about the new x_2 axis by an angle ϕ_2. Derive formulas for the values of the components of a vector $\mathbf{A}$ in the final rotated coordinate system. *(b).* Apply your results to the case where $\mathbf{A}$ is a vector parallel to the x_3 axis. What do you notice about the result? Can you explain it?

5. Suppose the values of all the components σ_{ij} of the stress tensor are known in a certain Cartesian (xyz) coordinate system. In this coordinate system, the pressure P on the $x-y$ plane due to a deformation is given by $P = -\sigma_{zz}$. What is P in a coordinate system rotated by an angle θ about the $y-$axis? In other words, derive the formula for $P' = -\sigma'_{zz}$. The rotation is counter-clockwise (the $z-$axis rotates towards the $x-$axis). Hint: use the transformation law for tensors of rank 2: $C'_{ij} = \sum_k \sum_\ell R_{ik} R_{j\ell} C_{k\ell}$. Does your answer reduce to the correct one for $\theta = 0^o$ and 90^o ?

6. For a certain isotropic medium, $c_{2211} = 4.78\, GPa$, $c_{3333} = 19.22\, GPa$, and the density is $2.0\, g/cm^3$ What are the P and S wave speeds in the medium?

7. Evaluate the sum $\sum_i \sum_j \sigma_{ij}\eta_{ij}$, where σ_{ij} and η_{ij} are the stress and rotation tensors, respectively. Hint: use the symmetry properties of these two tensors.

8. True or false?: For P waves, $\partial u_j / \partial x_i = \partial u_i / \partial x_j$, $(i, j = 1,2,3)$, which means that the strain tensor for P waves can be written as $e_{ij} = \partial u_j / \partial x_i = \partial u_i / \partial x_j$. Prove your answer.

9. It is known that W=K at all times for a plane harmonic P wave, where W and K are the potential (strain) and kinetic energy densities for an isotropic medium. But in the case of "simple harmonic motion" for a mass m connected to a spring, the potential and kinetic energies, W_m and K_m , of the mass m, are generally not equal – W_m is a maximum when K_m is a minimum, and vice versa, and W_m and K_m are equal only at a specific time instants (e.g., part-way through an oscillation cycle). Explain why $W = K$ at all times, whereas $W_m = K_m$ only at specific times.

10. Derive a formula for Young's modulus Y in terms of the P and S wave speeds α and β. Apply your formula to obtain the value of Y for a rock which is as "hard" as possible (i.e., Poisson's ratio $\sigma = 0$). What is Lamé's parameter λ for such a rock? What kind of materials have Y=0 ?

11. The speed of a compressional wave propagating down a long thin homogeneous rod is given by $v_{rod} = [Y/\rho]^{1/2}$, where Y is Young's modulus, i.e., $Y = \mu(3\lambda + 2\mu)/(\lambda + \mu)$. Note that v_{rod} is different from the compressional wave speed $\alpha = [(\lambda + 2\mu)/\rho]^{1/2}$ in an infinite homogeneous medium. True or false? : $v_{rod} < \alpha$ for all materials with a positive Poisson ratio $\sigma \equiv \lambda/[2(\lambda + \mu)]$. Prove your answer mathematically. What is the ratio v_{rod}/α for $\sigma = 1/4$?

12. ***(a)*** Calculate the mean value of the vector $\mathbf{V}$, where $V_i = -\sum_k (\partial u_k / \partial t) e_{ik}$ for a plane harmonic P wave propagating in the x direction in a homogeneous isotropic medium. e_{ik} is the strain tensor and $\boldsymbol{u}$ is the displacement vector.
(b) What would the mean value be if the wave was propagating in the z direction? (Hint: consider the nature of the medium).

13. Assume the z axis points down into the Earth. The displacement produced by a certain seismic plane wave is given by $\mathbf{u} = U \exp[i\omega(px + qz - t)]\mathbf{d}$, where "$\exp[a]$" means "$e^a$", and where

$p = 0.1272\, s/km,\quad q = -0.4158\, s/km,\quad \omega = 188.50\, Hz,$ and $\mathbf{d} = (0.2924, 0, -0.9563)$.

(a) What is the speed of the wave?
(b) In what direction is the wave propagating, and at what angle (relative to the vertical)?
(c) What kind of wave is it (P, SV, or SH)?
(d) What are the frequency and wavelength ?
(e) If the wave has a phase-delay of 30º and if the maximum value of $|\mathbf{u}|$ is 2 mm, what is U ? (express your answer in polar form, i.e., in the form "$re^{i\theta}$").
(f) What is the displacement $\mathbf{u}$ experienced by a material point (or particle) at $(x, z) = (0,0)$ at the time $t = 0$?

14. Consider an interface between two rock layers. The *P* and *S* wave speeds in the upper layer are 2.5 and $1.4\,km/s$, respectively, and in the lower layer they are 3.6 and $2.08\,km/s$, respectively. The density is $2.0\,g/cm^3$, in both layers. Consider an incident *P* wave in the upper layer. The angle of incidence is 30º. The displacement reflection coefficient for the reflected *SV* wave is $R_{PS} = -0.12$.

(a) What fraction of the incident wave's down-going energy flux is carried away by the reflected *SV* wave?

(b) What is the phase-delay for the reflected *SV* wave?

15. Consider a horizontal flat interface separating two rock layers. For our purposes, assume each layer is infinitely thick (i.e., it is a half-space). The lower layer has the higher acoustic impedance. Suppose a compressional wave source is located in the upper layer, well above the interface, and that a receiver is located vertically above the source and well-separated from it. Assume that the waves can be approximated by plane waves. Sketch the trace you would expect to see if the receiver was a geophone. Repeat for a hydrophone.

16. We have seen that the displacement reflection and transmission coefficients for a normally incident plane $P-$wave are R and T, and that the corresponding pressure coefficients are $\overline{R}$ and $\overline{T}$, where $\overline{R} = R$ and $\overline{T} = (Z_2 / Z_1)T$. What are the corresponding P wave potential coefficients? For the "ghost" reflection discussed in chapter 3 of the text (see Figure 3-3), if the geophone were replaced by a hydrophone, would the primary and ghost still have the opposite polarity? What about if the geophone were replaced by an instrument that could measure the P wave potential?

17. Consider a flat horizontal interface between two homogeneous materials.

(a). In the upper material (medium 1) the $P-$ and $S-$wave speeds are 2.0 and $1.3\,km/s$, respectively, and in the lower material (medium 2) they are 3.1 and $1.7\,km/s$, respectively. A plane *P* wave is incident in medium 1 at an angle of $20°$. What are the slowness and polarization vectors (**s** and **d**) for the two transmitted waves? If the waves have a frequency of $30\,Hz$, what are their wavelengths?

(b). Consider a *P* or *SV* plane wave incident at some non-zero angle. It generates four scattered (i.e., reflected and transmitted) plane waves. True or false? : if the scattered wave speed is greater than the incident wave speed, then the scattered wave is bent towards the horizontal upon scattering (i.e., the scattering angle is larger than the incidence angle). Prove your answer.

18. The components of particle displacement for a hypothetical surface wave are given approximately by $u_x = -\exp\left[-Az\right]\cos\left(\kappa x - \omega t\right)$, $u_y = 0$, and $u_z = \left(1 - Bz^2\right)\exp\left[-Bz^2/2\right]\sin\left(\kappa x - \omega t\right)$ where *A* and *B* are positive constants, and where *z* is the depth below the surface (*z* is positive downwards). Discuss the particle motion for this wave at the surface and at any depth *z*: is it elliptical? retrograde? prograde? What about nodal planes?

19. Consider a medium with *P* and *S* wave velocities of $1200\,m/s$ and $600\,m/s$ overlying another medium with *P* and *S* velocities of $2200\,m/s$ and $1000\,m/s$. Consider an *SV* wave incident from the upper medium. Calculate all the critical angles, and sketch a wavefront diagram showing all the head waves generated.

20. A set of Love-wave data in a layer over a half-space has been analyzed. It is found that the phase velocity c at $10 Hz$ is $1 km/s$. The layer is $20m$ thick. The shear wave speed β_1 in the layer is $0.7 km/s$. Calculate the shear wave speed β_2 in the half-space. Assume the layer and half-space have approximately the same densities.

21. Consider a subsurface model consisting of three flat horizontal homogeneous layers over a homogeneous half-space. A source is on the surface. The $P-$wave velocity values, from the top down, are 1, 2, and $3 km/s$. The layer thicknesses, from the top down, are 300, 500, and 800 m. Consider a ray for a primary reflection off the base of layer 3. The take-off angle is $15°$. Calculate the offset x and the travel-time t for the ray.

22. Consider a medium in which the wave speed varies continuously with depth z as $v(z) = v_0(2 - e^{-az})$, where v_0 and a are positive constants.

(a) Sketch a graph of *v(z)*.

(b) Without doing a calculation, sketch some typical ray paths with varying take-off angles that you might expect to see in this medium, and explain your results.

23. Consider a subsurface medium consisting of two horizontal homogeneous layers over a half-space. The $P-$wave speeds in the first and second layers are α_1 and α_2, with $\alpha_2 > \alpha_1$. Sketch a typical non-zero-offset ray path for the primary $P-$wave reflection from the second interface (all the ray segments are $P-$waves). Next, suppose that the value of α_2 is increased (with α_1 remaining the same), and sketch the ray path again for the same offset (superimpose it on your previous sketch). Which of these two ray paths has the larger ray parameter (p) value? Or do they have the same p value because they have the same offset? Explain your answer.

24. In a certain subsurface zone, the $P-$wave velocity is given by $v = v_0 + az$, where $v_0 = 2.0 km/s$ and $a = 0.5s^{-1}$. How deep does a $P-$wave penetrate for a source-receiver offset of $3 km$?

25. Consider a stack section containing a single flat dipping reflection event. It is produced by a flat subsurface reflector with dip angle δ. The body wave speed above the reflector is v_1. Let v be the apparent velocity of the event as measured on the stack section, i.e., v is the slope of the reflection event on the stack section (e.g., in *m/s*). Derive a formula for v in terms of v_1 and δ. How does it compare with the apparent velocity measured on a shot record?

26. The frequency spectrum of a signal *s(t)* is given by {1, for $-\omega_1 < \omega < +\omega_1$; 0, otherwise}. Derive the formulas for the signal and its Hilbert transform (HT), and sketch graphs of them. Does the HT look like what you would expect a 90^o-phase-advanced signal to look like? Explain.

27. A depth model consists of 31 flat horizontal alternating layers. Layers 1, 3, 5, …, 31 each have a density of $2.0\,g/cm^3$ and a $P-$wave speed of $1.0\,km/s$, and layers 2, 4, 6, …, 30 each have a density of $2.2\,g/cm^3$ and a $P-$wave speed of $2.0\,km/s$. The layers all have the same thickness, 20 *m*. What is the amplitude, measured by a surface geophone, of a zero-offset primary reflection from interface 30 if we consider ***(a)*** reflection losses only? ***(b)*** both reflection and transmission losses? ***(c)*** reflection, transmission and geometrical spreading losses? ***(d)*** Comment on what other possible depth models would give identical results for parts (a), (b) and (c).

28. Consider a medium consisting of two horizontal homogeneous absorbing layers, each of thickness 1 km, over a half-space. The $P-$wave speeds in the first layer, second layer, and half-space are 2, 3 and $4.0\,km/s$, respectively, and the $P-$wave Q values are 20, 30 and 40, respectively. The density is $2.0\,g/cm^3$ everywhere. Suppose a *plane* $P-$wave propagates vertically from the surface to the second reflector and back to the surface, i.e., the wave is a primary reflection. Suppose the amplitude spectrum of this wave as it leaves the surface is $A(\omega)=\exp\left\{-\left[c(\omega-\omega_0)/\omega_0\right]^2\right\}$ (for $\omega>0$), where $f_0=\omega_0/2\pi=50Hz$, and $c=\pi$. Sketch a graph of $A(\omega)$. What is the dominant frequency f_d? What is the amplitude spectrum $A_r(\omega)$ of the wave that arrives back at the surface, and what is its dominant frequency f_{dr}? Briefly discuss the physical meaning of your results. Ignore dispersion.

29. Consider a vertically transversely isotropic (*VTI*) medium that has the following parameter values: $\rho=2.2\,g/cm^3$, $\alpha_0=2\,km/s$, $\beta_0=1\,km/s$, $\varepsilon=0.33$, $\gamma=\delta=0.3$. Assume that this is not "weak" *VTI*. Compute the numerical value of the wave speed of the horizontally – travelling *qP* wave.

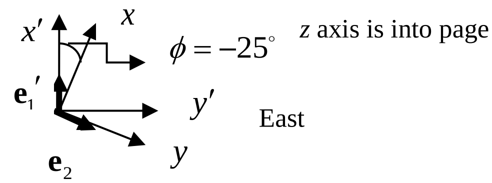
30. Consider two vertically transversely isotropic (*VTI*) half-spaces joined at a flat horizontal interface, and consider an incident plane *qSH* wave in the upper half-space. The vertical and horizontal *qSH* wave speeds are 1.0 and 1.3 *km/s*, respectively, in the upper half-space, and 2.0 and 2.2 *km/s*, respectively, in the lower half-space. If the phase angle of the incident wave is $20°$, what is the phase angle of the transmitted wave? What would it be if the two media were isotropic with *SH* wave speeds of 1.0 *km/s* and 2.0 *km/s* in the upper and lower media, respectively ?

ANSWERS begin on the next page

ANSWERS

1. ***(a).*** The rotation of the $x-y$ plane about the z axis, which is chosen as positive downwards, by some angle, ϕ, is described for an arbitrary point $\mathbf{x}=(x,y,z)^T$ as $\mathbf{x}'=\Re\mathbf{x}$, where

$$\Re=\begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}$$



and $\mathbf{x}'=(x',y',z')^T$ is the point in the rotated coordinate system. If the rotation is clockwise, so that the x axis moves towards the y axis, then ϕ is positive, while for counter clockwise rotation, as is the case here, ϕ is negative, i.e., $\phi=-25^o$. It follows that

$$\Re=\begin{bmatrix}\cos 25 & -\sin 25 & 0\\ \sin 25 & \cos 25 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0.9063 & -0.4226 & 0\\ 0.4226 & 0.9063 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Check: $R_{12}=\mathbf{e}_1'\bullet\mathbf{e}_2=\cos(90^\circ+25^\circ)=-0.42$

(b). With this and $\mathbf{u}=(u_x,u_y,u_z)^T=(-2,1,-1)^T$

$$\mathbf{u}'=\Re\mathbf{u}$$

$$\begin{bmatrix}u_x'\\ u_y'\\ u_z'\end{bmatrix}=\begin{bmatrix}0.9063 & -0.4226 & 0\\ 0.4226 & 0.9063 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-2\\ 1\\ -1\end{bmatrix}=\begin{bmatrix}-2.235\\ 0.0611\\ -1\end{bmatrix}$$

2.

$$L=\sum_{i=1}^{3}\sum_{j=1}^{3}t_{ij}\delta_{ij}=\sum_{i=1}^{3}t_{ii}=t_{11}+t_{22}+t_{33}$$

$$L=5+0+(-2)$$

$$L=3$$

3. The Fourier transform is defined as

$$\bar{g}(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{i\omega t}\, dt$$

For the $g(t)$ specified here

$$\bar{g}(\omega) = \int_{-a}^{0} (-1)\, e^{i\omega t} dt + \int_{0}^{a} (+1)\, e^{i\omega t} dt \;=\; -\left[\frac{e^{i\omega t}}{i\omega}\right]_{-a}^{0} + \left[\frac{e^{i\omega t}}{i\omega}\right]_{0}^{a}$$

$$= -\left[\frac{1 - e^{-i\omega a}}{i\omega}\right] + \left[\frac{e^{i\omega a} - 1}{i\omega}\right] \;=\; \frac{e^{i\omega a} - 2 + e^{-i\omega a}}{i\omega} = \frac{2\left[\cos\omega a - 1\right]}{i\omega} \;.$$

Using the definition $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ the following is obtained

$$\sin^2\theta = \frac{e^{2i\theta} - 2 + e^{-2i\theta}}{-4} \;.$$

Comparing this result with that obtained for $\bar{g}(\omega)$ has

$$\bar{g}(\omega) = -\frac{4\sin^2 \omega a/2}{i\omega} = \frac{4i\sin^2 \omega a/2}{\omega}$$

from which it follows that

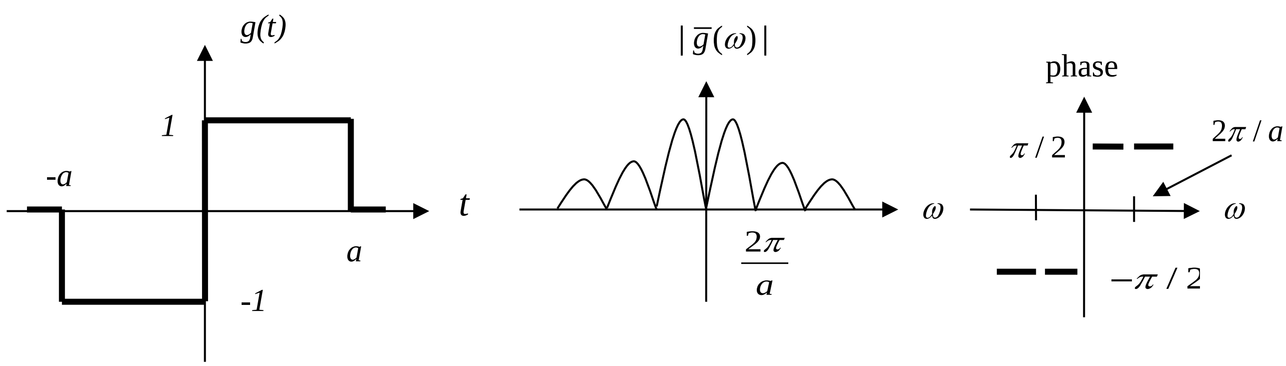
$$\left|\bar{g}(\omega)\right| = \left|-\frac{4\sin^2 \omega a/2}{i\omega}\right| = \frac{4\sin^2 \omega a/2}{|\omega|} .$$

It may appear that the above may be problematic at $\omega = 0$. If the small argument approximation for $\sin x$ is used, $\sin x \approx x$ for $|x| << 1$ then

$$\sin^2 \omega a/2 \approx \omega^2 a^2/4 \quad \rightarrow \quad \left|\bar{g}(\omega)\right| \approx |\omega| a^2 \quad \text{for} \quad |\omega| a/2 \ll 1.$$

Note that the amplitude spectrum has peaks (the peaks of $\sin^2(\omega a/2)$) which decrease as $|\omega|$ increases. As for the phase spectrum, inspection of the formula above for the Fourier transform $\bar{g}(\omega)$ shows that the phase spectrum is $\pi/2$ radians (or 90 deg) for positive ω and $-\pi/2$ radians (or -90 deg) for negative ω, except at $\omega = \pm 2\pi n/a$, n = 0, 1, 2, ... , where the phase spectrum is undefined (because $\bar{g}(\omega) = 0$ there).

The graphs are schematically shown below.



4. ***(a).*** The first rotation about the x_3 axis has the form

$$\Re_1 = \begin{bmatrix} \cos\phi_1 & \sin\phi_1 & 0 \\ -\sin\phi_1 & \cos\phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the second rotation about the initially rotated x_2 axis is given by

$$\Re_2 = \begin{bmatrix} \cos\phi_2 & 0 & \sin\phi_2 \\ 0 & 1 & 0 \\ -\sin\phi_2 & 0 & \cos\phi_2 \end{bmatrix}$$

After the two rotations

$$\mathbf{A}' = \Re_1 \mathbf{A},$$
$$\mathbf{A}'' = \Re_2 \mathbf{A}' = \Re_2 \Re_1 \mathbf{A} = \Re \mathbf{A}.$$

where

$$\Re = \Re_2 \Re_1 = \begin{bmatrix} \cos\phi_2 & 0 & \sin\phi_2 \\ 0 & 1 & 0 \\ -\sin\phi_2 & 0 & \cos\phi_2 \end{bmatrix} \begin{bmatrix} \cos\phi_1 & \sin\phi_1 & 0 \\ -\sin\phi_1 & \cos\phi_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi_1\cos\phi_2 & \sin\phi_1\cos\phi_2 & \sin\phi_2 \\ -\sin\phi_1 & \cos\phi_1 & 0 \\ -\cos\phi_1\sin\phi_2 & -\sin\phi_1\sin\phi_2 & \cos\phi_2 \end{bmatrix}$$

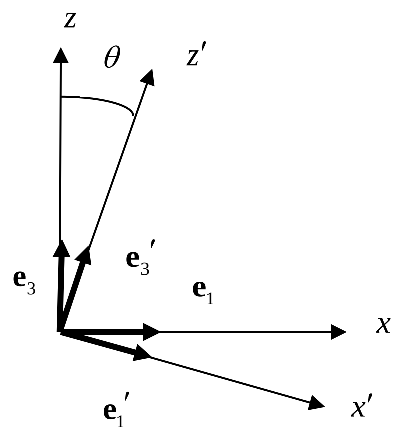
This produces the values

$$A_1'' = \cos\phi_1\cos\phi_2 A_1 + \sin\phi_1\cos\phi_2 A_2 + \sin\phi_2 A_3$$
$$A_2'' = -\sin\phi_1 A_1 + \cos\phi_1 A_2$$
$$A_3'' = -\cos\phi_1\sin\phi_2 A_1 - \sin\phi_1\sin\phi_2 A_2 + \cos\phi_2 A_3$$

(b). $A_1 = A_2 = 0 \quad \rightarrow \quad \mathbf{A}'' = \left[\sin\phi_2 A_3, 0, \cos\phi_2 A_3\right]^T$

The result depends only on ϕ_2, not ϕ_1. This makes sense because **A** is unaffected by the first rotation about the x_3 axis, as **A** is parallel to the x_3 axis.

(5). Refer to the following figure for the calculations below:



$R_{ij} = \mathbf{e}_i' \cdot \mathbf{e}_j \quad \rightarrow \quad R_{11} = \mathbf{e}_1' \cdot \mathbf{e}_1 = \cos\theta$, $R_{12} = \mathbf{e}_1' \cdot \mathbf{e}_2 = 0$,

$R_{13} = \mathbf{e}_1' \cdot \mathbf{e}_3 = \cos\left(90^o + \theta\right) = -\sin\theta$, $R_{21} = \mathbf{e}_2' \cdot \mathbf{e}_1 = 0$,

$R_{22} = \mathbf{e}_2' \cdot \mathbf{e}_2 = 1$, $R_{23} = \mathbf{e}_2' \cdot \mathbf{e}_3 = 0$,

$R_{31} = \mathbf{e}_3' \cdot \mathbf{e}_1 = \cos\left(90^o - \theta\right) = \sin\theta$, $R_{32} = \mathbf{e}_3' \cdot \mathbf{e}_2 = 0$,

$R_{33} = \mathbf{e}_3' \cdot \mathbf{e}_3 = \cos\theta$.

$$\Re = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$C_{ij}' = \sum_k \sum_\ell R_{ik} R_{j\ell} C_{k\ell}$$

$$\begin{aligned} C_{33}' = {} & R_{31}R_{31}C_{11} + R_{31}R_{32}C_{12} + R_{31}R_{33}C_{13} + \\ & R_{32}R_{31}C_{21} + R_{32}R_{32}C_{22} + R_{32}R_{33}C_{23} + \\ & R_{33}R_{31}C_{31} + R_{33}R_{32}C_{32} + R_{33}R_{33}C_{33} \end{aligned}$$

$$C_{33}' = C_{11}\sin^2\theta + C_{13}\sin\theta\cos\theta + C_{31}\sin\theta\cos\theta + C_{33}\cos^2\theta$$

$C_{ij} = \sigma_{ij}$ which is symmetric, so $C_{13} = C_{31}$.

Thus,

$\sigma_{33}' = \sin^2\theta\ \sigma_{11} + 2\sin\theta\cos\theta\ \sigma_{13} + \cos^2\theta\ \sigma_{33}$, and with $\sin 2\theta = 2\sin\theta\cos\theta$,

$\sigma_{33}' = \sigma_{11}\sin^2\theta + \sigma_{13}\sin 2\theta + \sigma_{33}\cos^2\theta$, and $P' = -\sigma_{33}'$.

$\theta = 0^o \quad \rightarrow \quad \sigma_{33}' = \sigma_{33}$, as expected for no rotation.

$\theta = 90^o \quad \rightarrow \quad \sigma_{33}' = \sigma_{11}$, as expected for a 90^o rotation.

The z' axis coincides with the xy plane and the x' axis is normal to the xy plane.

6.

The 81 stiffness coefficients required to describe a general elastic medium are given by the tensor $c_{ijk\ell}$ and the 21 independent quantities obtained from symmetry conditions may be written in Voigt notation as a 6×6 symmetric matrix, c_{mn}. In particular, for this problem $c_{3333} = c_{33} = \lambda + 2\mu = 19.22\,GPa$ and $c_{2211} = c_{21} = c_{12} = \lambda = 4.78\,GPa$. $\left[GPa = 10^9\, N/m^2 .\right]$ With the density, $\rho = 2000 kg/m^3 \left[2g/cm^2\right]$, it follows from using the definitions

$$\text{Shear } (S) \text{ wave speed:} \quad \beta = v_S = \sqrt{\mu/\rho}$$

$$\text{Compressional } (P) \text{ wave speed:} \quad \alpha = v_P = \sqrt{(\lambda + 2\mu)/\rho}$$

and

$$\mu = \frac{c_{33} - c_{12}}{2} = \frac{(19.22 - 4.78)}{2} GPa = 7.22\ GPa = 7.22 \times 10^9\ N/m^2 ,$$

so

$$\text{Shear } (S) \text{ wave speed:} \quad \beta = v_S = \sqrt{7.22 \times 10^9/2000}\, m/s = 1900\, m/s$$

$$\text{Compressional } (P) \text{ wave speed:} \quad \alpha = v_P = \sqrt{19.22 \times 10^9/2000}\, m/s = 3100\, m/s .$$

7. The symmetry properties for the stress and rotation tensors, σ_{ij} and η_{ij}, are $\sigma_{ij} = +\sigma_{ji}$ and $\eta_{ij} = -\eta_{ji}$, respectively. Using these properties, the summation

$$S = \sum_i \sum_j \sigma_{ij}\eta_{ij}$$

becomes

$$S = -\sum_i \sum_j \sigma_{ji}\eta_{ji} .$$

Switching the dummy summation indices produces

$$S = -\sum_j \sum_i \sigma_{ij}\eta_{ij} .$$

Reversing the order of summation has

$$S = -\sum_i \sum_j \sigma_{ij}\eta_{ij} .$$

For this to be true, S must equal $-S$, which is only satisfied if $S = 0$.

8. The relationships

$$\frac{\partial u_j}{\partial x_i} = \frac{\partial u_i}{\partial x_j}, \quad (i, j = 1, 2, 3) \qquad \Rightarrow \qquad \nabla \times \mathbf{u} = \mathbf{0} .$$

It is known that for $P-$wave propagation, for some displacement potential ϕ, $\mathbf{u} = \nabla\phi$ which yields

$$\nabla \times \mathbf{u} = \nabla \times \nabla\phi = \mathbf{0} \ .$$

Thus, the answer is "true". Further,

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\right) , \qquad e_{ij} = \frac{\partial u_j}{\partial x_i} = \frac{\partial u_i}{\partial x_j} \ .$$

Also, as

$$u_i = \frac{\partial \phi}{\partial x_i}, \ \text{ then } \ \frac{\partial u_i}{\partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i} = \frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial u_j}{\partial x_i} .$$

9. For single particle motion, like a mass on a spring, the kinetic energy KE and potential energy PE are not generally equal at all times, i.e., $KE \neq PE$, i.e., KE is large when PE is small, and vice versa. But for a solid body, the particles making up the body vibrate at different phases. For example, on one plane wavefront, the particles may be moving at peak speed (large KE and zero PE) but another plane wavefront a quarter of a cycle further on, they are moving at zero speed (zero KE and large PE). So overall, this suggests that with all the particles in the body contributing to KE and PE, it all balances out to give $K = W$.

But what about the fact that $K = W$ at any single time t at some specific point x? This means that at $t = x/v$, for instance, $K = W = 0$, which seems strange. To address this, note that for a mass on a spring, oscillating back and forth with a displacement u from the rest position, we have

$$F = -ku \ , \quad u = A\cos(\omega t) \ , \quad k = m\omega^2 \ ,$$

$$KE = \frac{1}{2}m\dot{u}^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t) = \frac{1}{2}kA^2 \sin^2(\omega t) \ ,$$

$$PE = \frac{1}{2}ku^2 = \frac{1}{2}kA^2 \cos^2(\omega t) \ .$$

So $KE \sim \sin^2(\omega t)$ whereas $PE \sim \cos^2(\omega t)$. KE is max when PE is zero, and vice versa. $KE \neq PE$. Note also that for a mass on spring, $PE \sim u^2$, but for a solid, $W \sim D^2$, i.e., $W \sim {e_{xx}}^2$, i.e., $W \sim (\partial u_x / \partial x)^2$, not ${u_x}^2$. In words, W involves strain, not displacement -- W is more correctly called the strain energy density, not the potential energy density, in this regard. This is also why both W and $K \sim \sin^2[\cdots]$, i.e., because both involve derivatives of u_x. So evidently one cannot directly compare W with the PE of a mass on a spring. One should not expect W and K to be 90° out of phase, like KE and PE for a mass on a spring. There are instants t where both K and W can be zero. So it may be better to look at time-averages, in which case $\langle K \rangle = \langle W \rangle$ for a solid, in agreement with $\langle KE \rangle = \langle PE \rangle$ for a mass on a spring.

10. Using the definitions

$$Y = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)} \quad , \quad \alpha^2 = \frac{\lambda + 2\mu}{\rho} \quad , \quad \beta^2 = \frac{\mu}{\rho}$$

it follows that:

$$\lambda = \rho\alpha^2 - 2\mu = \rho\alpha^2 - 2\rho\beta^2$$

from which

$$3\lambda + 2\mu = 3\rho\alpha^2 - 6\rho\beta^2 + 2\rho\beta^2 = 3\rho\alpha^2 - 4\rho\beta^2.$$

Thus,

$$Y \;=\; \frac{\rho\beta^2\left(3\rho\alpha^2 - 4\rho\beta^2\right)}{\rho\alpha^2 - 2\rho\beta^2 + \rho\beta^2} \;=\; \frac{\rho^2\beta^2\left(3\alpha^2 - 4\beta^2\right)}{\rho\alpha^2 - \rho\beta^2} \;=\; \frac{\rho\beta^2\left(3\alpha^2 - 4\beta^2\right)}{\alpha^2 - \beta^2}.$$

Equivalently, this could have been alternatively obtained by substituting

$$\mu = \rho\beta^2 \quad \text{and} \quad \sigma = \frac{(\beta/\alpha)^2 - \frac{1}{2}}{(\beta/\alpha)^2 - 1} \quad \text{into} \quad Y = 2\mu(1+\sigma) \quad \text{(eq. 2-38 in the text)}.$$

The value of Y for a rock which is as "hard" as possible requires Poisson's ratio to be specified as $\sigma = 0$. This has

$$\left(\frac{\beta}{\alpha}\right)^2 = \frac{1/2 - \sigma}{1 - \sigma} = \frac{1}{2} \quad \text{so that} \quad 2\beta^2 = \alpha^2. \quad \text{leading to}$$

$$Y = \frac{\rho\beta^2\left(6\beta^2 - 4\beta^2\right)}{2\beta^2 - \beta^2} = \frac{\left(\rho\beta^2\right)\left(2\beta^2\right)}{\beta^2}$$

from which it follows that

$$Y = 2\rho\beta^2 = \rho\alpha^2 \quad \text{for} \quad \sigma = 0.$$

Alternatively,

$$\sigma = \frac{\lambda}{2(\lambda + \mu)} = 0 \quad \text{implies} \quad \lambda = 0 \quad \text{for} \quad \sigma = 0.$$

In the strictest theoretical sense $Y > 0$ for all materials, which would preclude materials where $Y \le 0$ from existing. If $Y = 0$, then

$$\mu = 0, \rightarrow \beta^2 = 0; \quad \text{or} \quad 3\lambda + 2\mu = 0, \rightarrow 3\alpha^2 = 4\beta^2.$$

The condition that $\mu = 0$, i.e., $\beta^2 = 0$, indicates that the medium is a fluid. From this it may be seen that $\sigma = 1/2$ which is the maximum value of σ.

If the identity $3\lambda + 2\mu = 0$ holds, then $\mu = -3\lambda/2$ which yields

$$\sigma = \frac{\lambda}{2\lambda + (-3\lambda)} = \frac{\lambda}{-\lambda} = -1 \quad (\sigma < 0).$$

These material types are referred to as "exotic" materials.

11. $v_{rod} = \sqrt{Y/\rho}$, $\alpha = \sqrt{(\lambda + 2\mu)/\rho}$, $\sigma = \lambda/2(\lambda+\mu)$

$\{v_{rod} < \alpha \text{ for } \sigma > 0\}$?   i.e. $\{Y < \lambda + 2\mu \text{ for } \sigma > 0\}$?

Assume the above to be true and prove.

$Y = \dfrac{\mu(3\lambda + 2\mu)}{(\lambda+\mu)} < \lambda + 2\mu.$ For $\sigma > 0$ it is known that $\mu > 0$.

Also, $k_B = \lambda + 2\mu/3 > 0$, $\rightarrow$ $\lambda > -2\mu/3$, $\rightarrow$ $\lambda + \mu > \mu/3 > 0$.

So, $\sigma > 0 \quad \leftrightarrow \quad \lambda > 0.$

Continuing,

$\mu(3\lambda + 2\mu) < (\lambda + 2\mu)(\lambda+\mu)$ since $(\lambda + \mu) > 0$.

$(3\lambda\mu + 2\mu^2) < \lambda^2 + 3\lambda\mu + 2\mu^2$, $\rightarrow$ $0 < \lambda^2$ which is <u>true</u>.

Working this backwards proves $v_{rod} < \alpha$ for $\sigma > 0$ $\rightarrow$ <u>true</u>.

$\sigma = 1/4 \rightarrow \lambda = \mu \rightarrow Y = \dfrac{\mu(3\mu + 2\mu)}{\mu + \mu} = \dfrac{5\mu^2}{2\mu} = \dfrac{5\mu}{2}$ and $\lambda + 2\mu = 3\mu$.

Therefore,

$$\frac{v_{rod}}{\alpha} = \sqrt{\frac{Y}{\rho}\frac{\rho}{(\lambda + 2\mu)}} = \sqrt{\frac{5\mu}{2\cdot 3\mu}} = \sqrt{\frac{5}{6}} \quad \text{for } \sigma = 1/4.$$

$$\frac{v_{rod}}{\alpha} = \sqrt{\frac{5}{6}} = 0.913.$$

12. ***(a).*** The components $V_i ,(i=1,2,3)$ of the vector **V** are given by

$$V_i = -\sum_k \left(\frac{\partial u_k}{\partial t}\right) e_{ik} = -\frac{1}{2}\sum_k \left(\frac{\partial u_k}{\partial t}\right)\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

So that V_i is quadratic in u_i.

A plane harmonic $P-$wave is propagating in an isotropic homogeneous medium along the x_1 axis.

The only nonzero component of particle displacement is u_1 so that $\mathbf{u} = (u_1, 0, 0)$ $[u_2 = u_3 = 0]$ and u_1 may be written as

$$u_1 = Ae^{i\omega(x_1/v - t)} = |A| e^{i\omega(x_1/v - t + \psi/\omega)} \quad \left[A = |A| e^{i\psi} \text{ and } v = \text{speed} = \alpha\right].$$

$$V_i = -\sum_k \left(\frac{\partial u_k}{\partial t}\right) e_{ik} = -\frac{1}{2}\left(\frac{\partial u_1}{\partial t}\right)\left(\frac{\partial u_i}{\partial x_1} + \frac{\partial u_1}{\partial x_i}\right) = 0 \text{ for } i = 2 \text{ and } 3$$

so that

$$V_1 = -\frac{1}{2}\left(\frac{\partial u_1}{\partial t}\right)\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = -\left(\frac{\partial u_1}{\partial t}\right)\left(\frac{\partial u_1}{\partial x_1}\right)$$

and

$$V_2 = V_3 = 0.$$

Since V_1 is quadratic, i.e., not linear, in u_1, the complex exponential form of u_1 cannot be used, rather, the real part must first be taken, i.e., $u_1 = |A|\cos[\omega(x/v) - \omega t + \psi] \equiv |A|\cos\phi$, giving

$$V_1 = -\left[(-|A|\sin\phi)(-\omega)\right]\left[(-|A|\sin\phi)(\omega/v)\right] = |A|^2 (\omega^2/v)\sin^2\phi$$

where $\phi = \omega(x_1/v) - \omega t + \psi$ and

$$\langle V_1 \rangle = |A|^2 (\omega^2/v)\frac{1}{2}$$

since $\langle \sin^2\phi \rangle = 1/2$. From the above, the following is obtained

$$\langle V_1 \rangle = \frac{|A|^2 \omega^2}{2v} \quad , \quad \langle V_2 \rangle = 0 \quad , \quad \langle V_3 \rangle = 0.$$

which yields

$$\langle \mathbf{V} \rangle = \left(\frac{\omega^2 |A|^2}{2v}\right)\mathbf{e_1} \quad (v = \alpha)$$

(b). As the medium is isotropic, the mean value would be the same.

13. The slowness vector **s** in two dimensions with components *p* and *q* may be written as $\mathbf{s}=\left(p,0,q\right)$ with the slowness given by $|\mathbf{s}|=s=\left(p^2+q^2\right)^{1/2}$. Note that $p>0$ and $q<0$.

(a). The speed of wave propagation, *v*, is the inverse of the slowness, $v=1/s$, where the slowness is

$$s=\left(p^2+q^2\right)^{1/2}=\left[\left(0.1272\right)^2+\left(-0.4158\right)^2\right]^{1/2} s/km=0.434821 s/km \text{ so that } v=2.3 km/s.$$

(b). The plane wave travels in the direction of the unit vector $\boldsymbol{\xi}$, which is normal to the plane wave front. Since $p>0$ and $q<0$, the wave travels in the $+x$ and $-z$ directions, i.e., it travels upwards and to the right (the z axis points downwards, and the *x* axis points to the right) in the *xz* plane. This direction is determined by the slowness vector components, *p* and *q*.

$$p=\frac{\sin\phi}{v} \quad\to\quad \sin\phi=pv \quad\to\quad \phi=\sin^{-1}\left(pv\right)=17^o.$$

As $q=-\dfrac{\cos\phi}{v}$, one may use $\tan\phi=\dfrac{|p|}{|q|}$ to obtain the same result.

(c). The polarization vector is given by $\mathbf{d}=\left(d_x,d_y,d_z\right)=\left(0.2924,\,0,\,-0.9563\right)$.

i. As $d_y=0$, the wave cannot be an *SH* wave.

ii. If the wave was of the *SV* type, then $\mathbf{d}\cdot\mathbf{s}=0$, which is not the case.

iii. As $\mathbf{d}=\left(0.2924,\,0,\,-0.9563\right)=(\sin\phi,0,-\cos\phi)$, the wave is a *P* wave.

(d). The frequency, *f*, is given by $f=\dfrac{\omega}{2\pi}=\dfrac{188.5}{2\pi}Hz=30Hz.$

The wavelength, λ, is obtained in terms of a period, $T=1/f$, and speed, *v*, as

$\lambda=vT=v/f=\left(2300\,m/s\right)/\left(30s^{-1}\right)=76.67\,m.$

(e). The wave has a phase-delay of 30º and the maximum value of $|\mathbf{u}|$ is $2\,mm$.

$$|\mathbf{u}|=2\,mm \quad\to\quad |U|=2\,mm.$$

$$U=|U|e^{i\psi},\ \psi=30^o \quad\to\quad U=\left(2\,mm\right)e^{i30^o} \quad \text{or}$$

$$\mathbf{u}=|U|e^{i\omega\left(px+qz\right)}e^{-i\left(\omega t-30^o\right)}\mathbf{d} \quad\to\quad U=\left(2\,mm\right)e^{i30^o}.$$

(f).

$$\mathrm{Re}\left(\mathbf{u}\right)=\mathrm{Re}\left[Ue^{i\omega\left(px+qz-t\right)+i\psi}\mathbf{d}\right]=|U|\cos\left\{\omega\left(px+qz-t\right)+\psi\right\}\mathbf{d}$$

At $x=z=t=0$,

$$\mathrm{Re}\left(\mathbf{u}\right)=(|U|\cos\psi)\mathbf{d}=(2\,mm\cos30^o)\mathbf{d}=\left(1.732\,mm\right)\mathbf{d} \quad \text{or}$$

$$\mathrm{Re}\left(\mathbf{u}\right)=\left(0.5064,\,0,\,-1.656\right)mm.$$

(14). $\alpha_1 = 2.5\,km/s$, $\beta_1 = 1.4\,km/s$, $\alpha_2 = 3.6\,km/s$, $\beta_1 = 2.08\,km/s$, $\rho_1 = \rho_2 = 2\,g/cm^3$.

$$R_{PS} = \frac{B_R}{A_I} = -0.12.$$

(a). We want to determine the fraction $\Re_{PS} = \left(\frac{\rho_1\beta_1\cos\phi_1}{\rho_1\alpha_1\cos\theta_1}\right)\left|R_{PS}\right|^2$ where $\cos\theta_1 = 0.8660$.

$$\frac{\sin\phi_1}{\beta_1} = \frac{\sin\theta_1}{\alpha_1} \quad\rightarrow\quad \sin\phi_1 = 0.28 \quad\rightarrow\quad \phi_1 = 16.26^o$$

so that

$$\cos\phi_1 = \left(1-\sin^2\phi_1\right)^{1/2} = 0.96$$

$$\Re_{PS} = \left(\frac{\rho_1\beta_1\cos\phi_1}{\rho_1\alpha_1\cos\theta_1}\right)\left|-0.12\right|^2 = 0.00894.$$

Thus only a small fraction of the vertical component of the incident $P-$wave's energy flux is reflected up into the incident medium $\left[\Re_{PS} = 0.00894 \text{ or } 0.894\% < 1\%\right]$ as shear energy.

(b). $R_{PS} = -0.12 = (0.12)e^{i\pi} \quad\rightarrow\quad$ phase delay $= \pi = 180^o$.

$\left(u_x \propto R_{PS}d_x,\ u_z \propto R_{PS}d_z,\ \text{with } d_x > 0 \text{ and } d_z > 0\right)$

15. We will use the SEG convention stated in the caption of Figure 3-1 of the text. The direct wave going upward from the source is a compression, hence, the geophone is pushed upwards in the $-z$ direction, resulting in a pulse with a down-kicking first motion (as $u_z < 0$). The compression wave that goes down from the source reflects as a compression, because $Z_1 < Z_2$, and hence travels upwards to the receiver as a compression, resulting again in a pulse with a down-kicking first motion. Therefore the geophone trace consists of a down-kicking direct-wave pulse followed by a down-kicking reflected-wave pulse. The hydrophone is squeezed by both arriving compressions, resulting in negative down-kicking pulses. Therefore the hydrophone trace looks the same as the geophone trace in this case.

16. $R = \dfrac{A_R}{A_I}$ and $T = \dfrac{A_T}{A_I}$. With $Z_j = \rho_j \alpha_j$

$\bar{R} = R$ and $\bar{T} = \left(\dfrac{Z_2}{Z_1}\right) T$.

If ϕ is the P wave potential, then $\mathbf{u} = \nabla \phi$ so that the scalar quantity $u_z = \dfrac{\partial \phi}{\partial z}$.

$\phi_I = A'_I e^{-i\omega(t - z/\alpha_1)}, \quad \phi_R = A'_R e^{-i\omega(t + z/\alpha_1)}, \quad \phi_T = A'_T e^{-i\omega(t - z/\alpha_2)}$.

Potential reflection coefficient = $R' \equiv A'_R / A'_I$.

Potential transmission coefficient = $T' \equiv A'_T / A'_I$.

$$u_z^{(I)} = \frac{\partial \phi_I}{\partial z} = \frac{i\omega}{\alpha_1} A'_I e^{-i\omega(t - z/\alpha_1)}$$
$$= A_I e^{-i\omega(t - z/\alpha_1)} d_z^{(I)} = A_I e^{-i\omega(t - z/\alpha_1)} \quad \left(d_z^{(I)} = +1\right)$$

so that $(i\omega/\alpha_1) A'_I = A_I$.

$$u_z^{(R)} = \frac{\partial \phi_R}{\partial z} = -\frac{i\omega}{\alpha_1} A'_R e^{-i\omega(t + z/\alpha_1)}$$
$$= A_R e^{-i\omega(t + z/\alpha_1)} d_z^{(R)} = -A_R e^{-i\omega(t - z/\alpha_1)} \quad \left(d_z^{(R)} = -1\right)$$

and thus $(i\omega/\alpha_1) A'_R = A_R$.

$$u_z^{(T)} = \frac{\partial \phi_T}{\partial z} = \frac{i\omega}{\alpha_2} A'_T e^{-i\omega(t - z/\alpha_2)}$$
$$= A_T e^{-i\omega(t - z/\alpha_2)} d_z^{(T)} = A_T e^{-i\omega(t - z/\alpha_1)} \quad \left(d_z^{(T)} = +1\right)$$

resulting in $(i\omega/\alpha_2) A'_T = A_T$.

The three instances above lead to:

$$R' = \frac{A'_R}{A'_I} = \frac{(\alpha_1/i\omega) A_R}{(\alpha_1/i\omega) A_I} = \frac{A_R}{A_I} = R$$

and

$$T' = \frac{A'_T}{A'_I} = \frac{(\alpha_2/i\omega) A_T}{(\alpha_1/i\omega) A_I} = \frac{\alpha_2}{\alpha_1} \frac{A_T}{A_I} = \frac{\alpha_2}{\alpha_1} T.$$

The primary and ghost arrivals, a compression and rarefaction, have opposite polarities. A hydrophone would also record opposite polarities.

If a potential measuring instrument measured a wave based on the sign of the potential, then it also would measure opposite polarities.

For example, letting superscripts (P) and (G) denote the primary and the ghost, resp.,

If $u_z^{(P)} = -e^{-i\omega(t + z/\alpha_1)}$ then $u_z^{(G)} = +re^{-i\omega(t + z/\alpha_1)}$ where $0 < r < 1$.

Let $\phi^{(P)} = Ae^{-i\omega(t + z/\alpha_1)}$ and $\phi^{(G)} = Be^{-i\omega(t + z/\alpha_1)}$. Since $u_z = \partial\phi/\partial z$, A and B would have to have opposite signs so that $u_z^{(P)}$ and $u_z^{(G)}$ have opposite signs. ($\phi^{(P)}$ and $\phi^{(G)}$ have opposite signs.)

17.

(a). $\alpha_1 = 2.0\,km/s$, $\beta_1 = 1.3\,km/s$, $\alpha_2 = 3.1\,km/s$, $\beta_2 = 1.7\,km/s$.

Snell's Law: $\dfrac{\sin\theta_1}{\alpha_1} = \dfrac{\sin\theta_2}{\alpha_2} = \dfrac{\sin\phi_2}{\beta_2} = p$ (see the angles in Figure 3-5 of the text).

$$\sin\theta_2 = \frac{\alpha_2 \sin\theta_1}{\alpha_1} = \frac{3.1}{2}\sin\left(20^o\right) = 0.53013 \quad \rightarrow \quad \theta_2 = 32.014^o.$$

Similarly, $\phi_2 = \sin^{-1}\left[\dfrac{\beta_2 \sin\theta_1}{\alpha_1}\right] = 16.901^o.$

Transmitted P – wave :

$$\mathbf{s}^{(TP)} = \left(\frac{\sin\theta_2}{\alpha_2}, 0, \frac{\cos\theta_2}{\alpha_2}\right) s/km = \left(\frac{\sin\theta_1}{\alpha_1}, 0, \frac{\cos\theta_2}{\alpha_2}\right) s/km = \left(0.1710, 0, 0.2735\right) s/km$$

$$\mathbf{d}^{(TP)} = \left(\sin\theta_2, 0, \cos\theta_2\right) = \left(0.5301, 0, 0.8479\right)$$

Transmitted SV – wave :

$$\mathbf{s}^{(TS)} = \left(\frac{\sin\phi_2}{\beta_2}, 0, \frac{\cos\phi_2}{\beta_2}\right) s/km = \left(0.1710, 0, 0.5628\right) s/km$$

$$\mathbf{d}^{(TS)} = \left(\cos\phi_2, 0, -\sin\phi_2\right) = \left(0.9568, 0, -0.2907\right)$$

Wavelengths:

$$v = \lambda f \quad \rightarrow \quad \lambda^{(TP)} = \alpha_2 / f = 3.1/30 = 0.1033km = 103.3m$$

$$\lambda^{(TS)} = \beta_2 / f = 1.7/30 = 0.0567km = 56.7m$$

(b). Snell's Law: $\dfrac{\sin\theta_1}{\alpha_1} = \dfrac{\sin\phi_1}{\beta_1} = \dfrac{\sin\theta_2}{\alpha_2} = \dfrac{\sin\phi_2}{\beta_2}$

This implies that if the scattered wave speed is greater than the incident wave speed, then the scattered angle is greater than the incident angle. Thus, <u>true</u>.

Example:

If $\beta_2^{(Scat.)} > \alpha_1^{(Inc.)}$ then $\dfrac{\sin\phi_2}{\beta_2} = \dfrac{\sin\theta_1}{\alpha_1} \quad \rightarrow \quad \phi_2 > \theta_1$.

18.

$u_x = -e^{-Az}\cos(\kappa x - \omega t) = a(z)\cos(\kappa x - \omega t)$

$u_z = (1 - Bz^2)e^{-Bz^2/2}\sin(\kappa x - \omega t) = b(z)\sin(\kappa x - \omega t)$

It should be noted that $b(z)$ is of the form of a Ricker wavelet, but with a spatial coordinate as the dependent variable.

It does not matter at what value of *x* the particle motion is evaluated, so that at $(x, z) = (0, 0)$

$$u_x = -\cos(-\omega t) = -\cos(\omega t) \quad \text{and} \quad u_z = +\sin(-\omega t) = -\sin(\omega t).$$

At $t = 0, \quad u_x = -1 < 0 \quad \text{and} \quad u_z = 0.$

At $t = \dfrac{\pi}{2\omega}, \quad u_x = 0 \quad \text{and} \quad u_z = -1 < 0$ and so on at $t = \dfrac{\pi}{\omega}$ and $t = \dfrac{3\pi}{2\omega}$, etc.

The positive *z* direction is downwards and the positive *x* direction is to the right.
Hence, on the surface, the particle motion is prograde circular; circular because $|u_x| = |u_z| = 1,$ or equivalently as $|a(0)| = |b(0)| = 1.$

For $z > 0$ and $x = 0$, $a(z) \neq b(z)$ (generally) which implies elliptical particle motion $\left[(u_x/a)^2 + (u_z/b)^2 = 1 \quad \leftarrow \quad \text{ellipse}\right]$, except at points *z* where $a(z) = \pm b(z)$, where the motion is circular, because $|a(z)| = |b(z)|$. These points can be determined by numerically or graphically solving $a(z) = \pm b(z)$.

From the formula or graph of $b(z)$ it may be seen that $b(z) \leq 0$ for $z \geq z_0$ indicating a nodal plane at $z = z_0$. This requires $b(z_0) = 0$ or $1 - Bz_0^2 = 0$ yielding $z_0 = 1/\sqrt{B}$. In the nodal plane, $b(z) = 0$ so that $u_z = 0$, and the particle motion is linear and horizontal. For $0 < z < z_0$, it can be seen from a graphical analysis that there is a depth $z_c < z_0$ where $b(z_c) = -a(z_c)$ at which the motion is prograde circular. Otherwise, the motion for $0 < z < z_0$ is prograde elliptical.

For $z > z_0$, the partical motion is retrogade elliptical, $(b(z) < 0\,, a(z) < 0)$, although it is retrograde circular at depths where $a(z) = b(z)$.

The amplitude generally decreases with increasing *z*. As $z \to \infty$, the amplitude tends to zero.

19. $\beta_1 = 600\,m/s < \beta_2 = 1000\,m/s < \alpha_1 = 1200\,m/s < \alpha_2 = 2200\,m/s$:

Critical angles:

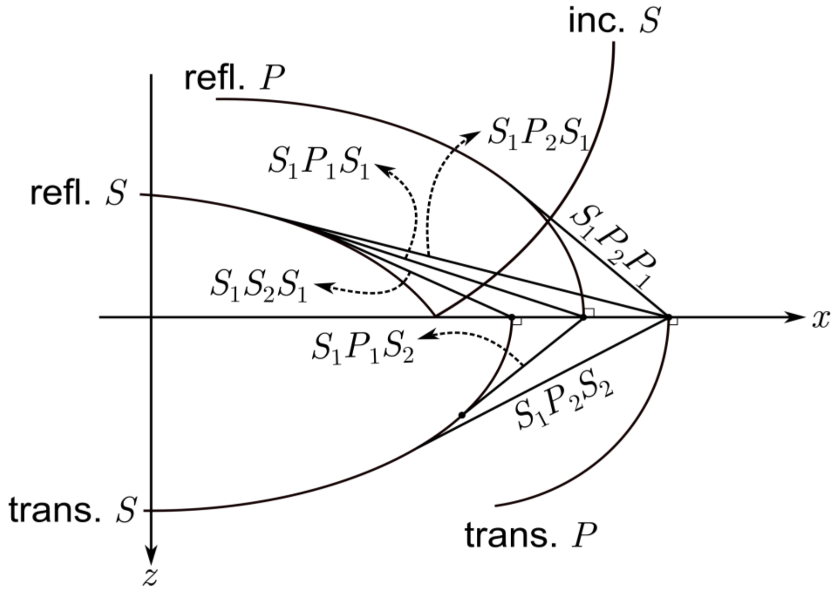
i. Transmitted P – wave: $\phi_{1C}(S_1P_2) = \sin^{-1}(\beta_1/\alpha_2) = 15.83^o$

ii. Reflected P – wave: $\phi_{1C}(S_1P_1) = \sin^{-1}(\beta_1/\alpha_1) = 30.00^o$

iii.Transmitted S – wave: $\phi_{1C}(S_1S_2) = \sin^{-1}(\beta_1/\beta_2) = 36.87^o$

The order in which the waves “go critical” is P_2, P_1, S_2 .

All of the head waves generated due to shear wave incidence are shown on the accompanying figure, in which it assumed that $\phi_1 > \max\left[\phi_{1C}(*)\right] \;\; \rightarrow \;\; \phi_1 > \phi_{1C}(S_1 S_2)$.



20. Solve $\tan\left[\omega\sqrt{1/\beta_1^2 - 1/c^2}\; h\right] = \dfrac{\mu_2\sqrt{1/c^2 - 1/\beta_2^2}}{\mu_1\sqrt{1/\beta_1^2 - 1/c^2}}$ for $\mu_2 = \rho_2\beta_2^2$.

$$\frac{\mu_2}{\mu_1} = \frac{\rho_2\beta_2^2}{\rho_1\beta_1^2} = \frac{\beta_2^2}{\beta_1^2} \text{ as } \rho_2 = \rho_1 .$$

From this it follows that

$$\sqrt{1/\beta_1^2 - 1/c^2}\;\beta_1^2 \tan\left[\omega\sqrt{1/\beta_1^2 - 1/c^2}\; h\right] = r\beta_1^2 \tan\left[\omega r h\right] = \beta_2^2\sqrt{1/c^2 - 1/\beta_2^2}$$

where $r = \sqrt{1/\beta_1^2 - 1/c^2}$.

Continuing,

$$\beta_2^4\left(1/c^2 - 1/\beta_2^2\right) - \left\{\beta_1^2 r \tan\left(\omega r h\right)\right\}^2 = 0, \text{ i.e., } \beta_2^4/c^2 - \beta_2^2 - S^2$$

where $S = \beta_1^2 r \tan\left(\omega r h\right)$, which is a quadratic equation in β_2^2 with solution

$$\beta_2^2 = \frac{c^2}{2}\left[-(-1) \pm \left(1 - (-1)4S^2/c^2\right)^{1/2}\right] = \frac{c^2}{2}\left[1 \pm \left(1 + 4S^2/c^2\right)^{1/2}\right].$$

Choose the "+" sign so that $\beta_2^2 > 0$.

With $\omega = 2\pi f$,

$$S = \beta_1^2 r \tan\left(2\pi f r h\right),\; \beta_1 = 0.7\,km/s,\; c = 1\,km/s,\; f = 10\,Hz,\; h = 0.02\,km,$$

then

$$r = 1.020204 \text{ and } S = 1.682746 \text{ so that } \beta_2 = 1.502\,km/s .$$

21. Isotropic 3 layered medium:

$$t = 2\sum_{j=1}^{3} \frac{h_j}{v_j r_j} \quad , \quad x = 2\sum_{j=1}^{3} \frac{h_j v_j p}{r_j} \quad \text{with} \quad r_j = \left(1 - p^2 v_j^2\right)^{1/2} \quad \leftarrow \left(\text{Snell's Law}\right)$$

$v_1 = 1\,km/s$, $v_2 = 2\,km/s$, $v_3 = 3\,km/s$, $h_1 = 0.3km$, $h_2 = 0.5km$, $h_3 = 0.8km$.

$\theta_1 = 15^o \quad \rightarrow \quad p = (\sin\theta_1)/v_1 = 0.258819\,s/km$.

$h_1/r_1 = 0.31058$, $h_2/r_2 = 0.58439$, $h_3/r_3 = 1.2695$.

so that

$$x = 2\,p\left[\frac{h_1}{r_1}v_1 + \frac{h_2}{r_2}v_2 + \frac{h_3}{r_3}v_3\right] = 2.737km$$

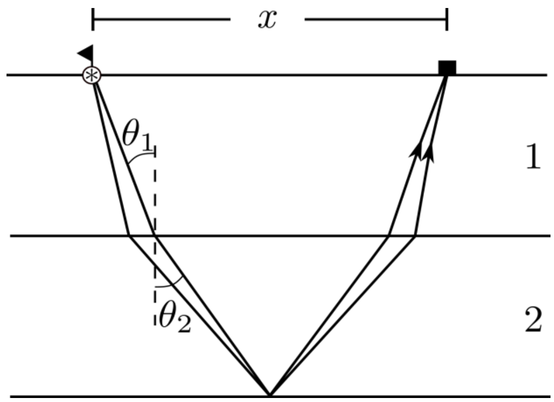
$$t = 2\left[\frac{h_1/r_1}{v_1} + \frac{h_2/r_2}{v_2} + \frac{h_3/r_3}{v_3}\right] = 2.052\,s$$

22. $v(z) = v_0\left(2 - e^{-az}\right)$, Both v_0 and a are constants, $\left(v_0 > 0\,, a > 0\right)$.

(a). By sketching a graph of the decaying exponential e^{-az}, and its negative below the x axis, and adding 2 to it (shifting the whole graph up 2 units), and multiplying by v_0 , yields a graph of $v(z)$ which shows that $v(z)$ increases monotonically from a value of v_0 at $z = 0$ to a value of $2v_0$ as $z \to \infty$.

(b). Near the surface, $\left(z \approx 0\right)$, $v(z) \approx v_0\left[2 - \left(1 - a\,z\right)\right] = v_0\left(1 + a\,z\right)$, therefore $v(z)$ increases linearly with z so that the ray paths are circular arcs, as in Figure 5-4b of the text. Deeper within the medium, $v(z) \to \text{constant} = v_0$ and the ray paths become straight lines. In the transition zone between these two limits, the ray paths are a "mix" of these. For large take – off angles, p is large, and one obtains curved ray paths that turn up back towards the surface (as in Figure 5-4b of the text). As p decreases, i.e., for smaller take-off angles, a ray will initially follow a curved path and curve up, but as it reaches a greater depth it will tend to straighten out as dv/dz is smaller there.

23. Two layered isotropic homogeneous media, $\alpha_2 > \alpha_1$, implies a symmetric ray path for a primary ray, where the ray path tends towards the horizontal, in the second medium for increasing θ_1. If α_2 is increased, while α_1 is held constant, and if the source-receiver offset *x* of the ray remains the same, then θ_2 increases while θ_1 decreases for a constant offset, which can be seen from Snell's law written as $\sin\theta_2 / \sin\theta_1 = \alpha_2 / \alpha_1$. Note that if θ_1 doesn't change, then the offset *x* is larger, therefore θ_1 must decrease. $p = (\sin\theta_1)/\alpha_1$ implies that as α_1 remains the same and θ_1 is smaller, then *p* is smaller. Thus the original ray path has the larger *p* value. See the figure below.



24. The ray path in a medium, with a speed linear in the vertical (*z*) direction, is an arc of a circle. Medium: $v = v_0 + az,\ v_0 = 2km/s\ \ , a = 0.5\dfrac{km/s}{km} = 0.5s^{-1},\ \ x = 3km.$

Refer to Figure 5.4a and the associated equations in the text.

The equation of the circle describing the arc is given by

$$\left(x-\overline{x}\right)^2 + \left(z-\overline{z}\right)^2 = R^2. \qquad \text{Also,}\ \ z_{\max} = R - \left|\overline{z}\right| = R - v_0/a \qquad \Rightarrow$$

$$z_{\max} = \frac{1}{pa} - \frac{v_0}{a} = \frac{v_0}{a\sin\theta_0} - \frac{v_0}{a} = \frac{v_0}{a}\left(\frac{1}{\sin\theta_0} - 1\right)$$

and

$$\overline{x} = \frac{1}{pa}\left(1 - p^2 v_0^2\right)^{1/2} = \frac{v_0}{a\sin\theta_0}\left(1 - \sin^2\theta_0\right)^{1/2} = \frac{v_0\cos\theta_0}{a\sin\theta_0}$$

$$\overline{x} = \frac{v_0}{a\tan\theta_0} = \frac{x}{2} \quad \rightarrow \quad \tan\theta_0 = \frac{2v_0}{ax} = \frac{8}{3}$$

so that

$$\theta_0 = 69.444^o \quad \rightarrow \quad z_{\max} = \frac{v_0}{a}\left(\frac{1}{\sin\theta_0} - 1\right) = 0.272\ km\,.$$

Alternately, the formula for $z_{\max}$ above can be derived as follows:

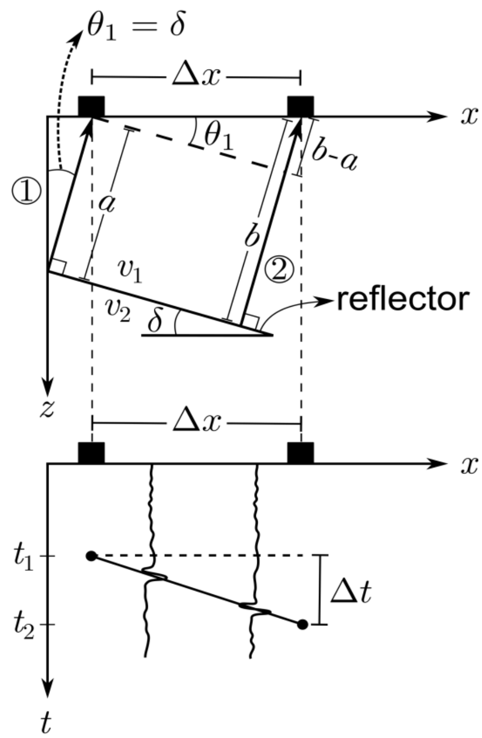
$$z = z_{\max} \text{ when } x = \overline{x}\ \text{ so that }\ \left(z_{\max} - \overline{z}\right)^2 = R^2\ \rightarrow z_{\max} = R + \overline{z} = R - v_0/a,\ \text{etc.}$$

25. Refer to the diagram below for the calculations that follow.
Zero offset ray 1: $t_1 = 2a/v_1$
Zero offset ray 2: $t_2 = 2b/v_1$

$$\sin\theta_1 = \sin\delta = \frac{b-a}{\Delta x}$$

$$v = \frac{\Delta x}{\Delta t} = \frac{\Delta x}{t_2 - t_1} = \frac{\Delta x}{2b/v_1 - 2a/v_1} = \frac{v_1 \Delta x}{2(b-a)} = \frac{v_1}{2\sin\theta_1} = \frac{v_1}{2\sin\delta}$$

Note that if $c = v_1/\sin\theta_1$ then $c = 2v$. $c = v_1/\sin\theta_1$ applies only to a shot record.



26. $g(t) + iH\left[g(t)\right] = \frac{1}{\pi}\int_0^\infty \bar{g}(\omega)e^{-i\omega t}dt$

$\left[H\left[\xi(t)\right] \;\leftarrow\; \text{Hilbert Transform}\right]$

$$\frac{1}{\pi}\int_0^\infty \bar{g}(\omega)e^{-i\omega t}dt = \frac{1}{\pi}\int_0^{\omega_1} \bar{g}(\omega)e^{-i\omega t}dt = \frac{1}{\pi}\left[\frac{e^{-i\omega t}}{-it}\right]_0^{\omega_1}$$

$$= \frac{i}{\pi t}\left[e^{-i\omega_1 t} - 1\right] = \frac{1}{\pi t}\left[i\left(\cos\omega_1 t - i\sin\omega_1 t\right) - i\right]$$

$$= \frac{1}{\pi t}\left[i\cos\omega_1 t - i + \sin\omega_1 t\right] = \frac{\sin\omega_1 t}{\pi t} + i\left[\frac{\cos\omega_1 t - 1}{\pi t}\right]$$

Thus,

$$g(t) = s(t) = \frac{\sin \omega_1 t}{\pi t} = \frac{\omega_1}{\pi}\left(\frac{\sin \omega_1 t}{\omega_1 t}\right) = \frac{\omega_1}{\pi}\operatorname{sinc}(\omega_1 t) \leftarrow (\text{definition})$$

and

$$H\left[g(t)\right] = \frac{\cos \omega_1 t - 1}{\pi t} = \frac{\omega_1}{\pi}\left(\frac{\cos \omega_1 t - 1}{\omega_1 t}\right).$$

It should be noted that as $t \to 0$, $g(t) \to 0/0$, which is undefined. Use the series expansion for $\sin \xi$ for $|\xi|$ near zero: $\sin \xi \approx \xi - \xi^3/6 + \ldots$, for $|\xi| \approx 0$,
or use l'Hopital's rule.

Thus,

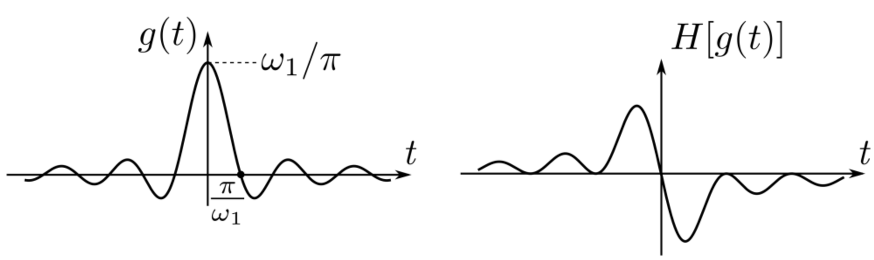
$$g(t)\Big|_{t\to 0} \to \frac{\omega_1}{\pi}\left[\frac{\omega_1 t - (\omega_1 t/6)^3 + \ldots}{\omega_1 t}\right]_{t\to 0} = \frac{\omega_1}{\pi}\left[1 - O(t^2)\right] = \frac{\omega_1}{\pi}.$$

Also, $H\left[g(t)\right]_{t\to 0} = \frac{\omega_1}{\pi}\left[\frac{1-1}{0}\right] = \frac{0}{0} \quad (\text{undefined}).$

For $|\xi|$ near zero: $\cos \xi \approx 1 - \xi^2/2 + \ldots$, for $|\xi| \approx 0$,

$$H\left[g(t)\right]\Big|_{t\to 0} \to \frac{\omega_1}{\pi}\left[\frac{1 - (\omega_1 t)^2/2 + \ldots - 1}{\omega_1 t}\right]_{t\to 0} = \frac{\omega_1}{\pi}\left[-\frac{\omega_1 t}{2} + O(t^2)\right]_{t\to 0} = -\frac{\omega_1^2 t}{2\pi}\Big|_{t\to 0} = 0.$$

It may be seen that $H\left[g(t)\right]$ looks like a 90^o phase advanced signal, very similar to the sinc function $g(t)$ shifted to the left. See the figure that follows.



27. ***(a).*** Reflection losses only.

Let R_j be the reflection coefficient at the base of layer *j*. Because of the alternating layers, we have

$R_1 = R_3 = R_5 = ... = R_{29} = R_1 = (Z_2 - Z_1)/(Z_2 + Z_1)$, and

$R_2 = R_4 = R_6 = ... = R_{30} = -R_1$. For reflection losses only:

$$A_{30} = A_0 R_{30} = -A_0 R_1 = -A_0 \left(\frac{Z_2 - Z_1}{Z_2 + Z_1} \right)$$

$Z_2 = \rho_2 \alpha_2 = 4.4, \quad Z_1 = \rho_1 \alpha_1 = 2.0 \quad \rightarrow \quad R_1 = 0.375$ and $A_{30} = -A_0 (0.375)$.

(b). As $T_k^{\pm} = (1 \pm R_k)$ with $"+" \rightarrow$ downward and $"-" \rightarrow$ upward then $T_k = T_k^+ T_k^- = (1 - R_k^2)$.

$$A_n = A_0 R_n \prod_{m=n-1}^{1} (1 - R_m^2)$$

From part (a), we have $R_1^2 = R_2^2 = R_3^2 = R_4^2 = ... = R_{29}^2$. So,

$A_{30} = A_0 R_{30} (1 - R_{29}^2)(1 - R_{28}^2)\ldots(1 - R_2^2)(1 - R_1^2) = -R_1 (1 - R_1^2)^{29} A_0$

With $R_1 = 0.375$ and $(1 - R_1^2)^{29} = 0.01234$ it may be determined that

$A_{30} = -0.004627 A_0$.

(c). Geometrical spreading:

$$L_{30} = \frac{2}{\alpha_1} (\alpha_1 h_1 + \alpha_2 h_2 + \ldots + \alpha_{30} h_{30}) \quad (h_j = h,\ j = 1, 2, \ldots 29, 30.)$$

$$L_{30} = \frac{2h}{\alpha_1} (\alpha_1 + \alpha_2 + \alpha_3 \ldots + \alpha_{29} + \alpha_{30})$$

$$= \frac{2h}{\alpha_1} ([\alpha_1 + \alpha_3 + \ldots + \alpha_{27} + \alpha_{29}] + [\alpha_2 + \alpha_4 \ldots + \alpha_{28} + \alpha_{30}])$$

$$L_{30} = \frac{2h}{\alpha_1} (15\alpha_1 + 15\alpha_2) = \frac{30h}{\alpha_1} (\alpha_1 + \alpha_2) = 30 (1 + \alpha_2 / \alpha_1) h = 90h = 90 \times 20 = 1800$$

Thus,

$$A_{30} = (-0.004627 A_0) \times \left(\frac{r_0}{L_{30}} \right) = -\frac{0.004627}{90h} (r_0 A_0) = -5.1415 \times 10^{-5} \left(\frac{r_0 A_0}{h} \right) = -2.571 \times 10^{-6} r_0 A_0 \ .$$

(d). The layers do not have to be alternating. As long as $R_j = R_1$ for all $j = 1, 2, \ldots, 30$ the same answer is obtained,as in ***(b).***, for reflection and transmission losses because $(-R_1)^2 = R_1^2$. So the $\alpha_3, \alpha_4, \ldots$ could be different. However, L_{30} would not be the same, as the α_j would have to be chosen so that $\sum_{j=1}^{30} \alpha_j = 15(\alpha_1 + \alpha_2)$.

28. At the source, $A(\omega) = e^{-\bar{c}^2(\omega-\omega_0)^2}$, $\omega > 0, \bar{c} = c/\omega_0$ (frequency shifted Gaussian). $\left[A(0) = e^{-\bar{c}^2\omega_0^2}\right]$

With the dominant frequency of $f_d = f_0 = 50Hz$ and $c = \pi$, the following results:

$$\omega_0 = 2\pi f_0 = 2\pi \times 50Hz = 314.159Hz,\ \bar{c} = 0.01s \text{ and } A(0) = 5.17\times10^{-5}.$$

As plane wave propagation is involved, there is no geometrical spreading losses $(L=1)$, only reflection, transmission and Q losses. For a different dominant frequencies only the quantities involving Q losses, $e^{-a\omega}$, need to be recomputed,

$$\left[e^{-a\omega}\right] \text{ where } a = \frac{h}{\alpha_1 Q_1} + \frac{h}{\alpha_1 Q_1} = \frac{1}{2(20)} + \frac{1}{3(30)} = 0.0361111s.$$

The amplitude specification at the receiver may be written as

$$A_r(\omega) = A(\omega)e^{-a\omega} \times R_2\left(1-R_1^2\right) = YA(\omega)e^{-a\omega} = Ye^{-\bar{c}^2(\omega-\omega_0)^2-a\omega}.$$

A new dominant frequency, ω_{dr}, is determined such that $\left[dA_r(\omega)/d\omega\right]_{\omega=\omega_{dr}} = 0.$

$$\frac{dA_r(\omega)}{d\omega} = Ye^{-\bar{c}^2(\omega-\omega_0)^2-a\omega}\left[-2\bar{c}^2(\omega-\omega_0)-a\right] = 0 \quad \rightarrow \quad -2\bar{c}^2(\omega-\omega_0) = a$$

$$\text{or } \omega_{dr} = \omega_0 - \frac{a}{2\bar{c}^2}.$$

Finally,

$$f_{dr} = f_0 - \frac{a}{4\pi\bar{c}^2} = f_0 - \frac{af_0^2}{\pi} = (50-28.736)Hz = 21.26Hz.$$

29. $\rho = 2.2\,g/cm^3 = 2200\,kg/m^2$, $\alpha_0 = 2000\,m/s$, $\beta_0 = 1000\,m/s$,
$\varepsilon = 0.33$, $\gamma = \delta = 0.3$.

$$\alpha_0 = \sqrt{c_{33}/\rho} \rightarrow \quad c_{33} = \rho\alpha_0^2 = (2000)(2200)^2\frac{kg}{m^3}\frac{m^2}{s^2}$$

so that

$$c_{33} = 8.8\times10^9\frac{kg}{ms^2} \quad (Pa)$$

Continuing,

$$\varepsilon = \frac{c_{11}-c_{33}}{2c_{33}} \quad \rightarrow \quad c_{11} = c_{33}(1+2\varepsilon) = \left(8.8\times10^9\right)(1.66)\,Pa$$

$$c_{11} = 14.608\times10^9\,Pa$$

$$\alpha_{90} = v_{qP}\left(90^\circ\right) = \sqrt{\frac{c_{11}}{\rho}} = \sqrt{\frac{14.608\times10^9}{2200}}\,m/s = 2576.8\,m/s.$$

30. *qSH* waves in *VTI* Media: $\beta_{v1} = 1\,km/s$, $\beta_{h1} = 1.3\,km/s$, $\beta_{v2} = 2\,km/s$, $\beta_{h2} = 2.2\,km/s$, $\theta_1 = 20^o$.

Snell's Law: $p = \dfrac{\sin\theta_1}{\beta_1(\theta_1)} = \dfrac{\sin\theta_2}{\beta_2(\theta_2)}$. p is known.

$\beta_2^2 = \beta_{h2}^2 \sin^2\theta_2 + \beta_{v2}^2 \cos^2\theta_2 \quad (\text{phase speed})$

$\sin^2\theta_2 = p^2\beta_2^2 \quad \rightarrow \quad \sin^2\theta_2 = \left[\beta_{h2}^2 \sin^2\theta_2 + \beta_{v2}^2\left(1-\sin^2\theta_2\right)\right]p^2$

$\sin^2\theta_2 = \beta_{h2}^2 p^2 \sin^2\theta_2 + \beta_{v2}^2 p^2 - \beta_{v2}^2 p^2 \sin^2\theta_2$

$\sin^2\theta_2\left[1-\beta_{h2}^2 p^2 + \beta_{v2}^2 p^2\right] = \beta_{v2}^2 p^2$

$\sin^2\theta_2 = \beta_{v2}^2 p^2 \Big/ \left[1-\beta_{h2}^2 p^2 + \beta_{v2}^2 p^2\right] = \beta_{v2}^2 \Big/ \left[p^{-2} - \beta_{h2}^2 + \beta_{v2}^2\right]$

For $\theta_1 = 20^o$, $\beta_1^2(\theta_1) = \beta_{h1}^2 \sin^2\theta_1 + \beta_{v1}^2\cos^2\theta_1 = 1.03957\,km^2/s^2$,

so that $p = 0.32900\,s/km \quad \rightarrow \quad \sin^2\theta_2 = 0.47627 \quad \rightarrow \quad \theta_2 = 43.640^o$.

For isotropic media,

$$\sin\theta_2 = \frac{\beta_2}{\beta_1}\sin\theta_1 = \frac{2}{1}\sin 20^o \quad \rightarrow \quad \theta_2 = 43.160.$$

Alternate Method: from Snell's Law,

$$p^2 = \frac{\sin^2\theta_1}{\beta_{h1}^2\sin^2\theta_1 + \beta_{v1}^2\cos^2\theta_1} = \frac{\sin^2\theta_2}{\beta_{h2}^2\sin^2\theta_2 + \beta_{v2}^2\cos^2\theta_2}$$

$$\frac{1}{\beta_{h1}^2 + \beta_{v1}^2\mathrm{ctn}^2\theta_1} = \frac{1}{\beta_{h2}^2 + \beta_{v2}^2\mathrm{ctn}^2\theta_2}$$

$$\mathrm{ctn}^2\theta_2 = \left[\beta_{h1}^2 - \beta_{h2}^2 + \beta_{v1}^2\mathrm{ctn}^2\theta_1\right]\Big/\beta_{v2}^2 \quad \rightarrow \quad \theta_2 = 43.640^o \ \text{ as above.}$$